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A CLASS OF OPTIMAL ROW-COLUMN DESIGNS WITH SOME EMPTY CELLS

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Abstract: Agrawal (1966) constructed a series of row column designs with row-column incidence structure $J - I$. We show that there exist designs which strongly dominate Agrawal's designs for all $v \geq 4$ and are therefore to be preferred with respect to any optimality criterion. These dominating designs are E-optimal within the entire class of such row-column designs and are also highly A- and D-efficient. Some methods for constructing such designs are also developed.

Key words and phrases: Structurally incomplete row-column design, row-column incidence structure, A-, D-, E-optimality, A- and D-efficiency.

1. Introduction

The study of optimality of structurally incomplete row-column designs where treatments are allocated to some but not all of the combinations of rows and columns has recently drawn considerable attention — see for example, Saharay (1986), Shah and Sinha (1990), Stewart and Bradley (1991), Heiligers and Sinha (1993). Shah and Sinha (1990) and Heiligers and Sinha (1993) investigated optimality aspects of the four types of structurally incomplete row-column designs, constructed by Agrawal (1966), for which the row-column incidence matrix is that of a *balanced incomplete block design* (BIBD). In most cases Agrawal designs admit a completely symmetric (c.s.) C-matrix for each classification (i.e. treatment, row or column). In view of the very strong optimality of BIBD's one would expect them to be optimal. Interestingly enough, Shah and Sinha (1990) and Heiligers and Sinha (1993) came up with better designs for some values of v . For the specific $v \times v$ row column set up with empty cells throughout the principal diagonal, used for comparing v treatments, Shah and Sinha (1990) obtained better designs for $v = 7$ with respect to (w.r.t) the D-optimality criterion and conjectured that Agrawal designs would be A-optimal for all v. In the present article we restrict our attention to this sort of structurally incomplete row-column set up and examine the prospect of A- and E-optimality of Agrawal designs. It is shown in Section 3 that there exist designs which strongly dominate Agrawal designs for all $v \geq 4$ and are therefore to be preferred to the latter designs w.r.t any meaningful optimality criterion. Finally, these dominating designs are proven to be E-optimal for $v \geq 4$ within the entire class and A- and D-optimal for $4 \leq v \leq 14$ within the binary and equireplicate class of connected row-column designs of the above structure. For larger values of v these designs are highly A- and D-efficient. The methods of constructing these designs are discussed in Section 4.

2. Preliminaries

Let d denote a design used for comparing v treatments applied to experimental units arranged in v rows and v columns such that in each row and in each column there is exactly one empty cell. Without loss of generality (w.l.o.g), the rows and the columns of the experimental set up can be rearranged so that the row-column incidence structure assumes the form $J - I$. Throughout the paper we assume the usual fixed effects additive model with uncorrelated and homoscedastic errors. For a design d, let $L_d = ((l_{dhj}))$, $M_d = ((m_{dhj}))$ and $N_d = J - I$ stand, respectively, for treatment-row, treatment-column and row-column incidence matrices. The C-matrix for treatment effects of such a design d can be written as (cf. Saharay (1986))

$$C_d = D_{r_d} - (L_d L_d')/v - (M_d M_d')/v - ((L_d + M_d)(L_d + M_d)')/v(v-2) + (r_d r_d')/(v-1)(v-2), \tag{2.1}$$

where $D_{r_d} = \text{diag}(r_{d1}, \ldots, r_{dv}), r_d = (r_{d1}, \ldots, r_{dv})$ and r_{di} = replication of treatment i.

Let Ω denote the class of all connected $v \times v\,(v \geq 4)$ row-column designs described above and V denote the set of v treatments. Any design d$\in \Omega$ is said to be binary if both L_d and M_d are (0,1)-matrices. Let $0 = \lambda_{d0} < \lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d(v-1)}$ denote the eigenvalues of the C_d matrix. A design $d^\star$ is said to be A-, D- and E-optimal in a relevant class Ω_0 if it minimizes $\sum_{i=1}^{v-1} \lambda_{di}^{-1}$, $\prod_{i=1}^{v-1} \lambda_{di}^{-1}$ and λ_{d1}^{-1} respectively among the designs in Ω_0. The A- and D-efficiency of $d \in \Omega_0$ is defined to be $\sum \lambda_{d^\star i}^{-1} / \sum \lambda_{di}^{-1}$ and $\{\prod \lambda_{d^\star i}^{-1} / \prod \lambda_{di}^{-1}\}^{1/(v-1)}$ respectively. A design $d_1 \in \Omega$ is said to *strongly dominate* another design d_2 if $C_{d_1} - C_{d_2}$ is n.n.d.. We refer to Shah and Sinha (1989) for a detailed discussion on optimality criteria.

