Statistica Sinica **9**(1999), 229-246

NONPARAMETRIC ESTIMATION OF THE SURVIVAL FUNCTION BASED ON CENSORED DATA WITH ADDITIONAL OBSERVATIONS FROM THE RESIDUAL LIFE DISTRIBUTION

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Abstract: We derive the nonparametric maximum likelihood estimator (NPMLE) of the distribution of the test items using a random, right-censored sample combined with an additional right-censored, residual-lifetime sample in which only lifetimes past a known, fixed time are collected. This framework is suited for samples for which individual test data are combined with left-truncated and randomly censored data from an operating environment. The NPMLE of the survival function using the combined sample is identical to the Kaplan-Meier product-limit estimator only up to the time at which the test items corresponding to the residual sample were known to survive. The limiting distribution for the NPMLE, discussed in detail, leads to confidence bounds for the survival function. For the uncensored case, we study the relative efficiency for the estimator based on the combined sample with respect to the analogous estimator based only on the simple random sample.

Key words and phrases: Counting processes, Nelson-Aalen estimator, relative efficiency, reliability, truncation.

1. Introduction

In various industrial settings it is not always possible to observe the uncensored lifetime of a test item, and this leads to analytical difficulties if inference for the entire lifetime of the system is pursued. The development of highly reliable test items, combined with limited product-testing budgets, have made random right-censoring increasingly common in many applied lifetesting experiments. Additionally, some settings dictate that only residual lifetimes are available for observation. For instance, a consumer might observe only a product's residual lifetime. If the item has a limited burn-in period, the manufacturer can choose to test the product for a limited time before making it available to the consumer, thus minimizing the item's perceptible failure rate.

In survival analysis of medical data, we might observe the recorded lifetimes of patients in preliminary stages of a terminal disease, often right censored due to the patients' recovery or their removal from the study. The data might include remaining lifetimes of patients who are known to survive time t_0 beyond the initial stage of the disease, thus the study fails to include some patients who were known to have died of the disease before time t_0. These latter observations constitute a residual-lifetime sample. Analysis of AIDs survival data involving residual lifetimes with right-censoring is discussed in Gross and Lai (1996), for example.

Naturally the underlying distribution of the test items is not necessarily identifiable if only (left-truncated) residual lifetime data is observed. To draw inference about the distribution of the test items extra sample information is required, such as an ordinary sample of independent and identically distributed (i.i.d.) lifetimes. If the censoring mechanism is independent of the lifetime distribution and has the same positive support, identifiability of the underlying distribution can be preserved.

Inference based on combining information from such related samples appears in many parts of the statistical literature. Vardi (1982) obtained the nonparametric maximum likelihood estimator (NPMLE) for a distribution function using a complete random sample supplemented with additional observations from a length biased sample. The length-biased sample distribution G is a functional of the original lifetime distribution F given by $G(t) = \frac{1}{\mu}\int_0^t x dF(x)$, $t > 0$, where $0 < \mu < \infty$ is the mean for $F(x)$. In such nonparametric estimation problems, we implicitly adopt the generalized maximum likelihood approach introduced by Kiefer and Wolfowitz (1956). Vardi (1985) later extended his previous results to general uncensored biased samples; the biased sample contains lifetimes with distribution $G(t) = \int_0^t w(x)dF(x)/\int_0^\infty w(x)dF(x)$, where $w(x)$ is a known, nonnegative bias function. Large sample theory for biased samples is explored by Gill, Vardi and Wellner (1988).

A sample of residual lifetimes in which test items have survived past a positive, known time t_0 can be written as a biased sample with $w(x) = I(x > t_0)$, where $I(A)$ is the indicator function of an event A. The residual lifetime survival function and the corresponding expected lifelength are

$$\bar{G}(t) = \bar{F}(t + t_0)/\bar{F}(t_0), \quad E_G(T) = \frac{1}{\bar{F}(t_0)}\int_{t_0}^{\infty} \bar{F}(u)du, \tag{1.1}$$

where for any distribution F, $\bar{F} = 1 - F$ denotes its survival function. For ease of notation in the sections to follow, residual lifetimes are characterized via a truncated distribution, which differs from the residual distribution in (1.1) by a location shift:

$$\bar{G}_0(t) = \bar{F}(t)/\bar{F}(t_0), \quad \text{for } t > t_0. \tag{1.2}$$

Barlow and Proschan (1975), Shaked and Shanthikumar (1994), among others, have emphasized the important role played by residual lifetimes in the analysis of system reliability and aging characteristics. Englehart, Williams and Bain (1993) discuss the analysis of residual reliability data using the Power Law process.

In this paper we consider the problem of estimating the underlying distribution function using a "conventional" sample of randomly right-censored lifetimes in addition to independent items generated from the residual lifetime in (1.2), which might also be right censored. The special case for which only uncensored samples are observed, which is featured by Vardi (1985), will be discussed here for purpose of illustration and to further study properties of the estimator that are not discussed in Vardi's paper. In Section 2, we derive the NPMLE of the underlying survival function $\bar{F}$ using the combined samples. The NPMLE is a product estimator that can be intuitively explained by treating the residual data as left-censored with missing values. We observe that the NPMLE of the survival function is identical to the Kaplan-Meier (1958) product-limit estimator for all values of $t < t_0$, but differs for values of $t > t_0$. In the case of no censoring, the estimator is an unbiased estimator of $\bar{F}(t)$ for all $t > 0$, with variance bounded below by the variance of the empirical distribution function (EDF) based on the conventional sample.

In Section 3, we derive the limiting distribution for the NPMLE of $\bar{F}$ using the Nelson-Aalen hazard function estimator, as first explained by Aalen (1975) in his Ph.D. thesis. The asymptotic results lead to the construction of variance estimates, confidence bands, goodness-of-fit tests, and quantile estimators for $\bar{F}$. In Section 4, we demonstrate the NPMLE for a combined sample, and investigate the estimator's efficiency (with respect to the conventional estimator) as observations from the residual life distribution are added to a simple random sample. For the reader's benefit, we have relegated proofs of theorems and corollaries to the appendix.

