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OPTIMAL EXPERIMENTAL DESIGNS FOR INVERSE QUADRATIC REGRESSION MODELS

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Abstract: In this paper optimal experimental designs for inverse quadratic regression models are determined. We consider two different parameterizations of the model and investigate local optimal designs with respect to the c-, D- and E-criteria, which reflect various aspects of the precision of the maximum likelihood estimator for the parameters in inverse quadratic regression models. In particular it is demonstrated that, for a sufficiently large design space, geometric allocation rules are optimal with respect to many optimality criteria. Moreover, in numerous cases the designs with respect to the different criteria are supported at the same points. Finally, the efficiencies of different optimal designs with respect to various optimality criteria are studied, and the efficiency of some commonly used designs are investigated.

Key words and phrases: Chebyshev systems, $E-, c-, D-$optimality, optimal designs, rational regression models.

1. Introduction

Inverse polynomials define a flexible family of nonlinear regression models which are used to describe the relationship between a response, say Y, and a univariate predictor, say u (see e.g., Nelder (1966)). The model is defined by the expected response

$$\mathrm{E}\,(Y|u) = \frac{u}{P_n(u,\theta)}, \quad u \geq 0, \tag{1.1}$$

where $P_n(u,\theta)$ is a polynomial of degree n with coefficients $\theta_0,\ldots,\theta_n$ defining the shape of the curve. Nelder (1966) compared the properties of inverse and ordinary polynomial models for analyzing data. In contrast to ordinary polynomials, inverse polynomial regression models are bounded and can be used to describe a saturation effect, in which case the response does not exceed a finite amount. Similarly, a toxic effect can be produced, in which case the response eventually falls to zero.

An important class of inverse polynomial models are defined by inverse quadratic regression models, which correspond to the case $n = 2$ in (1.1). These models have numerous applications, in particular in chemistry and agriculture (see Ratkowsky (1990), Sparrow (1979a,b), Nelder (1966), Serchand, McNew, Kellogg and Johnson (1995), and Landete-Castillejos and Gallego (2000), among others). For example, Sparrow (1979a,b) analyzed data from several series of experiments designed to study the relationship between crop yield and fertilizer input. He concluded that among several competing models the inverse quadratic model produced the best fit to data obtained from yields of barley and grass crops. Similarly, Serchand et al. (1995) argued that inverse polynomials can produce a dramatically steep rise and might realistically describe lactation curves.

While much attention has been paid to the construction of various optimal designs for the inverse linear or Michaelis-Menten model (see Song and Wong (1998), Lopez-Fidalgo and Wong (2002), Dette, Melas and Pepelyshev (2003), Dette and Biedermann (2003) among many others), optimal designs for the inverse quadratic regression model have not been studied in so much detail. Cobby, Chapman and Pike (1986) determined local D-optimal designs numerically, and Haines (1992) provided some analytical results for D-optimal designs in the inverse quadratic regression model. In particular, in these references it is demonstrated that geometric allocation rules are D-optimal. The present paper is devoted to a more systematic study of local optimal designs for inverse quadratic models. We consider the c-, D-, D_1- and E-optimality criteria and determine local optimal designs for two different parameterizations of the inverse quadratic regression model. In Section 2 we introduce two parameterizations of the inverse quadratic regression model and describe some basic facts of approximate design theory. In Section 3 we discuss several c-optimal designs. In particular D_1-optimal designs are determined, which are of particular importance if discrimination between an inverse linear and inverse quadratic model is one of the interests of the experiment. As a further special case of the c-optimality criterion, we determine optimal extrapolation designs. Section 4 deals with the local D-optimality and E-optimality criteria. It is shown that for all criteria under consideration, geometric designs are local optimal whenever the design space is sufficiently large. We also determine the structure of the local optimal designs in the case of a bounded design space. These findings extend the observations made by Cobby, Chapman and Pike (1986) and Haines (1992) for the D-optimality criterion to other optimality criteria, different design spaces, and a slightly different inverse quadratic regression model.

2. Preliminaries

We consider two parameterizations of the inverse quadratic regression model

$$\mathrm{E}\,(Y|u) = \eta(u, \theta), \tag{2.1}$$

where $\theta = (\theta_0, \theta_1, \theta_2)^T$ denotes the vector of unknown parameters and the expected response is given by one of

$$\eta_1(u,\theta) = \tfrac{u}{\theta_0+\theta_1 u+\theta_2 u^2}, \tag{2.2a}$$

$$\eta_2(u,\theta) = \tfrac{\theta_0 u}{\theta_1+u+\theta_2 u^2}. \tag{2.2b}$$

The explanatory variable varies in the interval $\mathcal{U} = [s,t]$, where $s \geq 0$ and $0 < s < t < \infty$, or in the unbounded set $\mathcal{U} = [s,\infty)$ with $s \geq 0$. The assumptions regarding the parameters vary with the different parameterizations and should assure that the numerator in (2.2a) and (2.2b) is positive on $\mathcal{U}$. Under such assumptions the regression functions have no points of discontinuity. Moreover, both functions are strictly increasing to a maximum of size $(\theta_1 + 2\sqrt{\theta_0\theta_2})^{-1}$ at the point $u_{\max_1} = \sqrt{\theta_0/\theta_2}$ for parameterization (2.2a), and to a maximum of size $\theta_0(1+\sqrt{\theta_1\theta_2})^{-1}$ at the point $u_{\max_2} = \sqrt{\theta_1/\theta_2}$ for parameterization (2.2b), after which the functions are strictly decreasing to a zero asymptote. A sufficient condition for the positivity of the numerator is $\theta_0, \theta_2 > 0$, $|\theta_1| \leq 2\sqrt{\theta_0\theta_2}$ for model (2.2a), and $\theta_0, \theta_1, \theta_2 > 0$, $2\sqrt{\theta_1\theta_2} > 1$ for model (2.2b), respectively. We assume that at each $u \in \mathcal{U}$ a normally distributed observation is available with mean $\eta(u,\theta)$ and variance $\sigma^2 > 0$, where the function η is either η_1 or η_2, and different observations are assumed to be independent. An experimental design ξ is a probability measure with finite support defined on the set $\mathcal{U}$ (see Kiefer (1974)). The information matrix of an experimental design ξ is

$$M(\xi,\theta) = \int_{\mathcal{U}} f(u,\theta) f^T(u,\theta) d\xi(u), \tag{2.3}$$

where

$$f(u,\theta) = \frac{\partial}{\partial\theta}\eta(u,\theta) \tag{2.4}$$

denotes the gradient of the expected response with respect to the parameter θ. For the parameterizations (2.2a) and (2.2b) the vectors of the partial derivatives are given by

$$f_1(u,\theta) = \frac{-u}{(\theta_0+\theta_1 u+\theta_2 u^2)^2}(1,u,u^2)^T, \tag{2.5}$$

$$f_2(u,\theta) = \frac{u}{\theta_1+u+\theta_2 u^2}\left(1, -\frac{\theta_0}{\theta_1+u+\theta_2 u^2}, -\frac{\theta_0 u^2}{\theta_1+u+\theta_2 u^2}\right)^T, \tag{2.6}$$

respectively.

