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A SAMPLING PLAN FOR SELECTING THE MOST RELIABLE PRODUCT UNDER THE ARRHENIUS ACCELERATED LIFE TEST MODEL

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Abstract: In product research and development, the decision maker may wish to select the most reliable product from among several competing products. This selection problem is very difficult for highly reliable products, since it may take a long time to observe failures under normal operating condition. To shorten the life-testing time, an accelerated life test is usually used. For life-stress relations following a Weibull-Arrhenius model, this paper proposes an MLR (Modified Likelihood Ratio) rule to select the most reliable product. A suitable sampling plan which is needed by this rule is also derived. An illustrative example is given to demonstrate the proposed rule.

Key words and phrases: Accelerated lift test, Weibull-Arrhenius model, MLR selection rule, sampling plan.

1. Introduction

At the research and development stage, the decision maker usually faces the problem of selecting the most reliable product from among several competing products (populations). Several selection rules have been proposed over the past thirty years. General reference may be found in Bechhofer, Kiefer and Sobel (1968), Gibbons, Olkin and Sobel (1977), Gupta and Panchapakesan (1979), and Gupta and Huang (1981). A comprehensive survey of selection procedures in reliability models was given by Gupta and Panchapakesan (1988). In particular, Berger and Kim (1985) and Kim (1988) proposed some subset selection procedures for exponential populations under Type-I, Type-II and random censored data. Besides, Kingston and Patel (1980, 1982) and Tseng and Chang (1989) studied the problem of selecting the best population from several Weibull populations using an intuitive selection rule under a Type-II censoring plan. Most of these selection rules are based on complete or censored data. However, for highly reliable designs, it is very difficult to measure product reliability since it may take a long time to perform life testing under normal operating condition. Consequently, most of those selection rules mentioned above are not applicable in such a situation.

The Accelerated Life Test (ALT) is a commonly used method for estimating the (product) reliability in a short time. The products are tested at higher stresses and the results are extrapolated, by an assumed model, to estimate the reliability under normal operating condition. When "temperature" is the accelerated factor under consideration, the Arrhenius reaction rate model is often used to describe the relationship of the product parameter (such as failure rate) as a function of operating stress (temperature). Thus, the life of certain products is suitably described by a Weibull distribution whose product characteristic life follows the Arrhenius model, e.g. capacitor dielectric and insulative tape (Nelson (1990, p.82)).

Chang, Huang and Tseng (1992) proposed an intuitive rule for selecting the most reliable design under Type-II ALT. One advantage of this rule is that it has a very clear and simple expression; but it requires heavy numerical computation to obtain a sampling plan. Besides, the information contained in the observed data is not efficiently used. To overcome these drawbacks, we propose a modified likelihood ratio (MLR) selection rule which is obtained by the MLR principle.

This paper is organized as follows. Problem formulation is presented in Section 2. The MLR selection rule is proposed in Section 3 and a suitable sampling plan called the MLR rule is given in Table I. For illustrative purposes, an example is given in Section 4 to demonstrate this MLR rule. A simulation study for the robustness of the proposed rule is presented in Section 5. The theoretical derivation of the MLR selection rule is given in Section 6.

2. Problem Formulation

Let $\Pi_1, \ldots, \Pi_k$ denote k competing products. For $1 \le i \le k$, let $R_i(t, S_0)$ denote the reliability function of Π_i under stress S_0, where S_0 denotes the normal operating condition. Product Π_i is said to be most reliable at time t^* if

$$R_i(t^*, S_0) = \max_{1 \le \ell \le k} R_\ell(t^*, S_0), \tag{1}$$

where t^* is a specific constant (e.g., one-year warranty period) which was predetermined by the experimenter. The experimenter is only interested in selecting the most reliable product at t^*.

Suppose the life testing was conducted at m values of accelerated stresses $\{S_j\}_{j=1}^m$, where $(S_0 \le)\ S_1 \le \cdots \le S_m$. It is assumed that life-stress relation follows a Weibull-Arrhenius model. That is, the life of product Π_i under stress S_j follows a Weibull distribution with an unknown product characteristic life (scale parameter) θ_{ij} and a known shape parameter β_i. Thus, the reliability function of Π_i under stress S_j can be expressed as $R_i(t, S_j) = \exp(-(t/\theta_{ij})^{\beta_i})$, for $t > 0$. Besides, the relationship of θ_{ij} and stress S_j can be expressed as

$$\theta_{ij} = \exp(A_i - B_i/S_j), \tag{2}$$

where (A_i, B_i) denote the unknown parameters of product Π_i in the Arrhenius model. For each combination of (Π_i, S_j), there are n_{ij} units which are put on test to perform an ALT. The experiment of (Π_i, S_j) terminates when r_{ij} failures occur and the ordered failure data $Y_{ij(1)} \leq \cdots \leq Y_{ij(r_{ij})}$ are recorded.