2. The Nonparametric Maximum Likelihood Estimator

Let $X_1^0, X_2^0, \ldots, X_m^0$ represent m independent observations from a distribution F, where $F(0) = 0$. Let $X_{m+1}^0, X_{m+2}^0, \ldots, X_{m+n}^0$ be n independent observations generated by the distribution defined in (1.2). Define $C_1, \ldots, C_m$ to be m i.i.d. random variables with distribution function H_1 having support on $[0, \infty)$, and $C_{m+1}, \ldots, C_{m+n}$ to be i.i.d. with distribution H_2 having support on (t_0, ∞). Here, H_1 and H_2 represent censoring distributions for the conventional and residual-lifetime samples, respectively. Define

$$\begin{aligned} X_i &= X_i^0 \wedge C_i \quad \text{and} \quad \delta_i = I(X_i^0 \le C_i), \quad i = 1, \ldots, m; \\ \tilde{X}_i &= X_i^0 \wedge C_i \quad \text{and} \quad \tilde{\delta}_i = I(\tilde{X}_i^0 \le C_i), \quad i = m+1, \ldots, m+n. \end{aligned} \tag{2.1}$$

We assume that F is absolutely continuous with density function $f(t)$ and hazard rate function $\alpha(t) = f(t)/\bar{F}(t)$ for all values of t for which $F(t) < 1$. The cumulative hazard function can be expressed as $\Lambda(t) = \int_0^t \frac{dF(u)}{\bar{F}(u)} = \int_0^t \alpha(u)du$. Let $Z_1 \leq \cdots \leq Z_{m+n}$ represent the ordered values of the combined $m+n$ observations in (2.1), and let $\delta^*_{(1)}, \ldots, \delta^*_{(m+n)}$ be the concomitant values of δ_i and $\tilde{\delta}_i$ associated with $Z_1, \ldots, Z_{m+n}$. For convenience, define $Z_0 = 0$. The NPMLE for F follows in Theorem 2.1.

Theorem 2.1. *Given a sample of $n+m$ random observations as at* (2.1)*, the nonparametric maximum likelihood estimator of the survival function $\bar{F}(t) = 1 - F(t)$ is given by*

$$\hat{\bar{F}}(t) = \begin{cases} \prod_{i:z_i \in [0,t]} (1 - \frac{\delta^*_{(i)}}{m-i+1}), & t \leq t_0 \\ \prod_{i=1}^{s} (1 - \frac{\delta^*_{(i)}}{m-i+1}) \prod_{i:z_{s+i} \in [t_0,t]} (1 - \frac{\delta^*_{(s+i)}}{m+n-s-i+1}), & t_0 < t < Z_{m+n} \\ 0, & Z_{m+n} < t \end{cases} \tag{2.2}$$

where $s \in \{0, 1, \ldots, m\}$ is such that $Z_s \leq t_0 < Z_{s+1}$.

For the uncensored case, $\hat{\bar{F}}(t)$ can be obtained from an intuitive argument based on left-censoring and missing data. If n observations are made on G_0 (or, equivalently, G), we may treat the sample as a left censored one, with an unknown number (say, w) of observations less than or equal to t_0 that were not observed, and therefore treated as missing values. Given $n+w$ observations from the completed residual-lifetime sample, the expected number of observations greater than t_0 is $(n+w)\bar{F}(t_0)$. Thus a natural estimator for w is $\hat{w} = nF_m(t_0)/\bar{F}_m(t_0) = ns/(m-s)$, where F_m is the EDF based on the conventional sample. The hypothetical EDF will assign mass $1/(m+n+\hat{w})$ to each of the $m+n$ actual observations $Z_1, \ldots, Z_{m+n}$. From the $\hat{w}$ left censored values, assign mass $1/(m+n+\hat{w})$ evenly (i.e., using $\hat{w}/s$ according to F_m) over the points $Z_1, \ldots, Z_s$ that are less than t_0. As a result, one estimates

$$d\hat{F}(z_i) = \begin{cases} \frac{1}{m+n+\hat{w}} + \frac{\hat{w}}{s(m+n+\hat{w})}, & i = 1, \ldots, s \\ \frac{1}{m+n+\hat{w}}, & i = s+1, \ldots, m+n. \end{cases} \tag{2.3}$$

From (2.3) it is easy to show that $d\hat{F}(z_i) = 1/m$ for $i = 1, \ldots, s$ and $d\hat{F}(z_i) = (m-s)/[m(m+n-s)]$ for $i = s+1, \ldots, m+n$, as in Theorem 2.1. The next result follows directly from Theorem 2.1 by taking $\delta_i = 1$ for $i = 1, \ldots, m+n$.

Corollary 2.2. (*Uncensored Case*) *Let X_i^0 be defined as before with $i = 1, \ldots, m+n$, and let Z_i represent the ordered values of X_i^0 so that $0 = Z_0 < Z_1 < \cdots < Z_{n+m}$. Suppose $0 \leq s \leq m$ is such that $Z_s \leq t_0 < Z_{s+1}$. The following hold.*

(a) *For $Z_j \le t < Z_{j+1}$, the NPMLE of F is given by*

$$\hat{F}(t) = \begin{cases} j/m, & j = 0, \ldots, s \\ \frac{s}{m} + \frac{(j-s)(m-s)}{m(m+n-s)}, & j = s+1, \ldots, m+n. \end{cases}$$

(b) *The NPMLE from (a) is an unbiased estimator of $F(t)$ for all $t > 0$.*

(c) $\mathrm{Var}\,[\hat{F}(t)] = F(t)\bar{F}(t)/m$ *for $t \le t_0$, and for $t > t_0$,*

$$\mathrm{Var}\,[\hat{F}(t)] = \frac{F(t_0)\bar{F}(t_0)\bar{G}_0(t)^2}{m} + \frac{G_0(t)\bar{G}_0(t)}{m^2} \sum_{k=0}^{m} \frac{(m-k)^2}{m+n-k} \binom{m}{k} \cdot F(t_0)^k [1 - F(t_0)]^{m-k}.$$

(d) $\mathrm{Var}\,[\hat{F}(t)] \le \mathrm{Var}\,[F_m(t)]$ *for $t > t_0$, where F_m is the EDF based on $X_1, \ldots, X_m$.*

(e) *Let $R(\hat{F}, F)$ be the average (integrated) variance of $\hat{F}$. Then,*

$$R(\hat{F}, F) = \frac{1}{6m} \Big\{ F(t_0)(2 - F(t_0)) + \frac{1}{m}(1 - F(t_0)) E\Big[\frac{(m-S)^2}{m+n-S} \Big] \Big\},$$

where S is Binomial $(m, F(t_0))$. Furthermore, $R(\hat{F}, F) \le R(F_m, F) = 1/6m$.

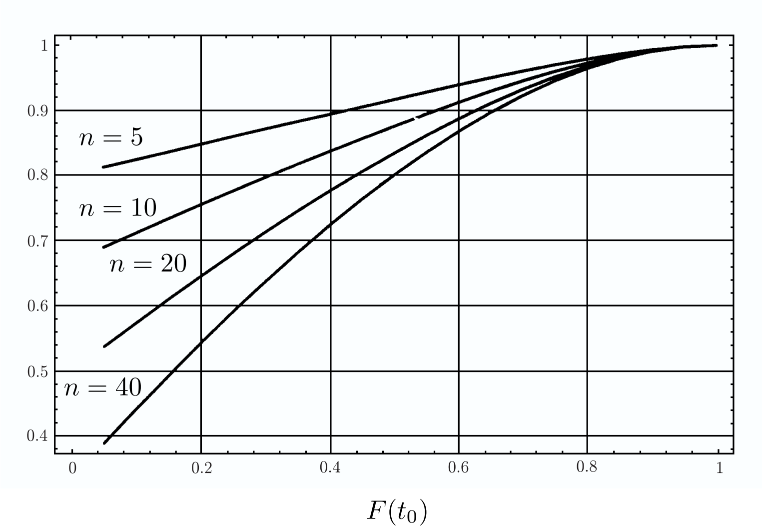


Figure 1. Relative risk of NPMLE for uncensored sample, based on $m = 20$ and various residual-lifetime sample sizes (n).

