Spectral Properties of Rescaled Sample Correlation Matrices

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Supplementary Material

The Supplementary Material contains detailed proofs of some theoretical results.

1. Some assumptions

Assumption A. Samples satisfy the following independent component structure

$$\mathbf{y}_i = \boldsymbol{\mu} + \boldsymbol{\Gamma}\mathbf{x}_i, i = 1, ..., n,$$

where $\mathrm{E}\mathbf{y}_i = \boldsymbol{\mu}$, $\boldsymbol{\Gamma} = [\mathrm{diag}(\boldsymbol{\Sigma})]^{1/2}\mathbf{R}^{1/2}$ and $\mathbf{x}_i = (x_{1i}, ..., x_{pi})^T$.

Assumption B. Assume that $\{x_{ji}, j = 1, ..., p, i = 1, ..., n\}$ are independent and identically distributed *(i.i.d.)* with

$$\mathrm{E}x_{ji} = 0, \mathrm{E}x_{ji}^2 = 1, E(|x_{ji}|^4(\log(|x_{ji}|)^{2+2\epsilon})) < \infty$$

for a small positive number $\epsilon > 0$.

Assumption C. The convergence regime is $\rho_n = p/n \to \rho \in (0, +\infty)$.

Assumption D. The functions $g_1, ..., g_K$ are analytic functions in a domain containing the support set $[a_\rho, b_\rho]$ of Marčenko-Pastur law in (2.1).

Assumption E. Assume that $\{x_{ji}, j = 1, ..., p, i = 1, ..., n\}$ are independent and identically distributed with

$$\mathrm{E}x_{ji} = 0,\ \mathrm{E}x_{ji}^2 = 1,\ \mathrm{E}x_{ji}^4 = \beta_x + 3 + o(1),\ \mathrm{E}(|x_{ji}|^4(\log(|x_{ji}|)^{2+2\epsilon})) < \infty.$$

Assumption F. Assume

$$a_g = \lim_{p\to\infty} p^{-1}\sum_{k=1}^{p}\sum_{h=1}^{p} g_{kh}^3 \mathbf{e}_h^T \mathbf{R}^{-1/2}\mathbf{e}_k, \qquad a_{\mathbf{R}} = \lim_{p\to\infty} p^{-1}\sum_{k,\ell=1}^{p} \mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{k\ell}^3,$$

$$c_g = \lim_{p\to\infty} p^{-1}\sum_{k=1}^{p}\sum_{h=1}^{p} g_{kh}^4, \qquad d_{\mathbf{R}} = \lim_{p\to\infty} p^{-1}\mathrm{tr}(\mathbf{R}^2),$$

$$h_{\mathbf{R}} = \lim_{p\to\infty} p^{-1}\sum_{k,\ell=1}^{p} \mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{k\ell}\sum_{h=1}^{p} g_{\ell h}^2 g_{kh}^2,$$

where $\mathbf{R}^{1/2} = (g_{kh})$, $\mathbf{R} = (r_{kh})$ and $\mathbf{e}_j$ is the jth column of $p\times p$ identity matrix for $j = 1, \cdots, p$.

2. Limiting spectral distribution

Theorem 2.1. Under Assumptions A-B-C, the empirical spectral distribution $F_n(x)$ of $\mathbf{R}^{-1}\widehat{\mathbf{R}}_n$ converges almost surely to the Marčenko-Pastur law with the index ρ as follows

$$f_\rho(x) = \begin{cases} \frac{1}{2\pi\rho x}\sqrt{(b_\rho - x)(x - a_\rho)}, & \text{if } a_\rho \le x \le b_\rho, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $a_\rho = (1 - \sqrt{\rho})^2$, and $b_\rho = (1 + \sqrt{\rho})^2$.

3. Central limit theorem

Theorem 3.1. Under Assumptions A-C-D-E-F, the random vector $(W(g_1), \ldots, W(g_K))$ weakly converges to a multivariate Gaussian random vector $(X_{g_1}, \ldots, X_{g_K})$ with

$$\mathrm{E}X_{g_\ell} = -\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}} g_\ell(z)\mathrm{E}M(z)\,dz$$

and

$$\mathrm{Cov}(X_{g_{\ell_1}}, X_{g_{\ell_2}}) = -\frac{1}{4\pi^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} g_{\ell_1}(z_1)g_{\ell_2}(z_2)\mathrm{Cov}(M(z_1), M(z_2))\,dz_2\,dz_1$$