We define the standardized stress v_j (Nelson and Meeker (1978)) as

$$v_j = \frac{(1/S_j) - (1/S_m)}{(1/S_0) - (1/S_m)}. \tag{3}$$

It is easily seen that $1 = v_0 > v_1 > \cdots > v_m = 0$. Further, (2) can be rewritten as

$$\ln \theta_{ij} = \alpha_{i0} + \alpha_{i1} v_j, \tag{4}$$

where $\alpha_{i0} = A_i + B_i(-1/S_m)$ and $\alpha_{i1} = -B_i(1/S_0 - 1/S_m)$.

Let $Z_{ij(\ell)} = \beta_i\{\ln(Y_{ij(\ell)}) - \alpha_{i0} - \alpha_{i1} v_j\}$. Then, the likelihood function of the ith product may be expressed as

$$\prod_{j=1}^{m} \left\{ \prod_{\ell=1}^{r_{ij}} \beta_i f(z_{ij(\ell)}) \right\} \left\{ (1 - F(z_{ij(r_{ij})}))^{n_{ij} - r_{ij}} \right\}, \tag{5}$$

where $f(\cdot)$ and $F(\cdot)$ denote the probability density function (pdf) and cumulative distribution function (cdf) for the standard extreme distribution, respectively. The maximum likelihood estimators (MLE) of α_{i0} and α_{i1}, $(\hat{\alpha}_{i0}, \hat{\alpha}_{i1})$, can be obtained by solving

$$\sum_{j=1}^{m} r_{ij} - \sum_{j=1}^{m} \left\{ \sum_{\ell=1}^{r_{ij}} \exp(z_{ij(\ell)}) + (n_{ij} - r_{ij}) \exp(z_{ij(r_{ij})}) \right\} = 0, \tag{6}$$

and

$$\sum_{j=1}^{m} r_{ij} v_j - \sum_{j=1}^{m} v_j \left\{ \sum_{\ell=1}^{r_{ij}} \exp(z_{ij(\ell)}) + (n_{ij} - r_{ij}) \exp(z_{ij(r_{ij})}) \right\} = 0. \tag{7}$$

Consequently, the MLE of θ_{i0}, $\hat{\theta}_{i0}$, can be obtained from the following equation

$$\hat{\theta}_{i0} = \exp(\hat{\alpha}_{i0} + \hat{\alpha}_{i1}). \tag{8}$$

By the assumption and results mentioned, we have the following lemma.

Lemma 1. *$\ln\hat{\theta}_{i0}^{\beta_i}$ is asymptotically normally distributed with mean $\ln\theta_{i0}^{\beta_i}$ and variance σ_{i0}^2, where*

$$\sigma_{i0}^2=\frac{\Big(\sum_{j=1}^m r_{ij}v_j^2\Big)-2\Big(\sum_{j=1}^m r_{ij}v_j\Big)+\Big(\sum_{j=1}^m r_{ij}\Big)}{\Big(\sum_{j=1}^m r_{ij}\Big)\Big(\sum_{j=1}^m r_{ij}v_j^2\Big)-\Big(\sum_{j=1}^m r_{ij}v_j\Big)^2}. \tag{9}$$

3. MLR Selection Rule and Sampling Plan

Without loss of generality, we assume that $t^*=1$. According to Kingston and Patel (1980), the observed failure time can be scaled so that $t^*=1$. Then (1) can be rewritten as

$$(\theta_{i0})^{\beta_i}=\max_{1\le\ell\le k}(\theta_{\ell 0})^{\beta_\ell}. \tag{10}$$

Based on the ALT data described above, we propose an MLR selection rule $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_k)$ as follows:

$$\delta_i:\text{Select }\Pi_i\text{ if and only if }\prod_{j\neq i}^{k}\left\{\hat{\theta}_{i0}^{\beta_i}/\hat{\theta}_{j0}^{\beta_j}\right\}\ge d. \tag{11}$$

This MLR rule may lead to selecting more than one product if d is too small. In order to select the most reliable one, we introduce a procedure to determine the values of $\{r_{ij},n_{ij}\}$ and d.

