

A NEW CONDITION FOR THE INVARIANCE PRINCIPLE FOR STATIONARY RANDOM FIELDS

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Abstract: We establish a central limit theorem and an invariance principle for stationary random fields, with projective-type conditions. Our result is obtained via an m -dependent approximation method. As applications, we establish invariance principles for orthomartingales and functionals of linear random fields.

Key words and phrases: Central limit theorem, invariance principle, linear random field, m -dependent approximation, orthomartingale.

1. Introduction

Markov (1910) proved a central limit theorem for a two-state Markov chain. That initiated one of the longest histories in probability theory, the central limit theorem for stationary processes. One successful approach is the *martingale approximation* method, first applied by Gordin (1969) and then developed by many other researchers. Along this line, Maxwell and Woodroffe (2000) proved the following result. Let $\{X_k\}_{k \in \mathbb{Z}}$ be a stationary process with $X_k = f \circ T^k$ for all $k \in \mathbb{Z}$, where f is a measurable function from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to \mathbb{R} , and T is a bimeasurable, measure-preserving, one-to-one and onto map on $(\Omega, \mathcal{A}, \mathbb{P})$. Consider

$$S_n(f) = \sum_{k=1}^n f \circ T^k. \quad (1.1)$$

Let $\{\mathcal{F}_k\}_{k \in \mathbb{Z}}$ be a filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $T^{-1}\mathcal{F}_k = \mathcal{F}_{k+1}$ for all $k \in \mathbb{Z}$. Suppose $\int f^2 d\mathbb{P} < \infty$, $\int f d\mathbb{P} = 0$, $f \in \mathcal{F}_0$ (i.e., the sequence is *adapted*), and $\mathbb{E}(f \mid \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k) = 0$. Maxwell and Woodroffe proved that if

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}(S_k(f) \mid \mathcal{F}_0)\|_2}{k^{3/2}} < \infty, \quad (1.2)$$

then $\sigma^2 = \lim_{n \rightarrow \infty} \mathbb{E}(S_n^2)/n$ exists, and

$$\frac{S_n(f)}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma^2).$$

Here ‘ \Rightarrow ’ denotes the weak convergence of the random variables (convergence in distribution), and the L^2 norm $\|\cdot\|_2$ is with respect to the measure \mathbb{P} . Note that (1.2) is implied by

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}(f \circ T^k \mid \mathcal{F}_0)\|_2}{k^{1/2}} < \infty. \quad (1.3)$$

Condition (1.2) is referred to as the Maxwell–Woodroffe condition. Later on, Peligrad and Utev (2005) showed that (1.2) also implies the invariance principle. Indeed, let $\{\mathbb{B}(t)\}_{t \in [0,1]}$ denote the standard Brownian motion. Then, (1.2) implies

$$\frac{S_{[n\cdot]}(f)}{\sqrt{n}} \Rightarrow \sigma \mathbb{B}(\cdot),$$

where $[x]$ denotes the largest integer smaller or equal to $x \in \mathbb{R}$ and ‘ \Rightarrow ’ is understood as the weak convergence in $C[0,1]$. Furthermore, Peligrad and Utev showed that (1.2) is the best possible (among conditions that only restrict the size of $\|\mathbb{E}(S_n(f) \mid \mathcal{F}_0)\|_2$). See also Dedecker, Merlevède, and Volný (2007) and Durieu and Volný (2008) for comparisons of Conditions (1.2) and (1.3) with other sufficient conditions for central limit theorems. For non-adapted sequences (i.e., $f \notin \mathcal{F}_0$), a similar condition guaranteeing the invariance principle is established by Volný (2007). Other important references on central limit theorems by martingale approximation include Gordin and Lifšic (1978), Kipnis and Varadhan (1986), Woodroffe (1992), Wu and Woodroffe (2004), Dedecker, Merlevède, and Volný (2007), Peligrad, Utev, and Wu (2007), among others, and Merlevède, Peligrad, and Utev (2006) for a survey. The martingale approximation can also be applied to establish invariance principle for empirical processes, see for example Wu (2003, 2008), and for random walks in random environment, see for example Rassoul-Agha and Seppäläinen (2005, 2007).

In this paper, we establish a central limit theorem and an invariance principle for stationary multiparameter random fields. We briefly mention a few results in the literature. Bolthausen (1982), Goldie and Morrow (1986) and Bradley (1989) studied this problem under suitable mixing conditions. Basu and Dorea (1979), Nahapetian (1995), and Poghosyan and Røelly (1998) considered the problem for *multiparameter martingales*. Another important result is due to Dedecker (1998, 2001), whose approach was based on the Lindeberg method. As a particular case, Cheng and Ho (2006) established a central limit theorem for *functionals of linear random fields*, based on a lexicographically ordered martingale approximation.

Here, we aim at establishing the so-called *projective-type* conditions such that the central limit theorem and invariance principle hold. Such conditions, often involving conditional expectations as in (1.2) and (1.3), have recently drawn much

attentions in central limit theorems for stationary sequences (see e.g., Dedecker, Merlevède, and Volný (2007)). In particular, such conditions are easy to verify when applying such results to stochastic processes from statistics and econometrics (see e.g., Wu (2011)). However, central limit theorems for stationary random fields based on projective conditions have been much less explored.

This problem is not a simple extension of a one-dimensional problem to a high-dimensional one. An important reason is that, the main technique for establishing central limit theorems with projective conditions in one dimension, the *martingale approximation* approach, does not apply to (high-dimensional) random fields as successfully as to (one-dimensional) stochastic processes. This obstacle has been known among researchers for more than 30 years. For example, Bolthausen (1982) remarked that ‘Gordin uses an approximation by martingales, but his method appears difficult to generalize to dimensions ≥ 2 .’

Our result, with a condition similar to (1.3), is a first attempt of extending central limit theorems with projective-type conditions to the multiparameter stationary random fields. The result is obtained by a different approximation approach, namely, approximation by m -dependent random fields.

To state our main result, we start with some notations. We consider a *product probability space* $(\Omega, \mathcal{A}, \mathbb{P})$, a \mathbb{Z}^d -indexed-product of i.i.d. probability spaces in form of

$$(\Omega, \mathcal{A}, \mathbb{P}) \equiv (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, P^{\mathbb{Z}^d}).$$

Write $\epsilon_k(\omega) = \omega_k$, for all $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $k \in \mathbb{Z}^d$. Then, $\{\epsilon_k\}_{k \in \mathbb{Z}^d}$ are i.i.d. random variables with distribution P . On such a space, we define the *natural filtration* $\{\mathcal{F}_k\}_{k \in \mathbb{Z}^d}$ by

$$\mathcal{F}_k := \sigma\{\epsilon_l : l \preceq k, l \in \mathbb{Z}^d\}, \text{ for all } k \in \mathbb{Z}^d. \quad (1.4)$$

Here and in the sequel, for all vector $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$ and for all $l, k \in \mathbb{R}^d$, let $l \preceq k$ stand for $l_i \leq k_i, i = 1, \dots, d$.

We focus on mean-zero stationary random fields, defined on a product probability space. Let $\{T_k\}_{k \in \mathbb{Z}^d}$ denote the group of shift operators on $\mathbb{R}^{\mathbb{Z}^d}$ with $(T_k \omega)_l = \omega_{k+l}$, for all $k, l \in \mathbb{Z}^d, \omega \in \mathbb{R}^{\mathbb{Z}^d}$. Then, we consider random fields in form of

$$\{f \circ T_k\}_{k \in \mathbb{Z}^d}, \text{ equivalently } \{f(\epsilon_{k+l} : l \in \mathbb{Z}^d)\}_{k \in \mathbb{Z}^d},$$

where f is in the class $\mathcal{L}_0^p = \{f \in L^p(\mathcal{F}_\infty), \int f d\mathbb{P} = 0\}, p \geq 2$, with $\mathcal{F}_\infty = \bigvee_{k \in \mathbb{Z}^d} \mathcal{F}_k$.

Throughout this paper, we consider a sequence $\{V_n\}_{n \in \mathbb{N}}$ of finite rectangular subsets of \mathbb{Z}^d ,

$$V_n = \prod_{i=1}^d \{1, \dots, m_i^{(n)}\} \subset \mathbb{N}^d, \text{ for all } n \in \mathbb{N}, \quad (1.5)$$

with $m_i^{(n)}$ increasing to infinity as $n \rightarrow \infty$ for all $i = 1, \dots, d$. Let

$$S_n(f) \equiv S(V_n, f) = \sum_{k \in V_n} f \circ T_k \tag{1.6}$$

denote the partial sums with respect to V_n . Moreover, write for $t \in [0, 1]$, $V_n(t) = \prod_{i=1}^d [0, m_i^{(n)} t] \subset \mathbb{R}^d$ and $R_k = \prod_{i=1}^d (k_i - 1, k_i] \subset \mathbb{R}^d$ for all $k \in \mathbb{Z}^d$.

