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A HIDDEN PROJECTION PROPERTY OF PLACKETT-BURMAN AND RELATED DESIGNS

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Abstract. Box and Hunter (1961) made an important observation that any fractional factorial design of resolution R has the property that when projected onto any $R-1$ factors it becomes a full factorial design. This has a significant implication for statistical analysis. We observe that the Plackett-Burman and related designs have a hidden projection property with an analogous implication for the analysis. Because of complex aliasing, these designs have traditionally been used for screening main effects only. The hidden projection property suggests that complex aliasing actually allows some interactions to be entertained and estimated without making additional runs and provides an explanation for the success of an analysis strategy due to Hamada and Wu (1992). We give a detailed study of the hidden projection property for 12-run and 20-run designs with two levels and an 18-run design with three levels.

Key words and phrases: Fractional factorial design, Plackett-Burman design, Hadamard matrix, orthogonal array, resolution, projective rationale, hidden projection, interaction, D efficiency, D_s efficiency.

1. Introduction

In the classic work of Box and Hunter (1961), they defined the resolution of a fractional factorial design to be the shortest wordlength R of the defining contrasts for the design. They further noted that the notion of resolution has an interesting statistical interpretation, that is, when the design is projected onto any subset of $R-1$ factors, it is a full factorial design. Therefore all the main effects and interactions among $R-1$ factors are orthogonal and estimable, if other factors have negligible effects. Although this geometric projection property is obvious from the notion of strength in Rao's (1947) earlier work on orthogonal arrays, its full statistical implications were recognized and exploited by Box and Hunter. Lin and Draper (1992), and, independently, Box and Bisgaard (1993), extended this projection property to Plackett-Burman (PB) designs, which cannot be defined by a group of defining contrasts. (Henceforth we shall refer to Lin and Draper (1992) as LD.)

The projective rationale has two aspects. The first and obvious one is the geometric projection, which was adopted by LD. Take the 12-run PB design as an example. LD showed that when projected onto any four factors, the design needs one additional run to make it into a 2^{4-1} design with resolution IV and five additional runs to make it into a full factorial design. The second and the more important aspect is the ability to entertain the estimation of interactions. For the usual 2^{n-k} designs (i.e., n factors with 2^{n-k} runs) with defining contrasts, these two aspects are equivalent. For the PB designs, however, the second aspect can be achieved even when geometric projection does not lead to a full factorial design or a fractional factorial design with high resolution . For the same example, we show in Section 2 that all the six two-factor interactions (2fi's) among the four factors can be estimated *without* adding runs. (The same observation was made by Lin and Draper (1993) in a different context.) By contrast the 2^{4-1} design obtained by adding one run has the defining relation $\mathbf{I} = \mathbf{1234}$ and only allows three out of the six 2fi's to be estimated because the six 2fi's are aliased in three pairs. A design is said to possess a *hidden projection* property if it allows some (or all) interactions to be estimated even when the projected design does not have the right resolution or other combinatorial design property for the same interactions to be estimated. For the PB designs their hidden projection property is a result of the complex aliasing patterns between the interactions and the main effects. For the 12-run design with four factors **1**, **2**, **3**, and **4**, any 2fi, say the interaction between **1** and **2**, is orthogonal to the main effects **1** and **2**, and is partially aliased with the main effects **3** and **4** with correlation $1/3$ or $-1/3$. Any other 2fi enjoys a similar property. No 2fi is fully aliased with any main effect, thus making it possible for all six 2fi's to be estimated along with the four main effects.

The hidden projection property provides an explanation for the success of an analysis strategy due to Hamada and Wu (1992) for entertaining and estimating interactions from PB-type experiments. Because of the complex aliasing, the PB designs have been used traditionally as a screening design, i.e., for estimating main effects only. The hidden projection property suggests that some interactions can be entertained and estimated without making additional runs at other settings.

In Section 2 we give a detailed analysis of the hidden projection property of the 12-run PB design for projections onto 3 to 6 factors. The results show that the number of estimable 2fi's is equal or close to the maximum degrees of freedom remaining for interactions. That is, no additional runs are necessary if only a moderate number of 2fi's are to be entertained. In Section 3 we extend the study to three 20-run designs including the PB design. The same approach can be applied to study the hidden projection property of 16-run designs based on the four Hadamard matrices of order 16 (Hall (1961)), which are not equivalent to the regular 16-run design defined by a group of defining contrasts. However, as shown by Sun and Wu (1993), some (but not all) of the factors in these 16-run designs also satisfy a group of defining contrasts, thus making it less attractive from the hidden-projective point of view. The hidden projection property of designs with more than two levels can also be studied. An important example is the 18-run orthogonal array with seven 3-level factors, which has complex aliasing patterns. In Section 4 we study the hidden projection property of this array. By following the same approach we can study the hidden projection property for many other designs with complex aliasing. Finally it is shown in Section 5 how the hidden projection rationale can be exploited in data analysis and for run size savings. Its advantages are demonstrated by reanalyzing the data in LD. In particular, we can identify the same 2fi as in the analysis of LD by adding only one run instead of the six runs required by the geometric projection approach.