for $\ell, \ell_1, \ell_2 \in \{1, ..., K\}$, where $\mathcal{C}, \mathcal{C}_1$, and $\mathcal{C}_2$ are three contours including $[a_\rho, b_\rho]$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-overlapping, the contour integral $\oint$ is anticlockwise, and $\mathrm{E}M(z)$ and $\mathrm{Cov}(M(z_1), M(z_2))$ are calculated as follows:

$$\begin{aligned}\mathrm{E}M(z) =& \frac{\rho \underline{s}^3(z)[1+\underline{s}(z)]^{-3}}{[1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}]^2} + \frac{\beta_x \rho \underline{s}(z)\underline{s}'(z)}{[1+\underline{s}(z)]^3} \\ &- \frac{\underline{s}'(z)}{[1+\underline{s}(z)]^2} \frac{[10-2a_{\mathbf{R}}+\beta_x(4a_g+c_g-h_{\mathbf{R}})]\rho}{4} \\ &+ \frac{\underline{s}'(z)}{[1+\underline{s}(z)]^3} \frac{[6-2a_{\mathbf{R}}+\beta_x(4a_g-c_g-h_{\mathbf{R}})]\rho}{2},\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathrm{Cov}(M(z_1), M(z_2)) &= 2\left[\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2)-\underline{s}(z_1))^2} - \frac{1}{(z_2-z_1)^2}\right] \\ &\quad +2(d_{\mathbf{R}}-2)\rho \frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2},\end{aligned} \tag{3.3}$$

where $'$ is the derivative notation, and $\underline{s}(z)$ is the unique solution to $z = -\underline{s}^{-1}(z) + \rho(1+\underline{s}(z))^{-1}$, which leads to $\underline{s}'(z) = \underline{s}^2(z)/\{1-\rho\underline{s}^2(z)[1+\underline{s}(z)]^{-2}\}$.

4. Proofs of Theorems 2.1 and 3.1

This part is devoted to the proofs of our main theorems. The main strategy of proving Theorems 2.1 and 3.1 relies on the Stieltjes transformation method. Section 4.2 lists notations that will be used below. Section 4.3 provides the proof of Theorem 2.1. Finally, Section 4.4 presents the proof of Theorem 3.1.

4.1 Remove the sample mean

First of all, let

$$\ddot{\mathbf{S}}_n = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^T,$$

where $\boldsymbol{\mu} = \mathrm{E}\mathbf{y}_i$. (1.1) shows that

$$\mathbf{S}_n = (n-1)^{-1}\sum_{i=1}^{n}(\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})^T, \quad \widehat{\mathbf{R}} = [\mathrm{diag}(\mathbf{S}_n)]^{-1/2}\,\mathrm{diag}(\mathbf{S}_n)[\mathrm{diag}(\mathbf{S}_n)]^{-1/2}.$$

We will show that

- Studying the limiting spectral distribution (*LSD*) of $\widehat{\mathbf{R}}$ is equivalent to studying the LSD of

$$[\mathrm{diag}(\ddot{\mathbf{S}}_n)]^{-1/2}\,\mathrm{diag}(\ddot{\mathbf{S}}_n)[\mathrm{diag}(\ddot{\mathbf{S}}_n)]^{-1/2}.$$

- Studying the CLT of LSS of $\widehat{\mathbf{R}}$ is equivalent to studying the CLT of LSS of

$$[\mathrm{diag}(\mathbf{S})]^{-1/2}\,\mathrm{diag}(\mathbf{S})[\mathrm{diag}(\mathbf{S})]^{-1/2},$$

where $\mathbf{S} = n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}$.

- Notice that $\bar{\mathbf{y}}\bar{\mathbf{y}}^T$ has rank one. Combining with Theorem A.43 in Bai and Silverstein (2010), we know that for large n, the difference between ESD of

$$\mathbf{S}_n = (n-1)^{-1}\sum_{i=1}^{n}(\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})^T$$

in (1.1) and ESD of $\ddot{\mathbf{S}}_n$ is neglectable.

Thus, in the proof of Theorem 2.1 in the next section, we only consider $\ddot{\mathbf{S}}_n$ instead of $\mathbf{S}_n$.

- Moreover, by the proof of substitution principle for CLT of LSS of sample covariance matrix in Zheng, Bai and Yao (2015), we know that the CLT of LSS of $\mathbf{S}_n$ is obtained only through replacing $y_n = p/n$ by $y_{n-1} = p/(n-1)$ in the CLT of LSS of $\ddot{\mathbf{S}}_n$.