We call the selection rule δ_i a correct selection (CS) if the selected Π_i is the most reliable product, and an error selection otherwise. Let $P_{\boldsymbol{\tau}}(\text{CS}|\delta_i)$ denote the probability of CS of rule δ_i under $\boldsymbol{\tau}=(\tau_{i1},\ldots,\tau_{ii-1},\tau_{ii+1},\ldots,\tau_{ik})$, where τ_{ij} denotes the measure of separation of products Π_i and Π_j. We can suitably define $\tau_{ij}=\ln\{\ln R_j(t^*)/\ln R_i(t^*)\}$. As $t^*=1$, τ_{ij} can be expressed as $\ln(\theta_{i0}^{\beta_i}/\theta_{j0}^{\beta_j})$. Thus, the ith preference region, Ω_i, may be suitably defined as

$$\Omega_i=\{\boldsymbol{\tau}|\tau_{ij}\ge\Delta,\ \text{for } j\neq i\},\quad \Delta>0. \tag{12}$$

Similarly, the indifference region, Ω_0, can be defined as

$$\Omega_0=\{\boldsymbol{\tau}|\tau_{ij}=0,\ \text{for } j\neq i\}. \tag{13}$$

It is usually required that the probability of CS have a minimum value P^* for $\boldsymbol{\tau}\in\Omega_i$ (P^*-condition) and the error probability have a maximum value α^* for

$\boldsymbol{\tau} \in \Omega_0$ (α^*-condition). That is, the P^* and α^* conditions can be expressed as (Tseng and Chang (1989))

$$\inf_{\boldsymbol{\tau} \in \Omega_i} \mathrm{P}_{\boldsymbol{\tau}}(\mathrm{CS}|\delta_i) \geq P^*,$$

and

$$\sup_{\boldsymbol{\tau} \in \Omega_0} \mathrm{P}_{\boldsymbol{\tau}}(\mathrm{CS}|\delta_i) \leq \alpha^*.$$

In case of $r_{ij} = r_j$ and $n_{ij} = n_j$, for $1 \leq i \leq k$ and $1 \leq j \leq m$, these two conditions are asymptotically approximated, utilizing Theorem 3 stated in Appendix, by

$$\Phi\left(\frac{\ln d - (k-1)\Delta}{\sqrt{k(k-1)\sigma_0^2}}\right) \leq (1 - P^*), \tag{14}$$

and

$$\Phi\left(\frac{\ln d}{\sqrt{k(k-1)\sigma_0^2}}\right) \geq (1 - \alpha^*), \tag{15}$$

where $\Phi(\cdot)$ denotes the cdf of a standard normal distribution. Furthermore, a decision maker usually wants to control the time-saving factor of life testing at a specific level ρ^* (ρ^*-condition). So, the time saving factor can be defined as follows (Tseng and Chang (1989)):

$$\frac{E(Y_{ij(r_{ij})})}{E(Y_{ij(n_{ij})})} \leq \rho^*, \quad \text{for each combination of} \quad (\Pi_i, S_j). \tag{16}$$

Let R_b denote the reliability of the most reliable product at normal operating condition and R_a denote the reliability of $(k-1)$ less reliable products at normal operating condition. Then Δ can be expressed as $\ln(\ln R_a / \ln R_b)$. Set $r_j = ra_j$, where $\{a_j\}_{j=1}^m$ are predetermined. We state an algorithm to compute $\{(n_j, r_j)\}_{j=1}^m$ and d under various combinations of $k, P^*, \alpha^*, \rho^*, \Delta$, $(a_1, \ldots, a_m)$ and stresses $S_1 < \cdots < S_m$ $(1 = v_0 \geq v_1 \geq \cdots \geq v_m = 0)$.

Step 1. Start with $r = 1$.

Step 2. Multiplying $(a_1, \ldots, a_m)$ by r, we get the associated value of $(r_1, \ldots, r_m)$.

Step 3. Compute d by (14).

Step 4. Check by (15). If it holds, then $(r_1, \ldots, r_m)$ is a feasible solution; go to Step 5. Otherwise, set $r = (r + 1)$ and return to Step 2.

Step 5. Compute the corresponding sample size $(n_1, \ldots, n_m)$ from (16).