2. Hidden Projections of the 12-Run Plackett-Burman Design

There are two general questions to be answered. (i) If n factors are to be studied, which n columns should be assigned to the n factors? Since any set of n columns are orthogonal, we must compare them in terms of their ability in entertaining k 2fi's in addition to the n main effects. (ii) For each assignment, main effect analysis may reveal that only n_1 factors (i.e. n_1 columns), $n_1 < n$, are significant. We can then raise the question (i) for these n_1 factors. Since the projection onto n_1 columns varies with the outcome of the analysis, it will be desirable to study this problem for all (or most) projections. The information obtained will be useful for experimenters in contemplating the choice of designs.

Beyond knowing the estimability of the main effects and a given set of 2fi's, it is desirable to have an estimation efficiency measure for the purpose of comparison. In this paper we adopt the following D criterion for measuring the overall efficiency for estimating the collection of effects:

$$|X^tX|^{1/k}, \tag{1}$$

where $X = [\boldsymbol{x}_1/\|\boldsymbol{x}_1\|, \ldots, \boldsymbol{x}_k/\|\boldsymbol{x}_k\|]$, and $\boldsymbol{x}_i$ is the coefficient vector of the ith effect. Because the columns of X are standardized, (1) achieves its maximum 1 if and only if the $\boldsymbol{x}_i$'s are orthogonal to each other, i.e., when the array is orthogonal. The vector $\mathbf{1}$ is not included in X since it is orthogonal to the $\boldsymbol{x}_i$'s. For the estimation of each individual effect, we use the following D_s criterion for measuring its efficiency:

$$\{\boldsymbol{x}_i^t\boldsymbol{x}_i - \boldsymbol{x}_i^tX_{(i)}(X_{(i)}^tX_{(i)})^{-1}X_{(i)}^t\boldsymbol{x}_i\}/\boldsymbol{x}_i^t\boldsymbol{x}_i, \tag{2}$$

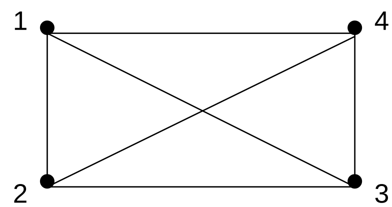
where $X_{(i)}$ is obtained from X by deleting $\boldsymbol{x}_i$. Note that (2) attains its upper bound 1 if and only if $\boldsymbol{x}_i$ is orthogonal to the other columns in X.

In this section we consider questions (i) and (ii) for the 12-run PB design (Table 1) with four to six factors. When collapsed onto any three factors, it consists of two parts: a 2^3 design with eight points and a 2^{3-1} design with four points (see LD). So the three main effects and four interactions can all be estimated with high efficiency. When collapsed onto four factors, in addition to the four main effects, there are still seven degrees of freedom left. Can they be used to estimate the six 2fi's without adding more runs? Before answering this question, we should address the issue of choice of columns for the factors. The choice of n factors is equivalent to the choice of a $12 \times n$ submatrix of the matrix given in Table 1. Two such matrices are said to be (combinatorially) equivalent if one can be obtained from the other by permutations of rows, columns and sign changes. In the context of design theory we refer to this equivalence as (combinatorial) equivalence of two factor assignments. It turns out that any four columns of the matrix in Table 1 can be chosen for the four factors because it is known (Draper (1985), Wang (1989)) that, except for $n = 5$ and 6, any $12 \times n$ submatrices are equivalent.

Table 1. 12-run Plackett-Burman Design

run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

For $n = 4$, we give in Table 2 the values of D and D_s for 10 cases. The last one consisting of four main effects and six 2fi's is the most comprehensive. From their D_s values we can see that all the 2fi's and the main effects can be estimated without adding runs. This estimability result was first observed by Lin and Draper (1993). The first nine cases in Table 2 represent all possible submodels of the comprehensive model of case 10. We further elaborate this point by the use of graphs. As in Wu and Chen (1992), we can represent any model consisting of main effects and selected 2fi's by a graph in which nodes represent factors and any line connecting two nodes represents the 2fi between the two factors represented by the two nodes. For example, case 10 can be represented by the following complete graph



Then the first nine cases correspond to all possible (non-isomorphic) subgraphs of this complete graph. (Two graphs are isomorphic if one can be obtained by permuting the nodes of the other graph.) We define a *graph model* to be the class of models (consisting of main effects and 2fi's) that can be represented by a graph. All the models within the same class are said to be *graphically equivalent*. For example, the model for case 4 and the model $\mathcal{M} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{14}, \mathbf{24}, \mathbf{34}\}$ are different but both can be represented by the following graph