We write also

$$B_{n,t}(f) \equiv B_{V_n,t}(f) = \sum_{k \in \mathbb{N}^d} \lambda(V_n(t) \cap R_k) f \circ T_k, \tag{1.7}$$

where λ is the Lebesgue measure on \mathbb{R}^d , and consider the weak convergence in the space $C[0, 1]^d$, the space of continuous functions on $[0, 1]^d$, equipped with the uniform metric. Recall that the standard d -parameter Brownian sheet on $[0, 1]^d$, denoted by $\{\mathbb{B}(t)\}_{t \in [0, 1]^d}$, is a mean-zero Gaussian random field with covariance $\mathbb{E}(\mathbb{B}(s)\mathbb{B}(t)) = \prod_{i=1}^d \min(s_i, t_i)$, $s, t \in [0, 1]^d$. Write $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^d$. In parallel to (1.3), our projective-type condition involves the term

$$\tilde{\Delta}_{d,p}(f) := \sum_{k \in \mathbb{N}^d} \frac{\|\mathbb{E}(f \circ T_k \mid \mathcal{F}_1)\|_p}{\prod_{i=1}^d k_i^{1/2}}, \tag{1.8}$$

and our main result is the following.

Theorem 1. *Consider a product probability space described above. If $f \in \mathcal{L}_0^2$, $f \in \mathcal{F}_0$ and $\tilde{\Delta}_{d,2}(f) < \infty$, then*

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(S_n(f)^2)}{|V_n|} < \infty$$

exists and

$$\frac{S_n(f)}{|V_n|^{1/2}} \Rightarrow \mathcal{N}(0, \sigma^2).$$

In addition, if $f \in \mathcal{L}_0^p$ and $\tilde{\Delta}_{d,p}(f) < \infty$ for some $p > 2$, then

$$\frac{B_{n,\cdot}(f)}{|V_n|^{1/2}} \Rightarrow \sigma \mathbb{B}(\cdot) \tag{1.9}$$

in $C[0, 1]^d$.

For the sake of simplicity, we will prove Theorem 1 in the case $d = 2$ in Sections 3 and 4.

We develop two applications of the main result. First, we obtain a central limit theorem for *orthomartingales*, a special class of multiparameter martingale (see e.g., Khoshnevisan (2002)), defined on a product probability space. To the

best of our knowledge, assuming the product structure of the probability space, this result is more general than existing central limit theorems for multiparameter martingales (Basu and Dorea (1979), Nahapetian (1995) and Poghosyan and Røelly (1998)). We provide a detailed discussion in Section 5. In particular, we demonstrate that one should not expect a central limit theorem even for general orthomartingales, without extra conditions on the structure of the underlying probability space.

Second, we obtain an invariance principle of functionals of stationary causal linear random fields in Section 6. This result extends the work of Wu (2002) in the one-dimensional case. Another central limit theorem for functional of stationary linear random fields has recently been developed by Cheng and Ho (2006), following the approach of Ho and Hsing (1997) and Cheng and Ho (2005) in the one-dimensional case. We provide simple examples where our condition is weaker.

Remark 1. After we finished this work, El Machkouri, Volný, and Wu (2013) obtained a central limit theorem and an invariance principle for stationary random fields, in the similar spirit as ours. They took also an m -approximation approach, based on the *physical dependence measure* introduced by Wu (2005). Their results are more general, in the sense that they established invariance principle for random fields indexed by arbitrary sets instead of rectangle ones. Their conditions are not directly comparable to ours. However, in the application to functionals of linear random fields, their condition on the coefficients is weaker (see Remark 9).

The paper is organized as follows. In Section 2 we provide preliminary results on m -dependent approximation. We establish the central limit theorem in Section 3 and the invariance principle in Section 4. Sections 5 and 6 are devoted to the applications to orthomartingales and functionals of stationary linear random fields, respectively. In Section 7, we prove a moment inequality that plays a crucial role in proving our limit results. Some other auxiliary proofs are given in Section 8.

2. m -Dependent Approximation

We describe the general procedure of m -dependent approximation in this section. In this section, we do not assume any structure on the underlying probability space, nor the filtration structure. Instead, we simply assume $f \in L_0^2 = \{f \in L^2(\Omega, \mathcal{A}, \mathbb{P}), \int f d\mathbb{P} = 0\}$, and $\{T_k\}_{k \in \mathbb{Z}^d}$ is an Abelian group of bimeasurable, measure-preserving, one-to-one and onto maps on $(\Omega, \mathcal{A}, \mathbb{P})$.

The notion of m -dependence was introduced by Hoeffding and Robbins (1948). We say a random variable f is m -dependent, if $f \circ T_k, f \circ T_l$ are independent whenever $|k - l|_\infty := \max_{i=1, \dots, d} |k_i - l_i| > m$. The following result

on the asymptotic normality of sums of m -dependent random variables is a consequence of Bolthausen (1982) (see also Rosén (1969)). Recall $\{V_n\}_{n \in \mathbb{N}}$ given at (1.5).

Theorem 2. *Suppose $f_m \in L_0^2$ is m -dependent and write*

$$\sigma_m^2 = \sum_{k \in \mathbb{Z}^d} \mathbb{E}[f_m(f_m \circ T_k)]. \tag{2.1}$$

Then,

$$\frac{S_n(f_m)}{|V_n|^{1/2}} \Rightarrow \mathcal{N}(0, \sigma_m^2).$$

Now, consider the function $f \in L_0^2(\mathbb{P})$ and define

$$\|f\|_{V,+} = \limsup_{n \rightarrow \infty} \frac{\|S_n(f)\|_2}{|V_n|^{1/2}}. \tag{2.2}$$

We refer to the pseudo norm defined by $\|\cdot\|_{V,+}$ as the *plus-norm*.

Lemma 1. *Suppose $f, f_1, f_2, \dots \in L_0^2(\mathbb{P})$ and f_m is m -dependent for all $m \in \mathbb{N}$. If*

$$\lim_{m \rightarrow \infty} \|f - f_m\|_{V,+} = 0, \tag{2.3}$$

then

$$\lim_{m \rightarrow \infty} \sigma_m = \lim_{m \rightarrow \infty} \|f_m\|_{V,+} =: \sigma < \infty \tag{2.4}$$

exists, and

$$\frac{S_n(f)}{|V_n|^{1/2}} \Rightarrow \mathcal{N}(0, \sigma^2). \tag{2.5}$$

Proof. It suffices to prove (2.4). We will show that $\{\sigma_m^2\}_{m \in \mathbb{N}}$ forms a Cauchy sequence in \mathbb{R}_+ . Observe that since f_m is m -dependent with zero mean,

$$\sigma_m = \lim_{n \rightarrow \infty} \frac{\|S_n(f_m)\|_2}{|V_n|^{1/2}}.$$

It then follows that

$$|\sigma_{m_1} - \sigma_{m_2}| \leq \limsup_{n \rightarrow \infty} \frac{\|S_n(f_{m_1} - f_{m_2})\|_2}{|V_n|^{1/2}} \leq \|f_{m_1} - f\|_{V,+} + \|f_{m_2} - f\|_{V,+},$$

which can be made arbitrarily small by taking m_1, m_2 large enough. We have thus shown that $\{\sigma_n^2\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R}_+ .

Remark 2. The idea of establishing the central limit theorem by controlling the quantity $\|f - f_m\|_{V,+}$ dates back to Gordin (1969), where f_m was selected from a

different subspace. In the one-dimensional case, when $V_n = \{1, \dots, n\}$, Zhao and Woodroffe (2008) named $\|\cdot\|_{V,+}$ the plus-norm, and established a necessary and sufficient condition for the martingale approximation, in term of the plus-norm. See Peligrad (2010) and Gordin and Peligrad (2011) for improvements and more discussions on such conditions.

In the next section, we establish conditions under which (2.3) holds.

3. A Central Limit Theorem

From this section on, we focus on stationary multiparameter random fields defined on product probability spaces. On such a space, any integrable function has a natural L^2 -approximation by m -dependent functions, and there is a natural commuting filtration.

For the sake of simplicity, we consider only the 2-parameter random fields in the sequel and simply say ‘random fields’ for short. We prove a central limit theorem here and an invariance principle in the next section. The arguments can be generalized easily to d -parameter random fields, and the result has been stated in Theorem 1.

We start with a product probability space with i.i.d. random variables $\{\epsilon_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$. Recall that $\{T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ is the group of shift operators on $\mathbb{R}^{\mathbb{Z}^2}$ and write $\mathcal{F}_{\infty,\infty} = \sigma(\epsilon_{i,j} : (i,j) \in \mathbb{Z}^2)$. We focus on the class of functions $\mathcal{L}_0^p = \{f \in L^p(\mathcal{F}_{\infty,\infty}) : \mathbb{E}f = 0\}$, $p \geq 2$. For all measurable function $f \in \mathcal{L}_0^2$, define, for all $m \in \mathbb{N}$,

$$f_m := \mathbb{E}(f | \mathcal{F}_{\langle m \rangle}) \quad \text{with} \quad \mathcal{F}_{\langle m \rangle} = \sigma(\epsilon_j : j \in \{-m, \dots, m\}^2). \quad (3.1)$$

Clearly, $f_m \in \mathcal{L}_0^2$, $\|f - f_m\|_2 \rightarrow 0$ as $m \rightarrow \infty$ and $\{f_m \circ T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ are m -dependent functions.

