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ON BLACKWELL'S MINIMAX THEOREM AND THE COMPOUND DECISION METHOD

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Abstract: Blackwell (1956a) proved a minimax theorem for games with a vector loss and characterized sets such that a player has a strategy under which, whatever strategy the other player uses, the average payoff approaches the set. Based on this result, Blackwell (1956b) described a strategy for a sequence of plays of a game, under which the average loss approaches the Bayes risk with respect to the relative frequencies of the opponent's actions. In both cases, the distance of the average loss from a set, in repeated plays of the game, was proved to converge to 0 with probability one. In both cases, we show that the rate of the convergence is better than $(n/\log^{1+t} n)^{-1/2}$ for every positive t, obtain bounds for the L_2 norm of the distance and extend the results to cases in which past losses and actions are only estimated.

Key words and phrases: Almost uniform convergence, Bayes risk, game, $L_2(E)$ convergence, sequence-compound statistical methods, sequence of plays, stochastic approximation, strategy, vector payoff.

1. Introduction

Blackwell (1956a) considers a sequence of plays of a game with vector payoffs and characterizes sets S such that Player I has a strategy under which, whatever Player's II strategy, the average payoff approaches S with probability 1. Blackwell (1956b) ingeniously uses this result to construct a strategy for Player I that guarantees, for a sequence of plays of a game with a one-dimensional payoff, that the average payoff is at least $\beta(p_n)-\eta_n$, where $\beta(p_n)$ is the Bayes expected payoff with respect to the relative frequencies p_n of Player II actions and η_n goes to 0, whatever strategy Player II uses.

Most authors with motivation in statistics reverse the names of Player I and II and this we shall also do now. Hannan (1957, addendum) has a bound for the expected average loss for the Blackwell strategy (see Remark 3.9 below). Cover and Gluss (1986) and Cover (1991) describe applications of the strategy in stock market decisions.

The Blackwell (1956b) result is similar to that obtained by Robbins (1951), with the main difference that Robbins considered sets of decision problems rather than a sequence of plays of a game, expected average loss rather than average loss and Player II only estimated Player I actions. Hannan (1956) obtained a similar result for sequences, again for expected average loss. Robbins (1956, 1983) obtained results for sequences of statistical problems in which nature plays independently according to a prior; this for expected average loss. Fabian and Špaček (1956) obtained such a result for average loss. Johns (1967), Cover and Shenhar (1977) and Vardeman (1975, 1982) study methods based on relative frequencies (or estimates of these) of overlapping s-tuples of Player I actions which is advantageous if these form a dependent sequence.

This paper determines a rate of convergence to the set S in Blackwell (1956a) and gives (non asymptotic) bounds on the expected squared distance of the average loss from the set S. At the same time the condition that the loss lies in a compact set X is weakened. These results are then used to strengthen results of Theorem 3.6 in Blackwell (1956b) on the convergence to 0 of the excess of the average loss over the Bayes risk. Again, we give rates of convergence and bounds and we weaken the assumption on the loss lying in a compact space. We also weaken considerably the assumptions in Blackwell (1956b) that the statistician is told the action of nature and the loss after each play. We show that a change of the geometry (see Remark 3.1 and Assumption 3.2) that is involved in Blackwell (1956b) makes it possible to obtain better bounds.

Jílovec (1970) obtained rates for the uniform almost everywhere convergence for the excess of the average loss over the Bayes risk for the Hannan (1957) compound sequential method; part of our Theorem 3.6 establishes the same result for the Blackwell method.

Neyman (1962) considers Robbins (1951, 1956) two breakthroughs in statistics and Efron (1994) wonders why this and some other principles do not get more attention in the current literature. The answer seems to lie in that theory does not develop in optimal way; attempts to improve this might make the situation worse.

The paper is organized into four sections, of which the first is this introduction and the next two are concerned with the results of the two Blackwell papers, respectively. The last section is devoted to an example. It may be worth mentioning that most of the tools used in proof evolved within the field of stochastic approximation methods.

Notation. If f is a function, and A is a subset of the domain of f, then $f[A]$ is the image of A under f. The symbols $\vee$ and $\wedge$ denote maximum and minimum, $a_+ = a \vee 0$, $a_- = (-a)_+$. The brackets $<>$ will be used to denote sequences; a finite sequence of numbers is considered a column vector, a row vector will be written using square brackets [ ]. In a context where a positive integer I is defined, e_i denotes, for each i in $\{1, \dots, I\}$, the I-dimensional vector with each sth component e_{is} equal to 1 if $s = i$ and 0 for all other s.

2. The Speed of Convergence to a Set

Definition 2.1. M is a game if M is an I by J matrix with elements $m(i,j)$ that are probability measures on $\mathbf{B}_k$, the σ-algebra of all Borel subsets of the k-dimensional Euclidean space R^k, and such that each $m(i,j)$ has a finite covariance matrix. $\{1,\ldots,I\}$ is the set of possible actions for nature $\mathfrak{N}$, $\{1,\ldots,J\}$ is the set of possible actions for the statistician $\mathfrak{S}$, the loss X is the random vector with distribution $m(i,j)$, if actions taken are i and j. A randomized action for $\mathfrak{N}$ means a probability $p = < p_1,\ldots,p_I >$ on $\{1,\ldots,I\}$, a randomized action for $\mathfrak{S}$ is a probability $q = < q_1,\ldots,q_J >$ on $\{1,\ldots,J\}$.

Let X denote a closed convex subset of R^k such that $m(i,j)(\mathrm{X}) = 1$ for all i and j (R^k is such a subset) and P and Q the set of all randomized actions for $\mathfrak{N}$ and $\mathfrak{S}$, respectively. With a game M we associate an I by J matrix L with components $l(i,j)$, the expected loss under $m(i,j)$, and

$$\sigma^2 = \sup\Big\{\int \|X - p'Lq\|^2 d(p'Mq); p \in P, q \in Q\Big\}. \tag{1}$$

For p in P, set $T(p) = \{p'Lq; q \in Q\}$, the set of expected losses available to $\mathfrak{S}$. Similarly, for any q in Q, set $R(q) = \{p'Lq; p \in P\}$.