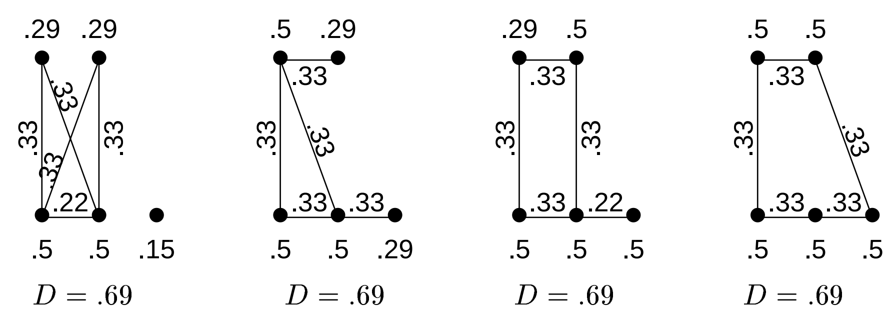
and therefore are graphically equivalent. Since a graph can represent different models, we do not usually put the factor labels for the nodes. An important question is whether graphically equivalent models have the same statistical efficiencies. Take, for example, the two models considered above. The D value for model $\mathcal{M}$ is .89 and the D_s values for the seven effects are .74, .74, .74, 1, .74, .74, .74, which are equivalent to the D and D_s values for case 4 of Table 2 after changing $\mathbf{1}$ to $\mathbf{4}$ and $\mathbf{4}$ to $\mathbf{1}$. We call two graphically equivalent models *efficiency equivalent* if the D and D_s values of one model are the same as the other model after relabelling the factor names. In general graphical equivalence does not imply efficiency equivalence. (One such example will be given in Figure 3.) For each of the 10 graph models in Table 2, graphical equivalence does imply efficiency equivalence. Therefore one set of D and D_s values represent all the models for the same graph.

Table 2. Estimation efficiency for four factors and h interactions, $h = 1, \ldots, 6$. The D_s efficiency is given for each effect.

case	D value	**1**	**2**	**3**	**4**	**12**	**13**	**14**	**23**	**24**	**34**
1	.95	1	1	.88	.88	.78					
2	.92	1	.87	.87	.75	.76	.76				
3	.89	.85	.85	.85	.85	.63					.63
4	.89	1	.74	.74	.74	.74	.74	.74			
5	.89	.87	.87	.87	.62	.74	.74		.74		
6	.87	.85	.85	.74	.74	.76	.63			.63	
7	.85	.85	.73	.73	.62	.73	.73	.62	.62		
8	.83	.73	.73	.73	.73	.63	.63			.63	.63
9	.82	.72	.72	.62	.62	.72	.62	.62	.62	.62	
10	.80	.62	.62	.62	.62	.62	.62	.62	.62	.62	.62

For $n = 5$, there are two non-isomorphic 12×5 submatrices: design 5.1 and design 5.2 in the notation of LD. Design 5.1 has two repeated runs, i.e., two runs with the same level combination. For example, in the design consisting of columns **1**, **2**, **3**, **4**, and **10**, runs 3 and 11 are identical. On the other hand, design 5.2 has two mirror image runs. For example, runs 7 and 11 are two mirror runs in the design consisting of columns **1** to **5**. We first study the properties of design 5.1. Because design 5.1 has two repeated runs, there are only 10 degrees of freedom for estimating effects. Consequently, it can entertain at most five 2fi's. It is easy to show that there are altogether four graphically non-equivalent models for estimating five 2fi's and the five main effects. As in the case of $n = 4$, all the graphically equivalent models (i.e., those having the same graph) have the same efficiencies. Therefore we can give one set of D and D_s values for each graph as is done in Figure 1. The total number of models for each graph (from left to right in Figure 1) is 30, 60, 60, and 12 respectively. For models with fewer 2fi's the efficiencies will be higher. The details are omitted.

Figure 1. Four graphically non-equivalent models with five 2fi's for design 5.1.



The situation for design 5.2 is more complicated. Models that are graphically equivalent may not be equivalent in terms of efficiency or even estimability. Before giving the details on the non-equivalence, we first note that, unlike design 5.1, there are 11 degrees of freedom for effect estimation, which allow six 2fi's to be estimated in addition to the five main effects. Altogether there are five, six and respectively six non-isomorphic graphs representing models with six, five and respectively four 2fi's (in addition to the five main effects). These graphs are given in Figure 2 as graphs 1 to 5, 6 to 11, and respectively 12 to 17. To illustrate the problem of non-equivalence we use graph 17 as an example. In Figure 3 we give representations of four models by using different factor labels for the five nodes in graph 17. The model for the left graph is not estimable. The models for the remaining graphs are estimable but have non-equivalent efficiencies.