Now, recall the natural filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ defined by $\mathcal{F}_{k,l} = \sigma(\epsilon_{i,j} : i \leq k, j \leq l)$. This is a 2-parameter filtration, i.e.,

$$\mathcal{F}_{i,j} \subset \mathcal{F}_{k,l} \quad \text{if} \quad i \leq k, j \leq l. \quad (3.2)$$

Also,

$$T_{-i,-j} \mathcal{F}_{k,l} = \mathcal{F}_{k+i,l+j}, \forall (i,j), (k,l) \in \mathbb{Z}^2. \quad (3.3)$$

The notion of commuting filtration is of importance to us.

Definition 1. A filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ is *commuting* if for all $\mathcal{F}_{k,l}$ -measurable bounded random variable Y , $\mathbb{E}(Y | \mathcal{F}_{i,j}) = \mathbb{E}(Y | \mathcal{F}_{i \wedge k, j \wedge l})$.

Since $\{\epsilon_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ are independent random variables, $\{\mathcal{F}_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ is commuting (see Proposition 2 in Section 8). This implies that the *marginal filtrations*

$$\mathcal{F}_{i,\infty} = \bigvee_{j\geq 0} \mathcal{F}_{i,j} \quad \text{and} \quad \mathcal{F}_{\infty,j} = \bigvee_{i\geq 0} \mathcal{F}_{i,j} \tag{3.4}$$

are commuting, in the sense that for all $Y \in L^1(\mathbb{P})$,

$$\mathbb{E}[\mathbb{E}(Y|\mathcal{F}_{i,\infty})|\mathcal{F}_{\infty,j}] = \mathbb{E}[\mathbb{E}(Y|\mathcal{F}_{\infty,j})|\mathcal{F}_{i,\infty}] = \mathbb{E}(Y|\mathcal{F}_{i,j}). \tag{3.5}$$

For more details on the commuting filtration, see Khoshnevisan (2002).

For all $\mathcal{F}_{0,0}$ -measurable function $f \in \mathcal{L}_0^2$, write

$$S_{m,n}(f) = \sum_{i=1}^m \sum_{j=1}^n f \circ T_{i,j}. \tag{3.6}$$

Thanks to the commuting structure of the filtration, by applying twice the maximal inequality in Peligrad, Utev, and Wu (2007), we can prove the following moment inequality with $p \geq 2$:

$$\|S_{m,n}(f)\|_p \leq Cm^{1/2}n^{1/2}\Delta_{(m,n),p}(f) \tag{3.7}$$

with

$$\Delta_{(m,n),p}(f) = \sum_{k=1}^m \sum_{l=1}^n \frac{\|\mathbb{E}(S_{k,l}(f) | \mathcal{F}_{1,1})\|_p}{k^{3/2}l^{3/2}}.$$

In fact, we will prove a stronger inequality without the assumptions of product probability space and the $\mathcal{F}_{0,0}$ -measurability of f . See Section 7, Proposition 1 and Corollary 2.

Recall that

$$\tilde{\Delta}_{2,p}(f) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\|\mathbb{E}(f \circ T_{k,l} | \mathcal{F}_{1,1})\|_p}{k^{1/2}l^{1/2}}. \tag{3.8}$$

Now we prove a central limit theorem for adapted stationary random fields.

Theorem 3. *Consider the product probability space above. Let $\{V_n\}_{n\in\mathbb{N}}$ be as in (1.5) with $d = 2$. Suppose $f \in \mathcal{L}_0^2$, $f \in \mathcal{F}_{0,0}$, and define f_m as in (3.1). If $\tilde{\Delta}_{2,2}(f) < \infty$, then*

$$\lim_{m\rightarrow\infty} \|f - f_m\|_{V,+} = 0.$$

Thus, $\sigma := \lim_{m\rightarrow\infty} \|f_m\|_{V,+} < \infty$ exists and $S_n(f)/|V_n|^{1/2} \Rightarrow \mathcal{N}(0, \sigma^2)$.

Proof. The second part follows immediately from Lemma 1. It suffices to prove $\|f - f_m\|_{V,+} \rightarrow 0$ as $m \rightarrow \infty$. First, by the fact that

$$\|\mathbb{E}(S_{k,l}(f) \mid \mathcal{F}_{1,1})\|_2 \leq \sum_{i=1}^k \sum_{j=1}^l \|\mathbb{E}(f \circ T_{i,j} \mid \mathcal{F}_{1,1})\|_2$$

and Fubini's Theorem, we have $\Delta_{(\infty,\infty),2}(f) \leq 9\tilde{\Delta}_{2,2}(f)$. So, by (2.2) and (3.7), it suffices to show

$$\tilde{\Delta}_{2,2}(f - f_m) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\|\mathbb{E}[(f - f_m) \circ T_{k,l} \mid \mathcal{F}_{1,1}]\|_2}{k^{1/2}l^{1/2}} \rightarrow 0 \tag{3.9}$$

as $m \rightarrow \infty$. Clearly, the summand in (3.9) converges to 0 for each k, l fixed, since (3.1) implies $\|f - f_m\|_2 \rightarrow 0$ as $m \rightarrow \infty$ and $\|\mathbb{E}[(f - f_m) \circ T_{k,l} \mid \mathcal{F}_{1,1}]\|_2 \leq \|f - f_m\|_2$. Moreover, observe that

$$\begin{aligned} \mathbb{E}(f_m \circ T_{k,l} \mid \mathcal{F}_{1,1}) &= \mathbb{E}[\mathbb{E}(f \circ T_{k,l} \mid T_{-k,-l}(\mathcal{F}_{\langle m \rangle})) \mid \mathcal{F}_{1,1}] \\ &= \mathbb{E}[\mathbb{E}(f \circ T_{k,l} \mid \mathcal{F}_{1,1}) \mid T_{-k,-l}(\mathcal{F}_{\langle m \rangle})], \end{aligned}$$

where in the second equality we can exchange the order of conditional expectations by the definitions of $\mathcal{F}_{1,1}$ and $T_{-k,-l}(\mathcal{F}_{\langle m \rangle})$ (see Proposition 2 in Section 8 for a detailed treatment). Therefore,

$$\begin{aligned} \|\mathbb{E}[(f - f_m) \circ T_{k,l} \mid \mathcal{F}_{1,1}]\|_2 &\leq \|\mathbb{E}(f \circ T_{k,l} \mid \mathcal{F}_{1,1})\|_2 + \|\mathbb{E}(f_m \circ T_{k,l} \mid \mathcal{F}_{1,1})\|_2 \\ &\leq 2\|\mathbb{E}(f \circ T_{k,l} \mid \mathcal{F}_{1,1})\|_2. \end{aligned}$$

Then, the condition $\tilde{\Delta}_{2,2}(f) < \infty$ combined with the Dominated Convergence Theorem yields (3.9). The proof is thus completed.

Remark 3. An ‘extension’ of Maxwell–Woodroffe condition (1.2) to high dimension remains an open problem. Namely if we replace $\tilde{\Delta}_{2,2}(f) < \infty$ by $\Delta_{(\infty,\infty),2}(f) < \infty$ in Theorem 3, do we have the same conclusion? The latter condition is weaker than the former one.

4. An Invariance Principle

Recall the space $C[0, 1]^2$ and the 2-parameter Brownian sheet $\{\mathbb{B}(t)\}_{t \in [0,1]^2}$.

Theorem 4. *Under the assumptions of Theorem 3, suppose in addition that $f \in \mathcal{L}_0^p$ and $\tilde{\Delta}_{2,p}(f) < \infty$ for some $p > 2$. Write $B_{n,t}(f)$ as in (1.7) with $d = 2$. Then,*

$$\frac{B_{n,\cdot}(f)}{|V_n|^{1/2}} \Rightarrow \sigma \mathbb{B}(\cdot),$$

where ‘ \Rightarrow ’ stands for weak convergence of probability measures on $C[0, 1]^2$.

Proof. It suffices to show that the finite-dimensional distributions converge, and $\{B_{n,t}(f)/|V_n|^{1/2}\}_{t \in [0,1]^2}$ is tight.