For illustrative purpose, three higher levels of accelerated stresses are considered (i.e., $m = 3$). Denote by L, M and H the low, middle and high stresses.

Meeker and Hahn (1985) suggested $(a_L, a_M, a_H) = (4, 2, 1)$ as an efficient allocation for censored ALT. Given $P^* = 0.90$, $\alpha^* = (1/k)$, $\rho^* = 0.50$ and $(v_L, v_M, v_H) = (0.5, 0.25, 0.0)$, we compute the number of failures (r_L, r_M, r_H), sample sizes (n_L, n_M, n_H) and critical value d for various combinations of (R_a, R_b) and k. The results are given in Table I.

4. An Illustrative Example

Problem description:

Suppose there are four competing highly reliable products of capacitor dielectric, where temperature is the accelerating variable (Nelson (1990)). From historical field data, it is known that the life-stress relation follows a Weibull-Arrhenius model with a common shape parameter which approximates 1.25. Suppose the operating condition S_0 is 85°c (358°k). Three higher stresses, say $S_L = 115°c$ (388°k), $S_M = 132°c$ (405°k), and $S_H = 150°c$ (423°k), are chosen for performing an accelerated life test. The corresponding standardized stresses are $v_L = 0.50$, $v_M = 0.25$ and $v_H = 0$. Based on the ALT data, the decision maker is interested in selecting the most reliable product. If we wish to control the quality of the decision such that the P^*-condition achieves 0.90, the α^*-condition achieves $(1/k) = (1/4)$ and the ρ^*-condition achieve 0.50, then two questions arise:

(1) How to determine the sample size and the number of failures for each combination of (Π_i, S_j)?
(2) What is the selection rule?

Answer:

Since Δ is unknown, we propose a two-stage procedure to determine a sampling plan as follows:

Stage 1

From Table I, we take the smallest sample sizes and the number of failures, among the combination of $(\beta, k) = (1.25, 4)$, to perform a pilot ALT. The corresponding sample sizes and the number of failures are $(n_L^0, n_M^0, n_H^0) = (34, 18, 10)$ and $(r_L^0, r_M^0, r_H^0) = (28, 14, 7)$. A pilot ALT is performed as follows:

Under 115°c, 132°c and 150°c, there are $30, 16$ and 9 items on test for each product. The experiment terminates when the number of failures for each product reaches $24, 12$ and 6 items. Based on the failure data, using (6)-(8) we may obtain $(\hat{\alpha}_{i0}, \hat{\alpha}_{i1})$ and $\hat{\theta}_{i0}$. Consequently, we obtain an approximate value of Δ. For illustrative purposes, it is assumed that $\Delta \approx 0.917$.

Stage 2

In Table I, as $\Delta = 0.9171$ and $(\beta, k) = (1.25, 4)$, we find $(n_L, n_M, n_H) = (56, 30, 16)$, $(r_L, r_M, r_H) = (48, 24, 12)$ and $d = 2.6484$. Thus, for each product, we shall add 22, 12 and 6 items on test under $115°c$, $132°c$ and $150°c$, respectively. The test will be terminated when the number of failures for each experiment reaches $r_L = 48$, $r_M = 24$ and $r_H = 12$.

Based on these failure data, we compute $\hat{\theta}_{i0}$ for $1 \le i \le 4$. The selection rule can be expressed as: Select Π_i if and only if

$$\Pi^4_{j\neq i}\left\{\hat{\theta}^{\beta_i}_{i0}/\hat{\theta}^{\beta_j}_{j0}\right\} \ge 2.65.$$

Using this selection procedure, we have at least 90% confidence to select the most reliable product.

5. A Simulation Study

The computation of the sampling plan in Table I is based on the asymptotic normal approximation. Besides, this sampling plan is affected by variation of β. Therefore, we perform a simulation to investigate the robustness of the proposed selection rule.

Suppose the shape parameter β follows a Beta distribution with parameter (5,5) on the interval $\beta^*(1\pm\epsilon)$, where β^* is the mean value of β. For each combination of (β^*, k) and (R_a, R_b), five hundred simulation runs are conducted by using the sampling plan given in Table I. A selection trial leads to a correct selection whenever (11) holds. The proportion of correct selection (PCS) is calculated. The results are given in Table II, where the PCS values are the average of all combinations of (R_a, R_b). From the results, it is seen that the total average PCS (0.865) is close to the predetermined value P^* (0.90). This suggests that the performance of the MLR rule is insensitive to moderate variations in β.