If N observations can be made and the design ξ concentrates mass w_i at the points u_i, $i = 1,\ldots,r$, the quantities $w_i N$ are rounded to integers such that $\sum_{j=1}^r n_i = N$ (see Pukelsheim and Rieder (1992)), and the experimenter takes

n_i observations at each point u_i, $i = 1, \ldots, r$. If the sample size N converges to infinity, then (under appropriate assumptions of regularity) the covariance matrix of the maximum likelihood estimator for the parameter θ is approximately proportional to the matrix $(\sigma^2/N)M^{-1}(\xi, \theta)$, provided that the inverse of the information matrix exists (see Jennrich (1969)). An optimal experimental design maximizes or minimizes an appropriate functional of the information matrix or its inverse, and there are numerous optimality criteria which can be used to discriminate between competing designs (see Silvey (1980) or Pukelsheim (1993)). In this paper we investigate the D-optimality criterion that maximizes the determinant of the inverse of the information matrix with respect to the design ξ, the c-optimality criterion that minimizes the variance of the maximum likelihood estimate for the linear combination $c^T\theta$, and the E-optimality criterion that maximizes the minimum eigenvalue of the information matrix $M(\xi, \theta)$.

3. Local c-optimal Designs

Recall that, for a given vector $c \in \mathbb{R}^{n+1}$, a design ξ_c is called c-optimal if the linear combination $c^T\theta$ is estimable by the design ξ_c, that is Range$(c) \subset$ Range$(M(\xi_c, \theta))$, and the design ξ_c minimizes

$$c^T M^-(\xi, \theta)c \tag{3.1}$$

among all designs for which $c^T\theta$ is estimable, where $M^-(\xi, \theta)$ denotes a generalized inverse of the matrix $M(\xi, \theta)$. It is shown in Pukelsheim (1993) that the expression (3.1) does not depend on the specific choice of the generalized inverse. Moreover, a design ξ_c is c-optimal if and only if there exists a generalized inverse G of $M(\xi_c, \theta)$ such that the inequality

$$(f'(u, \theta)Gc)^2 \leq c'M^-(\xi_c, \theta)c \tag{3.2}$$

holds for all $u \in \mathcal{U}$ (see Pukelsheim (1993)). A further important tool to determine c-optimal designs is the theory of Chebyshev systems, which is briefly described here for the sake of completeness.

Following Karlin and Studden (1966), a set of functions $\{g_0, \ldots, g_n\}$ defined on the set $\mathcal{U}$ is called Chebychev-system, if every linear combination $\sum_{i=0}^n a_i g_i(x)$ with $\sum_{i=0}^n a_i^2 > 0$ has at most n distinct roots on $\mathcal{U}$. This property is equivalent to the fact that

$$\det(g(u_0), \ldots, g(u_n)) \neq 0 \tag{3.3}$$

holds for all $u_0, \ldots, u_n \in \mathcal{U}$ with $u_i \neq u_j$ $(i \neq j)$, where $g(u) = (g_0(u), \ldots, g_n(u))^T$ denotes the vector of all functions (see Karlin and Studden (1966)). If the functions $g_0, \ldots, g_n$ constitute a Chebyshev-system on the set $\mathcal{U}$, then there exists a unique "polynomial"

$$\phi(u) := \sum_{i=0}^{n} \alpha_i^* g_i(u) \quad (\alpha_0^*, \ldots, \alpha_n^* \in \mathbb{R}) \tag{3.4}$$

with the properties

(i) $|\phi(u)| \leq 1 \quad \forall u \in \mathcal{U}$,

(ii) There exist $n+1$ points $s_0 < \cdots < s_n$ such that $\phi(s_i) = (-1)^{n-i}$ for $i = 0, \ldots, n$.

The function $\phi(u)$ is called the Chebychev-polynomial, and the points $s_0, \ldots, s_n$ are called Chebychev-points, these are not necessarily unique. Kiefer and Wolfowitz (1965) defined the set $A^* \subset \mathbb{R}^{n+1}$ as the set of all vectors $c \in \mathbb{R}^{n+1}$ satisfying

$$\begin{vmatrix} g_0(x_1) & \cdots & g_0(x_n) & c_0 \\ g_1(x_1) & \cdots & g_1(x_n) & c_1 \\ \vdots & \ddots & \vdots & \vdots \\ g_n(x_1) & \cdots & g_n(x_n) & c_n \end{vmatrix} \neq 0, \tag{3.5}$$

whenever the points $x_1, \ldots, x_n \in \mathcal{U}$ are distinct. They showed that for each $c \in A^*$, the c-optimal design that minimizes

$$c^T \left(\int_{\mathcal{U}} g(u) g^T(u) d\xi(u) \right)^{-1} c$$

among all designs on $\mathcal{U}$, is supported by the entire set of the Chebychev-points $s_0, \ldots, s_n$. The corresponding optimal weights $w_0^*, \ldots, w_n^*$ can then easily be found using Lagrange multipliers, and are given by

$$w_i^* = \frac{|v_i|}{\sum_{j=0}^{n} |v_j|} \qquad i = 0, \ldots, n, \tag{3.6}$$

where the vector v is $v = (XX^T)^{-1} X c$, and the $(n+1) \times (n+1)$-matrix X is given by $X = (g_j(s_i))_{i,j=0}^{n}$ (see also Pukelsheim and Torsney (1991)).

In the following discussion we use these results to determine local optimal design for two specific goals in the data analysis with inverse quadratic regression models: discrimination between inverse linear and quadratic models, and extrapolation or prediction at a specific point x_e. We begin with the discrimination problem that has been extensively studied for ordinary polynomial regression models (see Stigler (1971), Studden (1982) or Dette (1995), among many others). To our knowledge the problem of constructing designs for the discrimination between inverse rational models has not been studied in the literature. We consider the inverse quadratic regression model (2.2a) and are interested in determining a design, which can be used to discriminate between this and the inverse linear regression model $\eta(u, \theta) = u/(\theta_0 + \theta_1 u)$.

The decision on which model should be used could be based on the likelihood ratio test for the hypothesis $H_0 : \theta_2 = 0$ in the model (2.2a), and a standard calculation shows that the (asymptotic) power of this test is a decreasing function of the quantity (3.1), where the vector c is given by $c = (0, 0, 1)^T$. Thus a design maximizing the power of the likelihood ratio test for discriminating between the inverse linear and quadratic model is a local c-optimal design for the vector $c = (0, 0, 1)^T$. Following Stigler (1971), we call this design local D_1-optimal. Our first results determine the local D_1-optimal design for the two parameterizations of the inverse quadratic regression model explicitly.