We first show that, for all $\tilde{t} = (t^{(1)}, \dots, t^{(k)}) \subset [0, 1]^2$,

$$\left(\frac{B_{n,t^{(1)}}(f)}{|V_n|^{1/2}}, \dots, \frac{B_{n,t^{(k)}}(f)}{|V_n|^{1/2}} \right) \Rightarrow \sigma(\mathbb{B}(t^{(1)}), \dots, \mathbb{B}(t^{(k)})) =: \sigma \tilde{\mathbb{B}}_{\tilde{t}}. \tag{4.1}$$

Consider the m -dependent function f_m defined in (3.1). The convergence of the finite-dimensional distributions (4.1) with f replaced by f_m follows from the invariance principle of m -dependent random fields (see e.g., Shashkin (2003)). Furthermore, by Theorem 3, $\tilde{\Delta}_{2,2}(f) \leq \tilde{\Delta}_{2,p}(f) < \infty$, so that $\|f - f_m\|_{V,+} \rightarrow 0$ as $m \rightarrow \infty$, and therefore, letting $\tilde{B}_{n,\tilde{t}}(f)/|V_n|^{1/2}$ denote the left-hand side of (4.1), $\tilde{B}_{n,\tilde{t}}(f_m - f)/|V_n|^{1/2} \rightarrow (0, \dots, 0) \in \mathbb{R}^k$ in probability. The convergence of the finite-dimensional distribution (4.1) follows.

Now we prove the tightness of $\{B_{n,t}(f)\}_{t \in [0,1]^2}$. Fix n and consider

$$V_n = \{1, \dots, n_1\} \times \{1, \dots, n_2\}.$$

Write $B_{n,t} \equiv B_{n,t}(f)$ and $S_{m,n} \equiv S_{m,n}(f)$ for short. For all $0 \leq r_1 < s_1 \leq 1, 0 \leq r_2 < s_2 \leq 1$, set

$$B_n((r_1, s_1] \times (r_2, s_2]) := B_{n,(s_1,s_2)} - B_{n,(r_1,s_2)} - B_{n,(s_1,r_2)} + B_{n,(r_1,r_2)}.$$

We will show that there exists a constant C , independent of n, r_1, r_2, s_1 and s_2 , such that

$$(n_1 n_2)^{-1/2} \|B_n((r_1, s_1] \times (r_2, s_2])\|_p \leq C \sqrt{(s_1 - r_1)(s_2 - r_2)} \tilde{\Delta}_{2,p}(f). \tag{4.2}$$

Inequality (4.2) implies the tightness, by Nagai (1974/75), Theorem 1.

Now, we prove (4.2) to complete the proof. From now on, the constant C may change from line to line. Write $m_i = \lfloor n_i s_i \rfloor - \lfloor n_i r_i \rfloor, i = 1, 2$. If $m_i \geq 2, i = 1, 2$, then

$$\|B_n((r_1, s_1] \times (r_2, s_2])\|_p \leq \|S_{m_1, m_2}\|_p + 2\|S_{m_1, 1}\|_p + 2\|S_{1, m_2}\|_p + 4\|S_{1, 1}\|_p.$$

Here, we first write $B_n((r_1, s_1] \times (r_2, s_2])$ as the sum of S_{m_1, m_2} and the partial sums over boundary regions, and then we simply bound the boundary widths by 1. Thus, by (3.7),

$$\|B_n((r_1, s_1] \times (r_2, s_2])\|_p \leq C(m_1 m_2)^{1/2} \tilde{\Delta}_{2,p}(f) \tag{4.3}$$

for some constant C . Note that $m_i \geq 2$ also implies $n_i(s_i - r_i) > 1$. Therefore, $m_i \leq n_i(s_i - r_i) + 1 < 2n_i(s_i - r_i)$, and (4.3) can be bounded by $C(n_1 n_2)^{1/2} [(s_1 - r_1)(s_2 - r_2)]^{1/2} \tilde{\Delta}_{2,p}(f)$, which yields (4.2).

In case $m_1 < 2$ or $m_2 < 2$, to obtain (4.2) requires more careful analysis. We only show the case with $m_1 = 1, m_2 \geq 2$, as the proof for the other cases are similar. Suppose that $m_1 = 1$ and we exclude the case $n_1 r_1 = \lfloor n_1 r_1 \rfloor = \lceil n_1 r_1 \rceil$ (it is easy to see that this case can be eventually controlled by continuity). We have $n_1 r_1 < \lceil n_1 r_1 \rceil = \lfloor n_1 s_1 \rfloor \leq n_1 s_1$, and then

$$\begin{aligned} \|B_n((r_1, s_1] \times (r_2, s_2])\|_p &\leq n_1(s_1 - r_1)(\|S_{1,m_2}\|_p + 2\|S_{1,1}\|_p) \\ &\leq Cn_1(s_1 - r_1)m_2^{1/2}\tilde{\Delta}_{2,p}(f). \end{aligned}$$

Observe that $m_1 = 1$ also implies $n_1(s_1 - r_1) \in (0, 2)$. If $n_1(s_1 - r_1) \leq 1$, then $n_1(s_1 - r_1) \leq \lfloor n_1(s_1 - r_1) \rfloor^{1/2}$. If $n_1(s_1 - r_1) \in (1, 2)$, then $n_1(s_1 - r_1) < \sqrt{2}\lfloor n_1(s_1 - r_1) \rfloor^{1/2}$. It follows that (4.2) still holds.

Remark 4. To prove the invariance principle of stationary random fields, most of the results require a finite moment of order strictly larger than 2. See for example Berkes and Morrow (1981), Goldie and Greenwood (1986), and Dedecker (2001). This is in contrast to the one-dimensional case, where the invariance principle can be established with the finite second moment assumption. One of the key ingredients missing in high dimensions is a maximal inequality similar to the one established by Peligrad and Utev (2005) (see also Peligrad, Utev, and Wu (2007); Volný (2007)).

To the best of our knowledge, the only invariance principle so far for stationary random fields that assumes finite second moment is due to Shashkin (2003), where the random fields are assumed to be $BL(\theta)$ -dependent (including m -dependent stationary random fields). In general the $BL(\theta)$ -dependence is difficult to check. Besides, Basu and Dorea (1979) proved an invariance principle for martingale difference random fields with the finite second moment assumption, but they have different conditions on the filtration (see Remark 6 below). In our case, it remains an open problem: whether $\tilde{\Delta}_{2,2}(f) < \infty$ implies the invariance principle. See also a similar conjecture in Dedecker (2001), Remark 1.

5. Orthomartingales

The central limit theorems and invariance principles for multiparameter martingales are more difficult to establish than in the one-dimensional case. This is due to the complex structure of multiparameter martingales. We will focus on orthomartingales first and establish an invariance principle, and then compare the results on other types of multiparameter martingales.

The idea of orthomartingales is due to R. Cairoli and J. B. Walsh. See e.g., references in Khoshnevisan (2002), which also provides a nice introduction to the materials. For the sake of simplicity, we suppose $d = 2$. Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and recall the definition of 2-parameter filtration at (3.2). We restrict ourselves to the filtration indexed by \mathbb{N}^2 .

Definition 2. Given a commuting 2-parameter filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, we say a family of random variables $\{M_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ is a *2-parameter orthomartingale* on $(\Omega, \mathcal{A}, \mathbb{P})$, with respect to $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{N}^2}$, if for all $(i, j) \in \mathbb{N}^2$, $M_{i,j}$ is $\mathcal{F}_{i,j}$ -measurable, and $\mathbb{E}(M_{i+1,j} | \mathcal{F}_{i,\infty}) = \mathbb{E}(M_{i,j+1} | \mathcal{F}_{\infty,j}) = M_{i,j}$, almost surely.

In our case, for $\mathcal{F}_{0,0}$ -measurable function $f \in \mathcal{L}_0^2$, $M_{m,n} = S_{m,n}(f)$ as in (3.6) yields a 2-parameter orthomartingale if

$$\mathbb{E}(f \circ T_{i+1,j} | \mathcal{F}_{i,\infty}) = \mathbb{E}(f \circ T_{i,j+1} | \mathcal{F}_{\infty,j}) = 0 \text{ almost surely,} \quad (5.1)$$

for all $(i, j) \in \mathbb{N}^2$. In this case, we say $\{f \circ T_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ are *2-parameter orthomartingale differences*.

Remark 5. In our case, $\{M_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ is also a *2-parameter martingale* in the normal sense, i.e., $\mathbb{E}(M_{i,j} | \mathcal{F}_{k,l}) = M_{i \wedge k, j \wedge l}$, almost surely. Indeed,

$$\mathbb{E}(M_{i,j} | \mathcal{F}_{k,l}) = \mathbb{E}[\mathbb{E}(M_{i,j} | \mathcal{F}_{k,\infty}) | \mathcal{F}_{\infty,l}] = \mathbb{E}(M_{i \wedge k, j} | \mathcal{F}_{\infty,l}) = M_{i \wedge k, j \wedge l}.$$

In general, however, the converse is not true, i.e., multiparameter martingales are not necessarily orthomartingales (see e.g., Khoshnevisan (2002) p. 33). The two notions are equivalent when the filtration is commuting (see e.g., Khoshnevisan (2002, Chap. I, Thm. 3.5.1)).

Theorem 5. Consider a product probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{N}^2}$. Suppose $f \in \mathcal{L}_0^2$ and $f \in \mathcal{F}_{0,0}$. If $\{f \circ T_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ are 2-parameter orthomartingale differences (5.1) holds, then $\sigma^2 = \lim_{n \rightarrow \infty} \mathbb{E}(S_n(f)^2) / |V_n|^2 < \infty$ exists, and

$$\frac{S_n(f)}{|V_n|^{1/2}} \Rightarrow \sigma \mathcal{N}(0, 1).$$

In addition, if $f \in \mathcal{L}_0^p$ for some $p > 2$, then the invariance principle (1.9) holds.