Figure 2. Graphically non-equivalent models for design 5.2 with h 2fi's, $h = 6, 5, 4$.

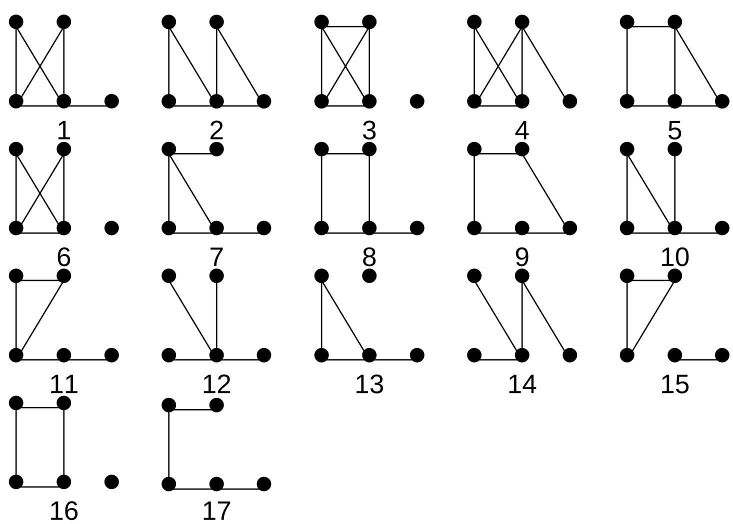
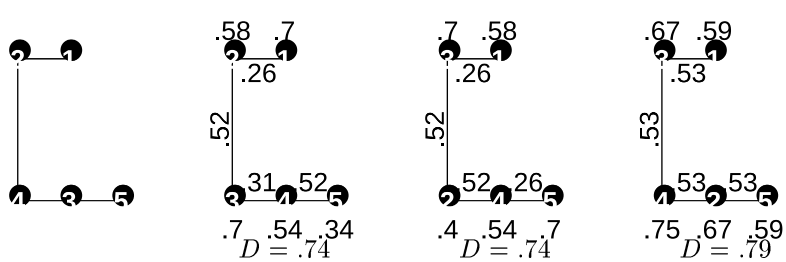


Figure 3. Four efficiency non-equivalent models for model 17 in Figure 2. The labels for the nodes indicate the column numbers in Table 1.



These 17 graphs are further studied in Table 3 under cases 1 to 17 using the same order. For the first five graphs (with 6 2fi's) only graph 3 (i.e., case 3) has equivalence in estimability. It happens to be a complete graph for four of its five nodes. Recall that case 10 of Table 2 has a complete graph with four nodes and also has efficiency-equivalence (which implies estimability-equivalence). Among the next six graphs, graphs 6 and 10 have estimability-equivalence. For graph 6 this property follows from the fact that it is a subgraph of graph 3. For graph 10 we cannot provide an explanation. For the last six graphs (with four 2fi's) only graph 17 does not have estimability equivalence. For graphs with three or fewer 2fi's all models are estimable. Its details are omitted. By examining the ratios n_e/n in the second column of Table 3 we find that the percent of non-estimable models drops from 99/170 to 5/21, 1/21 and 0 as the number of 2fi's drops from six to three. Because estimable models for the same graph may have different D and D_s values, we give, in Table 3, the range of such values over these models. Noting that the efficiency for estimating main effects tends to be higher than for estimating 2fi's, we give the range of D_s values for main effects and 2fi's in two separate columns.

Table 3. Efficiencies of graphically non-equivalent models for design 5.2 with h interactions, $h = 6, 5, 4$.

model number	n_e/n	range of D values	range of D_s for main effects	range of D_s for 2fi's
1	40/60	.69–.69	.20–.50	.20–.50
2	1/15	.69–.69	.33–.50	.20–.50
3	5/5	.69–.69	.17–.50	.20–.50
4	20/60	.69–.69	.29–.50	.20–.50
5	10/60	.69–.69	.29–.50	.22–.50
6	30/30	.71–.78	.21–.72	.22–.59
7	50/60	.71–.77	.24–.67	.22–.67
8	40/60	.71–.71	.29–.67	.22–.51
9	2/12	.76–.76	.58–.58	.52–.52
10	30/30	.71–.78	.24–.83	.22–.59
11	40/60	.71–.71	.33–.67	.22–.51
12	5/5	.81–.81	.59–1.0	.59–.59
13	60/60	.74–.81	.25–.84	.26–.67
14	60/60	.74–.81	.31–.83	.26–.67
15	10/10	.74–.74	.42–.70	.26–.52
16	15/15	.74–.81	.29–.73	.26–.59
17	50/60	.74–.79	.34–.75	.26–.53

Note: n_e = number of estimable models, n = total number of models for a given graph.