Theorem 3.1. *The local D_1-optimal design $\xi^*_{D_1}$ for the inverse quadratic regression model* (2.2a) *on the design space $\mathcal{U} = [0, \infty)$ is given by*

$$\xi^*_{D_1} = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_0}{\theta_2}} & \sqrt{\frac{\theta_0}{\theta_2}} & \rho\sqrt{\frac{\theta_0}{\theta_2}} \\ w_0 & w_1 & 1 - w_0 - w_1 \end{pmatrix}, \tag{3.7}$$

with weights $w_0 = (\sqrt{\theta_2}\theta_0 + \theta_1\sqrt{\theta_0}\rho + \sqrt{\theta_2}\theta_0\rho^2)^2/[(1+\rho)\lambda]$, $w_1 = (2\sqrt{\theta_2}\theta_0 + \theta_1\sqrt{\theta_0})^2\rho^2/\lambda$, and $\lambda = \theta_0(\theta_0\theta_2(1 + 6\rho^2 + \rho^4) + 2\theta_1\rho(\theta_1\rho + \sqrt{\theta_0\theta_2}(1+\rho)^2))$. Here the geometric scaling factor ρ is

$$\rho = \rho(\gamma) = 1 + \frac{2+\gamma}{\sqrt{2}} + \sqrt{2(1+\sqrt{2}) + (2+\sqrt{2})\gamma + \frac{\gamma^2}{2}} \tag{3.8}$$

with $\gamma = \theta_1/\sqrt{\theta_0\theta_2}$. This design is also local D_1-optimal on the design space $\mathcal{U} = [s, t]$ $(0 < s < t)$ if the inequalities $0 \le s \le \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \ge \rho\sqrt{\theta_0/\theta_2}$ are satisfied.

The local D_1-optimal design on the design space $\mathcal{U} = [s, t]$ for model (2.2a) *is of the form*

$$\xi^*_{D_1} = \begin{pmatrix} s & u'_1 & u'_2 \\ w'_0 & w'_1 & 1 - w'_0 - w'_1 \end{pmatrix} \tag{3.9}$$

if the inequalities $s \ge \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t > \rho\sqrt{\theta_0/\theta_2}$ hold, is of the form

$$\xi^*_{D_1} = \begin{pmatrix} u''_0 & u''_1 & t \\ w''_0 & w''_1 & 1 - w''_0 - w''_1 \end{pmatrix} \tag{3.10}$$

if the inequalities $s < \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \le \rho\sqrt{\theta_0/\theta_2}$ are satisfied, and is of the form

$$\xi^*_{D_1} = \begin{pmatrix} s & u'''_1 & t \\ w'''_0 & w'''_1 & 1 - w'''_0 - w'''_1 \end{pmatrix} \tag{3.11}$$

if the inequalities $s \geq \rho^{-1}\sqrt{\theta_0/\theta_2}$ *and* $t \leq \rho\sqrt{\theta_0/\theta_2}$ *hold.*

Proof. The proof is in three steps:

(A) identify a candidate for the local D_1-optimal design on the interval $[0, \infty)$ using the theory of Chebyshev polynomials;

(B) use the properties of the Chebyshev polynomial (3.4) to prove the local D_1-optimality of this candidate;

(C) consider the case of a bounded design space and determine how the constraints interfere with the support points of the local optimal design on the unbounded design space.

(A): Let $f(u, \theta)$ be the vector of the partial derivatives in parameterization (2.2a) defined in (2.5). It is easy to see that the components of the vector $f_1(u, \theta)$, say $\{ f_{10}(u, \theta), f_{11}(u, \theta), f_{12}(u, \theta) \}$, constitute a Chebyshev-system on any bounded interval $[s, t] \subset (0, \infty)$. Furthermore, for $y_0, y_1 > 0$ with $y_0 \neq y_1$, we get

$$\begin{vmatrix} f_{10}(y_0,\theta) & f_{10}(y_1,\theta) & 0 \\ f_{11}(y_0,\theta) & f_{11}(y_1,\theta) & 0 \\ f_{12}(y_0,\theta) & f_{12}(y_1,\theta) & 1 \end{vmatrix} \neq 0,$$

and it follows that the vector $(0, 0, 1)^T$ is an element of the set A^* defined in (3.5). Therefore we obtain from the results of Kiefer and Wolfowitz (1965) that the local D_1-optimal design is supported on the entire set of Chebyshev-points $\{u_0^*, u_1^*, u_2^*\}$ of the Chebyshev-system $\{f_{10}(u, \theta), f_{11}(u, \theta), f_{12}(u, \theta)\}$. If the support points are given, say u_0, u_1, u_2, the corresponding weights can be determined by (3.6) such that the function defined in (3.1) is maximal.

Now the D_1-optimality criterion can be expressed as a function of the points u_0, u_1, u_2 and optimized analytically. For this purpose we obtain, by a tedious computation,

$$T(\tilde{u}, \theta) := \frac{|M(\xi, \theta)|}{|\tilde{M}(\xi, \theta)|} = \frac{u_0^2(u_0 - u_1)^2 u_1^2 (u_1 - u_2)^2 u_2^2}{N}, \tag{3.12}$$

where $\tilde{M}(\xi, \theta)$ denotes the matrix obtained from $M(\xi, \theta)$ by deleting the last row and column, $\tilde{u} = (u_0, u_1, u_2)$, $\theta = (\theta_0, \theta_1, \theta_2)$, and

$$\begin{aligned} N = (&4\theta_0 u_0 u_1(\theta_1 + \theta_2 u_1)u_2 + \theta_0^2(u_1(u_2 - u_1) + u_0(u_1 + u_2)) \\ &+ u_0 u_1 u_2(2\theta_1^2 u_1 + 2\theta_1\theta_2(u_0(u_1 - u_2) + u_1(u_1 + u_2)) \\ &+ \theta_2^2(u_0^2(u_1 - u_2) + u_0(u_1 - u_2)u_2 + u_1(u_1^2 + u_2^2))))^2. \end{aligned}$$

We maximize $T(\tilde{u},\theta)$ with respect to u_0, u_1, u_2. The necessary conditions for a maximum yield the system of nonlinear equations

$$\begin{aligned}\frac{\partial T}{\partial u_0}(\tilde{u},\theta) = {}& 4\theta_0 u_0^2 u_1(\theta_1+\theta_2 u_1)u_2 + \theta_0^2(-2u_0u_1(u_1-u_2)\\ &+u_1^2(u_1-u_2)+u_0^2(u_1+u_2)) + u_0^2u_1u_2(2\theta_1^2u_1+4\theta_1\theta_2u_1^2\\ &+\theta_2^2(2u_0u_1(u_1-u_2)+u_0^2(u_2-u_1)+u_1^2(u_1+u_2)))\cdot R_1 = 0, \quad (3.13)\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial u_1}(\tilde{u},\theta) = {}& 4\theta_0u_0u_1^2u_2(-(\theta_2u_1^2)+\theta_2u_0u_2+\theta_1(u_0-2u_1+u_2))\\ &+\theta_0^2(u_1^2(u_1-u_2)^2-2u_0u_1(u_1^2+u_1u_2-u_2^2)+u_0^2(u_1^2+2u_1u_2-u_2^2))\\ &-u_0u_1^2u_2(2\theta_1^2(u_1^2-u_0u_2)+4\theta_1\theta_2u_1(u_0(u_1-2u_2)+u_1u_2)\\ &+\theta_2^2(u_0^2(u_1-u_2)^2+2u_0u_1(u_1^2-u_1u_2-u_2^2)\\ &+u_1^2(-u_1^2+2u_1u_2+u_2^2)))\cdot R_2 = 0, \quad (3.14)\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial u_2}(\tilde{u},\theta) = {}& 4\theta_0u_0u_1(\theta_1+\theta_2u_1)u_2^2+\theta_0^2(u_1(u_1-u_2)^2+u_0(-u_1^2+2u_1u_2+u_2^2))\\ &+u_0u_1u_2^2(2\theta_1^2u_1+4\theta_1\theta_2u_1^2+\theta_2^2(u_0(u_1-u_2)^2\\ &+u_1(u_1^2+2u_1u_2-u_2^2)))\cdot R_3 = 0, \quad (3.15)\end{aligned}$$