The accuracy of the estimated Δ (in Stage 1 of the illustrative example) also affects the determination of this sampling plan. For each Δ, we use the smallest sample sizes and the number of failures (as described in Section 4) to estimate the true value of Δ. Five hundred simulation runs were conducted under each combination of β and k. The results are given in Table III, where $E(\Delta, \hat{\Delta})$ denotes the absolute error (in percentage) of $\hat{\Delta}$ with respect to Δ. It is seen that all the values of $E(\Delta, \hat{\Delta})$ are within 3%. Thus, the procedure in Stage 1 provides a satisfactory estimation of Δ.

6. Theoretical Derivation of The MLR Selection Rule

Let $\Pi_1, \ldots, \Pi_k$ be k independent products and $S_1 \le S_2 \le \cdots \le S_m$ denote m different levels of accelerated stresses. For each cell of (Π_i, S_j), n_{ij} items are exposed to a life test. Let $Y_{ij(1)} \le \cdots \le Y_{ij(r_{ij})}$ denote the first r_{ij} ordered failure data. Suppose the life distribution for each cell of (Π_i, S_j) follows a Weibull distribution with scale parameter θ_{ij} and shape parameter β_i. The relationship between θ_{ij} and S_j in the Arrhenius model can be expressed as (2) and the MLE of θ_{i0} can be computed by (8). Since $\tau_{ij} = \ln(\theta_{i0}^{\beta_i}/\theta_{j0}^{\beta_j})$, we define $t_{ij} = \ln(\hat{\theta}_{i0}^{\hat{\beta}_i}/\hat{\theta}_{j0}^{\hat{\beta}_j})$. Under the assumption that $r_{ij} = r_j$, for $j = 1, \ldots, m$ and with the notation stated previously, we have the following result:

Theorem 1. *The joint* pdf *of* $\boldsymbol{T} = (t_{i1}, \ldots, t_{ii-1}, t_{ii+1}, \ldots, t_{ik})$ *under* $\boldsymbol{\tau}$ *is*

$$h_{\boldsymbol{T}}(\boldsymbol{t}) = \frac{1}{(2\pi)^{k-1/2}|\Sigma_T|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{t}-\boldsymbol{\tau})'\Sigma_T^{-1}(\boldsymbol{t}-\boldsymbol{\tau})\right], \tag{17}$$

where

$$\Sigma_T = \sigma_0^2 \begin{bmatrix} 2 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 2 \end{bmatrix}_{(k-1)\times(k-1)}$$

and

$$\sigma_0^2 = \frac{\left(\sum_{j=1}^m r_j v_j^2\right) - 2\left(\sum_{j=1}^m r_j v_j\right) + \left(\sum_{j=1}^m r_j\right)}{\left(\sum_{j=1}^m r_j\right)\left(\sum_{j=1}^m r_j v_j^2\right) - \left(\sum_{j=1}^m r_j v_j\right)^2}.$$

Proof. (see the appendix).

In order to construct a suitable selection rule, we condsider a family of hypotheses as follows:

$$H_0 : \boldsymbol{\tau} \in \Omega_0 \quad \text{vs} \quad H_i : \boldsymbol{\tau} \in \Omega_i, \quad i = 1, \ldots, k.$$

The MLR selection rule $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ can be defined as

$$\delta_i : \text{Select } \Pi_i \text{ if and only if } \frac{\inf_{\boldsymbol{\tau}\in\Omega_i} h_{\boldsymbol{T}}(\boldsymbol{t})}{\sup_{\boldsymbol{\tau}\in\Omega_0} h_{\boldsymbol{T}}(\boldsymbol{t})} \geq c, \tag{18}$$

where c is a constant. From Theorem 1, we have the following result.

Theorem 2. *The MLR selection rule can be expressed approximately as*

$$\delta_i : \textit{Select } \Pi_i \textit{ if and only if } \prod_{j\neq i}^{k} \left\{\hat{\theta}_{i0}^{\hat{\beta}_i}/\hat{\theta}_{j0}^{\hat{\beta}_j}\right\} \geq d, \tag{19}$$

where d is a constant to be determined.

Proof. (see the appendix).