Based on the previous information for designs 5.1 and 5.2, we conclude that design 5.2 has better overall properties for the following reasons:

1. It allows six 2fi's to be estimated in 71 models while design 5.1 does not allow any model with six 2fi's to be estimated. Note that 71 is obtained by adding up the n_e values in the first 5 rows of Table 3.
2. Design 5.2 has 192 estimable models with five 2fi's, which can be grouped into six graph models, while design 5.1 has 162 such models grouped into four graph models. Furthermore the four graphs are part of the six graphs.
3. For the same graph model design 5.2 has higher overall efficiencies than design 5.1.

If there are five factors in an experiment, we should choose design 5.2 for the five factors. If there are more than five factors, projections (after data analysis) onto five factors can lead to either design 5.1 or 5.2. In this case the key issue is the capacities of design 5.1 and 5.2 for estimating interactions, which were studied in the previous paragraphs. A similar remark can be made for $n = 6$ in this section and other comparisons in Sections 3 and 4.

By contrast the geometric projection approach would require adding six and respectively ten runs for designs 5.1 and 5.2 so that the augmented designs have resolution V (see LD). If only six or fewer 2fi's are to be entertained, the hidden projection approach will be preferred.

For $n = 6$, there are two non-isomorphic 12×6 submatrices (see LD). Design 6.1 is characterized by having no mirror runs (e.g. columns **1** to **6**) while design 6.2 has two mirror image runs (e.g. runs 7 and 11 are mirror image runs in columns **1** to **5** and **7**). To save space we only briefly discuss the hidden projection property of designs 6.1 and 6.2. Design 6.1 allows any model with three 2fi's to be estimated. (There are five such graph models.) It does not, however, allow all models with four 2fi's to be estimated. The estimable ones can be grouped into seven graphs. Estimable models with five 2fi's can be grouped into 15 graphs. The percent of estimable models varies greatly among the 15 graph models, ranging from 1/6 to 90/90. The D value ranges from .61 to .74. The D_s values vary wildly from .05 to .8 for the main effects, and from .06 to .5 for the 2fi's. The small D_s values for some models can be explained by the fact that these models are close to being non-estimable. For design 6.2 the estimable models with five 2fi's can be grouped into 7 graphs, which are part of the 15 graphs for design 6.1, and the percent of estimable models ranges from 30/180 to 6/6. Unlike design 6.1 it does not allow all models with three 2fi's to be estimated. Overall design 6.1 is preferred. The inferiority of design 6.2 may be explained as follows. Its two mirror image runs in design 6.2 provide no information about any 2fi because any 2fi column in the two runs is either $(1, 1)^t$ or $(-1, -1)^t$, and therefore is confounded with the grand mean.

Finally we note that the 8-run design can also be used to study four to seven factors and respectively three to zero 2fi's. The choice between the 12-run and the 8-run designs depends on the trade-off between run size and the desired number of estimable 2fi's.

3. Hidden Projections of Three 20-Run Designs

According to Hall (1965), there are three non-isomorphic Hadamard matrices of order 20, which he called class N, P and Q. Our computer search shows that class Q is equivalent to the cyclic design studied by Plackett and Burman (1946), a fact not pointed out by Hall.

Table 4. Class Q Hadamard matrix of order 20

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−
+	+	+	−	+	−	−	−	+	−	+	−	−	+	+	−	−	−	+	+
+	+	+	−	−	+	−	−	+	−	−	+	+	−	−	+	+	−	−	+
+	+	+	−	−	−	+	−	−	+	−	+	−	−	+	+	−	+	+	−
+	+	+	−	−	−	−	+	−	+	+	−	+	+	−	−	+	+	−	−
+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	−	−	−	−
+	+	−	+	+	+	−	−	+	+	−	−	−	+	−	+	+	+	+	−
+	+	−	+	+	−	−	+	−	−	+	+	−	−	+	+	+	+	−	+
+	+	−	+	−	+	+	−	−	+	+	−	+	−	+	−	+	−	+	+
+	+	−	+	−	−	+	+	+	−	−	+	+	+	−	−	−	+	+	+
+	−	+	−	+	+	+	+	+	+	+	+	−	−	−	−	+	+	+	+
+	−	+	+	+	+	−	−	−	+	−	+	+	+	+	−	−	+	−	+
+	−	+	+	+	−	+	−	−	−	+	+	+	+	−	+	+	−	+	−
+	−	+	+	−	+	−	+	+	−	+	−	+	−	+	+	−	+	+	−
+	−	+	+	−	−	+	+	+	+	−	−	−	+	+	+	+	−	−	+
+	−	−	+	−	−	−	−	+	+	+	+	−	−	−	−	−	−	−	−
+	−	−	−	+	−	+	−	+	−	−	−	+	−	+	−	+	+	−	−
+	−	−	−	+	−	−	+	−	+	−	−	+	−	−	+	−	−	+	+
+	−	−	−	−	+	+	−	−	−	+	−	−	+	−	+	−	+	−	+
+	−	−	−	−	+	−	+	−	−	−	+	−	+	+	−	+	−	+	−

Note that in Table 4, the column (labelled **0**) consisting of all +'s cannot be used to study a factor effect. Similarly, for Classes N and P, only 19 columns can be used for studying factor effects.