where R_1, R_2 and R_3 are rational functions that do not vanish for all u_0, u_1, u_2 with $0 < u_0 < u_1 < u_2$. In order to solve this system of equations, we assume that

$$u_0 = \frac{u_1}{r}, \quad u_2 = r\cdot u_1 \quad (3.16)$$

holds for some factor $r > 1$, to be specified later. Inserting this in (3.14) gives, as the only positive solution, $u_1^* = \sqrt{\theta_0/\theta_2}$. Substituting this term into (3.13) or (3.15) yields the following equation for the factor r:

$$2\theta_1(\theta_1+4\sqrt{\theta_0\theta_2})r^2 - \theta_0\theta_2(1-4r-2r^2-4r^3+r^4) = 0,$$

with four roots given by

$$r_{1/2} = 1 \pm \frac{(2+\gamma)}{\sqrt{2}} \pm \sqrt{2(1+\sqrt{2})+(2+\sqrt{2})\gamma+\frac{\gamma^2}{2}}, \quad (3.17)$$

$$r_{3/4} = 1 \pm \frac{(2+\gamma)}{\sqrt{2}} \mp \sqrt{2(1+\sqrt{2})+(2+\sqrt{2})\gamma+\frac{\gamma^2}{2}},$$

where $\gamma = \sqrt{\theta_1\theta_2}^{-1}$. The factor r has to be strict greater 1 according to our assumption on the relation between u_0, u_1 and u_2. This provides only the first solution in (3.17), and the geometric scaling factor is given by (3.8). Therefore it remains to justify assumption (3.16), which is done in the second part of the proof.

(B): Because the calculation of the support points $\rho^{-1}\sqrt{\theta_0/\theta_2}$, $\sqrt{\theta_0/\theta_2}$, and $\rho\sqrt{\theta_0/\theta_2}$ in step (A) is based on assumption (3.16), we still have to prove that these points are the support points of the local D_1-optimal design. For this purpose we show that the unique oscillating polynomial defined by (3.4) attends minima and maxima exactly in these support points. Recall that the vector of the partial derivatives of the regression function $f_1(u,\theta) = (f_{10}(u,\theta), f_{11}(u,\theta), f_{12}(u,\theta))$ is given by (2.5). We now set

$$t(u) = f_{10}(u,\theta) + \alpha_1 f_{11}(u,\theta) + \alpha_2 f_{12}(u,\theta) \tag{3.18}$$

and determine the factors α_1 and α_2 such that it is equioscillating, i.e.,

$$t'(u_i^*) = 0 \qquad i = 0,1,2, \tag{3.19a}$$

$$t(u_i^*) = c(-1)^{i-1} \qquad i = 0,1,2, \tag{3.19b}$$

for some constant $c \in \mathbb{R}$. By this choice, the polynomial t must be proportional to the polynomial ϕ defined in (3.4). For the determination of the coefficients calculate that

$$t'(u) = \frac{-(\theta_0(1+2u\alpha_1+3u^2\alpha_2)) + u(\theta_1(1-u^2\alpha_2) + \theta_2 u(3+2u\alpha_1+u^2\alpha_2))}{(\theta_0 + u(\theta_1+\theta_2 u))^3}. \tag{3.20}$$

Substituting the support points $u_1^* = \sqrt{\theta_0/\theta_2}$ and $u_2^* = \rho\sqrt{\theta_0/\theta_2}$ in (3.20), we obtain from (3.19a) the equations

$$0 = \frac{\sqrt{\theta_0}(\theta_1 + 2\sqrt{\theta_0\theta_2})(\theta_2 - \theta_0\alpha_2)}{\sqrt{\theta_2}\theta_2},$$

$$0 = \frac{\sqrt{\theta_0}\theta_2(\theta_1\rho + \sqrt{\theta_0\theta_2}(3\rho^2-1) + 2\theta_0\rho(\rho^2-1)\alpha_1)}{\sqrt{\theta_2}\theta_2} + \frac{\sqrt{\theta_0}\theta_0\rho^2(-(\theta_1\rho) + \sqrt{\theta_0\theta_2}(\rho^2-3))\alpha_2}{\sqrt{\theta_2}\theta_2}.$$

The solution with respect to α_1 and α_2 is

$$\alpha_1 = -\frac{\sqrt{\theta_0\theta_2} - \theta_1\rho + \sqrt{\theta_0\theta_2}\rho^2}{2\theta_0\rho}, \qquad \alpha_2 = \frac{\theta_2}{\theta_0},$$

which yields

$$t(u) = \frac{u(-2\theta_0\rho + \sqrt{\theta_0\theta_2}(1+\rho^2)u - \rho u(\theta_1 + 2\theta_2 u))}{2\theta_0\rho(\theta_0 + u(\theta_1+\theta_2 u))^2},$$

$$t'(u) = -\frac{(\sqrt{\theta_0} - \sqrt{\theta_2}u)(\sqrt{\theta_0}\rho - \sqrt{\theta_2}u)(\sqrt{\theta_0} + \sqrt{\theta_2}u)(\sqrt{\theta_0} - \sqrt{\theta_2}\rho u)}{\theta_0\rho(\theta_0 + u(\theta_1+\theta_2 u))^3}, \tag{3.21}$$

respectively. A straightforward calculation shows that the third support point $u_0^* = \rho^{-1}\sqrt{\theta_0/\theta_2}$ satisfies $t'(u_0^*) = 0$, and that the three equations in (3.19b) are satisfied. Therefore it only remains to prove that the inequality $|t(u)| \leq c$ holds on the interval $[0, \infty)$. In this case the polynomial t must be proportional to the equioscillating polynomial ϕ, and the design with support points $\rho^{-1}\sqrt{\theta_0/\theta_2}$, $\sqrt{\theta_0/\theta_2}$ and $\rho\sqrt{\theta_0/\theta_2}$ and optimal weights is local D_1-optimal.
Observing the representation (3.21) shows that the equation $t'(u) = 0$ is equivalent to

$$(\sqrt{\theta_0} - \sqrt{\theta_2}u)(\sqrt{\theta_0}\rho - \sqrt{\theta_2}u)(\sqrt{\theta_0} + \sqrt{\theta_2}u)(\sqrt{\theta_0} - \sqrt{\theta_2}\rho u) = 0, \tag{3.22}$$

with roots

$$n_0 = -\sqrt{\frac{\theta_0}{\theta_2}}, \quad n_1 = \frac{1}{\rho}\sqrt{\frac{\theta_0}{\theta_2}}, \quad n_2 = \sqrt{\frac{\theta_0}{\theta_2}} \quad \text{and} \quad n_3 = \rho\sqrt{\frac{\theta_0}{\theta_2}}.$$

Therefore t has exactly three extrema on $\mathbb{R}^+$. Furthermore if $u \to \infty$, we have $t(u) \to 0$ and it follows that $|t(u)| \leq c$ holds for all $u \geq 0$. Consequently, the functions t and ϕ are proportional and the points $u_0^* = \rho^{-1}\sqrt{\theta_0/\theta_2}$, $u_1^* = \sqrt{\theta_0/\theta_2}$, $u_2^* = \rho\sqrt{\theta_0/\theta_2}$ are the support points of the local D_1-optimal design. The explicit construction of the weights w_0 and w_1 is obtained by substituting the support points u_0^*, u_1^* and u_2^* into (3.6).