Due to $\mathrm{E}(\mathbf{y}_i - \boldsymbol{\mu}) = \mathbf{0}$, without loss of generality, we assume $\boldsymbol{\mu} = \mathbf{0}$ (that is, $\mathrm{E}\mathbf{y}_i = \mathbf{0}$) and use

$$\hat{\mathbf{S}} = n^{-1}\sum_{i=1}^{n}\mathbf{y}_i\mathbf{y}_i^T$$

to replace $\ddot{\mathbf{S}}_n$.

- By Assumption A, we have

$$\hat{\mathbf{S}} = [\mathrm{diag}(\mathbf{\Sigma})]^{1/2} n^{-1} \sum_{i=1}^{n} \mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}[\mathrm{diag}(\mathbf{\Sigma})]^{1/2}.$$

That is, we have

$$\mathrm{diag}(\hat{\mathbf{S}}) = \mathrm{diag}\left(n^{-1}\sum_{i=1}^{n}\mathbf{y}_i\mathbf{y}_i^T\right) = [\mathrm{diag}(\mathbf{\Sigma})]^{1/2}\,\mathrm{diag}\left(n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}\right)[\mathrm{diag}(\mathbf{\Sigma})]^{1/2},$$

and

$$[\mathrm{diag}(\hat{\mathbf{S}})]^{-1/2} = \mathrm{diag}\left(n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}\right)^{-1/2}[\mathrm{diag}(\mathbf{\Sigma})]^{-1/2}.$$

Let $\mathbf{S} = n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}$, and we have

$$\begin{aligned}
\widehat{\mathbf{R}} &= [\mathrm{diag}(\hat{\mathbf{S}})]^{-1/2}\,\mathrm{diag}(\hat{\mathbf{S}})[\mathrm{diag}(\hat{\mathbf{S}})]^{-1/2} \\
&= \mathrm{diag}\left(n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}\right)^{-1/2} n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}\,\mathrm{diag}\left(n^{-1}\sum_{i=1}^{n}\mathbf{R}^{1/2}\mathbf{x}_i\mathbf{x}_i^T\mathbf{R}^{1/2}\right)^{-1/2} \\
&= [\mathrm{diag}(\mathbf{S})]^{-1/2}\mathbf{S}[\mathrm{diag}(\mathbf{S})]^{-1/2}, \qquad (4.1)
\end{aligned}$$

where $\mathrm{Var}(\mathbf{R}^{1/2}\mathbf{x}_i) = \mathbf{R}$.

4.2 Some notation and preliminary results

Let $\mathbf{G} = \mathbf{R}^{1/2} = (g_{kh})$ and

$$\mathbf{r}_j = \mathbf{x}_j/\sqrt{n}, \quad \mathbf{A}(z) = \sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i^T - z\mathbf{I}_p, \quad \mathbf{A}_j(z) = \mathbf{A}(z) - \mathbf{r}_j\mathbf{r}_j^T,$$
$$\mathbf{A}_{jk}(z) = \mathbf{A}_j(z) - \mathbf{r}_k\mathbf{r}_k^T, \quad \mathbf{A}_{jk\ell}(z) = \mathbf{A}_{jk}(z) - \mathbf{r}_\ell\mathbf{r}_\ell^T, \quad \text{for } 1 \le j,k,l \le n.$$

Let $\breve{\mathbf{A}}_j(z) = \sum_{i<j}\mathbf{r}_i\mathbf{r}_i^T + \sum_{i>j}\breve{\mathbf{r}}_i\breve{\mathbf{r}}_i^T - z\mathbf{I}$, where $\breve{\mathbf{r}}_{j+1},\ldots,\breve{\mathbf{r}}_n$ are independent copies of $\mathbf{r}_{j+1},\ldots,\mathbf{r}_n$, and let E_j denote the conditional expectation given $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_j$. Moreover, define

$$b_j(z) = [1 + n^{-1}\mathrm{tr}(\mathbf{A}_j^{-1}(z))]^{-1}, \qquad b_{ij}(z) = [1 + n^{-1}\mathrm{tr}(\mathbf{A}_{ij}^{-1}(z))]^{-1}$$
$$b_{ijk}(z) = [1 + n^{-1}\mathrm{tr}(\mathbf{A}_{ijk}^{-1}(z))]^{-1}, \quad \breve{b}_j(z) = [1 + n^{-1}\mathrm{tr}(\breve{\mathbf{A}}_j^{-1}(z))]^{-1},$$

$b(z) = [1 + n^{-1}\mathrm{Etr}(\mathbf{A}^{-1}(z))]^{-1}$, $\beta_{i(j)}(z) = (1 + \mathbf{r}_i^T\mathbf{A}_{ij}^{-1}\mathbf{r}_i)^{-1}$,