For each of the three designs, projection onto any three columns consists of at least one 2^3, thus allowing all the factorial effects to be estimated with high efficiency. Details on the geometric projection property for $n = 3$ can be found in LD.

For $n = 4$ and 5, the hidden projection approach reveals some interesting aspects that are missed by the geometric projection approach. First we review the geometric projection property. According to LD, for each of N, P and Q, there are three non-isomorphic 20×4 submatrices:

1. Design 20-4.1. It has five runs with two repeats. As a result it has 15 out of the 16 ($= 2^4$) level combinations.
2. Design 20-4.2. It has one run with three repeats and six runs with two repeats and therefore has only 12 out of the 16 level combinations.
3. Design 20-4.3. It has one run with three repeats and six runs with two repeats and therefore has only 12 out of the 16 level combinations.

Designs 20-4.1 to 20-4.3 require the addition of 1, 4 and 4 runs respectively to complete a full factorial 2^4. If we are primarily interested in estimating the four main effects and six 2fi's, there is *no* need to add runs for estimating the ten effects. Using the hidden projection property, we have the estimation efficiencies for design 20-4.1: $D = .93$, $D_s = .86$ for each of the ten effects. For 20-4.2 (using columns **1**, **2**, **3**, **4** of Table 4), $D = .8$, $D_s = .81$ for **4**, **14**, **24**, **34**, and $D_s = .53$ for **1**, **2**, **3**, **12**, **13**, **23**. For design 20-4.3, $D = .8$, $D_s = .81$ for each main effect, $D_s = .53$ for each 2fi. So if the 3-factor and 4-factor interactions are of little interest, which is usually the case, the hidden projection property would allow us to estimate the 2fi's without adding runs.

Among the three projections, 20-4.1 is the best in terms of the geometric projection (i.e., smallest number of additional runs required) as well as the hidden projection property (i.e., high D and D_s values). For each of N, P and Q, 20-4.1 appears 2736 times, 20-4.2 912 times, and 20-4.3 228 times. So there is no difference among N, P and Q when $n = 4$.

For $n = 5$, there are respectively 10, 10, 9 non-equivalent 20×5 submatrices for N, P and Q corresponding to the same matrices, which Lin and Draper (1991) called design 20-5.1 to 20-5.10. Note that design 20-5.10 was missing in their collection for Q. (For brevity sake, we drop 20 in 20-5.i in the remainder of the section. To save space we refer to their paper for these designs.) The number of additional runs required to make a full factorial 2^5 ranges from 12 to 19 (Lin and Draper (1991)). If, however, the 2fi's are of primary interest, we need to add fewer runs to complete a 2^{5-1} resolution V design defined by $\mathbf{5} = \pm\mathbf{1234}$ because in any resolution V design, all the 2fi's are estimable. Our calculations show that, among the ten 20×5 submatrices, one requires adding three runs, two require adding four runs, and the rest require adding six to nine runs. These are substantially smaller than the 12 to 19 runs as previously indicated.

By contrast the hidden projection property would allow most or all of the 2fi's to be estimated without adding any run. Straight but tedious calculations show that for designs 5.1, 5.4, 5.3 and 5.5, all the 10 2fi's are estimable with 5.1 and 5.4 having higher overall estimation efficiencies than 5.3 and 5.5. For designs 5.2, 5.6, 5.7 and 5.8, nine 2fi's are estimable while for designs 5.9 and 5.10, only seven are estimable. The results can be explained by the structures of the designs. Design 5.1 has no run with repeats, 5.4 has one run with two repeats, and the rest have at least two runs with repeats. The worst are designs 5.9 and 5.10 with six and seven runs with repeats respectively.

We can compare N, P and Q in term of the frequencies of the best designs 5.1 and 5.4 among the projections. The best is Q, which has 1881 projections of design 5.1 and 1368 projections of design 5.4, while N has 1680 of design 5.1 and 1488 of design 5.2 and P has 1296 of design 5.1 and 1728 of design 5.2. Recall that Q does not have the "worst" design 5.10 among its projections, which may partially explain its superiority.