(C) Finally consider the cases (3.9), (3.10) and (3.11) in the second part of Theorem 3.1 that correspond to a bounded design space. For the sake of brevity we restrict ourselves to the case (3.9), all other cases are treated similarly. Obviously the assertion follows from the existence of a point $u_0^* > 0$, such that $T(u_0, u_1^*, u_2^*, \theta)$ is increasing in u_0 on the interval $(0, u_0^*)$ and decreasing on (u_0^*, u_1^*).
For a proof of this property we fix u_1, u_2, and note that $\bar{T}(u_0) := T(u_0, u_1, u_2, \theta)$ has minima in $u_0 = 0$ and $u_0 = u_1$, since the inequality $\bar{T}(u_0) \geq 0$ holds for all $u_0 \in [0, u_1]$ and $\bar{T}(0) = \bar{T}(u_1) = 0$. Because $\bar{T}(u_0)$ is not constant, there is at least one maximum in the interval $(0, u_1)$. In order to prove that there is exactly one maximum, we calculate

$$\bar{T}'(u_0) = \frac{\partial T}{\partial u_0}(u_0, u_1, u_2, \theta) = 2u_0(u_0 - u_1)u_1^2(u_1 - u_2)^2 u_2^2 \frac{P_4(u_0)}{P_9(u_0)}, \tag{3.23}$$

where P_9 is a polynomial of degree 9 (which is in the following discussion without interest) and the polynomial P_4 in the numerator is given by

$$\begin{aligned} P_4(u_0) = {} & 4\theta_0 u_0^2 u_1(\theta_1 + \theta_2 u_1)u_2 + \theta_0^2(-2u_0u_1(u_1 - u_2) + u_1^2(u_1 - u_2) \\ & + u_0^2(u_1 + u_2)) + u_0^2 u_1 u_2(2\theta_1^2 u_1 + 4\theta_1\theta_2 u_1^2 + \theta_2^2(2u_0u_1(u_1 - u_2) \\ & + u_0^2(u_2 - u_1) + u_1^2(u_1 + u_2))). \end{aligned}$$

The roots of the function $\bar{T}'$ are given by the roots of the polynomial P_4. Differentiating this polynomial yields

$$\frac{\partial P_4}{\partial u_0}(u_0) = 8\theta_0 u_0 u_1(\theta_1 + \theta_2 u_1)u_2 + 2\theta_0^2(u_1(u_2 - u_1) + u_0(u_1 + u_2)) \\ +2u_0 u_1 u_2(2\theta_1^2 u_1 + 4\theta_1\theta_2 u_1^2 + \theta_2^2(-2u_0^2(u_1 - u_2) \\ +3u_0 u_1(u_1 - u_2) + u_1^2(u_1 + u_2))),$$

which has only one real root. Consequently $P_4(u_0)$ has just one extremum and therefore at most two roots. The case of no roots has been excluded above. If $P_4(u_0)$ would have two roots, then the function $\bar{T}(u_0)$ has at most two extrema in the interval $(0, u_1)$. However, $\bar{T}(u_0)$ is zero in the two points 0 and u_1, and in the interval $(0, u_1)$ is strictly positive. Therefore the number of its extrema has to be odd and $\bar{T}(u_0)$ has exactly one maximum on $(0, u_1)$, attained for given $(u_1, u_2) = (u_1^*, u_2^*)$ at a point $u_0^* \in (0, u_1^*)$.
Assume that the design space is of the form $\mathcal{U} = [s, t]$. If the inequality $s < u_0^*$ holds, (3.7) remains the local D_1-optimal design. However if the inequality $s > u_0^*$ holds, the function $\bar{T}(u_0)$ is maximal in s, and it follows that (3.9) is the local D_1-optimal design.

Remark 3.1. Note that part (A) of the proof essentially follows the arguments presented in Haines (1992) for D-optimality criterion, under the model

$$\eta(u, \alpha, \beta_0, \beta_1, \beta_2) = \frac{u + \alpha}{\beta_0 + \beta_1(u + \alpha) + \beta_2(u + \alpha)^2}.$$

However, the proof presented by Haines (1992) is not complete. Here we present a tool for correcting this gap, as demonstrated in part (B) of the preceding proof. It is also worthwhile to mention that an analogue of Theorem 3.1 does not hold in the four-parameter model discussed in Haines (1992). For example if $\beta_0 = \beta_2 = 1$, $\beta_1 = -1.8$, and $\alpha = 0.1$, we obtain by numerical computation that the local D_1-optimal design is supported at the Chebyshev-points $\{ 0, 0.6272, 0.9861, 1.8714 \}$; there does not exist a similar geometric spacing behaviour as in the models considered in this paper.

The following theorem states the corresponding results for the inverse quadratic regression model with parameterization (2.2b). The proof is similar to the proof of the previous theorem and therefore omitted.

Theorem 3.2. *The local D_1-optimal design $\xi_{D_1}^*$ for the inverse quadratic regression model* (2.2b) *on the design space $\mathcal{U} = [0, \infty)$ is given by*

$$\xi_{D_1}^* = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_1}{\theta_2}} & \sqrt{\frac{\theta_1}{\theta_2}} & \rho\sqrt{\frac{\theta_1}{\theta_2}} \\ w_0 & w_1 & 1 - w_0 - w_1 \end{pmatrix} \tag{3.24}$$

with

$$w_0 = \frac{(\sqrt{\theta_2}\theta_1 + \sqrt{\theta_1}\rho + \sqrt{\theta_2}\theta_1\rho^2)^2(1 + \sqrt{\theta_1\theta_2}(1+\rho))}{[(1+\rho)\lambda]}$$
$$w_1 = \frac{(2\theta_1 + \sqrt{\theta_1\theta_2})^2\rho(\rho + \sqrt{\theta_1\theta_2}(1+\rho^2))}{\lambda},$$
$$\lambda = \theta_1(\rho(2\rho + 3\sqrt{\theta_1\theta_2}(1+\rho)^2) + \theta_1\theta_2(1 + 2\sqrt{\theta_1\theta_2}(1+\rho)^2(1+\rho^2) + \rho(8 + \rho(6 + \rho(8+\rho))))).$$

The geometric scaling factor ρ is given by (3.8) *with $\gamma = (\sqrt{\theta_1\theta_2})^{-1}$. This design is also local D_1-optimal on the design space $\mathcal{U} = [s,t]$ $(0 < s < t)$ if the inequalities $0 \le s \le \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \ge \rho\sqrt{\theta_1/\theta_2}$ are satisfied.*
The local D_1-optimal design on the design space $\mathcal{U} = [s,t]$ for the inverse quadratic regression model (2.2b) *is of the form* (3.9) *if the inequalities $s \ge \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t > \rho\sqrt{\theta_1/\theta_2}$ hold, of the form* (3.10) *if the inequalities $s < \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \le \rho\sqrt{\theta_1/\theta_2}$ are satisfied, and is of the form* (3.11) *if the inequalities $s \ge \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \le \rho\sqrt{\theta_1/\theta_2}$ hold.*

In the following discussion we concentrate on the problem of extrapolation in the inverse quadratic regression model. An optimal design for this purpose minimizes the variance of the estimate of the expected response at a point x_e and is therefore c-optimal for the vector $c_e = f_1(x_e, \theta)$ in the case of parameterization (2.2a), and for the vector $c_e = f_2(x_e, \theta)$ in the case of parameterization (2.2b), respectively. If x_e is an element of the design space $\mathcal{U}$, it is obviously optimal to take all observations at the point x_e, and therefore we assume for the remaining part of this section that $\mathcal{U} = [s,t]$, where $0 \le s < t$ and $0 < x_e < s$ or $x_e > t$. The following result specifies local optimal extrapolation designs for the inverse quadratic regression model that are called local c_e-designs in the following discussion. The proofs are similar to the proofs for D_1-optimality and are therefore omitted.