We shall say that a compact subset S of X has property BL if there exist functions π and ρ on X into S and into Q, respectively, with the following properties: For every x in S, $\pi(x) = x$. For every x in $\mathrm{X}-S$, $\pi(x)$ is a closest point in S to x, and the hyperplane containing $\pi(x)$ and orthogonal to $x - \pi(x)$ separates x and $R(\rho(x))$.

For a BL set S and π and ρ as above, denote by c the diameter of X ($c = \infty$ if X is not bounded) and set

$$C = \sup\{\|\pi(x) - r\|; x \in \mathrm{X}, r \in R(\rho(x))\}, \quad \theta = \frac{C}{\sqrt{\sigma^2 + C^2}}. \tag{2}$$

Remark 2.2. A game can be considered a statistical problem with I states of nature and J statistical methods. The finiteness of the covariances of X follows from the assumption in Blackwell (1956a) that X is bounded, and implies the boundedness of σ^2 in (1). Also C is finite, because S is compact. Note that $C \vee \sigma \le c$ and that $C > 0$ except in rather degenerate situations.

Consider now playing M at times $n = 1, 2, \ldots$ For the play at time $n+1$, $\mathfrak{N}$ selects his randomized action f_n in P, and $\mathfrak{S}$ its randomized action g_n in Q. Then the actions i_{n+1} and j_{n+1} are selected according to the densities f_n and g_n. Finally, the loss X_{n+1} is determined by the probability $m(i_{n+1}, j_{n+1})$. Blackwell (1956a) assumes f_n and g_n are functions of $< X_1, \ldots, X_n >$. In fact, his result holds even when allowing $\mathfrak{N}$ to use additional information, for example, allowing $\mathfrak{N}$ to choose the random action f_n to depend on the random action g_n of $\mathfrak{S}$. For certain arguments, we shall use the fact that the determination of X_{n+1} includes a preliminary stage, at which i_{n+1}, j_{n+1} are determined at a time n^+ between times n and $n+1$.

Definition 2.3. By a *sequence of plays of* M we mean a pair $<M, <\mathbf{F}_0, \mathbf{F}_{0+}, \mathbf{F}_1, \mathbf{F}_{1+}, \ldots>>$ where M is a game and $<\mathbf{F}_0, \mathbf{F}_{0+}, \ldots>$ is a non-decreasing sequence of sub-σ-algebras of a σ-algebra $\mathbf{F}$. For such a sequence of plays, with the notation for M as in Subsections 2.1 and 2.2, f is a *strategy* for $\mathfrak{N}$ (or $\mathfrak{N}$-strategy) if $f = <f_0, f_1, \ldots>$ where, for each $n = 0, 1, \ldots, f_n$ is an $\mathbf{F}_n$ measurable function into P and g is a strategy for $\mathfrak{S}$ (or $\mathfrak{S}$-strategy) if $g = <g_0, g_1, \ldots>$ where, for each $n = 0, 1, \ldots, g_n$ is an $\mathbf{F}_n$ measurable function into Q. X_n, i_n and j_n are the loss, $\mathfrak{N}$-action and $\mathfrak{S}$-action in play n, i_{n+1} and j_{n+1} are $\mathbf{F}_{n+}$ measurable, X_{n+1} is $\mathbf{F}_{n+1}$ measurable.

If the strategies f and g are given, $\mathrm{P}_{f,g}$ is a probability on $\mathbf{F}$ under which, for each $n = 0, 1, \ldots, i_{n+1}$ and j_{n+1} are, given $\mathbf{F}_n$, independent with discrete densities f_n and g_n, respectively. The (regular) conditional distribution of X_{n+1}, given $\mathbf{F}_{n+}$, is $m(i_{n+1}, j_{n+1})$. Instead of $\mathbf{F}_{m+}$ we shall also write $\mathbf{F}_{n-}$ if $n = m+1$. The average loss is $\bar{X}_n = (X_1 + \cdots + X_n)/n$.

If S has property BL, then an $\mathfrak{S}$-strategy g is called BL(S) if, with π and ρ as in Definition 2.1, $g_0 \in \rho[S]$ and $g_n = \rho(\bar{X}_n)$ for each $n>0$.

If g is a strategy for $\mathfrak{S}$, and d_n are $\mathbf{F}_n$ measurable random variables, then we say that $d_n \to 0$ a.u. (almost uniformly) if, for every ε positive, there exists an N such that

$$\mathrm{P}_{f,g}\{d_n \geq \varepsilon \text{ for some } n \geq N\} \leq \varepsilon$$

for every $\mathfrak{N}$-strategy f (in most cases we omit the subscripts f and g in the probability and expectation notation).

Finally, a sequence $< b_n >$ of positive numbers is a *norming sequence* if $<n^{-2}b_n>$ is nonincreasing and summable (e.g. $<n/\log^{1+t} n>$ with t a positive number).