We conclude the section with a summary of results for $n = 6$. There are 59, 56 and 50 non-equivalent 20×6 submatrices for N, P and Q respectively. The complexity may explain why Lin and Draper (1991) did not study the geometric projection for $n = 6$. Among the 59 submatrices, 20 have no repeated runs. At most 13 2fi's can be estimated. In Table 5 we give the percentages and cumulative percentages of projections that allow h 2fi's to be estimated, $h = 13, 12, 11, 10, 7$. Since at least 99.7% of the projections will allow 10 or more 2fi's to be estimated (with average efficiencies approximately $D = .65$, $D_s = .37$ for main effects, and $D_s = .32$ for 2fi's), the hidden projection property suggests that usually no additional runs are needed for studying important 2fi's. In practice, the number of important 2fi's seldom exceeds six. Design Q is again the best among N, P and Q because all of its 20×6 submatrices can entertain at least 10 2fi's!

Table 5. Percentages and cumulative percentages (in parentheses) of 20×6 submatrices of N, P and Q that can entertain h 2fi's, $h = 13, 12, 11, 10, 7$.

	h				
design	13	12	11	10	7
N	33.0(33.0)	47.9(80.9)	12.4(93.3)	6.6(99.9)	0.1(100)
P	39.0(39.0)	42.1(81.1)	11.2(92.3)	7.4(99.7)	0.3(100)
Q	29.4(29.4)	51.5(80.9)	13.2(94.1)	5.9(100)	

4. Hidden Projections of $L_{18}(3^7)$

The hidden projection property also holds for three-level designs with complex aliasing. To save space we only consider the orthogonal array $L_{18}(3^7)$ (Masuyama (1957)) given in Table 6. This array plays an ubiquitous role in practical experimentation and in theoretical research because it is the smallest orthogonal array with three levels and complex aliasing.

Table 6. 18-run Orthogonal Array

run	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1
3	0	2	2	2	2	2	2
4	1	0	0	1	1	2	2
5	1	1	1	2	2	0	0
6	1	2	2	0	0	1	1
7	2	0	1	0	2	1	2
8	2	1	2	1	0	2	0
9	2	2	0	2	1	0	1
10	0	0	2	2	1	1	0
11	0	1	0	0	2	2	1
12	0	2	1	1	0	0	2
13	1	0	1	2	0	2	1
14	1	1	2	0	1	0	2
15	1	2	0	1	2	1	0
16	2	0	2	1	2	0	1
17	2	1	0	2	0	1	2
18	2	2	1	0	1	2	0

For three factors (i.e. $n = 3$), there are three non-isomorphic 18×3 submatrices given as follows.

1. Design 18-3.1. It is a $2/3$ fraction of 3^3, consisting of one $1/3$ fraction defined by $\mathbf{C} = \mathbf{A} + \mathbf{B}(\text{mod } 3)$, and another $1/3$ fraction defined by $\mathbf{C} = \mathbf{A} + \mathbf{B} + 1(\text{mod } 3)$, where $\mathbf{A}$ and $\mathbf{B}$ are any two of the three columns. Any three columns in Table 6 not containing column **1** form a design of this type.
2. Design 18-3.2. It consists of two $1/3$ fractions of 3^3 in which the two fractions share three points in common. For instance, in columns **1**, **2** and **7**, run 7 through run 15 form a $1/3$ fraction defined by col $\mathbf{7} =$ col $\mathbf{2} +$ col $\mathbf{1}(\text{mod } 3)$, and the remaining runs form another $1/3$ fraction defined by col $\mathbf{7} =$ col $\mathbf{2} +$ $2\times$col $\mathbf{1}(\text{mod } 3)$, where col is an abbreviation for column. These two fractions share (0,0,0), (1,0,1) and (2,0,2).
3. Design 18-3.3. It contains two identical replicates of a $1/3$ fraction of 3^3. Only one choice of three columns (columns **1**, **3** and **4**) is of this type.

Obviously design 18-3.1 is the best because it has 17 df's for estimating effects while designs 18-3.2 and 18-3.3 have, respectively, 14 and 8 df's. For efficiency comparison we consider design 18-3.1 with quantitative factors only. For each quantitative factor, we use $\ell = (-1, 0, 1)$ for its linear effect and $q = (1, -2, 1)$ for its quadratic effect. The four df's for each 2fi are represented by the linear-by-linear ($\ell \times \ell$), linear-by-quadratic ($\ell \times q$), quadratic-by-linear ($q \times \ell$), and quadratic-by-quadratic ($q \times q$) effects. The model consists of the grand mean, the three main effects **A**, **B**, **C** (with 6 df's) and the $\ell \times \ell$, $\ell \times q$, and $q \times \ell$ components of **A**×**B**, **B**×**C**, **A**×**C** (with 9 df's). For this model, the overall D efficiency is 0.83. The individual D_s efficiencies are: 0.78 for ℓ, 0.76 for q, 0.48 for $\ell \times \ell$, and 0.6 for $\ell \times q$ and $q \times q$. Because the model has 16 df's, there are two remaining df's, which allow for estimating two of the three $q \times q$ effects.