Theorem 3.3. *Assume that $\mathcal{U} = [s,t]$, where $0 \le s < t$ and $0 < x_e < s$ or $x_e > t$, and let ρ denote the geometric scaling factor defined in* (3.8) *with $\gamma = \theta_1(\sqrt{\theta_0\theta_2})^{-1}$. If $0 \le s \le \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \ge \rho\sqrt{\theta_0/\theta_2}$, then the local c_e-optimal design $\xi^*_{c_e}$ for the inverse quadratic regression model* (2.2a) *is given by*

$$\xi^*_{c_e} = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_0}{\theta_2}} & \sqrt{\frac{\theta_0}{\theta_2}} & \rho\sqrt{\frac{\theta_0}{\theta_2}} \\ w_0 & w_1 & 1-w_0-w_1 \end{pmatrix} \tag{3.25}$$

where

$$w_0 = \Big(\sqrt{\theta_0} - x_e\sqrt{\theta_2}\Big)\Big(- x_e\sqrt{\theta_2} + \sqrt{\theta_0}\rho\Big)\Big(\theta_0\sqrt{\theta_2} + \theta_1\sqrt{\theta_0}\rho + \theta_0\sqrt{\theta_2}\rho^2\Big)^2 \times((1+\rho)\lambda)^{-1}$$

$$w_1 = \frac{(2\theta_0\sqrt{\theta_2} + \theta_1\sqrt{\theta_0})^2\rho(-x_e\sqrt{\theta_2} + \sqrt{\theta_0}\rho)(\sqrt{\theta_0} - x_e\sqrt{\theta_2}\rho)}{\lambda},$$

$$\lambda = \theta_0\Big(\theta_0^2\theta_2(1+6\rho^2+\rho^4) + \theta_0(2\theta_1^2\rho^2 + 2\theta_1\rho(\sqrt{\theta_0\theta_2}(1+\rho)^2 - 4x_e\theta_2(1+\rho^2)) + \theta_2 x_e(-2\sqrt{\theta_0\theta_2}(1+\rho)^2(1+\rho^2) + x_e\theta_2(1+6\rho^2+\rho^4))) + \theta_1 x_e\rho(2\sqrt{\theta_0\theta_2}\theta_2 x_e(1+\rho)^2 - \theta_1(\sqrt{\theta_0\theta_2} + \rho(-2x_e\theta_2 + \sqrt{\theta_0\theta_2}(2+\rho))))\Big).$$

The local c_e-optimal design for the inverse quadratic model (2.2a) *is of the form* (3.9) *if the inequalities $s \geq \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t > \rho\sqrt{\theta_0/\theta_2}$ hold, is of the form* (3.10) *if the inequalities $s < \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \leq \rho\sqrt{\theta_0/\theta_2}$ are satisfied, and is of the form* (3.11) *if the inequalities $s \geq \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \leq \rho\sqrt{\theta_0/\theta_2}$ hold.*

Theorem 3.4. *Assume that $\mathcal{U} = [s,t]$, where $0 \leq s < t$ and $0 < x_e < s$ or $x_e > t$, and let ρ denote the geometric scaling factor ρ defined in* (3.8) *with $\gamma = (\sqrt{\theta_1\theta_2})^{-1}$. If $0 \leq s \leq \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \geq \rho\sqrt{\theta_1/\theta_2}$, then the local c_e-optimal design $\xi^*_{c_e}$ for the inverse quadratic regression model* (2.2b) *on the design space $\mathcal{U} = [0,\infty)$ is given by*

$$\xi^*_{c_e} = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_1}{\theta_2}} & \sqrt{\frac{\theta_1}{\theta_2}} & \rho\sqrt{\frac{\theta_1}{\theta_2}} \\ w_0 & w_1 & 1-w_0-w_1 \end{pmatrix} \tag{3.26}$$

with

$$w_0 = \Big(\sqrt{\theta_1} - x_e\sqrt{\theta_2}\Big)\Big(- x_e\sqrt{\theta_2} + \sqrt{\theta_1}\rho\Big)\Big(\theta_1\sqrt{\theta_2} + \sqrt{\theta_1}\rho + \theta_1\sqrt{\theta_2}\rho^2\Big)^2 \times((1+\rho)\lambda)^{-1}$$

$$w_1 = \frac{(2\theta_1\sqrt{\theta_2} + \sqrt{\theta_1})^2\rho(-x_e\sqrt{\theta_2} + \sqrt{\theta_1}\rho)(\sqrt{\theta_1} - x_e\sqrt{\theta_2}\rho)}{\lambda},$$

$$\lambda = \theta_1\Big(\theta_1^2\theta_2(1+6\rho^2+\rho^4) + x_e\rho(-\sqrt{\theta_1\theta_2} + 2\sqrt{\theta_1\theta_2}\theta_2 x_e(1+\rho)^2 - \rho(-2x_e\sqrt{\theta_2} + \sqrt{\theta_1\theta_2}(2+\rho))) + \theta_1(2\rho^2 + 2\rho(\sqrt{\theta_1\theta_2}(1+\rho)^2 - 4x_e\sqrt{\theta_2}(1+\rho^2)) + x_e(-2\sqrt{\theta_1\theta_2}\theta_2(1+\rho)^2(1+\rho^2) + x_e\theta_2^2(1+6\rho^2+\rho^4)))\Big).$$

If the design space is given by a finite interval $[s,t]$, $0 < s < t$, then the local c_e-optimal design for model (2.2a) *is of the form* (3.9) *if the inequalities*

$s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t > \rho\sqrt{\theta_1/\theta_2}$ hold, is of the form (3.10) *if the inequalities $s < \rho^{-1}\sqrt{\theta_1\theta_2}$ and $t \leq \rho\sqrt{\theta_1/\theta_2}$ are satisfied, and is of the form* (3.11) *if the inequalities $s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \leq \rho\sqrt{\theta_1/\theta_2}$ hold.*

Note that for a sufficiently large design interval all designs presented in this section are supported at the same points, the Chebyshev points corresponding to the Chebyshev system of the components of the gradient of the regression function. In the next section we demonstrate that these points are also the support points of the local E-optimal design for the inverse quadratic regression model.

4. Local *D*- and *E*-optimal Designs

We begin by stating the corresponding result for the D-optimality criterion. The proof is omitted because it requires arguments that are similar to those presented in Haines (1992) and in the proof of Theorem 3.1.

