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ADAPTIVE TEST STATISTICS AND BAHADUR EFFICIENCY

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Abstract: A hypothesis testing problem is considered where the distribution of a random sample is specified only up to a nuisance parameter. Under the notion of asymptotic efficiency according to Bahadur, an adaptive test statistic is asymptotically efficient regardless of the value of the nuisance parameter. An adaptation condition is derived and the form of adaptive test statistics is discussed.

Key words and phrases: Bahadur efficiency, adaptive test statistics, stochastic comparison of tests, attained level, exact slope, linear inequalities, likelihood ratio test.

1. Introduction

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample drawn from one of two different probability distributions, P or Q, so that the hypothesis P versus the alternative Q is to be tested. We start here with simple hypotheses for conceptual simplicity.

If $T_n = T_n(\mathbf{x})$ is a test statistic which rejects P for large values and has the null distribution function $F_n(t) = P(T_n < t)$, then the level attained by T_n is

$$L(T_n) = L^P(T_n) = 1 - F_n(T_n(\mathbf{x})).$$

It is known (cf. Bahadur (1971, Theorem 7.5)) that with Q-probability one

$$\liminf_{n\to\infty} n^{-1} \log L(T_n) \geq -K(Q,P), \tag{1.1}$$

where $K(Q,P) = E^Q \log dQ/dP$ is the information number. The limit in the left-hand side of (1.1) is proportional to the so-called exact slope of T_n.

Equality in (1.1) is attained by the most powerful (likelihood ratio) test, and a test statistic is defined to be asymptotically Bahadur efficient if it attains equality in (1.1).

Suppose now that the different distributions P and Q are not known exactly but only up to a nuisance parameter α, so that for any fixed (but unknown) value of α one has to test the hypothesis P_α against Q_α.

A test statistic T_n is called adaptive if for every α

$$\lim_{n\to\infty} \inf n^{-1} \log L_\alpha(T_n) = -K(Q_\alpha, P_\alpha)$$

with Q_α-probability one. Here $L_\alpha(T_n) = L_\alpha^P(T_n)$ is the observed level of T_n calculated under P_α.

In other words an adaptive test statistic is asymptotically Bahadur efficient for every value of the nuisance parameter and is independent of this value. This problem can be also viewed as that of hypothesis testing of the family P_α versus the family Q_α when a Bahadur efficient test is desired for specified pairs (P_α, Q_α).

A notion of an adaptive test in the case of an infinite dimensional nuisance parameter has been introduced by Stein (1956). A related definition has been given by Rukhin (1982), and a necessary and sufficient condition for the existence of an adaptive test when asymptotic efficiency is defined by exponential rate of probability of type I error was obtained in Rukhin (1986) (see Mak and Rukhin (1991) for a more general adaptation definition).

It follows from the results of this paper that the existence of an adaptive test statistic is implied by the existence of an adaptive test in the sense of Rukhin (1982).

The present notion of an adaptive test statistic is based on stochastic comparison of tests which has been an important tool in asymptotic testing theory for the last thirty years. The value of the constant C_n, which determines the critical region $\{\mathbf{x} : T_n(\mathbf{x}) \geq C_n\}$, does not enter our definition of adaptation (as it does not enter the definition of asymptotic efficiency according to Bahadur), but this constant is involved in the definition of an adaptive test. For practical implementation of adaptive test statistics there remains the problem of specifying significance levels for different values of α.

In this paper in Sections 2 and 3 we derive sufficient conditions for the existence of an adaptive test statistic. These conditions are derived from asymptotic study of weighted likelihood ratio tests. Using some results of linear inequalities theory a simple condition for the existence of an adaptive test statistic is deduced in the case of a finite-valued nuisance parameter. In Section 4 a generalization of these results for composite hypotheses is provided.

2. Asymptotic Study of Weighted Likelihood Ratio Tests

We start with the situation where the nuisance parameter α takes only a finite number of values, say, $\alpha = 1, \dots, A$.