7. Conclusion

At R & D stage, a decision maker usually faces the problem of selecting the most reliable product from among several competing products. For highly reliable products, the ALT is usually used to get the information of product reliability within a short time of life test. Based on the ALT data, this paper proposes an MLR selection rule to select the most reliable product when the life-stress relation follows a Weibull-Arrhenius model where the shape parameter β_i is known. For the case where β_i is unknown, the reader may refer to the work of Tseng (1991). Besides, life-testing time needed by Type-II censoring is shorter than that of Type-I censoring (Tseng (1991)). Thus, in this paper we restrict our discussion to the case of Type-II censoring and derive a suitable sampling plan for selecting the most reliable design in this case.

Appendix

Proof of Lemma 1

The Fisher information matrix of $(\alpha_{i0}, \alpha_{i1})$ can be expressed as

$$\beta_i^2 \begin{bmatrix} \sum_{j=1}^{m} r_{ij} & \sum_{j=1}^{m} r_{ij} v_{ij} \\ \sum_{j=1}^{m} r_{ij} v_{ij} & \sum_{j=1}^{m} r_{ij} v_{ij}^2 \end{bmatrix}.$$

Since $\ln \hat{\theta}_{i0} = (\hat{\alpha}_{i0} + \hat{\alpha}_{i1})$, $\ln \hat{\theta}_{i0}^{\beta_i}$ is asymptotically normally distributed with mean $\ln \theta_{i0}^{\beta_i}$ and variance σ_{i0}^2.

Proof of Theorem 1

Let

$$\boldsymbol{Y} = \begin{bmatrix} \beta_1 \ln \hat{\theta}_{10} \\ \vdots \\ \beta_k \ln \hat{\theta}_{k0} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \beta_1 \ln \theta_{10} \\ \vdots \\ \beta_k \ln \theta_{k0} \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_{10}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{k0}^2 \end{bmatrix}.$$

By Lemma 1, it is seen that $\boldsymbol{Y}$ is asymptotically distributed with $N(\boldsymbol{\mu}, \Sigma)$. Since

$\boldsymbol{T} = H\boldsymbol{Y}$, where

$$H = \left[\begin{array}{ccccc|ccccc} -1 & & & & 1 & & & & \\ & -1 & & & \vdots & & & 0 & \\ & & \ddots & & \vdots & & & & \\ & & & -1 & 1 & & & & \\ \hline & & & & 1 & -1 & & & \\ & & & & \vdots & & -1 & & \\ & 0 & & & \vdots & & & \ddots & \\ & & & & 1 & & & & -1 \end{array}\right] \quad (\downarrow i\text{th column at the column of } 1\text{'s}).$$

So, $\boldsymbol{T}$ is asymptotically distributed with $N(\boldsymbol{\mu}_T, \Sigma_T)$, where

$$\boldsymbol{\mu}_T = H\boldsymbol{\mu} = \boldsymbol{\tau} = \begin{bmatrix} \beta_i \ln\theta_{i0} - \beta_1 \ln\theta_{10} \\ \vdots \\ \beta_i \ln\theta_{i0} - \beta_k \ln\theta_{k0} \end{bmatrix}$$

and

$$\Sigma_T = H\Sigma H' = \left[\begin{array}{ccc|ccc} \sigma_{10}^2 + \sigma_{i0}^2 & & \sigma_{i0}^2 & & & \sigma_{i0}^2 \\ & \ddots & & & & \\ \sigma_{i0}^2 & & \sigma_{i-10}^2 + \sigma_{i0}^2 & & & \\ \hline & & & \sigma_{i+10}^2 + \sigma_{i0}^2 & & \sigma_{i0}^2 \\ & & & & \ddots & \\ & \sigma_{i0}^2 & & \sigma_{i0}^2 & & \sigma_{k0}^2 + \sigma_{i0}^2 \end{array}\right].$$

Under the condition that $r_{ij} = r_j$, we have $\sigma_{i0}^2 = \sigma_0^2$, $1 \le i \le k$. So Σ_T can be reduced to

$$\Sigma_T = \sigma_0^2 \begin{bmatrix} 2 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 2 \end{bmatrix}_{(k-1)\times(k-1)}$$

Proof of Theorem 2

Since $\inf_{\boldsymbol{\tau}\in\Omega_i} h_{\boldsymbol{\tau}}(\boldsymbol{t})$ depends on the value of $\boldsymbol{t}$, it is rather complicated to derive the MLR rule. Instead of using Ω_i in (12), we restrict our attention to Ω_i^*, where

$$\Omega_i^* = \{\boldsymbol{\tau}|\tau_{ij} = \Delta, \text{ for } j \neq i\}, \quad \Delta > 0.$$

