Statistica Sinica 1(1991), 229-232

TWO CONSTRUCTIONS FOR BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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Abstract: Difference techniques are used to obtain two constructions for balanced incomplete block designs with nested rows and columns. They include some previously available series of designs as special cases.

Key words and phrases: Balanced incomplete block design, nested row-column design, method of differences, finite fields.

1. Introduction

In this note we give two constructions for balanced incomplete block designs with nested rows and columns (BIBRC's). These designs, introduced by Singh and Dey (1979), are arrangements of v treatments in b blocks of size $k = pq$, where each block is a $p \times q$ array of p rows and q columns (hence the 'nesting' of rows and columns within blocks), so that

(i) a treatment occurs at most once in each block,
(ii) each treatment occurs in exactly r blocks, and
(iii) $pqrI_v - pN_1N_1^T - qN_2N_2^T + NN^T = \lambda(vI_v - J_v)$.

Here I_v is the order v identity matrix, J_v is a $v \times v$ matrix of 1's, N_1, N_2 and N are respectively the incidence matrices for rows, columns and blocks, and $\lambda = r(p-1)(q-1)/(v-1)$.

Sufficient for the above conditions to hold are that the bp rows taken together form a balanced incomplete block design (BIBD) with bp blocks of size q, the bq columns taken together form a BIBD with bq blocks of size p, and the b blocks ignoring nested rows and columns form a BIBD with block size pq. Such BIBRC's have been said to belong to 'Series A' by Agrawal and Prasad (1982b), who used them in the construction of partially balanced incomplete block designs with nested rows and columns. Series A designs are also generally balanced in the sense of Houtman and Speed (1983). Both of our results are for Series A designs.

Series constructions for BIBRC's have previously been given by Street (1981), Agrawal and Prasad (1982a, 1983), Cheng (1986), Sreenath (1989), and Uddin and Morgan (1990). Some of their results occur as special cases of our two theorems. A few designs are included in the listings of Preece (1967) and Ipinyomi and John (1985). For analysis and discussion of BIBRC's the reader is referred to the original paper of Singh and Dey (1979).

2. The Constructions

Theorem 1. *Let $v = mq + 1$ be a prime or prime power. Then there exists a* BIBRC *with $b = mv$, $r = mpq$, p, q, and $\lambda = p(p-1)(q-1)$ for $2 \leq p \leq m$.*

Proof. For $i = 1, 2, \ldots, m$ let

$$A_i = \begin{pmatrix} x^{i-1} & x^{m+i-1} & \ldots & x^{(q-1)m+i-1} \\ x^{i} & x^{m+i} & \ldots & x^{(q-1)m+i} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p+i-2} & x^{m+p+i-2} & \ldots & x^{(q-1)m+p+i-2} \end{pmatrix},$$

where x is a primitive element of the Galois field GF_v of order v. We will show that $A_1, A_2, \ldots, A_m$ are initial blocks for the required BIBRC by showing that the Series A properties hold, that is, that the sets of mp rows, mq columns, and m blocks given by the A_i's each generate a BIBD. So, for instance, it must be demonstrated that the $mqp(p-1)$ symmetric differences arising from within the mq columns of the A_i's are every non-zero field element with frequency $p(p-1)$; similar comments apply for the differences within rows and within blocks.

Row h of A_i is $A_{ih} = (x^{i+h-2}, x^{m+i+h-2}, \ldots, x^{(q-1)m+i+h-2})$, so by comparison with Theorem 2.1 of Sprott (1954) the result for rows is established, $A_{1h}, A_{2h}, \ldots, A_{mh}$ being initial blocks for a BIBD.

The symmetric column differences for rows h and h' in A_i are $\pm(A_{ih} - A_{ih'}) = \pm(x^{h-1} - x^{h'-1})(x^{i-1}, x^{m+i-1}, \ldots, x^{(q-1)m+i-1})$. Letting i range from 1 to m to look over all m initial blocks, these become $\pm(x^{h-1} - x^{h'-1})(x^0, x^1, \ldots, x^{qm-1})$; that is, each non-zero element occurs exactly twice. So the columns of A_1, $A_2, \ldots, A_m$ generate a BIBD, since any two rows of these columns do.