Let p_k, q_k denote densities of (different) distributions P_k and Q_k, $k =$

$1,\dots,A$, all of which are supposed to have common support. Also let

$$\rho_\alpha(b_1,\dots,b_A,c) = \max_{1\le k\le A} \inf_{s_1,\dots,s_A>0}\{-\sum_i s_i(b_i+c)+\log E_\alpha^P \prod_i [q_k(X)/p_i(X)]^{s_i}\}.$$

As in Bahadur (1971, Lemma 3.3) it is easy to show that, for fixed $b_1,\dots,b_A$, the function ρ_α is a continuous function of c taking values in an open interval.

For fixed real constants $b_1,\dots,b_A$ define a test statistic $\hat{T}_n$ as

$$\hat{T}_n(\mathbf{x}) = n^{-1}\log\Big\{\sum_k u_k \prod_{j=1}^n q_k(x_j)/\sum_k w_k \prod_{j=1}^n p_k(x_j)\Big\}, \tag{2.1}$$

where $w_k = \exp(nb_k)$ and u_k are fixed positive numbers.

The asymptotic behavior of (2.1) is provided by the following result.

Lemma 2.1. *With Q_α probability one*

$$\lim_{n\to\infty} n^{-1}\log L_\alpha(\hat{T}_n) = \rho_\alpha(b_1,\dots,b_A,B_\alpha),$$

where

$$B_\alpha = \min_\beta[K_{\alpha\beta} - b_\beta], \quad K_{\alpha\beta} = K(Q_\alpha, P_\beta). \tag{2.2}$$

Proof. Lemma 5.1 in the Appendix implies that

$$\begin{aligned}&\lim_{n\to\infty} n^{-1}\log P_\alpha(\hat{T}_n(\mathbf{x}) \ge t)\\ &= \lim_{n\to\infty} n^{-1}\log P_\alpha\{\max_k u_k \prod_1^n q_k(x_j) \ge e^{nt}\max_k w_k \prod_1^n p_k(x_j)\}\\ &=\rho_\alpha(b_1,\dots,b_A,t).\end{aligned}$$

Here t is any number such that (5.2) holds with $c = t$. For every α with Q_α probability one the law of large numbers implies that

$$\begin{aligned}\lim_{n\to\infty}\hat{T}_n(\mathbf{x}) &= \lim_{n\to\infty} n^{-1}\{\max_k \sum_j \log q_k(x_j) - \max_k[nb_k + \sum_j \log p_k(x_j)\}\\ &= \max_k E_\alpha^Q \log q_k(X) - \max_k[b_k + E_\alpha^Q \log p_k(X)].\end{aligned}$$

Because of the known properties of information numbers one obtains

$$\lim_{n\to\infty}\hat{T}_n(\mathbf{x}) = \min_k\{E_\alpha^Q \log\frac{q_\alpha}{p_k}(X) - b_k\} = B_\alpha.$$

The conclusion of Lemma 2.1 now follows from Theorem 7.2 of Bahadur (1971), which, for completeness sake, is also given in the Appendix. Indeed, we have only to verify that with $c = B_\alpha$ for any probabilities v_i

$$P_\alpha\Big(\sum v_i\Big[\log\frac{q_k}{p_i}(X) - b_i\Big] > c\Big) > 0, \quad k = 1,\dots,A.$$

Since all distributions P_α and Q_α are mutually absolutely continuous it suffices to show that

$$Q_\alpha\Big(\sum v_i\Big[\log\frac{q_\alpha}{p_i}(X) - b_i\Big] > c\Big) > 0. \tag{2.3}$$

Because of the definition of B_α one has

$$E_\alpha^Q \sum\Big[v_i\log\frac{q_\alpha}{p_i}(X) - b_i - c\Big] = \sum v_i[K_{\alpha i} - b_i - B_\alpha] \geq 0.$$

Since the support of Q_α has at least two points, (2.3) follows.