Proof. Observe that, (5.1) implies $\mathbb{E}(f \circ T_{i,j} | \mathcal{F}_{1,1}) = 0$ if $i > 1$ or $j > 1$. Then for $f \in \mathcal{L}_0^p, p \geq 2$,

$$\tilde{\Delta}_{\infty,p}(f) = \|\mathbb{E}(f \circ T_{1,1} | \mathcal{F}_{1,1})\|_p = \|f\|_p < \infty.$$

The result then follows immediately from Theorem 1. Note that the argument holds for the general d -parameter orthomartingales ($d \geq 2$) defined in Khoshnevisan (2002).

Remark 6. Our result is more general than those in Basu and Dorea (1979), Nahapetian (1995) and Poghosyan and Roelly (1998) in the following sense. Let be $\{\epsilon_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ be i.i.d. random variables. In Nahapetian (1995), a central limit

theorem was established for the so-called *martingale-difference random fields* $\{M_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ with $M_{i,j} = \sum_{k=1}^i \sum_{l=1}^j D_{k,l}$, such that

$$\mathbb{E}[D_{i,j} \mid \sigma(\epsilon_{k,l} : (k,l) \in \mathbb{Z}^2, (k,l) \neq (i,j))] = 0, \text{ for all } (i,j) \in \mathbb{N}^2.$$

In Basu and Dorea (1979) and Poghosyan and Røelly (1998), the authors considered the multiparameter martingales $\{M_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ with respect to the filtration $\tilde{\mathcal{F}}_{i,j} = \sigma(\epsilon_{k,l} : k \leq i \text{ or } l \leq j)$. It is easy to see, in both these cases, the assumptions are stronger in the sense that they imply that $\{M_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ is an orthomartingale with the natural filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{N}^2}$ (1.4). On the other hand, the results in Basu and Dorea (1979) and Poghosyan and Røelly (1998) only assume that $\{\epsilon_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ is a stationary random field, which is weaker than our assumption.

Remark 7. By assumption, the σ -algebra of $\{T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ -invariant sets is \mathbb{P} trivial. Therefore, our results are restricted to ergodic random fields, and exclude the simple case

$$X_{i,j} = Y \epsilon_{i,j}, (i,j) \in \mathbb{Z}^2,$$

where Y is a random variable independent of $\{\epsilon_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$. Clearly, if $\epsilon_{0,0}$ has zero mean and finite variance σ^2 , then

$$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n X_{i,j} \Rightarrow YZ,$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is independent of Y . For central limit theorems on non-ergodic random fields, see for example Dedecker (1998, 2001).

Finally, we point out that the product structure of the probability space plays an important role. We provide an example of an orthomartingale with a different underlying probability structure. In this case, the limit behavior is quite different from the case that we have studied so far.

Example 1. Suppose $\{\epsilon_k\}_{k \in \mathbb{Z}}$ and $\{\eta_k\}_{k \in \mathbb{Z}}$ are two families of i.i.d. random variables. Define $\mathcal{G}_i = \sigma(\epsilon_j : j \leq i)$ and $\mathcal{H}_i = \sigma(\eta_j : j \leq i)$ for all $i \in \mathbb{N}$. Then $\mathcal{G} = \{\mathcal{G}_n\}_{n \in \mathbb{N}}$ and $\mathcal{H} = \{\mathcal{H}_n\}_{n \in \mathbb{N}}$ are two filtrations.

Now, let $\{Y_n\}_{n \in \mathbb{N}}$ and $\{Z_n\}_{n \in \mathbb{N}}$ be two arbitrary martingales with stationary increment with respect to the filtration \mathcal{G} and \mathcal{H} , respectively. Suppose $Y_n = \sum_{i=1}^n D_i, Z_n = \sum_{i=1}^n E_i$, where $\{D_n\}_{n \in \mathbb{N}}$ and $\{E_n\}_{n \in \mathbb{N}}$ are stationary martingale differences. Then, $\{D_i E_j\}_{(i,j) \in \mathbb{N}^2}$ is a stationary random field and

$$M_{m,n} := \sum_{i=1}^m \sum_{j=1}^n D_i E_j = Y_m Z_n$$

is an orthomartingale with respect to the filtration $\{\mathcal{G}_i \vee \mathcal{H}_j\}_{(i,j) \in \mathbb{N}^2}$. Clearly,

$$\frac{M_{n,n}}{n} = \frac{Y_n}{\sqrt{n}} \frac{Z_n}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma_Y^2) \times \mathcal{N}(0, \sigma_Z^2),$$

where the limit is the distribution of the product of two independent normal random variables (a Gaussian chaos). That is, $M_{n,n}/n$ has an asymptotically non-normal distribution.

One can also take $\widetilde{M}_{m,n} = Y_m + Z_n$, which again gives an orthomartingale, and $\{D_i + E_j\}_{(i,j) \in \mathbb{N}^2}$ is the corresponding stationary random field. This time, one can show that

$$\frac{\widetilde{M}_{n,n}}{\sqrt{n}} = \frac{Y_n}{\sqrt{n}} + \frac{Z_n}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma_Y^2 + \sigma_Z^2).$$

Here the limit is a normal distribution, but the normalizing sequence is \sqrt{n} instead of n .

This example demonstrates that for general orthomartingales, to obtain a central limit theorem one must assume extra conditions on the structure of the underlying probability space. For the structure mentioned above, there is no m -dependent approximation for the random fields. Indeed, the example corresponds to the sample space $\Omega = (\mathbb{R}^{\mathbb{Z}}, \mathbb{R}^{\mathbb{Z}})$ with $[T_{k,l}(\epsilon, \eta)]_{i,j} = (\epsilon_{i+k}, \eta_{j+l})$, and if we define f_m similarly as in (3.1) with

$$\mathcal{F}_{\langle m \rangle} := \sigma(\epsilon_i, \eta_j : -m \leq i, j \leq m),$$

then f and $f \circ T_{k,l}$ are independent if and only if $\min(k, l) > m$. That is, the dependence can be very strong along the horizontal (the vertical resp.) direction of the random field.

Remark 8. We thank an anonymous referee for drawing our attention to a recent paper by Gordin (2009), where conditions are established for sequences of stationary random fields so that they can be represented as the sum of two sequences, one a multiparameter martingale and the other a *co-boundary*. Thus, one could establish a central limit theorem for stationary random fields through an approximation by multiparameter random fields.

6. Stationary Causal Linear Random Fields

We establish a central limit theorem for functionals of stationary causal linear random fields. We focus on $d = 2$. Consider the stationary linear random field $\{Z_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ with

$$Z_{i,j} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{r,s} \epsilon_{i-r, j-s} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{i-r, j-s} \epsilon_{r,s}, \quad (6.1)$$

where the coefficients $\{a_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ satisfy $\sum_{(i,j) \in \mathbb{Z}^2} a_{i,j}^2 < \infty$, and the $\{\epsilon_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ are i.i.d. random variables with zero mean and finite variance. We restrict ourselves to *causal* linear random fields, that is, $a_{i,j} = 0$ unless $i \geq 0$ and $j \geq 0$. They are also referred to as *adapted* to the filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$.

Now, consider the random fields $\{f \circ T_{k,l}\}_{(k,l) \in \mathbb{Z}^2}$ with a more specific form $f = K(\{Z_{i,j}\}_h^{0,0})$, where h is a fixed strictly positive integer, K is a measurable function from \mathbb{R}^{h^2} to \mathbb{R} and, for all $(k,l) \in \mathbb{Z}^2$,

$$\{Z_{i,j}\}_h^{k,l} := \{Z_{i,j} : k - h + 1 \leq i \leq k, l - h + 1 \leq j \leq l\}$$

is viewed as a random vector in \mathbb{R}^{h^2} with coordinates lexicographically ordered. In the sequel, the same definition applies similarly to $\{x_{i,j}\}_h^{k,l}$ given $\{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$. Assume that

$$\mathbb{E}K(\{Z_{i,j}\}_h^{0,0}) = 0 \quad \text{and} \quad \mathbb{E}K^p(\{Z_{i,j}\}_h^{0,0}) < \infty \tag{6.2}$$

for some $p \geq 2$. In this way,

$$f \circ T_{k,l} = K(\{Z_{i,j}\}_h^{k,l}). \tag{6.3}$$

The model (6.3) is a natural extension of the functionals of causal linear processes considered by Wu (2002).