Let P be a permutation matrix of order v which can be written in the form $P = [e_{i_1} : e_{i_2} : \ldots : e_{i_v}]$ where e_i is a $v \times 1$ vector with 1 at the ith position and 0's at other positions. Then there exists a permutation $\pi^{(P)}$ of $\{1, \ldots, v\}$ taking j to $i_j, 1 \leq j \leq v$. Whenever there is exactly one cycle in $\pi^{(P)}$, we call P a cyclic permutation matrix. Using the properties of P and $\pi^{(P)}$ (see e.g. Birkhoff and MacLane (1977) and Hohn (1957)) the following theorems, helpful in the sequel, are immediate.

Theorem 2.1. *The eigenvalues of a cyclic permutation matrix P of order v are given by* $\mu_j = \cos 2\pi j/v + i \sin 2\pi j/v$, $0 \leq j \leq v-1$.

Theorem 2.2. *Let P be a permutation matrix of order v and $\pi^{(P)}$ be a product of k disjoint cycles of s distinct lengths, n_t being the multiplicity of cycle length $l_t, 1 \leq t \leq s, \sum_{i=1}^{s} n_i = k$ and $\sum_{i=1}^{s} n_i l_i = v$. Then the eigenvalues of P are given by $\mu_{tj} = \cos 2\pi j/l_t + i \sin 2\pi j/l_t$, $1 \leq t \leq s$, $0 \leq j \leq l_t - 1$, with multiplicity n_t.*

Lemma 2.3. *Let P be a permutation matrix of order v. Then $2I - (P + P^{'})$ is n.n.d..*

Proof. The result follows immediately using the Cauchy Schwartz inequality for all $x \in \mathbb{R}^v$,

$$x'(P + P')x = x'Px + x'P'x \leq \| x \| \| Px \| + \| x \| \| P'x \| = 2x'x.$$

3. Designs Dominating Agrawal Designs

Agrawal (1966) constructed designs for $v \geq 4$, (henceforth denoted by d_A) based on his method 3 with $N_{d_A} = L_{d_A} = M_{d_A} = J - I$, which yields

$$C_{d_A} = (v(v-3)/(v-2))(I - J/v). \tag{3.1}$$

A simple way to construct d_A is to start with a *Latin Square* design of order v with diagonal elements all different and then to delete the diagonal.

Let $d^\star \in \Omega$ be a design with

$$L_{d^\star} = J - I = L_{d_A} \text{ and } M_{d^\star} = J - P, \tag{3.2}$$

where P is a cyclic permutation matrix of order v different from the identity matrix. From (2.1) and (3.1),

$$C_{d^\star} = C_{d_A} + (1/v(v-2))(2I - (P + P')). \tag{3.3}$$

Using *Lemma* 2.3, it follows that $C_{d^\star} - C_{d_A}$ is n.n.d. and nonzero. This suggests that $d^\star$ is to be preferred to d_A with respect to any meaningful optimality criterion. In the following, we establish that $d^\star$ is E-optimal and highly A- and D-efficient. The various bounds that have been used in this context are listed below:

$$\lambda_{d1} \leq (v/(v-1)) \min_{h \in V} C_{dhh}. \tag{3.4}$$

Lemma 3.1. *For given positive integers p and t, the minimum of $\sum_{i=1}^{p} x_i^2$ subject to $\sum_{i=1}^{p} x_i = t$, where the x_i's are nonnegative integers, is obtained when $t - p[t/p]$ of the x_i's are equal to $[t/p] + 1$, and $p - t + p[t/p]$ of the x_i's are equal to $[t/p]$. ($[t/p]$ is the greatest integer $\leq t/p$).*

Lemma 3.2. *For given positive integer $p \geq 3$, the minimum value of $\sum_{i=1}^{p} x_i^2$ subject to $\sum_{i=1}^{p} x_i = p-1$, where the x_i's are nonnegative integers and $x_i \geq 2$ for at least one i, is equal to $p+1$. It is attained when two of the x_i's are 0, $p-3$ of the x_i's are 1 and one x_i is 2.*

Proof. For any feasible $x = (x_1, \ldots, x_p)$

$$\sum_{i=1}^{p} x_i^2 = \underbrace{\sum_{i=1}^{p} x_i(x_i-1)}_{\geq 2} + p - 1 \geq p+1$$

and $\sum_{i=1}^{p} x_i^2 = p+1$ for the particular x_i's described in the lemma.

Theorem 3.3. *Suppose that for given $v, d^\star$ defined by (3.2) exists. Then $d^\star$ is* E-*optimal in Ω.*

Proof. Let us partition the collection of designs of Ω in the following way:

$$\Omega_1 = \{d : d \in \Omega, d \text{ is equireplicate and binary}\},$$
$$\Omega_2 = \{d : d \in \Omega, d \text{ is equireplicate and nonbinary}\},$$
$$\Omega_3 = \{d : d \in \Omega, d \text{ is nonequireplicate}\}.$$

We organize our proof in three steps.