Corollary 1. *An adaptive test exists if for some $b_1,\dots,b_A$ and all $\alpha = 1,\dots,A$*

$$\min_\beta[K_{\alpha\beta} - b_\beta] = K_{\alpha\alpha} - b_\alpha.$$

Proof. The condition of Corollary 1 means that for all α

$$B_\alpha = K_{\alpha\alpha} - b_\alpha,$$

so that

$$\rho_\alpha(b_1,\dots,b_A,B_\alpha) \leq \max_k \inf_{s\geq 0}\{-s(b_\alpha + B_\alpha) + \log E_\alpha^P[q_k(X)/p_\alpha(X)]^s\}$$
$$\leq -b_\alpha - B_\alpha = -K_{\alpha\alpha}.$$

Since always

$$\lim_{n\to\infty}\inf n^{-1}\log L_\alpha(\hat{T}_n) \geq -K_{\alpha\alpha},$$

one concludes that because of Lemma 2.1 test statistic (2.2) is adaptive.

To obtain an analogue of Corollary 1 in a more general situation we introduce the following:

Assumption A. The set Θ of nuisance parameters is a compact topological space and the function $E_\gamma^P \log q_\alpha(X)$, γ fixed, is a continuous function of α.

Notice that Assumption A can be replaced by the condition of existence of a suitable compactification of Θ as in Bahadur (1971, Section 9).

Theorem 2.1. *Under Assumption A an adaptive test statistic exists if there exists a continuous function b on Θ such that*

$$\min_\beta[K_{\alpha\beta} - b(\beta)] = K_{\alpha\alpha} - b(\alpha). \tag{2.4}$$

Proof. Let

$$\hat{T}_n(\mathbf{x}) = n^{-1}\log\big[\max_\alpha \prod q_\alpha(x_j)/\max_\beta e^{nb(\beta)}\prod p_\alpha(x_j)\big].$$

As in the proof of Lemma 2.1 we see that with probability one

$$\lim_{n\to\infty}\hat{T}_n(\mathbf{x}) = \min_\beta[K_{\alpha\beta} - b(\beta)] = B_\alpha.$$

For any positive ϵ and any fixed γ one can find points $\alpha_1,\ldots,\alpha_A$ of Θ such that

$$\max_\alpha E^P_\gamma \log q_\alpha(X) \le \max_{\alpha_1,\ldots,\alpha_A} E^P_\gamma \log q_\alpha(X) + \epsilon/2.$$

It follows from our assumptions for all sufficiently large n

$$\max_\alpha n^{-1}\sum_1^n \log q_\alpha(x_j) \le \max_{\alpha_1,\ldots,\alpha_A} n^{-1}\sum_1^n \log q_\alpha(x_j) + \epsilon$$

and

$$\begin{aligned}
&P_\gamma(\hat{T}_n(\mathbf{x}) \ge t)\\
&= P_\gamma\Big(\max_\alpha \sum_1^n \log q_\alpha(x_j) \ge nt + \max_\beta[nb(\beta) + \sum_1^n \log p_\beta(x_j)]\Big)\\
&\le P_\gamma\Big(\max_{\alpha_1,\ldots,\alpha_A} \sum_1^n \log q_\alpha(x_j) \ge n(t-\epsilon) + \max_{\beta_1,\ldots,\beta_A}[nb_i + \sum_1^n \log p_\beta(x_j)]\Big),
\end{aligned}$$

where $\beta_1,\ldots,\beta_A$ are arbitrary elements of Θ and $b_i = b(\beta_i)$.

Now Lemma 5.1 in the Appendix implies that

$$\limsup_{n\to\infty} n^{-1}\log P_\gamma(\hat{T}_n(\mathbf{x}) \ge t) \le \rho_\gamma(b_1,\ldots,b_A,t-\epsilon)$$

so that

$$\limsup_{n\to\infty} n^{-1}\log L_\alpha(\hat{T}_n) \le \rho_\alpha(b_1,\ldots,b_A,B_\alpha-\epsilon).$$

Because of continuity of the function ρ_α

$$\begin{aligned}
\limsup_{n\to\infty} n^{-1}\log L_\alpha(\hat{T}_n) &\le \max_k \inf_{s\ge 0}[-s(b_\alpha + B_\alpha) + \log E^P_\alpha[\frac{q_k}{p_\alpha}(X)]^s]\\
&\le -b_\alpha - B_\alpha = -K_{\alpha\alpha}
\end{aligned}$$

and, as in Corollary 1, we conclude that $\hat{T}_n$ is adaptive.