$\beta_{i(jk)}(z) = (1 + \mathbf{r}_i^T\mathbf{A}_{ijk}^{-1}\mathbf{r}_i)^{-1}$, $\beta_j(z) = [1 + \mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j]^{-1}$,

$\widehat{\gamma}_j(z) = \mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j - n^{-1}\mathrm{tr}\mathbf{A}_j^{-1}(z)$, $\widehat{\gamma}_{i(j)}(z) = \mathbf{r}_i^T\mathbf{A}_{ij}^{-1}(z)\mathbf{r}_i - n^{-1}\mathrm{tr}\mathbf{A}_{ij}^{-1}(z)$,

$\widehat{\gamma}_{i(jk)}(z) = \mathbf{r}_i^T\mathbf{A}_{ijk}^{-1}(z)\mathbf{r}_i - n^{-1}\mathrm{tr}\mathbf{A}_{ijk}^{-1}(z)$.

By eqn.(3.4) of Bai et al. (1998), we have

$$\max(|b_j(z)|, |b_{ij}(z)|, |b(z)|, |\beta_{i(j)}(z)|, |\beta_{i(jk)}(z)|, |\beta_j(z)|) \leq |z|/\,\mathrm{Im}(z), \tag{4.1}$$

where $\mathrm{Im}(z)$ stands for the imaginary part of z. With the notations above, $\mathbf{A}^{-1}(z)$ can be decomposed as

$$\mathbf{A}^{-1}(z) = \mathbf{A}_j^{-1}(z) - \beta_j(z)\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z), \tag{4.2}$$

see, e.g., the expression above eqn.(2.2) in Bai and Silverstein (2004); and $\beta_j(z)$ can be further written as

$$\beta_j(z) = b_j(z) - \beta_j(z)b_j(z)\widehat{\gamma}_j(z) = b_j(z) - b_j^2(z)\widehat{\gamma}_j(z) + \beta_j(z)b_j^2(z)\widehat{\gamma}_j^2(z), \tag{4.3}$$

and

$$\beta_{i(jk)}(z) = b_{ijk}(z) - \beta_{i(jk)}(z)b_{ijk}(z)\widehat{\gamma}_{i(jk)}(z). \tag{4.4}$$

(see line 9 on Page 569 of Bai and Silverstein (2004)). The matrix $\mathbf{A}_1^{-1}(z)$ can be further decomposed as

$$\mathbf{A}_1^{-1}(z) = \mathbf{B}_1(z) + \mathbf{B}_2(z) + \mathbf{B}_3(z) + \mathbf{B}_4(z), \tag{4.5}$$

where

$$\mathbf{B}_1(z) = -\left(z - (n-1)/n \cdot b(z)\right)^{-1} \mathbf{I}_p,$$

$$\mathbf{B}_2(z) = b(z) \sum_{i \neq 1} \left(z - (n-1)/n \cdot b(z)\right)^{-1} (\mathbf{r}_i \mathbf{r}_i^T - n^{-1}\mathbf{I}_p) \mathbf{A}_{i1}^{-1}(z),$$

$$\mathbf{B}_3(z) = \sum_{i \neq 1} (\beta_{i(1)}(z) - b(z))(z - (n-1)/n \cdot b(z))^{-1} \mathbf{r}_i \mathbf{r}_i^T \mathbf{A}_{i1}^{-1}(z), \quad \text{and}$$

$$\mathbf{B}_4(z) = n^{-1} b(z)(z - (n-1)/n \cdot b(z))^{-1} \sum_{i \neq 1} (\mathbf{A}_{i1}^{-1}(z) - \mathbf{A}_1^{-1}(z)),$$

$$= -n^{-1} b(z)(z - (n-1)/n \cdot b(z))^{-1} \sum_{i \neq 1} \mathbf{A}_{i1}^{-1} \mathbf{r}_i \mathbf{r}_i^T \mathbf{A}_{i1}^{-1} \beta_{i(1)},$$

according to eqn.(2.9) of Bai and Silverstein (2004).

We also note here that throughout this paper, C and $C_{(\cdot)}$ stand for constants that may take different values from one appearance to another.