Theorem 4.1. *The local D-optimal design ξ_D^* for the inverse quadratic regression model* (2.2a) *on the design space $\mathcal{U} = [0, \infty)$ is given by*

$$\xi_D^* = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_0}{\theta_2}} & \sqrt{\frac{\theta_0}{\theta_2}} & \rho\sqrt{\frac{\theta_0}{\theta_2}} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{4.1}$$

with the geometric scaling factor

$$\rho = \frac{\delta + \sqrt{\delta^2 - 4}}{2}, \tag{4.2}$$

where the constants δ and γ are defined by $\delta = (1/2)(\gamma + 1 + \sqrt{\gamma^2 + 6\gamma + 33})$ and $\gamma = \theta_1(\sqrt{\theta_0\theta_2})^{-1}$, respectively. This design is also local D-optimal on the design space $\mathcal{U} = [s, t]$ $(0 < s < t)$ if the inequalities $0 \leq s \leq \rho^{-1}\sqrt{\theta_0/\theta_2}$ and $t \geq \rho\sqrt{\theta_0/\theta_2}$ are satisfied.
The local D-optimal design on the design space $\mathcal{U} = [s, t]$ for the inverse quadratic regression model (2.2b) *is of the form* (3.9) *if the inequalities $s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t > \rho\sqrt{\theta_1/\theta_2}$ hold, is of the form* (3.10) *if the inequalities $s < \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \leq \rho\sqrt{\theta_1/\theta_2}$ are satisfied, and is of the form* (3.11) *if the inequalities $s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ and $t \leq \rho\sqrt{\theta_1/\theta_2}$ hold.*

Theorem 4.2. *The local D-optimal design ξ_D^* for the inverse quadratic regression model* (2.2b) *on the design space $\mathcal{U} = [0, \infty)$ is given by*

$$\xi_D^* = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_1}{\theta_2}} & \sqrt{\frac{\theta_1}{\theta_2}} & \rho\sqrt{\frac{\theta_1}{\theta_2}} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{4.3}$$

with the geometric scaling factor ρ *given by* (4.2) *with* $\gamma = (\sqrt{\theta_1\theta_2})^{-1}$. *This design is also* D*-optimal on the design space* $\mathcal{U} = [s, t]$ $(0 < s < t)$ *if the inequalities* $0 \le s \le \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \ge \rho\sqrt{\theta_1/\theta_2}$ *are satisfied.*
The local D*-optimal design on the design space* $\mathcal{U} = [s, t]$ *for the inverse quadratic regression model* (2.2b) *is of the form* (3.9) *if the inequalities* $s \ge \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t > \rho\sqrt{\theta_1/\theta_2}$ *hold, is of the form* (3.10) *if the inequalities* $s < \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \le \rho\sqrt{\theta_1/\theta_2}$ *are satisfied, and is of the form* (3.11) *if the inequalities* $s \ge \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \le \rho\sqrt{\theta_1/\theta_2}$ *hold.*

We conclude this section with the discussion of the E-optimality criterion. For this purpose recall that a design ξ_E is local E-optimal if and only if there exists a matrix $E \in \text{conv}(S)$ such that the inequality

$$f'(u,\theta)Ef(u,\theta) \le \lambda_{\min} \tag{4.4}$$

holds for all $u \in \mathcal{U}$, where $\lambda_{\min}$ denotes the minimum eigenvalue of the matrix $M(\xi_E, \theta)$ and

$$S = \left\{ zz' \mid \|z\|_2 = 1,\ z \text{ is an eigenvector of } M(\xi_E,\theta) \text{ corresponding to } \lambda_{\min} \right\}. \tag{4.5}$$

The following two results specify the local E-optimal designs for the inverse quadratic regression models with parameterization (2.2a) and (2.2b). Because both statements are proved similarly, we restrict ourselves to a proof of the first theorem.

Theorem 4.3. *The local* E*-optimal design* ξ_E^* *for the inverse quadratic regression model* (2.2a) *on the design space* $\mathcal{U} = [0, \infty)$ *is given by*

$$\xi_E^* = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_0}{\theta_2}} & \sqrt{\frac{\theta_0}{\theta_2}} & \rho\sqrt{\frac{\theta_0}{\theta_2}} \\ w_0 & w_1 & 1 - w_0 - w_1 \end{pmatrix}, \tag{4.6}$$

where the weights w_0, w_1 *are given by* (3.6) *and* c *is the vector with components given by the coefficients of the Chebyshev polynomial, that is*

$$c = \Bigg(-\frac{\sqrt{\theta_0}(2\theta_1^2\rho^2 + 2\sqrt{\theta_0}\theta_1\sqrt{\theta_2}\rho(1+\rho)^2 + \theta_0\theta_2(1+6\rho^2+\rho^4))}{\sqrt{\theta_2}(-1+\rho)^2\rho},$$
$$\frac{\theta_1^2\rho(1+\rho)^2 + 8\sqrt{\theta_0}\theta_1\sqrt{\theta_2}\rho(1+\rho^2) + 2\theta_0\theta_2(1+\rho)^2(1+\rho^2)}{(-1+\rho)^2\rho},$$
$$-\frac{\sqrt{\theta_2}(2\theta_1^2\rho^2 + 2\sqrt{\theta_0}\theta_1\sqrt{\theta_2}\rho(1+\rho)^2 + \theta_0\theta_2(1+6\rho^2+\rho^4))}{\sqrt{\theta_0}(-1+\rho)^2\rho}\Bigg)^T.$$

The geometric scaling factor is given by (3.8) *with* $\gamma = \theta_1(\sqrt{\theta_0\theta_2})^{-1}$. *This design is also local* E*-optimal on the design space* $\mathcal{U} = [s, t]$ $(0 < s < t)$ *if the inequalities* $0 \le s \le \rho^{-1}\sqrt{\theta_0/\theta_2}$ *and* $t \ge \rho\sqrt{\theta_0/\theta_2}$ *are satisfied.*

The local E-optimal design on the design space $\mathcal{U} = [s,t]$ *for model* (2.2a) *is of the form* (3.9) *if the inequalities* $s \geq \rho^{-1}\sqrt{\theta_0/\theta_2}$ *and* $t > \rho\sqrt{\theta_0/\theta_2}$ *hold, is of the form* (3.10) *if the inequalities* $s < \rho^{-1}\sqrt{\theta_0/\theta_2}$ *and* $t \leq \rho\sqrt{\theta_0/\theta_2}$ *are satisfied, and is of the form* (3.11) *if the inequalities* $s \geq \rho^{-1}\sqrt{\theta_0/\theta_2}$ *and* $t \leq \rho\sqrt{\theta_0/\theta_2}$ *hold.*

Proof. It is straightforward to show that for every subset of $\{f_{10}(u,\theta), f_{11}(u,\theta), f_{12}(u,\theta)\}$, the components of the vector $f_1(u,\theta)$ that consists of 2 elements is a (weak) Chebychev-system. Therefore it follows from Theorem 2.1 in Imhof and Studden (2001) that the local E-optimal is supported at the Chebyshev points. The assertion regarding the weights finally follows from (3.6) by observing that the results of Imhof and Studden (2001) imply that the local E-optimal design is also c-optimal for the vector c with components given by the coefficients of the Chebyshev polynomial.