Step 1. Let d be a design in Ω_1. Since there are exactly $v-1$ feasible cells in each row and in each column and $r_{di} = \bar{r} = v-1$ for all i, note that

$$L_d = J - P_{1d} \text{ and } M_d = J - P_{2d}, \tag{3.5}$$

where P_{1d} and P_{2d} are permutation matrices of order v. Using (2.1) and *Lemma* 2.3 it can be easily verified that

$$\begin{aligned} C_d &= (v(v-3)/(v-2))(I - J/v) + (1/v(v-2))(2I - (P_{1d}P_{2d}^{'} + P_{2d}P_{1d}^{'})) \\ &= C_{d_A} + A_d \geq C_{d_A}, \end{aligned}$$

where $A_d = (1/v(v-2))(2I - (P_{1d}P_{2d}^{'} + P_{2d}P_{1d}^{'}))$ and $P_{1d}P_{2d}^{'} = Q_d$ (say) is again a permutation matrix. Let the notations used in the context of *Theorem* 2.2 hold for Q_d. Then the eigenvalues of A_d are

$$\theta_{ij} = (2/v(v-2))(1 - \cos 2\pi j/l_i), \quad 0 \leq j \leq l_i - 1, \quad 1 \leq i \leq s,$$

with multiplicity n_i. Note that $\theta_{i0} = 0$, $1 \leq i \leq s$ and $\sum_{i=1}^{s} n_i = k$. Since C_{d_A} and A_d commute, these two matrices can be simultaneously diagonalised and hence the nonzero eigenvalues of C_d are obtained by adding $v(v-3)/(v-2)$ to the θ_{ij}'s, $(i,j) \neq (1,0)$ (assuming w.l.o.g that the eigenvectors corresponding to the eigenvalue 0 of C_{d_A} and the eigenvalue $\theta_{10}(=0)$ of A_d are the same). Thus,

$$\lambda_{d1} = \begin{cases} v(v-3)/(v-2) = \lambda_{d_A 1}, & \text{for } k \geq 2, \\ v(v-3)/(v-2) + (2/v(v-2))(1-\cos 2\pi/v), & \text{for } k = 1. \end{cases} \quad (3.6)$$

Note that $d^\star$ corresponds to $k = 1$. Thus $d^\star$ is E-best in Ω_1.
Step 2. Suppose d is a design in Ω_2. Relying on (2.1), we obtain

$$C_{dhh} = \bar{r} - \sum_{j=1}^{v} l_{dhj}^2/v - \sum_{j=1}^{v} m_{dhj}^2/v - \sum_{j=1}^{v}(l_{dhj}+m_{dhj})^2/v(v-2) + \bar{r}^2/(v-1)(v-2). \quad (3.7)$$

Since d is nonbinary, using *Lemmas* 3.1 *and* 3.2, we get for some treatment $h_0 \in V$,

$$\sum_{j=1}^{v} l_{dh_0j}^2 + \sum_{j=1}^{v} m_{dh_0j}^2 \geq (v-1)+(v+1) = 2v$$

and

$$\sum_{j=1}^{v}(l_{dh_0j}+m_{dh_0j})^2 \geq 4v-6.$$

Applying these two bounds in (3.7), note that

$$C_{dh_0h_0} \leq (v^3-4v^2+v+6)/v(v-2) \leq (v^3-4v^2+3v)/v(v-2) = C_{d_Ah_0h_0}$$

for all $v \geq 4$. Thus, (3.4) and the fact that C_{d_A} is c.s. yield

$$\lambda_{d1} \leq (v/(v-1))C_{d_Ah_0h_0} = \lambda_{d_A1} < \lambda_{d^\star 1}. \quad (3.8)$$

Thus $d^\star$ is E-better than any design in Ω_2.
Step 3. Finally, for a design $d \in \Omega_3$ we see that there exists a treatment h_0, such that $r_{dh_0} < \bar{r} = v-1$ and, using (3.4),

$$\lambda_{d1} \leq (v/(v-1))C_{dh_0h_0}. \quad (3.9)$$

We now obtain an upper bound for $C_{dh_0h_0}$ referring to Saharay (1986), p.51. Let g be a function defined by

$$g(r) = \max_{\{d: d\in\Omega, r_{dh}=r\}} C_{dhh} \quad (3.10)$$

Saharay (1986) derived that

$$g(r) = \begin{cases} g_1(r), & \text{if } 1 \leq r < v/2, \\ g_2(r), & \text{if } v/2 \leq r \leq v-1, \end{cases} \quad (3.11)$$

where

$$g_1(r) = r^2/(v-1)(v-2) + r(v^2-4v+2)/v(v-2),$$
$$g_2(r) = r^2/(v-1)(v-2) + r(v^2-4v-2)/v(v-2) + 2/(v-2).$$