Figure 1 illustrates the potential decrease in risk (in Corollary 2.2(e)) earned by adding residual lifetime data to the conventional sample. The relative risk, $R(\hat{F}, F)/R(F_m, F)$, is plotted as a function of the population quantile $F(t_0)$. In this figure, $m = 20$ and residual lifetime samples of size $n = \{5, 10, 20, 40\}$ are considered. The curves suggest that significant reductions in risk are realized even with modest additions of residual data. For example, if $(m, n) = (20, 5)$, risk is reduced over 15% if t_0 is such that $F(t_0) \leq 0.20$.

3. Asymptotic Properties

To further examine properties of the NPMLE derived in Section 2, we reframe our discussion in terms of counting processes and product integrals. By expressing the NPMLE estimator from (2.2) in terms of its corresponding intensity process, we produce a product integral that makes limit properties more easily attainable from existing martingale theory. Several clarifying examples of this technique are discussed in Chapter IV of Andersen, Borgan, Gill and Keiding (1993).

We consider the counting process $N_1(t) = \sum_{i=1}^{m} I(X_i \leq t, \delta_i = 1)$, denoting the number of failures up to time t from the conventional sample. Let $Y_1(t) = \sum_{i=1}^{m} I(X_i \geq t)$ represent the corresponding process for the number of items "at risk" just before time t. We define similar counting processes for the residual-lifetime data: $N_2(t) = \sum_{i=m+1}^{m+n} I(t_0 \leq \tilde{X}_i \leq t, \tilde{\delta}_i = 1)$ and $Y_2(t) = \sum_{i=m+1}^{m+n} I(\tilde{X}_i \geq t)$, and denote the combined processes $N = N_1 + N_2$, $Y = Y_1 + Y_2$.

Using these processes, we can express the NPMLE of survival function in terms of the product integral

$$\hat{\bar{F}}(t) = \begin{cases} \prod_{[0,t]} (1 - \frac{dN_1}{Y_1}), & t \leq t_0 \\ \prod_{[0,t_0]} (1 - \frac{dN_1}{Y_1}) \prod_{[t_0,t]} (1 - \frac{dN}{Y}), & t_0 < t < \tau, \end{cases} \tag{3.1}$$

where $\tau = \tau_1 \wedge \tau_2$ with $\tau_1 = \sup\{u : \bar{F}(u)\bar{H}_1(u) > 0\}$ and $\tau_2 = \sup\{u : \bar{G}_0(u)\bar{H}_2(u) > 0\} = \sup\{u : \bar{F}(u)\bar{H}_2(u) > 0\}$.

For convenience, we examine the cumulative hazard function estimator corresponding to $\hat{\bar{F}}(t)$, which is analogous to the Nelson-Aalen estimator presented in Chapter IV of Andersen, Borgan, Gill and Keiding (1993):

$$\hat{\Lambda}(t) = \begin{cases} \int_0^t \frac{dN_1}{Y_1}, & t \leq t_0 \\ \int_0^{t_0} \frac{dN_1}{Y_1} + \int_{t_0}^t \frac{dN}{Y}, & t_0 < t < \tau. \end{cases} \tag{3.2}$$

Let m and n tend to infinity so that $m/(n+m) \to \rho$, $0 < \rho < 1$. Let $D[a, b]$ be the space of functions on $[a, b]$ that are right continuous with left limits, equipped with the Skorohod topology. We shall assume that conditions $(C.1) : \int_0^\tau \frac{dF(u)}{H(u^-)} < \infty$, and $(C.2) : \int_0^\tau \frac{dF(u)}{H_2(u^-)} < \infty$ hold. To further characterize the asymptotic distribution for (3.2), we need the following lemma:

Lemma 3.1. *For $t \in (t_0, \tau)$, the asymptotic covariance between $\int_0^{t_0} \frac{dN_1}{Y_1}$ and $\int_{t_0}^{t} \frac{dN}{Y}$ is zero.*

Asymptotically, the two distinct processes included in (3.2) have independent increments. Under $(C.1)$, $(C.2)$, and employing Lemma 3.1, $(m+n)^{1/2}(\hat{\Lambda} - \Lambda) \xrightarrow{d} W_1$ on $D[0, t_0]$ as m, $n \to \infty$, where W_1 is a mean zero Gaussian process with $\mathrm{Cov}\,(W_1(u), W_1(v)) = \sigma_1^2(u \wedge \nu)$, and

$$\sigma_1^2(t) = \int_0^{t\wedge t_0} \frac{\alpha(u)du}{\rho\bar{F}(u)\bar{H}_1(u)} = -\int_0^{t\wedge t_0} \frac{d\bar{F}(u)}{\rho\bar{F}(u)^2\bar{H}_1(u)}. \tag{3.3}$$

Also, $(m+n)^{1/2}(\hat{\Lambda} - \Lambda) \xrightarrow{d} W_2$ on $D(t_0, \tau]$ as m, $n \to \infty$, where W_2 is a mean Gaussian process with $\mathrm{Cov}\,(W_2(u), W_2(v)) = -\int_0^{t_0} \frac{d\bar{F}(u)}{\rho\bar{F}(u)^2\bar{H}_1(u)} + \sigma_2^2(u \wedge \nu)$, and

$$\begin{aligned}\sigma_2^2(t) &= \int_{t_0}^{t} \frac{\alpha(u)du}{\rho\bar{F}(u)\bar{H}_1(u) + (1-\rho)\bar{G}_0(u)\bar{H}_2(u)} \\ &= -\int_{t_0}^{t} \frac{d\bar{F}(u)}{\rho\bar{F}(u)^2\bar{H}_1(u) + (1-\rho)\bar{F}(u)\bar{G}_0(u)\bar{H}_2(u)} \\ &= -\int_{t_0}^{t} \frac{d\bar{G}_0(u)}{\rho\bar{F}(u)\bar{G}_0(u)\bar{H}_1(u) + (1-\rho)\bar{G}_0(u)^2\bar{H}_2(u)}. \end{aligned} \tag{3.4}$$

From (3.2) - (3.4), using the relationship between F and Λ (see IV.3.1 of Andersen, Borgan, Gill and Keiding (1993)), the following theorem and corollary follow.