Theorem 2.4. *Consider a sequence of plays of a game M, a set S with property BL and assume g is a BL(S) strategy. Then the distance δ_n of $\bar{X}_n$ from S satisfies, for all $n = 1, 2, \ldots$ and all $\mathfrak{N}$-strategies f*

$$E\delta_n^2 \leq \frac{1}{n}(c^2 \wedge \psi_n^2) \text{ where } \psi_n > 0 \text{ and } \psi_n^2 = (\sigma^2 + C^2)\Big(1 + \frac{5.0527}{\sqrt{n}}\theta\Big). \tag{3}$$

Also, for every norming sequence $<b_n>$,

$$b_n\delta_n^2 \to 0 \ a.u. \tag{4}$$

Proof. For easier writing of recurrence relations set $\delta_0 = 0$. Let f be a strategy for $\mathfrak{N}$. Let n be a non-negative integer. Suppose U is an $\mathbf{F}_n$ measurable k-dimensional random vector and $Z = E(X_{n+1}|\mathbf{F}_n)$. Then, since $Z - U$ is $\mathbf{F}_n$

measurable and because of (1),

$$E(\|X_{n+1}-U\|^2|\mathbf{F}_n)=E(\|X_{n+1}-Z\|^2|\mathbf{F}_n)+\|Z-U\|^2\le\sigma^2+\|Z-U\|^2. \quad (5)$$

Next we shall show that

$$E(\delta_{n+1}^2|\mathbf{F}_n)\le\frac{n-1}{n+1}\delta_n^2+\frac{c^2\wedge[\sigma^2+(\delta_n+C)^2]}{(n+1)^2}\quad\text{for } n=0,1,\ldots \quad (6)$$

Consider $n=0$. Note first that $\delta_1\le c$. Secondly, $g_0=\rho(s)$ for an s in S and, since $Z\in R(g_0)$, $\|Z-s\|\le C$ by (2). Thus, from (5) with $U=s$, we obtain $E(\delta_1^2|\mathbf{F}_0)\le\sigma^2+C^2$ and (6) holds for $n=0$.

For $n>0$, set $U=\bar{X}_n$. Note that Z is in $R(\rho(\bar{X}_n))$, thus $\|\pi(\bar{X}_n)-Z\|\le C$ again by (2). It follows that

$$\|Z-\bar{X}_n\|\le\|\bar{X}_n-\pi(\bar{X}_n)\|+\|\pi(\bar{X}_n)-Z\|\le\delta_n+C.$$

This, (5) and $\|X_{n+1}-\bar{X}_n\|\le c$ give

$$E(\|X_{n+1}-\bar{X}_n\|^2|\mathbf{F}_n)\le c^2\wedge[\sigma^2+(\delta_n+C)^2]. \quad (7)$$

On $\{\delta_n=0\}$, we have $\delta_{n+1}\le\|\bar{X}_{n+1}-\bar{X}_n\|$, $\bar{X}_{n+1}-\bar{X}_n=(X_{n+1}-\bar{X}_n)/(n+1)$ and the inequality in (6) follows from (7). On $\{\delta_n>0\}$, (6) follows by retracing the proof of Theorem 1 in Blackwell (1956a) up to the relation (5) there, and using our (7) instead of the bound c^2 there.

It is easy to show that lim sup $\sup_f nE(\delta_n^2)\le c^2\wedge(\sigma^2+2C^2)$ by replacing, in (6), $(\delta_n+C)^2$ by $2\delta_n^2+2C^2$, rearranging terms, taking expectation and using a Chung Lemma (see Lemma 1 in Chung (1954) or relation (4.2.2) in Fabian (1967)). The stronger nonasymptotic assertion (3) needs a different proof.

First, use (6) with the minimum replaced by c^2, take expectations and obtain

$$(n+1)E(\delta_{n+1}^2)\le\frac{n-1}{n}nE(\delta_n^2)+\frac{c^2}{n+1};$$

by induction, for all n, $nE(\delta_n^2)\le c^2$.

Second, replace the minimum in (6) by the other term than c^2, take expectations, replace $E(\delta_n)$ by the bound $\sqrt{E(\delta_n^2)}$ and rearrange to obtain that $v_n=n^2E(\delta_n^2)/(\sigma^2+C^2)$ satisfies $v_1\le1$, $v_{n+1}\le v_n+1+(2\theta/n)\sqrt{v_n}$. If $\theta=0$, it follows that $v_n\le n$ for all n and (3) holds.

It remains to consider the case $\theta>0$. Set $w_n=n^{-1/2}(v_n-n)/\theta$ and verify that

$$w_1\le0,\ w_{n+1}\le T_n(w_n)\text{ where } T_n(w)=\sqrt{\frac{n}{n+1}}w+\frac{2}{\sqrt{(n+1)n}}\sqrt{1+\frac{w_+}{\sqrt{n}}},$$

where $w_+ = w \vee 0$ was used as an upper bound for θw. If we set $W_1 = 0$ and $W_{n+1} = T_n(W_n)$, we obtain upper bounds for w_n; from now on restrict T_n to the domain $[0, \infty]$. Set $\varphi_n = 2(\sqrt{(n+1)/n} + 1)$. In the inequality $T_n(w) \le w$, subtract the first term in the expression for $T_n(w)$ and multiply by $(n+1)(1 + \sqrt{n/(n+1)})$ to obtain an equivalent set of inequalities $w \ge 0$, $\varphi_n \sqrt{(1 + w/\sqrt{n})} \le w$. We obtain

$$T_n(w) \le w \quad \text{if and only if } \frac{\varphi_n^2}{2\sqrt{n}} + \sqrt{\varphi_n^2 + \frac{\varphi_n^4}{4n}} \le w.$$

Write the last inequality as $r_n \le w$. In particular, $T_n(r_n) \le r_n$ (in fact, $T_n(r_n) = r_n$), and, since T_n is isotone we obtain $T_n(w) \le w \vee r_n$. Direct computation establishes that $W_n \le W_{75} \le 5.0527$ for all $n = 1, \ldots, 75$ and $r_{75} < W_{75}$. Note that $< r_n >$ is a decreasing sequence. If $W_n \le W_{75}$ is true for an $n \ge 75$, then $W_{n+1} = T_n(W_n) \le W_{75} \vee r_n = W_{75}$ and it follows that $W_n \le W_{75}$ for all $n = 1, 2, \ldots$

Since $nE(\delta_n^2) = (\sigma^2 + C^2)(1 + \theta w_n/\sqrt{n})$, this completes the proof of (3).