It can be easily checked that g_1 and g_2 are nondecreasing functions in r, $1 \leq r \leq v-1$ and

$$\begin{array}{ll} g_1(v/2) = g_2(v/2), & \text{if } v \text{ is even,} \\ g_1((v-1)/2) < g_2((v+1)/2), & \text{if } v \text{ is odd.} \end{array} \tag{3.12}$$

Using (3.9), (3.10), (3.11) and the properties of g_1 and g_2 we conclude the proof as follows:

$$\lambda_{d1} \leq (v/(v-1))g(r_{dh_0}) \leq (v/(v-1))g_2(v-2) \leq v(v-3)/(v-2) = \lambda_{d_A 1} < \lambda_{d^\star 1}.$$

4. Construction of $d^\star$

In this section, we develop systematic methods of constructing $d^\star$. For convenience, the rows, columns and treatments are numbered as $0, 1, \ldots, v-1$.

Case(i): v even

We construct $d^\star$ by assigning treatment symbol $x(\text{mod } v)$ to the (i, j)th cell as indicated below:

cell		x
$0 \leq i < v-2$	$0 \leq j < i$	$i+j+1$
	$i < j \leq v-1$	$i+j$
$i = v-2$	$0 \leq j \leq v/2-2$	$2j$
	$v/2-1 \leq j \leq v-3$	$2j+1$
	$j = v-1$	$v-3$
$i = v-1$	$0 \leq j \leq v/2-2$	$2j+1$
	$v/2-1 \leq j \leq v-2$	$2j$

Case(ii): v odd

In this case, we first make an array $A_{v \times v}$ by assigning treatment symbols to the first $(v-2)$ rows following the rule described just above. The remaining cells of A are filled as shown below:

cell		x
$i = v-2$	$0 \leq j \leq v-3, j \neq (v-5)/2$	$2j$
	$j = (v-5)/2$	$v-4$
	$j = v-1$	$v-3$
$i = v-1$	$0 \leq j \leq v-3, j \neq (v-5)/2$	$2j+1$
	$j = (v-5)/2$	$v-5$
	$j = v-2$	$v-4$

Finally, $d^\star$ is obtained from A by interchanging treatment symbols $v-5$ and $v-3$ in the column numbers $0,2,4,\ldots,2(p-1)$ and $4p$ when $v=4p+1$, and in the column numbers $2p,2p+2,\ldots,4p$ when $v=4p+3$.

In both cases, it can be easily verified that in $d^\star$, treatment i does not occur in the ith row as well as in the $(i+1)$th $(\bmod v)$ column, $0\le i\le v-1$, and hence (3.2) is ensured.

Example 1.

$v=8$

$d^\star$:

-	1	2	3	4	5	6	7
2	-	3	4	5	6	7	0
3	4	-	5	6	7	0	1
4	5	6	-	7	0	1	2
5	6	7	0	-	1	2	3
6	7	0	1	2	-	3	4
0	2	4	7	1	3	-	5
1	3	5	6	0	2	4	-

$v=7$

$d^\star$:

-	1	4	3	2	5	6
2	-	3	4	5	6	0
3	4	-	5	6	0	1
4	5	6	-	0	1	2
5	6	0	1	-	2	3
0	3	2	6	1	-	4
1	2	5	0	4	3	-

Remark. It is interesting to note that for $v=7, d^\star$ is A- and D-better than the nonequireplicate design proposed by Shah and Sinha (1990).

5. Concluding Remarks

It is clear that $d^\star$ is not completely symmetric and has maximum $\mathrm{tr}(C_d)$ in the binary and equireplicate class Ω_1. A computer search for A- and D-optimal designs for $v\le 20$ within Ω_1 indicates that $d^\star$ remains A- and D-optimal for $4\le v\le 14$, whereas from $v=15$ onwards the structures of A- and D-optimal designs depend on v . For example, for $v=15, d_0$ with $L_{d_0}=J-I, M_{d_0}=J-P_0, \pi^{(P_0)}=\pi_6\pi_9$ is A- and D-optimal, whereas for $v=16, d^\star$ is A-optimal, d_0 with $L_{d_0}=J-I, M_{d_0}=J-P_0, \pi^{(P_0)}=\pi_5\pi_{11}$ is D-optimal. However, the A- and D-efficiencies of $d^\star$ in Ω_1 turn out to be greater than 0.99 for $v\ge 15$. The determination of exact A- and D-optimal designs in Ω is still an open problem.

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