3. Linear Inequalities and Adaptive Test Statistics

We prove here the following result for the finite case.

Theorem 3.1. *Let $\Theta = \{1, \dots, A\}$ be a finite parameter set. An adaptive test statistic exists if for any permutation σ of $\{1, \dots, A\}$*

$$\sum_i K_{i\sigma(i)} \geq \sum_i K_{ii}. \tag{3.1}$$

Proof. To prove Theorem 3.1 we use Corollary 1, whose condition can be written in the form of simultaneous linear inequalities

$$b_i - b_k \geq K_{ii} - K_{ik}. \tag{3.2}$$

According to Theorem 1 of Ky Fan (1956, p.100), the system (3.2) has a solution if (and only if) for any matrix C with non-negative elements c_{ik}, the identity

$$\sum_{i,k} c_{ik}(b_i - b_k) = 0, \tag{3.3}$$

which holds for all $b_1, \dots, b_A$, implies that

$$\sum_{i,k} c_{ik}(K_{ii} - K_{ik}) \leq 0. \tag{3.4}$$

Clearly (3.3) means that for any k

$$\sum_i c_{ik} = \sum_i c_{ki}. \tag{3.5}$$

It is obvious that if (3.4) and (3.5) are satisfied for a matrix C then these conditions are also satisfied for a matrix with zero diagonal and all other elements coinciding with those of C. Therefore one can assume that $c_{ii} = 0$. The set of all matrices C under condition (3.5) and, such that $c_{ik} \leq 1$, $c_{ii} = 0$, is a convex compact subset of the space of all $A \times A$ matrices. The set of its extreme points is formed by matrices of the form, $E_{r\sigma} = \{\delta_{i\sigma(i)}\}$ where σ is a permutation of an r-element subset of $\{1, \dots, A\}$, such that for all i, $\sigma(i) \neq i$ and δ_{ik} is Kronecker symbol.

By the Krein-Milman Theorem (see for example Rudin (1991)) any matrix C under condition (3.3) and with zero diagonal can be represented as a convex combination of the matrices $E_{r\sigma}$ above.

It follows that (3.4) is equivalent to (3.1), which proves Theorem 3.1.

A heuristic interpretation of condition (3.1) is that summarily the hypotheses testing problems P_α versus Q_α are at least as difficult as problems of testing P_α against Q_β for $\alpha \neq \beta$.

Theorem 3.2. *Under Assumption A an adaptive test statistic exists if*

$$K_{\alpha\alpha} = \min_{\beta} K_{\alpha\beta}. \tag{3.6}$$

Proof. Put

$$b(\alpha) = K_{\alpha\alpha}$$

and observe that condition (2.4) is satisfied because of (3.6).

Corollary 2. *If $P_\alpha = P$ for all α, then an adaptive test statistic exists.*

Proof. Indeed, in this case $K_{\alpha\alpha} = K_{\alpha\beta}$ for all α, β, so that (3.6) is met.

Formula (3.6) coincides with the condition for the existence of an adaptive test in Rukhin (1986) for asymptotic efficiency defined by the exponential rate of Type II error with guaranteed significance level. It implies (3.1) and one can take $b_i = K_{ii}$ in (3.2). However an adaptive test statistic can exist when there is no adaptive test. The choice of the function b in Theorem 3.2 corresponds to the traditional maximum likelihood ratio test statistic.

We also note that an argument similar to the one used to prove Theorem 3.1 has been used by Kendall (1960), who extended the representation of double stochastic matrices by means of permutation matrices to the infinite-dimensional case.

To conclude this Section let us consider the following example studied earlier by Cox (1962).

Let P_β be the Poisson distribution with parameter β, and let Q_α be the geometric distribution with probabilities

$$Q_\alpha(x) = \frac{\alpha^x}{(1+\alpha)^{x+1}};\ x = 0, 1, \dots,$$

so that its mean is equal to α.