It remains to prove (4). Set $z_n = b_n \delta_n^2$ and $\xi_n = n^{-2} b_n$. Since $< b_n >$ is a norming sequence, $< \xi_n >$ is summable and non-increasing. From the latter, we obtain $b_{n+1}/b_n \le [(n+1)/n]^2$ and

$$\frac{b_{n+1}}{b_n}\Big(\frac{n-1}{n+1} + \frac{2}{(n+1)^2}\Big) \le 1 + n^{-2}.$$

Use this, set $A = \sigma^2 + 2C^2$, replace the minimum in (6) by $2\delta_n^2 + A$ and obtain $E(z_{n+1}|\mathbf{F}_n) \le (1 + n^{-2}) z_n + A\xi_{n+1}$. This implies, by Proposition 2 in Robbins and Siegmund (1971), that for every ε positive and every positive integer N

$$\mathrm{P}\{z_n \ge \varepsilon \text{ for some } n \ge N\} \le \varepsilon^{-1}\Big[E(z_N) + \sum_{n=N}^{\infty} (A\xi_n + n^{-2})\Big]. \tag{8}$$

Note that $N\xi_N \to 0$, since $(N - n)\xi_N \le \xi_{n+1} + \cdots + \xi_N$. From (3) (we could use the weaker statement we deduced by using the Chung Lemma), we have that $\sup_{f,n} nE(\delta_n^2) < \infty$. Thus $E(z_N) = N\xi_N NE(\delta_N^2)$ converges uniformly (in f) to 0. The other terms in the brackets in (8) do not depend on f, and, for N large enough, the right-hand side is smaller than ε for all f. This proves (4).

3. The Compound Statistical Problem

Remark 3.1. Blackwell (1956b) considers a sequence of plays of a game M with a one-dimensional loss. He constructs a method for $\mathfrak{S}$, for which the excess of the average loss $\bar{X}_n$ over the Bayes risk $\beta(p_n)$ converges to 0 no matter what is the $\mathfrak{N}$-strategy, where p_n denotes the frequency distribution of the $\mathfrak{N}$-actions $i_1, \ldots, i_n$.

Methods of this type are called sequence-compound statistical methods. The ingenious use, in Blackwell (1956b), of the results in Blackwell (1956a), consists of changing the loss X in such a way that the changed loss contains information about actions used by $\mathfrak{N}$.

It is assumed in Blackwell (1956b) that $\mathfrak{S}$ learns the $\mathfrak{N}$-action and the loss after each play. We shall show a weaker assumption suffices under which $\mathfrak{S}$ has available estimates of the $\mathfrak{N}$-actions and losses. The construction of such estimates is well known (see, e.g. Van Ryzin (1966)): Consider I probability measures μ_i on a σ-algebra and assume that they are linearly independent, i.e., $c_1\mu_1 + \cdots + c_I\mu_I = 0$ only if $c_1 = \cdots = c_I = 0$. Then there exists a random vector ξ such that the expectation of ξ with respect to μ_i is e_i (see *Notation* at the end of Section 1). Indeed, let φ_i be a density of μ_i with respect to $\mu = \mu_1 + \cdots + \mu_I$. Then, under μ_i, $\xi = < V_1, \ldots, V_I >$ has the required expectation e_i if, in $L_2(\mu)$, $(V_s, \varphi_i) = e_{is}$. It is enough to determine V_s as a projection of φ_s on $\{\varphi_1, \ldots, \varphi_{s-1}, \varphi_{s+1}, \ldots, \varphi_I\}^\perp$ multiplied by a suitable constant to have $(V_s, \varphi_s) = 1$.

The densities φ_i can be chosen such that the I-tuple $< \varphi_1, \ldots, \varphi_I >$ has values in P, V_s are linear combinations of the $\varphi_1, \ldots, \varphi_I$ and so there is a number K_0 such that $\|\xi\|_\infty \leq K_0$.

If $\mathfrak{S}$ is able to observe the loss X in the game M, and if, for each j, the measures $m(1,j), \ldots, m(I,j)$ are linearly independent, then $\mathfrak{S}$, knowing his action j, can estimate the action i by $\mathfrak{N}$ using the estimate ξ_j described above.

Often, $\mathfrak{S}$ is unable to observe the loss, but observes another random vector γ, whose distribution depends on the $\mathfrak{N}$-action i (usually, γ is the information provided to the statistical method $\mathfrak{S}$ is using). Again, if these I probability distributions are linearly independent, there is an estimate $U = \zeta(\gamma)$ with the expectation e_i, if $\mathfrak{N}$ plays i. In this case, set Y equal to the jth column of $U'L$ if $\mathfrak{S}$ plays j; the expectation of Y, if the two actions are i and j, is $l(i,j)$.

We shall assume below that estimates for i_n and X_n are used; this includes, as a special case, the situation where $\mathfrak{S}$ knows i_n or X_n.

In the assumption below, the game $\tilde{M}$ with the changed loss will depend on a positive number κ that determines how the distance in the space that contains the values of the loss is measured. Introducing this constant helps to obtain better results in Theorem 3.6.

Assumption 3.2. Assume κ is a positive number. Assume M is a game with one-dimensional non-negative loss. Set

$$\beta(p) = \mathrm{Min}\{p'Lq; q \in Q\} \text{ for every } p \text{ in } P,$$

and

$$B = \{< p, x >; p \in P, x \in \mathrm{X}, 0 \leq x \leq \beta(p)\}.$$

Assume that $m(i,j)$ are restrictions of probability measures μ_{ij} on $\mathbf{F}_1$, and that U is an $\mathbf{F}_1$ measurable I-dimensional random vector and Y an $\mathbf{F}_1$ measurable random variable such that

$$\int U d\mu_{ij} = e_i, \quad \int \|U - e_i\|^2 d\mu_{ij} \le K_U^2, \tag{9}$$

$$\int Y d\mu_{ij} = l(i,j), \quad \int (Y - X)^2 d\mu_{ij} \le K_Y^2 \text{ and } K^2 = \kappa^2 K_U^2 + K_Y^2. \tag{10}$$

Assume $\tilde{M}$ is a game with the same I and J as M and with loss $<U,Y>$. Note that $\tilde{m}(i,j)$ are the distributions of $<U,Y>$ under μ_{ij}. For the range of U we select the inner product $\boldsymbol{(}u_1,u_2\boldsymbol{)} = \kappa^2(u_1,u_2)$ and for $\tilde{\mathrm{X}}$ the innerproduct $\boldsymbol{(}<u_1,y_1>,<u_2,y_2>\boldsymbol{)} = \boldsymbol{(}u_1,u_2\boldsymbol{)} + y_1y_2$. The corresponding norms are $\boldsymbol{\|}u\boldsymbol{\|} = \kappa\|u\|$ and $\boldsymbol{\|}<u,y>\boldsymbol{\|}^2 = \kappa^2\|u\|^2 + y^2$.