Then (18) can be approximately expressed as

$$\left\{ \exp\left(-\frac{1}{2}(\boldsymbol{t}-\boldsymbol{\Delta})'\Sigma_T^{-1}(\boldsymbol{t}-\boldsymbol{\Delta})\right) \Big/ \exp\left(-\frac{1}{2}(\boldsymbol{t}-\boldsymbol{0})'\Sigma_T^{-1}(\boldsymbol{t}-\boldsymbol{0})\right) \right\} \geq c.$$

It can rewritten as

$$2\boldsymbol{t}'(\Sigma_T)^{-1}\boldsymbol{\Delta} - \boldsymbol{\Delta}'(\Sigma_T)^{-1}\boldsymbol{\Delta} \geq c_1,$$

where c_1 is a constant. Since

$$\boldsymbol{t}'(\Sigma_T)^{-1}\boldsymbol{\Delta} = \frac{\Delta}{\sigma_0^2}\frac{1}{k}\left(\sum_{j\neq i}^{k} t_{ij}\right) \quad \text{and} \quad \boldsymbol{\Delta}'(\Sigma_T)^{-1}\boldsymbol{\Delta} = \frac{\Delta^2}{\sigma_0^2}\left(\frac{k-1}{k}\right),$$

(18) can be expressed as

$$\sum_{j\neq i} t_{ij} \geq c_2,$$

where c_2 is a constant. Since $t_{ij} = \ln(\hat{\theta}_{i0}^{\beta_i}/\hat{\theta}_{j0}^{\beta_j})$, we have

$$\prod_{j\neq i}^{k}\left\{\hat{\theta}_{i0}^{\beta_i}/\hat{\theta}_{j0}^{\beta_j}\right\} \geq d,$$

where d is a constant to be determined.

Theorem 3. *The P^*-condition and α^*-condition are equivalent to* (14) *and* (15), *respectively.*

Proof of Theorem 3

$$\begin{aligned}
P_{\boldsymbol{\tau}}(\text{CS}|\delta_i) &= P_{\boldsymbol{\tau}}\left(\prod_{j\neq i}\left((\hat{\theta}_{i0})^{\beta_i}/(\hat{\theta}_{j0})^{\beta_j} \geq d\right)\right) \\
&= P_{\boldsymbol{\tau}}\left((k-1)\ln(\hat{\theta}_{i0})^{\beta_i} - \sum_{j\neq i}^{k}\ln(\hat{\theta}_{j0})^{\beta_j} \geq \ln d\right) \\
&= P_{\boldsymbol{\tau}}\left(Z \geq \frac{\ln d - \sum_{j\neq i}\left\{\ln(\theta_{i0})^{\beta_i} - \ln(\theta_{j0})^{\beta_j}\right\}}{\sqrt{k(k-1)\sigma_0^2}}\right),
\end{aligned}$$

where Z denotes the standard normal density. Thus P^*-condition and α^*-condition can be expressed by (14) *and* (15), *respectively.*

Table I. (r_L, r_M, r_H), (n_L, n_M, n_H) and d, in the MLR selection rule, under various combinations of β and (R_a, R_b)