One has

$$K_{\alpha\beta} = \alpha \log(\alpha/\beta) - (\alpha+1)\log(\alpha+1) + E_\alpha^Q \log(X!).$$

It is easy to check that (3.6) is satisfied and the likelihood ratio test for testing the Poisson distribution against the geometric distribution is fully Bahadur efficient as a test of P_α versus Q_α for any α. It can be shown that under any other parametrization of the family Q_α there is no adaptive test statistic.

4. Adaptation for Composite Hypotheses

In the case of a composite hypothesis $\mathcal{P}$ versus $\mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are disjoint, the level attained by a test statistic T_n, whose large values are significant, is defined as

$$L(T_n) = 1 - F_n(T_n(\mathbf{x}))$$

where $F_n(t) = \inf\{P(T_n(\mathbf{x}) < t), P \in \mathcal{P}\}$.

It is shown in Bahadur (1971) that for any test statistic T_n with Q probability one

$$\liminf n^{-1} \log L(T_n) \geq -K(Q, \mathcal{P}) \tag{4.1}$$

where $K(Q, \mathcal{P}) = \inf\{K(Q, P), P \in \mathcal{P}\}$.

Under some regularity conditions equality in (4.1) is attained by the likelihood ratio test statistic of $\mathcal{P}$ versus $\mathcal{Q}$.

Assume now that the hypotheses are determined only up to a finite-valued nuisance parameter α. In other terms one has to test $\mathcal{P}_\alpha$ against $\mathcal{Q}_\alpha$ with some unknown α. Define L_α as above with P replaced by P_α. A test statistic T_n is called adaptive if for any α with Q_α probability one

$$\lim_{n\to\infty} \inf n^{-1} \log L_\alpha(T_n) = -K(Q_\alpha, \mathcal{P}_\alpha).$$

Let

$$\begin{aligned} \lambda_n(\mathcal{P}|\mathbf{x}) &= \sup\{\sqcap_1^n p(\mathbf{x}_j) : p \in \mathcal{P}\}, \\ \hat{T}_n(\mathbf{x}) &= n^{-1} \log\{\max_\alpha \lambda_n(\mathcal{P}_\alpha|\mathbf{x})\}, \end{aligned} \tag{4.2}$$

where $w_\alpha = \exp\{nb(\alpha)\}$. Also suppose that the following quantities are finite

$$\begin{aligned} K_{\alpha\alpha} &= \sup\{K(Q_\alpha, \mathcal{P}_\alpha), Q_\alpha \in \mathcal{Q}_\alpha\}, \\ K_{\alpha\beta} &= \inf\{K(Q_\alpha, \mathcal{P}_\beta), Q_\alpha \in \mathcal{Q}_\alpha\}, \quad \alpha \neq \beta. \end{aligned}$$

In the following theorem we assume that for all α, β the likelihood ratio test statistic for testing the hypotheses $\mathcal{P}_\alpha$ versus $\mathcal{Q}_\beta$ satisfies conditions 1 and 2 from Theorem 5.2 in the Appendix and that the assumptions of Theorem 2.1 are met.

Theorem 4.1. *Under the assumptions above, an adaptive test statistic exists if*

$$K_{\alpha\alpha} = \min_\beta K_{\alpha\beta}.$$

If $\Theta = \{1, \ldots, A\}$ *is a finite set, then an adaptive test statistic exists if*

$$\sum_\alpha [K_{\alpha\sigma(\alpha)} - K_{\alpha\alpha}] \geq 0 \tag{4.3}$$

for any permutation σ *of* $\{1, \ldots, A\}$.

Proof. We consider here only the case of finite Θ. By Theorem 1 in Ky Fan (1956), (4.3) implies the existence of real numbers $b_1, \dots, b_A$ such that

$$K_{\alpha\beta} - K_{\alpha\alpha} \geq b_\beta - b_\alpha. \tag{4.4}$$

We now show that the test statistic (4.2) is adaptive with this choice of b's.