For repeated plays of M and $\tilde{M}$ we assume a common sequence $<\mathbf{F}_0, \mathbf{F}_1, \ldots>$. Definition 2.3 specifies the meaning of $<U_n, Y_n> = \tilde{X}_n$. We assume that $\tilde{m}(i,j)$ is the conditional distribution of $<U_n, Y_n>$, given $\mathbf{F}_{n-}$, on $\{i_n = i, j_n = j\}$. The concept of the two strategies is the same for both games. We assume g is a BL(B) strategy for $\tilde{M}$ (B has property BL for $\tilde{M}$, see below). For each $n = 1, 2, \ldots, \bar{U}_n$ and $\bar{Y}_n$ are the arithmetic means of the first n members of the sequences $<U_n>$ and $<Y_n>$ and p_n is the arithmetic mean of $e_{i_1}, \ldots, e_{i_n}$.

Remark 3.3. Let Assumption 3.2 hold. The constants σ, c and C and the set X in Definition 2.1, but referring to $\tilde{M}$, will be denoted by $\tilde{\sigma}, \tilde{c}, \tilde{C}$ and $\tilde{\mathrm{X}}$. For the game $\tilde{M}$ with $\kappa = 1$, B is a compact convex set with property BL as noted by Blackwell (1956b): the intersection of B and $T(p)$ is non-empty for every $p \in P$ which implies property BL (see Theorem 3 and its proof in Blackwell (1956a)). However the case with $\kappa \ne 1$ corresponds to the change of $<U,Y>$ to $\kappa<U, Y/\kappa>$, so that what we stated above assuming $\kappa = 1$ holds without this assumption.

The hypotheses of Theorem 2.4 hold and results there apply to the distance $\tilde{\delta}_n$ mutatis mutandis (we shall use $\tilde{\psi}_n$ from Theorem 2.4).

Consider a point $\tilde{x} = <p, x>$ with p in P and $x > \beta(p)$. The point $\tilde{\pi}(\tilde{x})$ can be written as $\tilde{\beta}(p_0)$ where $\tilde{\beta}(p)$ denotes $<p, \beta(p)>$. We shall show that $p_0' L \tilde{\rho}(\tilde{x}) = \beta(p_0)$, i.e., $\tilde{\rho}(\tilde{x})$ is a randomized Bayesian action against p_0. This makes the determination of $\tilde{\rho}(\tilde{x})$ easy if there is only one such action. If not, a suitable $\tilde{\rho}(\tilde{x})$ can be found (as implicitly described in the proof of Theorem 3 in Blackwell (1956a)) based on a minimax randomized action for $\mathfrak{S}$ for the game with payoff the inner product $\boldsymbol{(}\tilde{x} - \tilde{\beta}(p_0), \tilde{l}(i,j)\boldsymbol{)}$. Note that this payoff is proportional to the coefficient α of the projection $\alpha(\tilde{x} - \tilde{\beta}(p_0))$ of the expected loss $\tilde{l}(i,j)$ on the line containing the segment joining $\tilde{x}$ and $\tilde{\beta}(p_0)$.

To prove the property $p_0' L\tilde{\rho}(\tilde{x}) = \beta(p_0)$, begin with the identity

$$(x-\beta(p))(x-\beta(p_0)) = (\tilde{x}-\tilde{\beta}(p), \tilde{x}-\tilde{\beta}(p_0)) = (\tilde{\beta}(p_0)-\tilde{\beta}(p), \tilde{x}-\tilde{\beta}(p_0)) + \| \tilde{x}-\tilde{\beta}(p_0)\|^2.$$

On the right-hand side, the inner product is nonnegative, because the hyperplane H (through $\tilde{\beta}(p_0)$ and orthogonal to $\tilde{x}-\tilde{\beta}(p_0)$) separates $\tilde{x}$ from $\tilde{\beta}(p)$. It follows that $x > \beta(p_0)$. H also separates $\tilde{x}$ from $p_0'\tilde{L}\tilde{\rho}(\tilde{x})$, thus $0 \geq (\tilde{x}-\tilde{\beta}(p_0), < p_0, p_0' L\tilde{\rho}(\tilde{x}) > -\tilde{\beta}(p_0)) = (x-\beta(p_0))(p_0' L\tilde{\rho}(\tilde{x}) - \beta(p_0))$ and the property holds since $x > \beta(p_0)$.