Theorem 3.2. $(m+n)^{1/2}(\hat{\bar{F}}(t) - \bar{F}(t)) \xrightarrow{d} \begin{cases} -\bar{F}(t)W_1(t) & t \le t_0 \\ -\bar{F}(t)W_2(t) & t_0 < t < \tau \end{cases}$ *as $m, n \to \infty$ with the covariance kernel given by*

$$\mathrm{Cov}\,(-\bar{F}(u)W_1(u), -\bar{F}(\nu)W_1(\nu)) = \bar{F}(u)\bar{F}(\nu)\sigma_1^2(u \wedge \nu), \text{ and}$$
$$\mathrm{Cov}\,(-\bar{F}(u)W_2(u), -\bar{F}(\nu)W_2(\nu)) = \bar{F}(u)\bar{F}(\nu)\Big[-\int_0^{t_0} \frac{d\bar{F}(u)}{\rho\bar{F}^2(u)\bar{H}_1(u)} + \sigma_2^2(u \wedge \nu)\Big].$$

Corollary 3.3. *For fixed $t \le \tau$, as $m, n \to \infty$,*

$$(m+n)^{1/2}(\hat{\bar{F}}(t) - \bar{F}(t)) \xrightarrow{d} \begin{cases} N\Big(0, \bar{F}(t)^2\sigma_1(t)\Big), & t \le t_0 \\ N\left(0, \bar{F}(t)^2\left[\sigma_2^2(t) - \int_0^{t_0} \frac{d\bar{F}(u)}{\rho\bar{F}(u)^2\bar{H}_1(u)}\right]\right), & t_0 < t < \tau. \end{cases} \tag{3.5}$$

If we allow $\rho \to 1$, the asymptotic variance of $\hat{\bar{F}}$ from (3.5) is $-\bar{F}(t)^2 \int_0^t \frac{d\bar{F}(u)}{\bar{F}(u)^2\bar{H}_1(u)}$ for all $t \in (0, \tau_1)$. On the other hand, if $\rho \to 0$, so that only the residual-lifetime sample is accessible, we can estimate $\bar{G}_0(t)$ on (t_0, τ_2), using its product limit estimator, $\prod_{(t_0,t]}(1 - \frac{dN_2}{Y_2})$, which has asymptotic variance

$-\bar{G}_0(t)^2 \int_{t_0}^{t} \frac{d\bar{G}_0(u)}{G_0(u)^2 H2(u)}$. For the uncensored case, define

$$\sigma_0^2(t) = \frac{\bar{F}(t)F(t)}{\rho}\Big(1 - \frac{G_0(t)}{F(t)}\Big(\frac{(1-\rho)}{(1-\rho)+\rho\bar{F}(t_0)}\Big)\Big). \tag{3.6}$$

Corollary 3.4. (*Uncensored Case*) *If there is no right censoring mechanism (i.e.,* $\bar{H}_1 = \bar{H}_2 = 1$*), then for fixed* $t > 0$*, and as* $m, n \to \infty$*,*

$$(m+n)^{1/2}(\hat{\bar{F}}(t) - \bar{F}(t)) \xrightarrow{d} \begin{cases} N\Big(0, F(t)\bar{F}(t)/\rho\Big), & t \le t_0 \\ N\Big(0, \sigma_0^2(t)\Big) & t > t_0. \end{cases} \tag{3.7}$$

If $\rho \to 1$, the asymptotic variance equals $F(t)\bar{F}(t)$ for all $t > 0$. If $\rho \to 0$, the variance increases to infinity for $t > 0$. The next corollary follows easily using basic inequalities.

Corollary 3.5. (*Uncensored Case*) *Let* σ_0^2 *be defined in* (3.6)*. For* $t > t_0$*, and* $0 < \rho < 1$*,* $F(t)\bar{F}(t)\bar{F}(t_0) \le \sigma_0^2(t) \le F(t)\bar{F}(t)/\rho$*.*

Corollaries 3.4 and 3.5 can be deduced from the general results of Gill, Vardi and Wellner (1988), where the asymptotic efficiency is deduced for the uncensored case. Using $\hat{\rho} = m/(m+n)$, the variance components $\sigma_1^2(t)$, $\sigma_2^2(t)$ are estimated consistently by

$$\hat{\sigma}_1^2(t) = \int_0^t \frac{dN_1(u)}{\hat{\rho}Y_1^2(u)}, \text{ and } \hat{\sigma}_2^2(t) = \int_0^{t_0} \frac{dN_1(u)}{\hat{\rho}y_1^2(u)} + \int_{t_0}^{t} \frac{dN_1(u)}{\hat{\rho}Y_1^2(u) + (1-\hat{\rho})Y_1(u)Y_2(u)}. \tag{3.8}$$

Likewise, we estimate $\sigma_0^2(t)$ by substituting $(\hat{\bar{F}}, \hat{\bar{G}}_0)$ for $(\bar{F}, \bar{G}_0)$ in (3.6). Construction of confidence bands for the underlying distribution can be carried out directly using the asymptotic limits produced above, as well as bootstrap procedures explained by Akitas (1986) for the Kaplan-Meier estimator. See also Wells and Tiwari (1994), Lu, Wells and Tiwari (1994), and Li, Tiwari and Wells (1996).

In studies of the survival function, quantiles have particular importance. For $0 < p < 1$, define the pth quantile functional of F to be $\xi_p = F^{-1}(p) = \inf\{x : F(x) \ge p\} = \inf\{x : \bar{F}(x) \le 1 - p\}$. Accordingly, define the estimated quantile functional $\hat{\xi}_p = \inf\{x : \hat{\bar{F}}(x) \le 1 - p\}$. We can write $\xi_p = \phi(F)$, where ϕ is the mapping from the space of distribution functions to the real line that is defined by $\phi(F) = F^{-1}(\rho) = \inf\{x : F(x) \ge p\}$; thus ϕ is compactly differentiable at F with derivative $d\phi(F) \cdot h = \frac{-h(F^{-1}(p))}{f(F^{-1}(p))}$, provided $f(\xi_p) > 0$. From Theorem IV.3.2 of Andersen Borgan, Gill and Keiding (1993), we get the following.

Theorem 3.6. *If* F *has a positive derivative on* $(0, \tau)$*,*

$$(m+n)^{1/2}(\hat{\xi}_p - \hat{\xi}_p) \xrightarrow{w} \begin{cases} \frac{-(1-p)W_1(F^{-1}(p))}{f(F^{-1}(p))}, & \xi_p \le t_0 \\ \frac{-(1-p)W_2(F^{-1}(p))}{f(F^{-1}(p))}, & t_0 < \xi_p \end{cases} \tag{3.9}$$

with $p \in (0, F(\tau))$*, as* m*,* $n \to \infty$*. In particular, when there is no censoring, we have for fixed* p *in* $(0,1)$

$$(m+n)^{1/2}(\hat{\xi}_p - \xi_p) \xrightarrow{d} N\Big(0, \frac{(1-p)^2\tilde{\sigma}^2(\xi_p)}{f(\xi_p)^2}\Big), \tag{3.10}$$

where

$$\tilde{\sigma}^2(\xi_p) = \begin{cases} F(\xi_p)\bar{F}(\xi_p)/\rho, & \xi_p \le \tau_0 \\ \sigma^2(\xi_p), & \xi_p > \tau_0. \end{cases}$$

The variance $\sigma_0^2(\xi_p)$ is estimated by $\hat{\sigma}_0^2(\hat{\xi}_p)$, and the density $f(\xi_p)$ from (3.9) and (3.10) can be estimated by the kernel estimate $\hat{f}(t) = \frac{-1}{b_n^2}\int K'(\frac{t-u}{b_n})d\hat{\bar{F}}(u)$, where K' is the derivative of the uniform kernel function, and $b_n = b_{n,m}$ is the selected bandwidth. See Example IV.2.1 from Andersen, Borgan, Gill and Keiding (1993) for further description of this approach. Confidence regions can be constructed using these estimates. An alternative method that does not require the density estimation of f is to construct a critical region based on a hypothesis test for $H_0 : \xi_p = \xi_p^*$ vs. $H_A : \xi_p \ne \xi_p^*$. Analogous test statistics, based on suitably transforming the Kaplan-Meier estimator, are discussed by Brookmeyer and Crowly (1982).