We introduce a few notations similar to those in Ho and Hsing (1997) and Wu (2002). Our ultimate goal is to translate Condition (3.8) into a condition on the regularity of K and the summability of $\{a_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$. For all $(i,j) \in \mathbb{Z}^2$, let

$$\Gamma(i,j) = \{(r,s) \in \mathbb{Z}^2 : r \leq i, s \leq j\}, \tag{6.4}$$

and write

$$\begin{aligned} Z_{i,j} &= \sum_{(r,s) \in \Gamma(i,j)} a_{i-r,j-s} \epsilon_{r,s} \\ &= \sum_{(r,s) \in \Gamma(i,j) \setminus \Gamma(1,1)} a_{i-r,j-s} \epsilon_{r,s} + \sum_{(r,s) \in \Gamma(1,1)} a_{i-r,j-s} \epsilon_{r,s} \\ &=: Z_{i,j,+} + Z_{i,j,-}. \end{aligned}$$

Write $W_{k,l,-} = \{Z_{i,j,-}\}_h^{k,l}$ and define, for all $(k,l) \in \mathbb{Z}^2$,

$$K_{k,l}(\{x_{i,j}\}_h^{k,l}) = \mathbb{E}K(\{Z_{i,j,+} + x_{i,j}\}_h^{k,l}).$$

In this way,

$$\mathbb{E}(f \circ T_{k,l} \mid \mathcal{F}_{1,1}) = K_{k,l}(\{Z_{i,j,-}\}_h^{k,l}) =: K_{k,l}(W_{k,l,-}). \tag{6.5}$$

Plugging (6.5) into (3.8), we obtain a central limit theorem for functionals of stationary causal linear random fields.

Theorem 6. *Consider the functionals of stationary causal linear random fields (6.3). If (6.2) holds and*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\|K_{k,l}(W_{k,l,-})\|_p}{k^{1/2}l^{1/2}} < \infty \tag{6.6}$$

for $p = 2$, then $\sigma^2 = \lim_{n \rightarrow \infty} \mathbb{E}(S_n^2)/n^2 < \infty$ exists and $S_n/|V_n|^{1/2} \Rightarrow \mathcal{N}(0, \sigma^2)$. If the conditions hold with $p > 2$, then the invariance principle (1.9) holds.

Next, we provide conditions on K and $\{a_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ such that (6.6) holds. For all $\Lambda \subset \mathbb{Z}^2$, write

$$Z_{\Lambda} = \sum_{(i,j) \in \Lambda} a_{i,j} \epsilon_{-i,-j} \quad \text{and} \quad A_{\Lambda} = \sum_{(i,j) \in \Lambda} a_{i,j}^2. \tag{6.7}$$

In particular, our conditions involves summations of $a_{i,j}$ over the following type of region:

$$\Lambda(k, l) := \{(i, j) \in \mathbb{Z}^2 : i \geq k, j \geq l\}, (k, l) \in \mathbb{Z}^2.$$

For the sake of simplicity, we write $A_{k,l} \equiv A_{\Lambda(k,l)}$. The following lemma is a simple extension of Lemma 2, part (b), in Wu (2002).

Lemma 2. *Suppose that there exist $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \leq 1 \leq \beta < \infty$ and $\mathbb{E}(|\epsilon|^{2\beta}) < \infty$. If*

$$\mathbb{E}M_{\alpha,\beta}^2(W_{1,1}) < \infty \text{ with } M_{\alpha,\beta}(x) = \sup_{y \in \mathbb{R}^h, y \neq x} \frac{|K(x) - K(y)|}{|x - y|^{\alpha} + |x - y|^{\beta}}, \tag{6.8}$$

then, for all $p \geq 2$,

$$\|K_{k,l}(W_{k,l,-})\|_p = O(A_{k+1-h,l+1-h}^{\alpha/2}). \tag{6.9}$$

The proof is deferred to Section 8. Consequently, Condition (6.6) can be replaced by specific ones on $A_{k,l}$.

Corollary 1. *Assume there exist $\alpha, \beta \in \mathbb{R}$ as in Lemma 2. Consider the functionals of stationary linear random fields in form of (6.3). Suppose (6.8) holds and*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{A_{k+1-h,l+1-h}^{\alpha/2}}{k^{1/2}l^{1/2}} < \infty. \tag{6.10}$$

If $\mathbb{E}(|\epsilon|^p) < \infty$ and (6.2) hold with $p = 2$, then $S_n/n \Rightarrow \mathcal{N}(0, \sigma^2)$ for some $\sigma < \infty$. If $\mathbb{E}(|\epsilon|^p) < \infty$ and (6.2) holds with $p > 2$, then the invariance principle (1.9) holds.

We compare our Condition (6.10) on the summability of $\{a_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$, to the one considered by Cheng and Ho (2006). They only established central limit theorems for functionals of stationary linear random fields, so we restrict to the case $p = 2$. Cheng and Ho (2006) assumed

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{i,j}|^{1/2} < \infty, \tag{6.11}$$

and $\sup_{\Lambda \subset \mathbb{Z}^2} \mathbb{E}K^2(x + Z_{\Lambda}) < \infty$ for all $x \in \mathbb{R}$ with Z_{Λ} defined in (6.7), and that for any two independent random variables X and Y with $\mathbb{E}(K^2(X) + K^2(Y) + K^2(X + Y)) < M < \infty$,

$$\mathbb{E}[(K(X + Y) - K(X))^2] \leq C[\mathbb{E}(Y^2)]^{\gamma} \tag{6.12}$$

for some $\gamma \geq 1/2$. In general, Cheng and Ho (2006)'s condition and ours on the regularity K are not comparable and thus have different ranges of applications. We focus on the simple case that $h = 1$ and K is Lipschitz, covered by both works. This corresponds to $\alpha = \beta = 1$ in (6.8) and $\gamma = 1$ in (6.12). In the following two examples, our (6.10) is weaker than (6.11).

Example 2. Let $a_{i,j} = (i + j + 1)^{-q}$ for all $i, j \geq 0$ and some $q > 1$. Then, $A = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}^2 < \infty$ and

$$A_{k,l} = \sum_{j=1}^{\infty} j(k + l + j)^{-2q} = O((k + l)^{2-2q}).$$

Here (6.10) is bounded by, up to a multiplicative constant,

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(k + l)^{1-q}}{k^{1/2}l^{1/2}} < \sum_{k=1}^{\infty} \frac{k^{(1-q)/2}}{k^{1/2}} \sum_{l=1}^{\infty} \frac{l^{(1-q)/2}}{l^{1/2}} \leq \left(\sum_{k=1}^{\infty} k^{-q/2} \right)^2.$$

Therefore, (6.10) requires $q > 2$, while (6.11) requires $q > 4$.

Example 3. Let $a_{i,j} = (i + 1)^{-q}(j + 1)^{-q}$, for all $i, j \geq 0$ for some $q > 1$. Then $A = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}^2 < \infty$ and

$$A_{k,l} = \sum_{i=k}^{\infty} \sum_{j=l}^{\infty} a_{i,j}^2 = O(k^{-(2q-1)}l^{-(2q-1)}). \tag{6.13}$$

One can thus check that (6.10) requires $q > 3/2$ while (6.11) requires $q > 2$.

Remark 9. For the central limit theorem for functionals of linear random fields, the weakest condition known is due to El Machkouri, Volný, and Wu (2013) (Example 1 and Theorem 1), who showed that it suffices to require K to be Lipschitz and

$$\sum_{i,j} |a_{i,j}| < \infty.$$

Furthermore, this result and the one of Cheng and Ho (2006) do not assume the linear random field to be causal.

7. A Moment Inequality

We establish a moment inequality for stationary 2-parameter random fields on general probability spaces without assuming the product structure. We first review the Peligrad–Utev inequality, a maximal L^p -inequality in dimension one, with $p \geq 2$. Recall the partial summation in (1.1) and the related probability space. Let C denote a constant that may change from line to line. It is known that for all $f \in L^p(\mathcal{F}_\infty)$ with $\mathbb{E}(f | \mathcal{F}_{-\infty}) = 0$,

$$\begin{aligned} \left\| \max_{1 \leq k \leq n} |S_k(f)| \right\|_p &\leq Cn^{1/2} \left(\|\mathbb{E}(f | \mathcal{F}_0)\|_p + \|f - \mathbb{E}(f | \mathcal{F}_0)\|_p \right. \\ &\quad \left. + \sum_{k=1}^n \frac{\|\mathbb{E}(S_k(f) | \mathcal{F}_0)\|_p}{k^{3/2}} + \sum_{k=1}^n \frac{\|S_k(f) - \mathbb{E}(S_k(f) | \mathcal{F}_k)\|_p}{k^{3/2}} \right). \end{aligned} \quad (7.1)$$

This inequality was first established for adapted stationary sequences in Peligrad and Utev (2005), then extended to L^p -inequality for $p \geq 2$ in Peligrad, Utev, and Wu (2007). The case $p \in (1, 2)$ was addressed by Wu and Zhao (2008). The non-adapted case for $p \geq 2$ was addressed by Volný (2007).