For four factors (i.e. $n = 4$), there are four non-isomorphic 18×4 submatrices given as follows:

1. Design 18-4.1. Any four columns not containing column **1** form a design of this type. Note that any three columns of this design form a design 18-3.1.
2. Design 18-4.2. One set of its three columns is a design 18-3.2 and the remaining three sets are design 18-3.1. Columns **1**, **2**, **3**, and **6** form a design of this type.
3. Design 18-4.3. One set of its three columns is a design 18-3.3 and the remaining three sets are design 18-3.1. Columns **1**, **2**, **3**, and **7** form a design of this type.
4. Design 18-4.4. One set of its three columns is a design 18-3.1 and the remaining three sets are design 18-3.2. Columns **1**, **2**, **4**, **7** form a design of this type.

Any of the four designs allows the four main effects (8 df's altogether) and the $\ell \times \ell$ components of the six interactions to be estimated. Their efficiencies are given in Table 7. Overall designs 18-4.1 and 18-4.2 are better than the other two.

Table 7. D and D_s efficiencies for 18 runs in 4 factors

design		effect													
		linear				quadratic				linear×linear					
	D	**A**	**B**	**C**	**D**	**A**	**B**	**C**	**D**	**AB**	**AC**	**AD**	**BC**	**BD**	**CD**
18-4.1	.82	.62	.62	.62	.62	.87	.87	.87	.87	.53	.54	.54	.54	.54	.53
18-4.2	.84	.85	.78	.70	.85	.61	.91	.72	.74	.55	.49	.49	.51	.58	.57
18-4.3	.73	.44	.27	.54	.27	.65	.82	.78	.82	.33	.30	.33	.58	.35	.58
18-4.4	.72	.42	.51	.75	.51	.41	.57	.47	.57	.33	.25	.33	.49	.27	.49

Note: We use columns **2**, **3**, **4**, **5** for 18-4.1, columns **1**, **2**, **3**, **6** for 18-4.2, columns **1**, **2**, **3**, **4** for 18-4.3, and columns **1**, **2**, **4**, **7** for 18-4.4.

For $n = 5$ the same method can be used to find out how many $\ell \times \ell$ effects (in addition to the main effects) can be estimated. The details are omitted.

5. An Illustrative Example

In this section we illustrate the hidden projection approach to analysis by reanalyzing an example reported in LD. To save space we refer to LD for the data and design matrix. Initially a 12-run PB design was used to study 10 factors. Using standard analysis they identified five significant main effects **1**, **3**, **7**, **8** and **10**. Then they added six more runs, labelled 13 to 18, to make it a resolution V design. Based on the augmented data, they found the **7**×**8** interaction to be significant.

Using the hidden projection property, we can entertain and analyze some 2fi's based on the original 12 runs. Since columns **1**, **3**, **7**, **8** and **10** form design 5.1, the collapsed design can entertain at most five 2fi's whose graphs are given in Figure 1. It is known from the discussion in Section 2 that any set of three 2fi's among these five factors are estimable. So we can study any three 2fi's without adding runs. Using forward selection in regression analysis with the five main effects and the 10 2fi's as candidate variables, we did not find any significant 2fi since their partial t values are small. Although the hidden projection property allows us to estimate any three 2fi's, the original data is sufficiently noisy to mask the significance of any 2fi. In order to detect the significance of some 2fi's, we need to add runs that give the maximum amount of information for this purpose. Suppose the objective is to be able to estimate one more 2fi, say, xy. Since design 5.1 has 11 distinct runs, there are 21 remaining factor level combinations in the 2^5 design. To select one additional run from among the 21 factor level combinations, we use the D criterion for the overall model (five main effects plus xy) and the D_s criterion for estimating xy. It turns out that the combination in run no. 14 of LD is the best, with $D = .96$ and $D_s = .83$ for any xy. (Note that run no. 14 is the mirror image of the two repeated runs, no. 5 and no. 10 in the design matrix of LD.) By adding this run to the original data and repeating the forward selection for model search, the variables are entered in the order: **10**, **7**, **8**, **1**, **3**, **7**×**8** and the partial t value for **7**×**8** is 2.55. The fitted model is

$$\hat{y} = 72.3 + 22.1x_{10} + 16.4x_7 + 11.5x_8 - 8.9x_1 - 6.4x_3 + 2.4x_7x_8$$

and the adjusted R^2 increases from 98.3% to 99.1% by adding x_7x_8. So we can reproduce results and conclusions very close to those in LD by adding only one run.

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