Other functionals, including the mean survival time, can be estimated similarly. As a practical application of the quantile estimator, we might test $H_0 : F = F_0$ vs. $H_A : F \ne F_0$ based on $\hat{\xi}_{0.5}$. It is possible to extend tests to several classes of distributions that might be pertinent in a given situation, such as tests for increasing failure rate (IFR) (Proschan and Pyke (1967)), "new better than used" (NBU) (Hollander and Proschan (1972)), and "new better than used in expectation" (NBUE) (Koul (1978), among others). Test statistics for each of these hypothesis tests are based on relatively simple functionals of $\hat{\bar{F}}(t)$. For example, $F \in NBU$ if $\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y)$, which can be examined by using $\int_0^\infty \int_0^\infty \hat{\bar{F}}(x+y)d\hat{\bar{F}}(x)d\hat{\bar{F}}(y)$ as the test statistic.

4. Relative Efficiency of the NPMLE to the EDF

To further study the performance of the NPMLE, we compare its properties to those of F_m. Previously it was shown that the two estimators are identical up to t_0, the time at which the test items from the residual sample were known to survive. For any value of $t > 0$, define the relative efficiency of the estimator F_1 with respect to F_2 as $e(F_1, F_2) = \mathrm{Var}\,(F_2)/\mathrm{Var}\,(F_1)$. The estimators' relative risk, directly related to $e(\hat{F}, F_m)$, was illustrated in Figure 1 for the uncensored case. In the case of censored data, one can compare the two estimators via the asymptotic relative efficiency (ARE) of $\hat{F}$ with respect to F_m, defined as the ratio of their respective asymptotic variances. The asymptotic variance of $\hat{F}$

is inferred from (3.5), and the asymptotic variance of F_m is $F(t)\bar{F}(t)/\rho$. The loss in efficiency due to censoring is also of practical interest. The potential improvement due to adding observations from the residual-lifetime distribution can be measured using the relative efficiency of the NPMLE with respect to F_m at values of $t > t_0$, as long as both estimators are (asymptotically) unbiased.

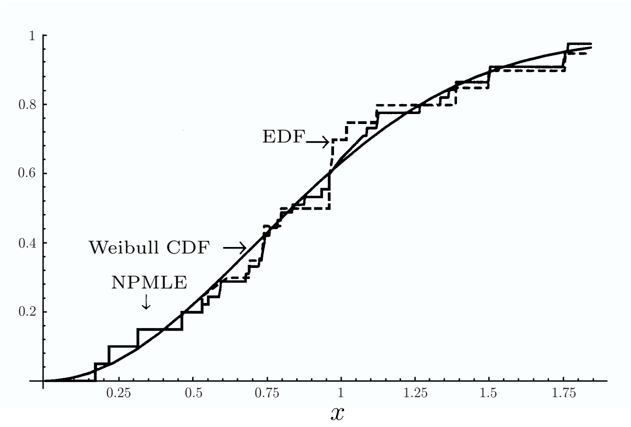


Figure 2. NPMLE (solid-line step function), EDF (dashed-line step function), and true Weibull(2,1) distribution function for generated data with $n = m = 20$ and $t_0 = 0.5$.

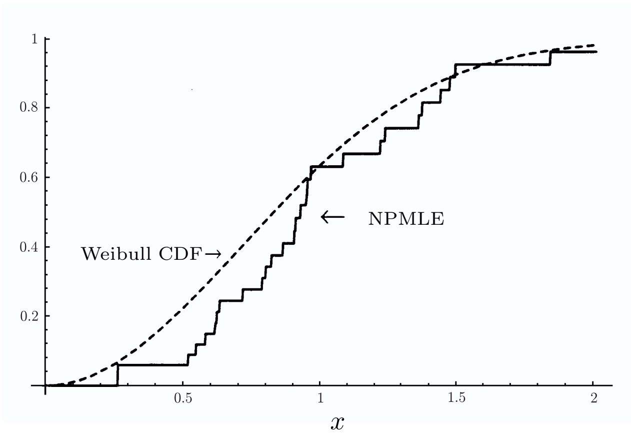


Figure 3. NPMLE (solid-line step function) and true Weibull(2,1) distribution function for generated data with $n = m = 20$, $t_0 = 0.5$, and exponential censoring distributions $\bar{H}_1(t) = e^{-t/2}$, $t > 0$, and $\bar{H}_2(t) = e^{-(t-t_0)/2}$, $t > t_0$.

Simulated estimation results are displayed in Figures 2 and 3. Both simulations are based on $t_0 = 0.5$ and data $(m = 20, n = 20)$ generated from a Weibull distribution (shape parameter "a"= 2 and scale parameter "b"= 1). For the Weibull(2,1) distribution, t_0 is the 0.2212 quantile of F. In Figure 2 the Weibull lifetimes are uncensored, and in Figure 3 exponential censoring distributions are applied: $\bar{H}_1(t) = e^{-t/2}$, $t > 0$ and $\bar{H}_2(t) = e^{-(t-t_0)/2}$, $t > t_0$. In the latter simulation, 12 of the 40 generated lifetimes are censored, including 7 out of 20 from the conventional sample. For Figure 2, the improvement gained in using added residual-lifetime data is subtle, but apparent at values of t to the right of $t_0 = 0.5$.