Lemma 3.4. *Under Assumption* 3.2*, for every strategy f of $\mathfrak{N}$ and for all n,*

$$E(\| < \bar{U}_n, \bar{Y}_n > - < p_n, \bar{X}_n > \|^2) \leq K^2/n; \tag{11}$$

for every norming sequence $< b_n >$,

$$b_n \| < \bar{U}_n, \bar{Y}_n > - < p_n, \bar{X}_n > \|^2 \to 0 \ a.u. \tag{12}$$

Proof. We have

$$\bar{U}_{n+1} - p_{n+1} = \frac{n}{n+1}(\bar{U}_n - p_n) + \frac{U_{n+1} - e_{i_{n+1}}}{n+1}.$$

The conditional expectation, given $\mathbf{F}_{n+}$, of the last term is 0, and thus

$$E(\|\bar{U}_{n+1} - p_{n+1}\|^2 | \mathbf{F}_{n+}) \leq \Big(\frac{n}{n+1}\Big)^2 \|\bar{U}_n - p_n\|^2 + \frac{K_U^2}{(n+1)^2}.$$

Taking expectations of both sides and a simple induction gives $E(\|\bar{U}_n - p_n\|^2) \leq K_U^2/n$; the same property holds for $\bar{Y}_n - \bar{X}_n$ with K_U replaced by K_Y, and (11) follows. Then (12) follows by Proposition 2 in Robbins and Siegmund (1971) in a way shown in more detail in the proof of Theorem 2.4.

Lemma 3.5. *Under Assumption* 3.2*, the distance d_n of $< p_n, \bar{X}_n >$ from the set B satisfies, for every $\mathfrak{N}$-strategy f and all n, $E(d_n^2) \leq (1/n)(K + \tilde{c} \wedge \tilde{\psi}_n)^2$ and, for every norming sequence $< b_n >$, $b_n d_n^2 \to 0$ a.u.*

Proof. Both assertions follow from Lemma 3.4 and Theorem 2.4, applied as described in Remark 3.3, because $d_n \leq \tilde{\delta}_n + \| < \bar{U}_n, \bar{Y}_n > - < p_n, \bar{X}_n > \|$. The second assertion follows immediately, and the first assertion by the Minkowski inequality.

Theorem 3.6. *Let Assumption* 3.2 *hold, $\eta(p, x) = (x - \beta(p))_+$, K_η be the Lipschitz constant of η and let $\eta_n = \eta(p_n, \bar{X}_n)$. Then, for every $\mathfrak{N}$-strategy f and every n,*

$$E(\eta_n^2) \leq \frac{1}{n} K_\eta^2 (K + \tilde{c} \wedge \tilde{\psi}_n)^2. \tag{13}$$

and, for every norming sequence $<b_n>$, $b_n\eta_n^2 \to 0$ *a.u.*

Proof. The assertion follows from the preceding lemma and the relation $\eta_n \leq K_\eta d_n$.

Remark 3.7. Bounds for some constants. If K_β is the Lipschitz constant for β, then

$$K_\eta^2 \leq 1 + K_\beta^2. \tag{14}$$

Indeed, set $x = x_1 - x_2$, $y = \| p_1 - p_2 \|$, $z = \beta(p_2) - \beta(p_1)$ and obtain $z^2/y^2 \leq K_\beta^2$; the Schwartz inequality, applied to $< x, y >$ and $< 1, z/y >$, gives $(x + z)^2 \leq (x^2 + y^2)(1 + z^2/y^2)$ and (14) follows.

Next, let l_j denote the jth column of L and set $l_j^* = l_j - a$ where a is the average of the components of l_j. Consider p_1 and p_2 in P and a Bayes action j corresponding to p_2. Then $\beta(p_1) - \beta(p_2) \leq (p_1 - p_2, l_j) = (p_1 - p_2, l_j^*) \leq \|p_1 - p_2\| \, \|l_j^*\| = \kappa^{-1} \| p_1 - p_2 \| \, \|l_j^*\|$. By symmetry, and with $\lambda = \operatorname{Max}_j \|l_j^*\|$,

$$K_\beta \leq \kappa^{-1}\lambda, \ \lambda \leq \sqrt{I/4} \ \operatorname*{Max}_j [\operatorname*{Max}_i l(i,j) - \operatorname*{Min}_i l(i,j)]. \tag{15}$$

The square diameter of P is $2\kappa^2$, because for $y = p_1 - p_2$ with p_i in P, the convex function $y \to \|y\|^2$ attains its maximum at an extremal point of its domain. It follows that

$$\tilde{c}^2 = 2\kappa^2 + c^2. \tag{16}$$

Remark 3.8. Choice of κ. Suppose Assumption 3.2 holds and $K = 0$. Then Theorem 3.6 implies

$$nE(\eta_n^2) \leq K_\eta^2 \tilde{c}^2. \tag{17}$$

Using (14), (15) and (16), we obtain $nE(\eta_n^2) \leq (1 + \kappa^{-2}\lambda^2)(2\kappa^2 + c^2) = (\lambda\sqrt{2} + c)^2 + (\kappa\sqrt{2} - c\lambda/\kappa)^2$. Using the value of κ that minimizes the last expression, we obtain

$$nE(\eta_n^2) \leq (\lambda_1\sqrt{2} + c)^2 \text{ if } c\lambda/\sqrt{2} \leq \kappa^2 \leq c\lambda_1/\sqrt{2}. \tag{18}$$

Indeed, the assertion for the two extreme choices of κ follows by the argument above and its modification in which λ is replaced by a larger number λ_1. (18) then follows easily. The value κ determines the innerproduct in $\tilde{\mathrm{X}}$ and thus the BL(B) strategy g.

A similar improvement by a choice of κ is possible even if K is not zero and if $\tilde{\psi}_n$ is used instead of $\tilde{c}$. This would usually require the knowledge of the numerical values of the constants involved and then easy numerical minimization.

Remark 3.9. A special case. Consider now the special case of the loss equal to its expectation (i.e., degenerate $m(i,j)$) and $\mathfrak{S}$ being told the loss and $\mathfrak{N}$-actions after each play. This means that $K = 0$ and $c = \mathrm{Max}_{ij}\, l(i,j) - \mathrm{Min}_{ij}\, l(i,j)$ is the diameter of the set $\{l(i,j)\}$. By (15), $\lambda^2 \le c^2 I/4$ and (18) gives

$$nE(\eta_n^2) \le c^2(\sqrt{I/2}+1)^2 \text{ if } c\lambda/\sqrt{2} \le \kappa^2 \le c^2\sqrt{I/8}; \tag{19}$$

in a specific situation when λ is known rather than its upper bound only, we would get a better bound than (19) for a different κ^2.