To see that the A_i's themselves are a set of initial blocks for a BIBD, we need only look at elements in different rows and different columns, since it has already been established that the row and column differences are balanced. The symmetric differences between elements of different columns of A_{ih} and $A_{ih'}$ are

$$\begin{aligned} &\pm (x^0, x^m, \ldots, x^{(q-1)m})(x^{i+h-2} - x^{lm+i+h'-2}) \\ &= \pm(x^i, x^{m+i}, \ldots, x^{(q-1)m+i})(1 - x^{lm+h'-h})x^{h-2}, \quad l = 1, 2, \ldots, q-1. \end{aligned}$$

Letting i range from 1 to m shows that these are $2(q-1)$ copies of the non-zero elements of GF_v. This establishes the result.

As special cases of Theorem 1 we get several series of designs with the same parameters as some previously appearing in the literature. Setting $m = tp$ gives Theorem 4 of Agrawal and Prasad (1982a). Theorem 3 of Agrawal and Prasad (1983) is the case $p = q = 2$. Theorems 1(a) and 1(c) of Uddin and Morgan (1990) follow for $m \geq p$ by setting $q = 2t$ and $p = 2t$ and $2t+1$, respectively. Theorem 2.5 of Sreenath (1989) is generalized by setting $m = 2$ and $q = (v-1)/2$.

Theorem 2. *Let* $v = 2mq + 1$ *be a prime or prime power where* q *is odd. Then there exists a* BIBRC *with* $b = mv$, $r = mpq$, p, q, *and* $\lambda = p(p-1)(q-1)/2$ *for* $2 \leq p \leq 2m$.

Proof. The initial blocks are, for $i = 1, 2, \ldots, m$,

$$A_i = \begin{pmatrix} x^{i-1} & x^{2m+i-1} & \cdots & x^{2(q-1)m+i-1} \\ x^{i} & x^{2m+i} & \cdots & x^{2(q-1)m+i} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p+i-2} & x^{2m+p+i-2} & \cdots & x^{2(q-1)m+p+i-2} \end{pmatrix}.$$

That the row differences are balanced follows from Theorem 3.1 of Sprott (1954). Again writing A_{ih} for row h of A_i, the symmetric column differences for rows h and h' of A_i are

$$\begin{aligned} \pm(A_{ih} - A_{ih'}) &= \pm x^{i-1}(x^{h-1} - x^{h'-1})(x^0, x^{2m}, \ldots, x^{2(q-1)m}) \\ &= x^{i-1}(x^{h-1} - x^{h'-1})(x^0, x^m, \ldots, x^{(2q-1)m}) \end{aligned}$$

since $-1 = x^{qm}$ and q is odd. As i ranges from 1 to m these give each non-zero element exactly once. Finally, the symmetric differences for elements in different columns of A_{ih} and $A_{ih'}$ are

$$\begin{aligned} &\pm (x^{i+h-2} - x^{2lm+i+h'-2})(x^0, x^{2m}, \ldots, x^{2(q-1)m}) \\ &= (x^i, x^{m+i}, \ldots, x^{(2q-1)m+i})(1 - x^{2lm+h'-h})x^{h-2}, \quad l = 1, 2, \ldots, q-1, \end{aligned}$$

which again for $i = 1, 2, \ldots, m$ gives a balanced set.

Theorem 6 of Street (1981) gives designs with the same parameters as Theorem 2, but is restricted to $p \leq m$. For $m \geq p/2$, the series of Theorems 2(a), 2(c), 3(a), and 3(b) of Uddin and Morgan (1990) result from Theorem 2 by setting $q = t$ and $p = t, t+1, 2t$, and $2t+1$ respectively. Sreenath's (1989) Theorem 2.2

is generalized by taking $m = 1$ and $q = (v-1)/2$, while his Theorem 2.3 is the case $p = 2$ and $q = 3$.

Example. For $v = 19$, a BIBRC with $p = 5$, $q = 3$ is obtained by developing these $m = 3$ initial blocks constructed using $x = 2$.

1	7	11	2	14	3	4	9	6
2	14	3	4	9	6	8	18	12
4	9	6	8	18	12	16	17	5
8	18	12	16	17	5	13	15	10
16	17	5	13	15	10	7	11	1

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(Received September 1989; accepted June 1990)