Theorem 4.4. *The local* E*-optimal design* ξ_E^* *for the inverse quadratic regression model* (2.2b) *on the design space* $\mathcal{U} = [0,\infty)$ *is given by*

$$\xi_E^* = \begin{pmatrix} \frac{1}{\rho}\sqrt{\frac{\theta_1}{\theta_2}} & \sqrt{\frac{\theta_1}{\theta_2}} & \rho\sqrt{\frac{\theta_1}{\theta_2}} \\ w_0 & w_1 & 1-w_0-w_1 \end{pmatrix}, \tag{4.7}$$

where the weights w_0, w_1 *are given by* (3.6) *and* c *is the vector with components given by the coefficients of the Chebyshev polynomial, that is*

$$c = \Bigg(-1 - 2\sqrt{\theta_1\theta_2} - \frac{2(2\rho+\sqrt{\theta_1\theta_2}(1+\rho)^2)(\rho+\sqrt{\theta_1\theta_2}(1+\rho^2))}{(-1+\rho)^2\rho},$$
$$-\frac{\sqrt{\theta_1}(1+2\sqrt{\theta_1\theta_2})(2\rho+\sqrt{\theta_1\theta_2}(1+\rho)^2)(\rho+\sqrt{\theta_1\theta_2}(1+\rho^2))}{\theta_0\sqrt{\theta_2}(-1+\rho)^2\rho},$$
$$-\frac{\sqrt{\theta_2}(1+2\sqrt{\theta_1\theta_2})(2\rho+\sqrt{\theta_1\theta_2}(1+\rho)^2)(\rho+\sqrt{\theta_1\theta_2}(1+\rho^2))}{\theta_0\sqrt{\theta_1}(-1+\rho)^2\rho}\Bigg)^T.$$

The geometric scaling factor is given by (3.8) *with* $\gamma = (\sqrt{\theta_1\theta_2})^{-1}$. *This design is also local E-optimal on the design space* $\mathcal{U} = [s,t]$ $(0 < s < t)$ *if the inequalities)* $0 \leq s \leq \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \geq \rho\sqrt{\theta_1/\theta_2}$ *are satisfied.*

The local E-optimal design on the design space $\mathcal{U} = [s,t]$ *for model* (2.2a) *is of the form* (3.9) *if the inequalities* $s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t > \rho\sqrt{\theta_1/\theta_2}$ *hold, is of the form* (3.10) *if the inequalities* $s < \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \leq \rho\sqrt{\theta_1/\theta_2}$ *are satisfied, and is of the form* (3.11) *if the inequalities* $s \geq \rho^{-1}\sqrt{\theta_1/\theta_2}$ *and* $t \leq \rho\sqrt{\theta_1/\theta_2}$ *hold.*

5. Further Discussion

In this Section we discuss some practical aspects of the local optimal designs derived in the previous sections. In particular, we calculate the efficiency of a

Table 5.1. D-, E-, D_1- and c_e-optimal designs for parametrization (2.2a).

Criterion	**Optimal design**				
D	points	:	1	3.4089	14
	weights	:	1/3	1/3	1/3
E	points	:	1	3.3561	14
	weights	:	0.3972	0.3914	0.2114
$\mathbf{D_1}$	points	:	1	3.3561	14
	weights	:	0.1239	0.2884	0.5877
$\mathbf{c_e}$	points	:	1	3.3561	14
	weights	:	0.0582	0.1535	0.7883

design that has recently been used in practice, and investigate the efficiency of local optimal designs with respect to other optimality criteria. Throughout, the efficiency of a design ξ is defined by $\text{eff}_\Phi(\xi) = \Phi(\xi)/\sup_\eta \Phi(\eta)$, where Φ denotes the particular optimality criterion under consideration and the optimal design maximizes Φ.

Landete-Castillejos and Gallego (2000) used the inverse quadratic regression model to analyze data that were obtained from lactating red deer hinds (Cervus elaphus). They concluded that inverse quadratic polynomials with parameterization (2.2a) can adequately describe the common lactation curves. The design space was given by the interval $\mathcal{U} = [1, 14]$, and the design used by these authors was a uniform design with support points $(1, 2, 3, 4, 5, 6, 10, 14)$, denoted by ξ_u throughout this section. The estimates for the parameters of model (2.2a) are given by $\hat{\theta}_0 = 0.0002865$, $\hat{\theta}_1 = 0.0002117$, and $\hat{\theta}_2 = 0.0000301$. Table 5.1 shows the local optimal designs for the different optimality criteria considered in Section 3 and 4, where we used the point $x_e = 21$ for the calculation of the optimal extrapolation design.

The efficiencies of the different designs are shown in Table 5.2. We observe that the design of Landete-Castillejos and Gallego (2000) yields rather low efficiencies with respect to all optimality criteria, and the efficiency of the statistical analysis could have been improved by allocating observations according to local optimal design (see the first row in Table 5.2). For example, a confidence interval based on the local D_1-optimal design would yield 66% shorter confidence intervals for the parameter c than the design actually used by Landete-Castillejos and Gallego (2000). The advantages of the local optimal designs are also clearly visible for the other criteria.

Note that the data are usually used for several purposes, for example for discrimination between a linear and a quadratic inverse polynomial, and for extrapolation using the identified model. Therefore it is important that an optimal design for a specific optimality criterion also yields reasonable efficiencies with

Table 5.2. Efficiencies of local optimal designs and the uniform design ξ_u for the inverse quadratic model (parameterization (2.2a)) with respect to various alternative criteria (in percent). The design space is the interval $\mathcal{U} = [1, 14]$, and the estimates of the parameters are given by $\hat{\theta}_0 = 0.0002865$, $\hat{\theta}_1 = 0.0002117$, and $\hat{\theta}_2 = 0.0000301$. The local extrapolation optimal design is calculated for the point $x_e = 21$.

	$\mathbf{D}$	$\mathbf{E}$	$\mathbf{D_1}$	$\mathbf{c_e}$
$\xi_\mathbf{u}$	69.92	50.33	45.85	33.82
$\xi^*_\mathbf{D}$	100	94.18	75.28	43.60
$\xi^*_\mathbf{E}$	93.96	100	51.89	25.71
$\xi^*_{\mathbf{D_1}}$	74.63	53.05	100	80.40
$\xi^*_{\mathbf{c_e}}$	51.23	33.24	85.73	100

respect to alternative criteria that reflect other aspects of the statistical analysis. In Table 5.2 we compare the efficiency of a given local optimal design with respect to the other optimality criteria. For example, the local D-optimal design has efficiencies 94.18%, 75.28%, and 43.60% with respect to the E-, D_1-, and c_e-optimality criterion, respectively. Thus this design is rather efficient for the D_1- and E-optimality criterion, but less efficient for extrapolation. The situation for the D_1-optimal design is similar, where the role of the c_e- and E-criterion have to be interchanged. On the other hand, the performance of the local E- and c_e-optimal design depends strongly on the underlying optimality criterion. The local E-optimal design yields only a satisfactory D-efficiency, but is less efficient with respect to the c_e- and D_1-optimality criterion, while the local c_e-optimal design yields only a satisfactory D_1-efficiency.

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