An improvement is possible. It is known that η_n does not change if the loss X in game M is changed to $X - t_i$ in case i is the $\mathfrak{N}$-action. If $t_i = \mathrm{Min}_j\, l(i,j)$, then the change leaves c unchanged or, in most cases, makes it smaller. Playing the BL(B) strategy in the sense of the changed game would result in an improvement of bound (19). (A similar improvement might be possible even when the loss is random, but the above choice of t_i may fail.)

In this special case, and within the context of $\mathrm{Min}_j\, l(i,j) = 0$ for each i, Hannan (1957, addendum) wrongly claimed $\sqrt{n}E(\eta_n) \le c(\sqrt{2I}+1)$ for the strategy with $\kappa = 1$. The inequality is weaker than that in (19), thus holds for the strategy with κ as in (19) or, by (18), even κ in a larger interval. Hannan's claim is cited and used in Cover and Shenhar (1977) in a situation where it is correct.

4. An Example

4.1. The games M and $\tilde{M}$

We shall consider a 2 by 2 game M with the loss

Loss for game M		
$i \backslash j$	1	2
1	$\gamma_{.2} - .2$	$\gamma_{.2} + .8$
2	$\gamma_{.8} + .2$	$\gamma_{.8} - .8$

where γ_t is a Bernoulli(t) random variable and $\mathfrak{S}$ observes $\gamma = \gamma_{.2}$ if $\mathfrak{N}$ selects action $i = 1$ and $\gamma = \gamma_{.8}$ if $\mathfrak{N}$ selects $i = 2$. The matrices of expected losses for M and $\tilde{M}$ are

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{L} = \begin{bmatrix} <1,0,0>, & <1,0,1> \\ <0,1,1>, & <0,1,0> \end{bmatrix}.$$

We shall treat special cases with various estimates U and Y. However, without these specified, we obtain that $\tilde{R}(q_1, q_2)$ is the segment connecting $<0,1,q_1>$ and $<1,0,q_2>$ and that

$$\begin{gathered} \beta(p_1,p_2) = p_1 \wedge p_2,\ B = \{<p,x>; p \in P, 0 \le x \le \beta(p)\} \\ \lambda^2 = 1/2, \quad K_\eta^2 \le 1 + \kappa^{-2}/2, \tilde{C}^2 = 1 + 2\kappa^2. \end{gathered} \tag{20}$$

The expressions for β, B and λ are obvious, the bound for K_η^2 follows from (15) and (14). In the present case, K_η^2 is equal to the bound, as seen by setting $\alpha = 1/(2\kappa^2)$ and considering two points $<p_1, 1-p_1, 1/2>$, $<p_1-\alpha\varepsilon,\ 1+\alpha\varepsilon-p_1,\ 1/2+\varepsilon>$ with $p_1 = 1/4$ and sufficiently small ε. $\tilde{C}$ is at most the maximum distance $\sqrt{1+2\kappa^2}$ of two elements of $\tilde{L}$; this distance is actually achieved if $\tilde{\pi}(\tilde{x}) =<0,1,0>$.

In the special cases that follow we shall assume that $\mathfrak{S}$ uses a BL(B) strategy g for the corresponding $\tilde{M}$; the property $b_n\eta_n^2 \to 0$ a.u. always holds. The bounds for $E(\eta_n^2)$ differ, however each such bound holds again for all n and all $\mathfrak{N}$-strategies f. We shall determine κ for which the limiting bound is optimal. In most cases, such a κ has to be determined numerically, and will be selected optimal among values of the form $m/100$ with m integers.

The BL(B) strategy is determined by the function ρ which can be described now. We shall assume that $U =<U_1, 1-U_1>$, a relation satisfied by all specific choices of U below. For any point $\tilde{x} =<a, 1-a, x>$ in $\tilde{\mathrm{X}} - B$ and with $a \le 1/2$ (the case $a > 1/2$ is symmetric), $\tilde{\rho}(\tilde{x})$ depends on the projection $\tilde{\pi}(\tilde{x})$ of $\tilde{x}$ on B and is as follows:
(i) $\tilde{\pi}(\tilde{x}) =<b, 1-b, 0>$, with $0 < b \le 1/2 : \tilde{\rho}(\tilde{x}) =<1/2, 1/2>$
(ii) $\tilde{\pi}(\tilde{x}) =<b, 1-b, b>$, with $0 \le b < 1/2 :\ \tilde{\rho}(\tilde{x}) =<0, 1>$
(iii) $\tilde{\pi}(\tilde{x}) =<1/2, 1/2, 1/2>$: $\tilde{\rho}(\tilde{x}) =<q, 1-q>$ with $q = \frac{2\kappa^2(2a-1)+2x-1}{2(2x-1)}$.
Verification is somewhat easier if we consider a two-dimensional graph obtained by omitting the second component in points of $\tilde{\mathrm{X}}$. If we extend the unit of the horizontal axis by a factor of $\kappa\sqrt{2}$, then distances and orthogonality will be preserved in the graph.

Call H_* the hyperplane containing the base of the triangle B and H_q the hyperplane containing $\tilde{R}(q)$. Thus the left-hand side of the triangle B is contained in $\tilde{R}(0)$ and that is contained in H_0. Let H be the hyperplane going through $\tilde{\pi}(\tilde{x})$ and orthogonal to $\tilde{x} - \tilde{\pi}(\tilde{x})$. In case (i), verify that H is equal to H_* which separates $\tilde{x}$ from every $\tilde{R}(q)$. In case (ii), verify that H can be obtained by rotating H_* clockwise around the point $<0,0>$, but not past H_0 and that $\tilde{\rho}(\tilde{x}) = 0$ is a possible, and if H has a non empty intersection with $(0,1]^2$ the only possible, choice. In case (iii) $\tilde{x} - \tilde{\pi}(\tilde{x})$ is orthogonal to H_q with q given.