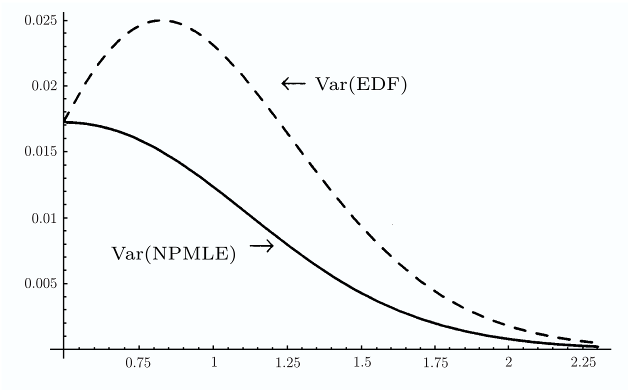
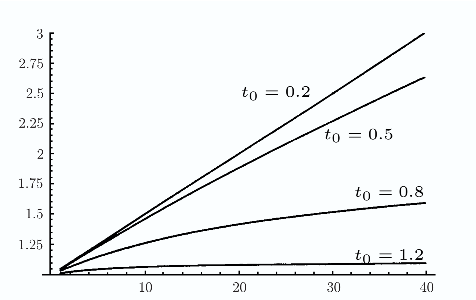


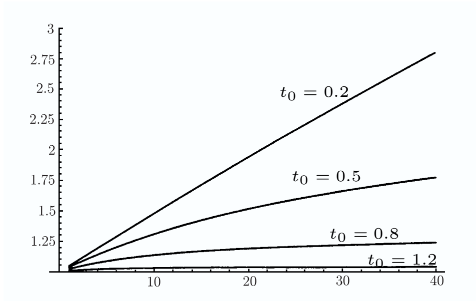
Figure 4. Variance of NPMLE (solid line) vs. Variance of EDF (dashed line) for uncensored Weibull(2,1) distribution function with $n = m = 10$ and $t_0 = 0.5$.

For the uncensored case, one can compare the estimators by using variance formulas from Corollary 2.2. For values of $t > t_0$, these variances are plotted in Figure 4, using the same distribution from Figure 2, and sample sizes of $m = n = 10$. Results using alternative lifetime distributions (e.g., Normal, Lognormal, Gamma) are similar. From the EDF, the variance of the NPMLE is reduced by nearly 50%, which seems intuitive because $m = n$, and the added residual-lifetime data are less informative than lifetimes from the conventional sample.

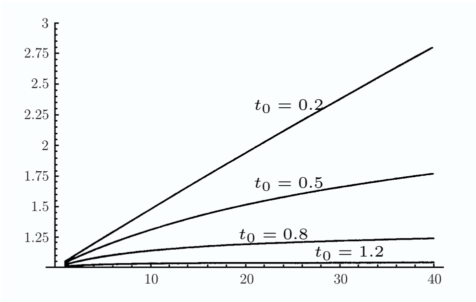
Figure 5 displays relative efficiency graphs of the NPMLE to the EDF for three lifetime distributions of interest to the reliability engineer: (a) Weibull, (b) Lognormal and (c) Gamma. The parameters are selected in order to make the means and variances approximately equal for all three distributions. The conventional sample size is kept constant at $m = 20$, and the residual sample size is varied ($n = 1, \ldots, 40$). The relative efficiencies are similar for all three distributions, and the loss of efficiency is clear as to increases from 0.2 to 1.2. In case (A), $F(t_0) = \{0.04, 0.22, 0.47, 0.76\}$ at the values of $t_0 = \{0.2, 0.5, 0.8, 1.2\}$. For all three distributions, the gain in efficiency seems modest for values of t_0 past the 50th percentile.



(a) Weibull(2,1)



(b) Lognormal(0,0.4)



(c) Gamma(3.5, 0.25)

Figure 5. Relative efficiency of the NPMLE to the EDF using various values to t_0 and uncensored distributions: (a) Weibull(2,1), (b) Lognormal(0,0.4), and (c) Gamma(3.5, 0.25). The horizontal axis represents the residual-lifetime sample size (n), and $m = 20$.

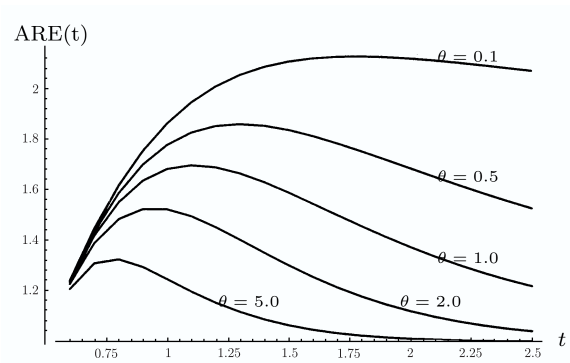


Figure 6. Asymptotic relative efficiency for the NPMLE (with respect to the conventional sample MLE) as a function of $t > t_0 = 0.5$, with equal sample sizes ($\rho = 1/2$). Computations based on Weibull(2,1) lifetimes, $\bar{H}_2(t) = e^{-\theta(t-t_0)}$, $t > t_0$ and different values of $\theta > 0$.

For studying the effect of censoring on the performance of the NPMLE, we make use of its asymptotic relative efficiency. A setting similar to those used in the simulations of Figure 2 and Figure 3 are considered: conventional data are generated from an uncensored Weibull(2,1) distribution ($\bar{H}_1(t) = 1$) with an equal number of observations ($\rho = 1/2$) from a variably censored residual lifetime sample with $t_0 = 0.5$ and $\bar{H}_2(t) = e^{-\theta(t-t_0)}$ for $t > t_0$, $\theta > 0$. Figure 6 illustrates the ARE as a function of t and θ. For small rates of censoring ($\theta = 0.10$), the ARE can exceed two at some values of $t > t_0$. At higher rates of right-censoring ($\theta = 5.0$), the ARE is scarcely over one. According to this model, the probability that any residual-lifetime observation is censored equals $\{0.0524, 0.2246, 0.3789, 0.5700, 0.7932\}$ at respective values of $\theta = \{0.1, 0.5, 1.0, 2.0, 5.0\}$.

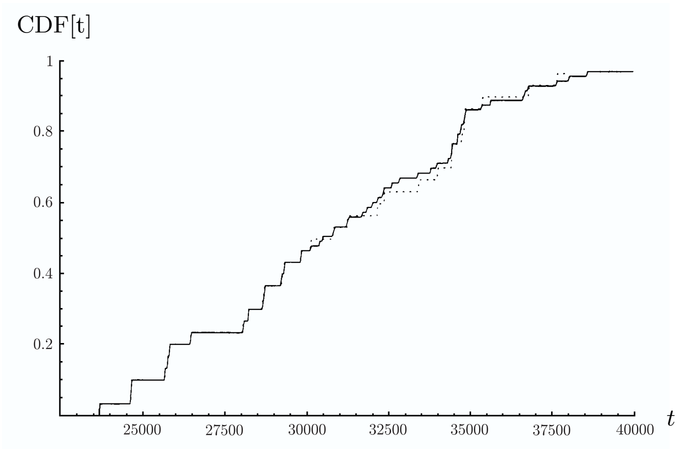


Figure 7. NPMLE (solid line) and EDF (dotted line) for the distribution of aluminum specimens' tensile strength.