For the sake of simplicity, we simplify the bound in (7.1) by regrouping the summations. Observe that $\|\mathbb{E}(S_k(f) | \mathcal{F}_0)\|_p \leq \|\mathbb{E}(S_k(f) | \mathcal{F}_1)\|_p$, $\|\mathbb{E}(f | \mathcal{F}_0)\|_p = \|\mathbb{E}(S_1(f) | \mathcal{F}_1)\|_p$, and $\|f - \mathbb{E}(f | \mathcal{F}_0)\|_p = \|S_1(f) - \mathbb{E}(S_1(f) | \mathcal{F}_1)\|_p$. Thus, we find that

$$\begin{aligned} \left\| \max_{1 \leq k \leq n} |S_k(f)| \right\|_p &\leq Cn^{1/2} \left(\sum_{k=1}^n \frac{\|\mathbb{E}(S_k(f) | \mathcal{F}_1)\|_p}{k^{3/2}} + \sum_{k=1}^n \frac{\|S_k(f) - \mathbb{E}(S_k(f) | \mathcal{F}_k)\|_p}{k^{3/2}} \right). \end{aligned} \quad (7.2)$$

Now consider a general probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and suppose there exists a commuting 2-parameter filtration $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ and an Abelian group of bimeasurable, measure-preserving, one-to-one and onto maps $\{T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ on $(\Omega, \mathcal{A}, \mathbb{P})$,

such that (3.3) holds. Define $\mathcal{F}_{\infty,\infty} = \bigvee_{(i,j) \in \mathbb{Z}^2} \mathcal{F}_{i,j}$, $\mathcal{F}_{-\infty,\infty} = \bigcap_{i \in \mathbb{Z}} \mathcal{F}_{i,\infty}$, and $\mathcal{F}_{\infty,-\infty} = \bigcap_{j \in \mathbb{Z}} \mathcal{F}_{\infty,j}$. Note that when $(\Omega, \mathcal{A}, \mathbb{P})$ is a product probability space, $\mathcal{F}_{-\infty,\infty}$ and $\mathcal{F}_{\infty,-\infty}$ are trivial by Kolmogorov’s zero-one law.

Recall the definition of $S_{m,n}(f)$ in (3.6). Given f , write $S_{m,n} \equiv S_{m,n}(f)$ for the sake of simplicity.

Proposition 1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$, $\{T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$, and $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ be as above. Suppose $p \geq 2$, $f \in L^p(\mathcal{F}_{\infty,\infty})$, and $\mathbb{E}(f \mid \mathcal{F}_{-\infty,\infty}) = \mathbb{E}(f \mid \mathcal{F}_{\infty,-\infty}) = 0$. Then*

$$\|S_{m,n}\|_p \leq Cm^{1/2}n^{1/2} \sum_{k=1}^m \sum_{l=1}^n \frac{d_{k,l}(f)}{k^{3/2}l^{3/2}}$$

with

$$\begin{aligned} d_{k,l}(f) &= \|\mathbb{E}(S_{k,l} \mid \mathcal{F}_{1,1})\|_p + \|\mathbb{E}(S_{k,l} \mid \mathcal{F}_{1,\infty}) - \mathbb{E}(S_{k,l} \mid \mathcal{F}_{1,l})\|_p \\ &\quad + \|\mathbb{E}(S_{k,l} \mid \mathcal{F}_{\infty,1}) - \mathbb{E}(S_{k,l} \mid \mathcal{F}_{k,1})\|_p \\ &\quad + \|\mathbb{E}(S_{k,l} - \mathbb{E}(S_{k,l} \mid \mathcal{F}_{k,\infty}) - \mathbb{E}(S_{k,l} \mid \mathcal{F}_{\infty,l}) + \mathbb{E}(S_{k,l} \mid \mathcal{F}_{k,l})\|_p. \end{aligned}$$

Corollary 2. *Suppose the assumptions in Proposition 1 hold.*

(i) *If $f \in L^p(\mathcal{F}_{0,0})$, then*

$$\|S_{m,n}(f)\|_p \leq Cm^{1/2}n^{1/2} \sum_{k=1}^m \sum_{l=1}^n \frac{\|\mathbb{E}(S_{k,l}(f) \mid \mathcal{F}_{1,1})\|_p}{k^{3/2}l^{3/2}}.$$

(ii) *If $\{f \circ T_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ are two-dimensional martingale differences, in the sense that $f \in L^p(\mathcal{F}_{0,0})$ and $\mathbb{E}(f \mid \mathcal{F}_{0,-1}) = \mathbb{E}(f \mid \mathcal{F}_{-1,0}) = 0$, then*

$$\|S_{m,n}(f)\|_p \leq Cm^{1/2}n^{1/2}\|f\|_p.$$

The proof of Corollary 2 is trivial. We only remark that the second case recovers the Burkholder’s inequality for multiparameter martingale differences established in Fazekas (2005).

Proof of Proposition 1. Fix f . Define $\tilde{S}_{0,n} = \sum_{j=1}^n f \circ T_{0,j}$. Clearly,

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n f \circ T_{i,j} = \sum_{i=1}^m \left(\sum_{j=1}^n f \circ T_{0,j} \right) \circ T_{i,0} = \sum_{i=1}^m \tilde{S}_{0,n} \circ T_{i,0}. \tag{7.3}$$

Fix n . Observe that $\mathbb{E}\tilde{S}_{0,n} = 0$ and $\tilde{S}_{0,n} \circ T_{i,0}$ is a stationary sequence. Furthermore, $\{\mathcal{F}_{i,\infty}\}_{i \in \mathbb{Z}}$ is a filtration, $T_{i,0}^{-1}\mathcal{F}_{j,\infty} = T_{-i,0}\mathcal{F}_{j,\infty} = \mathcal{F}_{i+j,\infty}$ and $\mathbb{E}(\tilde{S}_{0,n} \mid \mathcal{F}_{-\infty,\infty}) = 0$. Therefore, we can apply the Peligrad–Utev inequality (7.2) to obtain

$$\|S_{m,n}\|_p \leq Cm^{1/2} \left(\sum_{k=1}^m k^{-3/2} \underbrace{\|\mathbb{E}(S_{k,n} \mid \mathcal{F}_{1,\infty})\|_p}_{\Lambda_1} + \sum_{k=1}^m k^{-3/2} \underbrace{\|S_{k,n} - \mathbb{E}(S_{k,n} \mid \mathcal{F}_{k,\infty})\|_p}_{\Lambda_2} \right).$$

We first deal with Λ_1 . Define $\tilde{S}_{m,0} = \sum_{i=1}^m f \circ T_{i,0}$. Similarly as in (7.3), $S_{k,n} = \sum_{j=1}^n \tilde{S}_{k,0} \circ T_{0,j}$, and

$$\mathbb{E}(S_{k,n} | \mathcal{F}_{1,\infty}) = \sum_{j=1}^n \mathbb{E}(\tilde{S}_{k,0} \circ T_{0,j} | \mathcal{F}_{1,\infty}) = \sum_{j=1}^n \mathbb{E}(\tilde{S}_{k,0} \circ T_{0,j} | T_{0,-j}(\mathcal{F}_{1,\infty})),$$

where in the last equality we used the fact that $T_{0,j}(\mathcal{F}_{i,\infty}) = \mathcal{F}_{i,\infty}$, for all $i, j \in \mathbb{Z}$. Now, by the identify $\mathbb{E}(f | \mathcal{F}) \circ T = \mathbb{E}(f \circ T | T^{-1}(\mathcal{F}))$, we have

$$\mathbb{E}(S_{k,n} | \mathcal{F}_{1,\infty}) = \sum_{j=1}^n \mathbb{E}(\tilde{S}_{k,0} | \mathcal{F}_{1,\infty}) \circ T_{0,j}. \tag{7.4}$$

Observe that (7.4) is again a summation in the form of (1.1). Then, applying the Peligrad–Utev inequality (7.2) again, we obtain that

$$\begin{aligned} \Lambda_1 \leq Cn^{1/2} & \left(\sum_{l=1}^n l^{-3/2} \|\mathbb{E}[\mathbb{E}(S_{k,l} | \mathcal{F}_{1,\infty}) | \mathcal{F}_{\infty,1}]\|_p \right. \\ & \left. + \sum_{l=1}^n l^{-3/2} \|\mathbb{E}(S_{k,l} | \mathcal{F}_{1,\infty}) - \mathbb{E}[\mathbb{E}(S_{k,l} | \mathcal{F}_{1,\infty}) | \mathcal{F}_{\infty,l}]\|_p \right). \end{aligned}$$

By the commuting property of the marginal filtrations (3.5), this inequality becomes

$$\Lambda_1 \leq Cn^{1/2} \left(\sum_{l=1}^n l^{-3/2} \|\mathbb{E}(S_{k,l} | \mathcal{F}_{1,1})\|_p + \sum_{l=1}^n l^{-3/2} \|\mathbb{E}(S_{k,l} | \mathcal{F}_{1,\infty}) - \mathbb{E}(S_{k,l} | \mathcal{F}_{1,l})\|_p \right). \tag{7.5}$$

Similarly, one can show

$$\begin{aligned} \Lambda_2 &= \left\| \sum_{j=1}^n [S_{k,0} - \mathbb{E}(S_{k,0} | \mathcal{F}_{k,\infty})] \circ T_{0,j} \right\|_p \\ &\leq Cn^{1/2} \left(\sum_{l=1}^n l^{-3/2} \|\mathbb{E}(S_{k,l} | \mathcal{F}_{\infty,1}) - \mathbb{E}(S_{k,l} | \mathcal{F}_{k,1})\|_p \right. \\ &\quad \left. + \sum_{l=1}^n l^{-3/2} \|S_{k,l} - \mathbb{E}(S_{k,l} | \mathcal{F}_{k,\infty}) - \mathbb{E}(S_{k,l} | \mathcal{F}_{\infty,l}) + \mathbb{E}(S_{k,l} | \mathcal{F}_{k,l})\|_p \right). \end{aligned}$$

Combining (), (7.5), and (7.6), we have thus proved Proposition 1.