4.2. Case 1

Assume $\mathfrak{S}$ estimates the action i of $\mathfrak{N}$ by $U =<U_1, 1-U_1>$ and the loss by Y where, with j the action by $\mathfrak{S}$, $U_1 = \frac{1}{3}(4-5\gamma)$, $Y = (1-U_1)\chi_{\{j=1\}} + U_1\chi_{\{j=2\}}$. We shall show that

$$\tilde{c}^2 = (25/9)(1+2\kappa^2),\ K^2 = (.16/9)(64+50\kappa^2),\ \tilde{\sigma}^2 = (25/36)(1+2\kappa^2),\ \theta^2 = 36/61.$$

The range of U_1 is $\{-1/3, 4/3\}$ and the expression for $\tilde{c}^2$ follows. The random variable $Y - X$ is of the form $a + b\gamma$ with different numbers a and b depending on i and j. For $j = 1$, $b = 2/3$ and for $j = 2$, $b = -8/3$; thus (10) is satisfied with $K_Y^2 = (64/9)(.16)$. Similarly, (9) is satisfied with $K_U^2 = (50/9)(.16)\kappa^2$ and K^2 is as asserted.

To determine $\tilde{\sigma}^2$ (cf. (1)), let E and V denote, temporarily, the expectation and variance under $p'\tilde{M}q$ and E_{ij} and V_{ij} the conditional expectation and variance given actions i and j. Under $p'\tilde{M}q$, γ is Bernoulli with the maximal variance 1/4 when $p =< 1/2, 1/2 >$ and the maximal value of $E\| < U, 1-U > -E < U, 1-U >\|^2$ is $(50/36)\kappa^2$. Next, $V(Y) = E(V_{ij}(Y)) + V(E_{ij}(Y))$. But $V_{ij}(Y) = (25/9)(.16)$ and $E_{ij}(Y)$ is Bernoulli with the maximum variance 1/4 if $p =< 1/2, 1/2 >$; for this p, $V(Y) = (25/9)(.16) + 1/4 = 25/36$. This proves the relation above for $\tilde{\sigma}^2$.

Since Assumption 3.2 holds, the assertions of Theorem 3.6 hold. In particular,

$$nE(\eta_n^2) \le (1+\kappa^{-2}/2)\Big\{\frac{.4}{3}\sqrt{64+50\kappa^2}+\sqrt{1+2\kappa^2}\Big(\frac{5}{3}\wedge\sqrt{\frac{61}{36}(1+\frac{3.882}{\sqrt{n}})}\Big)\Big\}^2.$$

The minimum of the two terms is 5/3 for $n \le 36$, and the second term for all other n. $\kappa = .77$ minimizes the limit of the right-hand side above. For this κ,

$$nE(\eta_n^2) \le 5.098^2 \wedge \Big[4.365 + 2.613\Big(\sqrt{1+\frac{3.882}{\sqrt{n}}} - 1\Big)\Big]^2.$$

(The κ minimizing the bound for $n \le 36$ is .76 with the same bound as above when rounded up to three decimals.)

4.3. Case 2

We assume $\mathfrak{S}$ is told the past $\mathfrak{N}$-actions, and that he chooses $U_n = e_{i_n}$, $Y_n = l(i_n, j_n)$, $\tilde{\mathrm{X}} = P \times [0,1]$. We obtain, similarly as in Case 1,

$$\tilde{c}^2 = 1 + 2\kappa^2,\ K^2 = .16,\ \tilde{\sigma}^2 = .25(1+2\kappa^2),\ \theta^2 = .8.$$

Since $\tilde{c}^2 = \tilde{C}^2$, we have $\tilde{c} < \tilde{\psi}_n$ and the bound in Theorem 3.6 becomes $nE(\eta_n^2) \le (1+\kappa^{-2}/2)(.4+\sqrt{1+2\kappa^2})^2$. This is minimal for $\kappa = .80$, when it becomes

$$nE(\eta_n^2) \le 2.550^2 = 6.5025. \tag{21}$$

4.4. Case 3

We assume $\mathfrak{S}$ is told the past $\mathfrak{N}$-actions and losses and he chooses U as in Case 2, and $Y = X$. We have $\tilde{\mathrm{X}} = P \times [-.8, 1.8]$ and obtain

$$c=2.6,\ \tilde{c}^2=6.76+2\kappa^2,\ K=0, \tilde{\sigma}^2=.41+.5\kappa^2, \theta^2 = (1+2\kappa^2)/(1.41+2.5\kappa^2);$$

the loss for $\tilde{M}$ is now $< e_i, X >$ under $\tilde{m}_{ij}$; the variance of the first component is at most $.5\kappa^2$, of the second component at most $.16 + .25$, both attained at $p =< 1/2, 1/2 >$. In the limit, the bound involving ψ_n is better than the other and minimal for $\kappa = .73$. We obtain

$$nE(\eta_n^2) \leq 15.169 \wedge [5.316(1 + 4.386/\sqrt{n})] \text{ if } \kappa = .73.$$

For $n \geq 387$, the bound is better than that in (21), obtained for another estimate of the loss.

4.5. Case 4

Consider the game M but changed by assuming the same L but a non random loss (i.e., the loss for M is $l(i, j)$ under m_{ij}). Assume $\mathfrak{S}$ is told past $\mathfrak{N}$-actions and uses U and Y as in Case 2. Then $c = 1$, $\tilde{c}^2 = 1 + 2\kappa^2$, $\tilde{\sigma}^2 = .5\kappa^2$, $\theta^2 = .8$, $\tilde{c} \wedge \tilde{\psi}_n = \tilde{c}$, and (18) gives $nE(\eta_n^2) \leq 4$ if $\kappa = 1/2$. The same bound holds for the original game M without the change that X is non-random, if U and Y are as above and $\eta_n = (E(\bar{X}_n) - p_n)_+$.

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