Indeed, for each P_α

$$\begin{aligned}
&P_\alpha\{\hat{T}_n(\mathbf{x}) \geq t\} \\
&\leq P_\alpha\{\max_\beta n^{-1} \log[\lambda_n(\mathcal{Q}_\beta|\mathbf{x})/w_\alpha \lambda_n(\mathcal{P}_\alpha|\mathbf{x})] \geq t\} \\
&\leq \sum_\beta P_\alpha\{n^{-1} \log[\lambda_n(\mathcal{Q}_\beta|\mathbf{x})/\lambda_n(\mathcal{P}_\alpha|\mathbf{x})] \geq t + b_\alpha\} \\
&\leq A \max_\beta P_\alpha\{n^{-1} \log[\lambda_n(\mathcal{Q}_\beta|\mathbf{x})/\lambda_n(\mathcal{P}_\alpha|\mathbf{x})] \geq t + b_\alpha\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sup\{P_\alpha(\hat{T}_n(\mathbf{x}) \geq t), P_\alpha \in \mathcal{P}_\alpha\} \\
&\leq A \max_\beta \sup\{P_\alpha(n^{-1} \log[\lambda_n(\mathcal{Q}_\beta|\mathbf{x})/\lambda_n(\mathcal{P}_\alpha|\mathbf{x})] \geq t + b_\alpha), P_\alpha \in \mathcal{P}_\alpha\} \\
&= A \max_\beta [1 - G_\alpha^\beta(t + b_\alpha)],
\end{aligned}$$

where

$$G_\alpha^\beta(t) = \sup\{P_\alpha(n^{-1} \log[\lambda_n(\mathcal{Q}_\beta|\mathbf{x})/\lambda_n(\mathcal{P}_\alpha|\mathbf{x})] \geq t), P_\alpha \in \mathcal{P}_\alpha\}.$$

For fixed positive ϵ and τ, $0 < \tau < 1$, Condition 2 of Appendix implies that for some positive constant C, independent of t,

$$1 - G_\alpha^\beta(t) \leq C[(1+\epsilon)\exp\{-\tau t\}]^n.$$

It follows that

$$\liminf_{n\to\infty} n^{-1} \log \sup\{P_\alpha(\hat{T}_n(\mathbf{x}) \geq t), P_\alpha \in \mathcal{P}_\alpha\} \leq -\tau(t + b_\alpha) + \log(1+\epsilon).$$

Letting $\epsilon \to 0$, $\tau \to 1$, one obtains

$$\liminf_{n\to\infty} n^{-1} \log \sup P_\alpha(\hat{T}_n(\mathbf{x}) \geq t) \leq -t - b_\alpha.$$

Also for any α

$$\begin{aligned}
\hat{T}_n(\mathbf{x}) &\geq n^{-1} \log\{\lambda_n(\mathcal{Q}_\alpha|\mathbf{x})/ \max_\beta [w_\beta \lambda_n(\mathcal{P}_\beta|\mathbf{x})]\} \\
&\geq \min_\beta [n^{-1} \log(\lambda_n(\mathcal{Q}_\alpha|\mathbf{x})/\lambda_n(\mathcal{P}_\beta|\mathbf{x})) - b_\beta].
\end{aligned} \tag{4.5}$$

By Theorem 5.2 of Appendix with Q_α probability one as $n \to \infty$

$$n^{-1} \log(\lambda_n(\mathcal{Q}_\alpha|\mathbf{x})/\lambda_n(\mathcal{P}_\beta|\mathbf{x})) \to K(Q_\alpha, \mathcal{P}_\beta)$$

and, therefore, (4.5) implies

$$\lim_{n\to\infty} \inf \hat{T}_n(\mathbf{x}) \geq \min_\beta [K(Q_\alpha, \mathcal{P}_\beta) - b_\beta].$$

It follows that

$$\lim_{n\to\infty} \inf n^{-1} \log L_\alpha(\hat{T}_n) \leq - \min_\beta [K(Q_\alpha, \mathcal{P}_\beta) - b_\beta] - b_\alpha.$$

If

$$\min_\beta [K(Q_\alpha, \mathcal{P}_\beta) - b_\beta] = K(Q_\alpha, \mathcal{P}_\alpha) - b_\alpha,$$

then

$$\lim_{n\to\infty} \inf n^{-1} \log L_\alpha(\hat{T}_n) \leq -K(Q_\alpha, \mathcal{P}_\alpha).$$

Therefore with Q_α probability one

$$\lim_{n\to\infty} \inf n^{-1} \log L_\alpha(\hat{T}_n) \leq -K_{\alpha\alpha} < -K(Q_\alpha, \mathcal{P}_\alpha),$$

so that $\hat{T}_n$ is indeed an adaptive test statistic.