β	$(R_a, R_b), \Delta$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.75	(0.9975, 0.999) $\Delta = 0.917$	32, 16, 8 35, 18, 10 1.03	44, 22, 11 47, 25, 13 1.69	48, 24, 12 51, 27, 14 2.65	52, 26, 13 56, 29, 15 4.32	56, 28, 14 60, 31, 16 7.27
	(0.9970, 0.999) $\Delta = 1.097$	24, 12, 6 27, 14, 8 1.08	28, 14, 7 31, 16, 9 1.74	36, 18, 9 39, 20, 11 3.48	36, 18, 9 39, 20, 11 5.75	40, 20, 10 43, 22, 12 11.25
	(0.9965, 0.999) $\Delta = 1.254$	20, 10, 5 22, 12, 6 1.14	24, 12, 6 27, 14, 8 2.08	28, 14, 7 31, 16, 9 4.20	28, 14, 7 31, 16, 9 7.48	32, 16, 8 35, 18, 10 16.92
1.00	(0.9975, 0.999) $\Delta = 0.917$	32, 16, 8 37, 19, 10 1.03	44, 22, 11 49, 26, 14 1.69	48, 24, 12 53, 28, 15 2.65	52, 26, 13 58, 30, 16 4.32	56, 28, 14 62, 32, 17 7.27
	(0.9970, 0.999) $\Delta = 1.097$	24, 12, 6 28, 15, 8 1.08	28, 14, 7 42, 17, 9 1.74	36, 18, 9 41, 21, 12 3.48	36, 18, 9 41, 21, 12 5.75	40, 20, 10 45, 24, 13 11.25
	(0.9965, 0.999) $\Delta = 1.254$	20, 10, 5 24, 13, 7 1.14	24, 12, 6 28, 15, 8 2.08	28, 14, 7 32, 17, 9 4.20	28, 14, 7 32, 17, 9 7.48	32, 16, 8 37, 19, 10 16.92
1.25	(0.9975, 0.999) $\Delta = 0.917$	32, 16, 8 39, 21, 11 1.03	44, 22, 11 52, 27, 15 1.69	48, 24, 12 56, 30, 16 2.65	52, 26, 13 61, 32, 17 4.32	56, 28, 14 65, 34, 18 7.27
	(0.9970, 0.999) $\Delta = 1.097$	24, 12, 6 30, 16, 9 1.08	28, 14, 7 34, 18, 10 1.74	36, 18, 9 42, 23, 12 3.48	36, 18, 9 43, 23, 12 5.75	40, 20, 10 47, 25, 14 11.25
	(0.9965, 0.999) $\Delta = 1.254$	20, 10, 5 25, 14, 8 1.14	24, 12, 6 30, 16, 9 2.08	28, 14, 7 34, 18, 10 4.20	28, 14, 7 34, 18, 10 7.48	32, 16, 8 39, 21, 11 16.92
1.50	(0.9975, 0.999) $\Delta = 0.917$	32, 16, 8 41, 22, 12 1.03	44, 22, 11 55, 29, 16 1.69	48, 24, 12 60, 32, 17 2.65	52, 26, 13 64, 34, 18 4.32	56, 28, 14 69, 36, 20 7.27
	(0.9970, 0.999) $\Delta = 1.097$	24, 12, 6 32, 17, 9 1.08	28, 14, 7 36, 20, 11 1.74	36, 18, 9 46, 25, 13 3.48	36, 18, 9 46, 25, 13 5.75	40, 20, 10 50, 27, 15 11.25
	(0.9965, 0.999) $\Delta = 1.254$	20, 10, 5 27, 15, 8 1.14	24, 12, 6 32, 17, 9 2.08	28, 14, 7 36, 20, 11 4.20	28, 14, 7 36, 20, 11 7.48	32, 16, 8 41, 22, 12 16.92

Table II. A simulation study of PCS for the MLR rule under Beta distribution for β

β^*	$\varepsilon(\%)$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.75	0.01	.887	.872	.892	.886	.873
	0.03	.882	.865	.869	.891	.855
	0.05	.862	.833	.856	.857	.886
1.00	0.01	.863	.858	.892	.872	.850
	0.03	.843	.811	.852	.855	.869
	0.05	.829	.918	.902	.883	.891
1.25	0.01	.884	.875	.866	.864	.876
	0.03	.817	.879	.843	.818	.854
	0.05	.888	.866	.821	.856	.856
1.50	0.01	.891	.861	.903	.864	.901
	0.03	.861	.844	.856	.865	.884
	0.05	.887	.862	.865	.810	.855

Note: average PCS across the table is 0.865

Table III. A simulation study of the accuracy of Δ

β	(R_a, R_b)	Δ	$\hat{\Delta}$	$E(\Delta, \hat{\Delta})(\%)$
0.75	(.9975, .999)	0.917	0.930	1.42
	(.9970, .999)	1.097	1.113	1.46
	(.9965, .999)	1.254	1.265	0.88
1.00	(.9975, .999)	0.917	0.906	1.20
	(.9970, .999)	1.097	1.089	0.73
	(.9965, .999)	1.254	1.245	0.72
1.25	(.9975, .999)	0.917	0.892	2.73
	(.9970, .999)	1.097	1.082	1.37
	(.9965, .999)	1.254	1.234	1.60
1.50	(.9975, .999)	0.917	0.904	1.42
	(.9970, .999)	1.097	1.116	1.73
	(.9965, .999)	1.254	1.259	0.40

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