8. Auxiliary Proofs

For arbitrary σ -fields \mathcal{F}, \mathcal{G} , let $\mathcal{F} \vee \mathcal{G}$ denote the smallest σ -field that contains \mathcal{F} and \mathcal{G} .

Proposition 2. *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be mutually independent sub- σ -fields of \mathcal{B} . Then for all random variable $X \in \mathcal{B}$, $\mathbb{E}|X| < \infty$, we have*

$$\mathbb{E}[\mathbb{E}(X \mid \mathcal{F} \vee \mathcal{G}) \mid \mathcal{G} \vee \mathcal{H}] = \mathbb{E}(X \mid \mathcal{G}) \text{ a.s.} \tag{8.1}$$

Proposition 2 is closely related to the notion of conditional independence (see e.g., Chow and Teicher (1978, Chap. 7.3)). Namely, provided a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and sub- σ -fields $\mathcal{G}_1, \mathcal{G}_2$, and \mathcal{G}_3 of \mathcal{F} , \mathcal{G}_1 and \mathcal{G}_2 are said to be *conditionally independent* given \mathcal{G}_3 if for all $A_1 \in \mathcal{G}_1, A_2 \in \mathcal{G}_2$, $\mathbb{P}(A_1 \cap A_2 \mid \mathcal{G}_3) = \mathbb{P}(A_1 \mid \mathcal{G}_3)\mathbb{P}(A_2 \mid \mathcal{G}_3)$ almost surely.

Proof of Proposition 2. First we show that $\mathcal{F} \vee \mathcal{G}$ and $\mathcal{G} \vee \mathcal{H}$ are conditionally independent, given \mathcal{G} . By Theorem 7.3.1 (ii) in Chow and Teicher (1978), it is equivalent to show, for all $F \in \mathcal{F}, G \in \mathcal{G}$, $\mathbb{P}(F \cap G \mid \mathcal{G} \vee \mathcal{H}) = \mathbb{P}(F \cap G \mid \mathcal{G})$ almost surely. This is true since

$$\mathbb{P}(F \cap G \mid \mathcal{G} \vee \mathcal{H}) = \mathbf{1}_G \mathbb{E}(\mathbf{1}_F \mid \mathcal{G} \vee \mathcal{H}) = \mathbf{1}_G \mathbb{E}(\mathbf{1}_F \mid \mathcal{G}) = \mathbb{P}(F \cap G \mid \mathcal{G}) \text{ a.s.}$$

Next, by Theorem 7.3.1 (iv) in Chow and Teicher (1978), this conditional independence yields $\mathbb{E}(X \mid \mathcal{G} \vee \mathcal{H}) = \mathbb{E}(X \mid \mathcal{G})$ almost surely, for all $X \in \mathcal{F} \vee \mathcal{G}$, $\mathbb{E}|X| < \infty$. Replacing X by $\mathbb{E}(X \mid \mathcal{F} \vee \mathcal{G})$, we have thus proved (8.1).

Proof of Lemma 2. Take $W_{k,l} = \{Z_{i,j}\}_h^{k,l}$. Define (and recall that) $\widetilde{W}_{k,l,\pm} = \{Z_{i,j,\pm}\}_h^{k,l}$. Let $\widetilde{W}_{k,l,-}$ be a copy of $W_{k,l,-}$, independent of $W_{k,l,\pm}$. Set $\widetilde{W}_{k,l} := W_{k,l,+} + \widetilde{W}_{k,l,-}$.

Recall that $K_{k,l}(W_{k,l,-}) = \mathbb{E}(K(W_{k,l}) \mid \mathcal{F}_{1,1})$ in (6.5). Observe that, by (6.5), $W_{k,l,-} \in L^p(\mathcal{F}_{1,1})$, and $W_{k,l,+}, \widetilde{W}_{k,l,-}$ are independent of $\mathcal{F}_{1,1}$. Therefore, $\mathbb{E}(K(\widetilde{W}_{k,l}) \mid \mathcal{F}_{1,1}) = \mathbb{E}(K(\widetilde{W}_{k,l})) = 0$, and

$$|K_{k,l}(W_{k,l,-})| = |\mathbb{E}(K(W_{k,l}) - K(\widetilde{W}_{k,l}) \mid \mathcal{F}_{1,1})| \leq \mathbb{E}(|K(W_{k,l}) - K(\widetilde{W}_{k,l})| \mid \mathcal{F}_{1,1}).$$

Observe that by (6.8),

$$|K(W_{k,l}) - K(\widetilde{W}_{k,l})| \leq M_{\alpha,\beta}(\widetilde{W}_{k,l})(|W_{k,l,-} - \widetilde{W}_{k,l,-}|^\alpha + |W_{k,l,-} - \widetilde{W}_{k,l,-}|^\beta).$$

Write $U_{k,l} = W_{k,l,-} - \widetilde{W}_{k,l,-}$. By Cauchy–Schwartz’s inequality, and noting that $\mathbb{E}(|M_{\alpha,\beta}(\widetilde{W}_{k,l})|^2 \mid \mathcal{F}_{1,1}) = \|M_{\alpha,\beta}(\widetilde{W}_{k,l})\|_2^2 = \|M_{\alpha,\beta}(\widetilde{W}_{1,1})\|_2^2$, we have

$$|K_{k,l}(W_{k,l,-})| \leq \|M_{\alpha,\beta}(\widetilde{W}_{1,1})\|_2 \{\mathbb{E}[(|U_{k,l}|^\alpha + |U_{k,l}|^\beta)^2 \mid \mathcal{F}_{1,1}]\}^{1/2},$$

whence for $p \geq 2$,

$$\begin{aligned} \|K_{k,l}(W_{k,l,-})\|_p &\leq \|M_{\alpha,\beta}(\widetilde{W}_{1,1})\|_2 \| |U_{k,l}|^\alpha + |Y_{k,l}|^\beta \|_p \\ &\leq \|M_{\alpha,\beta}(\widetilde{W}_{1,1})\|_2 (\| |U_{k,l}|^\alpha \|_p + \| |Y_{k,l}|^\beta \|_p). \end{aligned}$$

Finally, since for all $\gamma > 0$ and $n \in \mathbb{N}$ there exists a constant $C(\gamma, n) > 0$ such that and for all vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n$,

$$|w|^{2\gamma} = \left(\sum_{i=1}^n w_i^2 \right)^\gamma \leq C(\gamma, n) \left(\sum_{i=1}^n w_i^{2\gamma} \right),$$

it follows that for all $\gamma > 0$,

$$\begin{aligned} \mathbb{E}(|U_{k,l}|^{2\gamma}) &= \mathbb{E}(|W_{k,l,-} - \widetilde{W}_{k,l,-}|^{2\gamma}) = \mathbb{E}(|\{Z_{i,j,-} - \widetilde{Z}_{i,j,-}\}_h^{k,l}|^{2\gamma}) \\ &= O \left[\mathbb{E} \left(\sum_{\substack{k-h < i \leq k \\ l-h < j \leq l}} (Z_{i,j,-} - \widetilde{Z}_{i,j,-})^{2\gamma} \right) \right]. \end{aligned}$$

By Wu (2002), Lemma 4, under the notation (6.7), $\mathbb{E}(|\epsilon|^{2\nu 2\gamma}) < \infty$ implies that for all $\Lambda \subset \mathbb{Z}^2$, $\mathbb{E}(|Z_\Lambda|^{2\gamma}) \leq CA_\Lambda^\gamma$ for some universal constant C . It then follows that $\mathbb{E}(|U_{k,l}|^{2\gamma}) = O(A_{k+1-h, l+1-h}^\gamma)$. Consequently, (8.2) yields

$$\begin{aligned} \|K_{k,l}(W_{k,l,-})\|_p &\leq \|M_{\alpha,\beta}(W_{1,1})\|_2 \left[O(A_{k+1-h, l+1-h}^{\alpha/2}) + O(A_{k+1-h, l+1-h}^{\beta/2}) \right] \\ &= O(A_{k+1-h, l+1-h}^{\alpha/2}). \end{aligned}$$

The proof is thus completed.

Acknowledgement

The authors thank Stilian Stoev for many constructive and helpful discussions and in particular for his suggestion of considering the m -dependent approximation approach. The author was partially supported by the NSF grant DMS-0806094 during his Ph.D. study at the University of Michigan.

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(Received April 2012; accepted October 2012)