We notice that the main assumption of Theorem 4.1 concerns the asymptotic Bahadur efficiency of the likelihood ratio statistic for each testing problem $\mathcal{P}_\alpha$ versus $\mathcal{Q}_\alpha$. Validity of this assumption has been studied by Hsieh (1979) and Kourouklis (1988). Kallenberg (1978) and Kourouklis (1984) investigated the role of this condition for exponential families.

Appendix

In this appendix we collect the results needed in the proofs of Theorems 2.1 and 4.1. We start with the following theorems due to Bahadur (see Bahadur (1971)).

Theorem A.1. *Assume that for any α with Q_α probability one as $n \to \infty$*

$$T(\mathbf{x}) \to B_\alpha$$

and the limit

$$f(t) = - \lim_{n\to\infty} n^{-1} \log P_\beta(T(\mathbf{x}) \geq t)$$

exists for t taking values in an open interval, which contains the set $\{B_\alpha, \alpha \in \Theta\}$, and is a continuous function there. Then under distribution Q_α

$$\lim_{n\to\infty} n^{-1} \log L_\alpha(T_n) = f(B_\alpha).$$

The next result is due to Bahadur and Raghavachari (1972, p.139).
Introduce the following two conditions:
Condition 1. For any α with Q_α probability one

$$\liminf_{n\to\infty} T_n(\mathbf{x}) \geq \min_\beta K_{\alpha\beta}. \tag{A.1}$$

Condition 2. For any n and any positive numbers ϵ and τ there exists a positive constant $k_n = k_n(\epsilon, \tau)$ such that

$$P(T_n(\mathbf{x}) \geq t) \leq \exp(-n\tau t)(1+\epsilon)^n k_n$$

for all positive t and such that as $n \to \infty$

$$n^{-1} \log k_n \to 0.$$

Theorem A.2. *If statistic T_n satisfies conditions* 1 *and* 2 *then* (A.1) *is an equality, i.e. T_n is Bahadur efficient.*

Now we state a proposition whose proof is similar to the one of Lemma in Rukhin (1982).

Lemma A.1. *Let c_n, $n = 1, 2, \ldots$ be a sequence of positive numbers such that $n^{-1} \log c_n$ converges to a finite limit c. Assume that u_i are positive probabilities, p_i and q_i are positive measurable functions, $i = 1, \ldots, A$ and for real numbers $b_1, \ldots, b_A$*

$$w_i = \exp(nb_i).$$

Also suppose that for all positive probabilities v_i

$$\Pr\Big\{ \sum_i v_i \big[\log(p_k(X)/q_i(X)) - b_i\big] > c \Big\} > 0, \quad k = 1, \ldots, A. \tag{A.2}$$

If $x_1, x_2, \ldots$ is a sequence of i.i.d. random variables, then

$$\begin{aligned}
&\lim_{n\to\infty} n^{-1} \log \Pr\Big\{ \sum_k u_k \prod_1^n q_k(x_j) \geq c_n \sum_k w_k \prod_1^n p_k(x_j) \Big\} \\
&= \lim_{n\to\infty} n^{-1} \log \Pr\Big\{ \max_k u_k \prod_1^n q_k(x_j) \geq c_n \max_k w_k \prod_1^n p_k(x_j) \Big\} \\
&= \max_{1\leq k\leq A} \inf_{s_1,\ldots,s_A \geq 0} \Big\{ -\sum_i s_i(b_i + c) + \log E \prod_i [q_k(X)/p_i(X)]^{s_i} \Big\}.
\end{aligned}$$

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