Finally, we illustrate the NPMLE using uncensored data from an experiment investigating the tensile strength of die-cast aluminum specimens, described in Freeman (1947). The units of the measurements are $pounds/inch^2$. Each observation in this data set is matched with covariates that can be used for a linear regression. Because we disregard the covariates associated with tensile strength, the analysis that follows is limited to the marginal inference of tensile strength, thus its purpose is illustrative. Table 1 lists the two groups of strength data; the left column contains a conventional sample of 30 i.i.d. measurements, and the right column contains a set of 23 measurements left-truncated at $t_0 = 30,000$ $pounds/inch^2$. Figure 7 displays the NPMLE, based on all 53 measurements, for the distribution of aluminum specimens' tensile strength (solid line). The EDF based on just the 30 i.i.d. measurements (dotted line) is also plotted alongside it. There is some disagreement between the estimators in the interval (30000, 35000), but the difference is not substantial. The maximum difference between the estimators never exceeds 0.05; such a test statistic would be insignificant at test levels less than 0.25 when comparing independent samples of the same size using the common Smirnov test.

Table 1. Uncensored measurements of tensile strength of die-cast aluminum specimens.

$m = 30$ conventional observations		$n = 23$ left-truncated observations	
29314	25810	31852	38580
34860	26460	31698	35636
36818	28070	30844	34332
30120	24640	31988	34750
30824	25770	36640	40578
35396	23690	41578	34648
31260	28650	30496	31244
32184	32380	32622	33802
33424	28210	32822	34850
37694	34002	30380	36690
34876	34470	34440	32344
24660	34470	34650	
34760	29248		
38020	28710		
25680	29830		

Acknowledgement

The authors would like to thank the referees whose helpful comments led to several improvements in the manuscript.

Appendix

Proof of Theorem 2.1. Define s to be the number of observations less than or equal to t_0. Based on the observed censored data in (2.1), the corresponding term of the likelihood function that only involves F can be expressed as

$$\begin{aligned} L(F) &= \prod_{i=1}^{m+n} [F(z_i) - F(z_{i-1})]^{\delta^*_{(i)}} \bar{F}(z_i)^{1-\delta^*_{(i)}} / \bar{F}(z_s)^n \\ &= \prod_{i=1}^{m+n} [\bar{F}(z_{i-1}) - \bar{F}(z_i)]^{\delta^*_{(i)}} \bar{F}(z_i)^{1-\delta^*_{(i)}} / \bar{F}(z_s)^n \\ &= \prod_{i=1}^{m+n} \bar{F}(z_{i-1}) \Big[1 - \frac{\bar{F}(z_i)}{F(z_{i-1})}\Big]^{\delta^*_{(i)}} \Big[\frac{\bar{F}(z_i)}{\bar{F}(z_{i-1})}\Big]^{1-\delta^*_{(i)}} / \bar{F}(z_s)^n. \end{aligned}$$

Set $q_0 = 1$, and define $q_i = \bar{F}(z_i)/\bar{F}(z_{i-1})$ for $i = 1, \ldots, m+n$. Then,

$$L(F) = \Big\{ \prod_{i=1}^{s} (1-q_i)^{\delta^*_{(i)}} q_i^{m-i+1-\delta^*_{(i)}} \Big\}\Big\{ \prod_{j=1}^{m+n-s} (1-q_{s+j})^{\delta^*_{(s+j)}} q_{s+j}^{m+n-s-j+1-\delta^*_{(s+j)}} \Big\}.$$

The likelihood is maximized at $\hat{q}_i = 1 - \frac{\delta^*_{(i)}}{m-i+1}$ for $i = 1, \ldots, s$, and for $j = 1, \ldots, m+n-s$, $\hat{q}_{s+j} = 1 - \frac{\delta^*_{(s+j)}}{m+n-s-j+1}$. The NPMLE of the survival function in (2.2) thus follows.

Proof of Corollary 2.2. Part (b): For $t \le t_0$, $\hat{F}(t) = F_m(t)$ and the unbiasedness of $\hat{F}$ follows from that of F_m. For $t > t_0$, $\hat{F}(t) = S/m + R(m-S)/[m(m+n-S)]$, where R equals the number of the $n+m$ observations in $(t_0, t]$. For fixed $S = s$, $E(R|S=s) = (m+n-s)G_0(t)$, thus $E(\hat{F}(t)|S=s) = s/m + (1-s/m)G_0(t)$. Because $E(S) = mF(t_0)$, it follows that $E[\hat{F}(t)] = F(t_0) + \bar{F}(t_0)G_0(t) = F(t)$.

Part (c): Given s, $\mathrm{Var}\,[R|S=s] = (m+n-s)G_0(t)\bar{G}_0(t)$, thus $\mathrm{Var}\,[\hat{F}(t)|S = s] = G_0(t)\bar{G}_0(t)(m-s)^2/[m^2(m+n-s)]$. As $\mathrm{Var}\,[E(\hat{F}(t)|S=s)] = \bar{G}_0(t)^2 F(t_0)\bar{F}(t_0)/m$, we have $\mathrm{Var}\,[\hat{F}(t)] = \mathrm{Var}\,[E(\hat{F}(t)|S=s)] + E[\mathrm{Var}\,(\hat{F}(t)|S=s)] = \bar{G}_0(t)^2 F(t_0)\bar{F}(t_0)/m + G_0(t)\bar{G}_0(t)E[(m-S)^2/(m+n-S)]/m^2$. The rest follows using the fact that S is a binomial random variable with parameters $(m, F(t_0))$.

Part (d): If $t \le t_0$, $\mathrm{Var}\,\hat{F}[(t)] = \mathrm{Var}\,[F_m(t)]$. For $t > t_0$, we proceed as in Part (c) above to show that $\mathrm{Var}\,[\hat{F}(t)] \le G_0(t)^2 F(t_0)\bar{F}(t_0)/m + G_0(t)\bar{G}_0(t)E[m-S]/m^2 = \bar{G}_0(t)^2 F(t_0)\bar{F}(t_0)/m + G_0(t)\bar{G}_0(t)\bar{F}(t_0)/m = \bar{G}_0(t)\bar{F}(t_0)[1-\bar{G}_0(t)\bar{F}(t_0)]/m = F(t)\bar{F}(t)/m = \mathrm{Var}\,[F_m(t)]$. This proves the right-hand side of the inequality.

Part (e): The integrated variance can be obtained directly from the NPMLE in (a) using $R(\hat{F}, F) = \int_0^\infty \mathrm{Var}\,(\hat{F}(t))dF(t)$.

Proof of Lemma 3.1. For $t < t_0$, define $M_1(t) = N_1(t) - \int_0^t Y_1(u)d\Lambda(u)$, and for $t > t_0$, $M_2(t) = N_2(t) - \int_{t_0}^t Y_2(u)d\Lambda(u)$. For each equation, the right hand side contains the difference between a counting process and its compensator, thus $M_1(t)$ and $M_2(t)$ are local square integrable martingales. Note the following limits:

$$\begin{aligned} &Y_1/(m+n) \xrightarrow{p} \rho y_1 = \rho(1-F)(1-H_1) \text{ as } \mathrm{Min}(m,n) \to \infty, \\ &\ \text{uniformly in } t \in (0,\tau), \\ &Y_2/(m+n) \xrightarrow{p} (1-\rho)y_2 = (1-\rho)(1-G)(1-H_2) \text{ as } \mathrm{Min}(m,n) \to \infty, \\ &\ \text{uniformly in } t \in (t_0,\tau). \end{aligned}$$

We can write $(m+n)^{1/2}(\int_0^t \frac{dN_1(u)}{Y_1(u)} - \int_0^t d\Lambda(u)) = (m+n)^{1/2}\int_0^t \frac{dM_1(u)}{Y_1(u)}$, and for $t > t_0$,

$$(m+n)^{1/2}\Big(\int_{t_0}^t \frac{dN(u)}{Y(u)} - \int_0^t d\Lambda(u)\Big)$$

$$=(m+n)^{1/2}\Big(\int_{t_0}^{t}\frac{Y_1(u)}{Y_1(u)+Y_2(u)}\cdot\frac{dN_1(u)}{Y_1(u)}+\int_{t_0}^{t}\frac{Y_2(u)}{Y_1(u)+Y_2(u)}\cdot\frac{dN_2(u)}{Y_2(u)}-\int_0^t d\Lambda(u)\Big)$$
$$=(m+n)^{1/2}\Big(\int_{t_0}^{t}\frac{\rho y_1}{\rho y_1+(1-\rho)y_2}\cdot\frac{dM_1}{Y_1}+\int_{t_0}^{t}\frac{(1-\rho)y_2}{\rho y_1+(1-\rho)y_2}\cdot\frac{dM_2}{Y_2}\Big)+o_p(1).$$

Hence for $t>t_0$,

$$\lim_{m,n\to\infty}\mathrm{Cov}\ \Big[(m+n)^{1/2}\Big(\int_0^{t_0}\frac{dN_1(u)}{Y_1(u)}-\int_0^{t_0}d\Lambda(u)\Big),(m+n)^{1/2}\Big(\int_{t_0}^{t}\frac{dN(u)}{Y(u)}.$$
$$-\int_{t_0}^{t}d\Lambda(u)\Big)\Big]$$
$$=\lim_{m,n\to\infty}\mathrm{Cov}\ \Big[(m+n)^{1/2}\int_0^{t_0}\frac{dM_1(u)}{Y_1(u)},(m+n)^{1/2}\int_{t_0}^{t}\frac{\rho y_1 dM_1(u)}{(\rho y_1+(1-\rho)y_2)Y_1(u)}\Big]$$
$$=\lim_{m,n\to\infty}E\Big\{(m+n)^{1/2}\int_0^{t_0}\frac{dM_1(u)}{Y_1(u)}E\Big((m+n)^{1/2}\int_{t_0}^{t}\frac{\rho y_1 dM_1(u)}{(\rho y_1+(1-\rho)y_2)Y_1(u)}\Big|\mathcal{F}_{t_0}\Big)\Big\}$$
$$=0,$$

where $\mathcal{F}_u=\sigma\{N_1(s):s\le u\}$, for all u, is the filtration generated by the process $N_1(\cdot)$. Note that $\int_0^{t_0}\frac{dM_1}{Y_1}$ is $\mathcal{F}_{t_0}$-measurable, and $E(\frac{dM_1}{Y_1}|\mathcal{F}_{t_0^-})=E(dM_1|\mathcal{F}_{t_0^-})/Y=0$. Also, because $\int_{t_0}^{t}\frac{\rho y_1 dM_1(u)}{(\rho y_1+(1-\rho)y_2)Y_1(u)}$ is a stochastic integral with respect to the local square integrable martingale M_1, Y_1 is predictable and locally bounded, and $\rho y_1/(\rho y_1+(1-\rho)y_2)$ is also bounded. As a result, $\int_{t_0}^{t}\frac{\rho y_1 dM_1(u)}{(\rho y_1+(1-\rho)y_2)Y_1(u)}$ is a locally square integrable martingale for $t>t_0$, with $E(\int_{t_0}^{t}\frac{\rho y_1 dM_1(u)}{(\rho y_1+(1-\rho)y_2)Y_1(u)}-|\mathcal{F}_{t_0^-})=\int_{t_0}^{t}\frac{\rho y_1 E(dM_1(u)|\mathcal{F}_{t_0^-})}{(\rho y_1+(1-\rho)y_2)Y_1(u)}=0$.

Proof of Corollary 3.4. If there is no random right-censoring, $\bar H_1=\bar H_2=1$ and for $t\le t_0$, $\bar F(t)^2\sigma_1^2(t)=-\bar F(t)^2\int_0^t\frac{d\bar F(u)}{\rho\bar F(u)^2}=-\frac{\bar F(t)^2}{\rho}\Big[\frac{-1}{\bar F(u)}\Big]_0^t=\frac{\bar F(t)^2}{\rho}[\frac{F(t)}{\bar F(t)}=\frac{1}{\rho}F(t)\bar F(t)$. For $t>t_0$, the corresponding asymptotic variance in (3.5) simplifies to

$$-\bar F(t)^2\Big[\int_0^{t_0}\frac{d\bar F(u)}{\rho\bar F(u)^2}+\int_{t_0}^{t}\frac{d\bar F(u)}{\rho\bar F(u)^2+(1-\rho)\bar F\bar G_0}\Big]$$
$$=\bar F(t)^2\Big[\frac{F(t_0)}{\rho\bar F(t_0)}-\frac{\bar F(u)}{\rho+(1-\rho)\bar F(t_0)^{-1}}\int_{t_0}^{t}\frac{d\bar F(u)}{\bar F(u)^2}\Big]$$
$$=\frac{\bar F(t)F(t_0)\bar G_0(t)}{\rho}+\frac{\bar F(t)G_0(t)}{\rho+(1-\rho)\bar F(t_0)^{-1}}$$
$$=\frac{\bar F(t)F(t)}{\rho}-\bar F(t)G_0(t)\Big(\frac{1}{\rho}-\frac{\bar F(t_0)}{(1-\rho)+\rho\bar F(t_0)}\Big)$$
$$=\frac{\bar F(t)F(t)}{\rho}-\bar F(t)G_0(t)\Big(\frac{(1-\rho)}{\rho(1-\rho)+\rho^2\bar F(t_0)}\Big)$$

$$= \frac{\bar{F}(t)F(t)}{\rho}\Big(1 - \frac{G_0(t)}{F(t)}\Big(\frac{(1-\rho)}{(1-\rho)+\rho\bar{F}(t_0)}\Big)\Big).$$

Proof of Corollary 3.5. The left hand inequality is obtained by noting that $\sigma_0^2 \geq \frac{\bar{F}(t)F(t)}{\rho}(1 - \frac{1-\rho}{1-\rho+\rho\bar{F}(t_0)}) = \frac{\bar{F}(t)F(t)\bar{F}(t_0)}{1-\rho+\rho F(t_0)} \geq \bar{F}(t)F(t)\bar{F}(t_0)$. The right hand inequality is clear.

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(Received May 1997; accepted April 1998)