4.3 Proof of Theorem 2.1

By Lemma 4 in Karoui (2009), we know that $||\operatorname{diag}(\mathbf{S}) - \mathbf{I}_p|| \to 0$ and $||[\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p|| \to 0$ almost surely, where throughout this paper $\|\cdot\|$ denote the spectral norm of a matrix. Let $\mathbf{X} = (\mathbf{x}_1, ...\mathbf{x}_n)$. Therefore, follow the same strategy in the proof of Lemma 1 in the same article, we know that the LSD of $\mathbf{R}^{-1}\widehat{\mathbf{R}}$ is the same as that of $n^{-1}\mathbf{X}\mathbf{X}^T$, which we know is the specified Marcenko-Pastur law. In fact, notice that $\|n^{-1}\mathbf{X}\mathbf{X}^T\| \to (1+\sqrt{\rho})^2, a.s.$ as $n \to \infty$, we obtain

$$\begin{aligned}
&\left\|\mathbf{R}^{-1/2}\widehat{\mathbf{R}}\mathbf{R}^{-1/2} - n^{-1}\mathbf{X}\mathbf{X}^T\right\| \qquad (4.1)\\
=&\left\|\mathbf{R}^{-1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p + \mathbf{I}_p\right) n^{-1}\mathbf{R}^{1/2}\mathbf{X}\mathbf{X}^T\mathbf{R}^{1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p + \mathbf{I}_p\right)\mathbf{R}^{-1/2} - n^{-1}\mathbf{X}\mathbf{X}^T\right\|\\
\leq&\left\|\mathbf{R}^{-1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right) n^{-1}\mathbf{R}^{1/2}\mathbf{X}\mathbf{X}^T\mathbf{R}^{1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right)\mathbf{R}^{-1/2}\right\|\\
&+\left\|n^{-1}\mathbf{X}\mathbf{X}^T\mathbf{R}^{1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right)\mathbf{R}^{-1/2}\right\|\\
&+\left\|\mathbf{R}^{-1/2}\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right)\mathbf{R}^{1/2}n^{-1}\mathbf{X}\mathbf{X}^T\right\|\\
\leq&\left\|\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right)\right\|^2\left\|n^{-1}\mathbf{X}\mathbf{X}^T\right\| + 2\left\|\left([\operatorname{diag}(\mathbf{S})]^{-1/2} - \mathbf{I}_p\right)\right\|\left\|n^{-1}\mathbf{X}\mathbf{X}^T\right\| \to 0 \quad a.s.
\end{aligned}$$

4.4 Proof of Theorem 3.1

Observe that by Cauchy integral formula, for any function g that is analytic in a domain containing the interval $[\lambda_1, \lambda_p]$, we have

$$\begin{aligned} W(g) &= \sum_{j=1}^{p} g(\lambda_j) - p \int g(x)\, dF_{\rho_n}(x) \\ &= -\frac{1}{2\pi i} \oint_{\mathcal{C}} g(z) M_n(z)\, dz, \text{ where } M_n(z) := p(s_n(z) - s_{\rho_n}(z)), \end{aligned} \tag{4.1}$$

where $s_{\rho_n}(z)$ is the Stieltjes transformation of the distribution F_{ρ_n}, and $\mathcal{C}$ is any contour inside the domain and surrounding the interval $[\lambda_1,\ \lambda_p]$. Thus to derive the CLT of the LSS of $\mathbf{R}^{-1}\widehat{\mathbf{R}}$, the analysis of its Stieltjes transformation $s_n(z)$ is required.

Truncation, Centralization and Rescaling the variables

Recall that $\mathbf{S} = n^{-1}\mathbf{G}\mathbf{X}_n\mathbf{X}_n^T\mathbf{G}^T$ and

$$\mathbf{R}^{-1}\widehat{\mathbf{R}} = \mathbf{R}^{-1}\left[\mathrm{diag}(\mathbf{S})\right]^{-1/2} n^{-1}\mathbf{R}^{1/2}\mathbf{X}\mathbf{X}^T\mathbf{R}^{1/2}\left[\mathrm{diag}(\mathbf{S})\right]^{-1/2}.$$

Denote $\check{\mathbf{X}} = (\check{x}_{ij})$ with $\hat{x}_{ij} = x_{ij} I_{\{|x_{ij}|<\eta_n\sqrt{n}\}}$, $\check{\mathbf{S}} = n^{-1}\mathbf{R}^{1/2}\check{\mathbf{X}}\check{\mathbf{X}}^T\mathbf{R}^{1/2}$,

$$\mathbf{R}^{-1}\check{\mathbf{R}} = \mathbf{R}^{-1}\left[\mathrm{diag}(\check{\mathbf{S}})\right]^{-1/2} n^{-1}\mathbf{R}^{1/2}\check{\mathbf{X}}\check{\mathbf{X}}^T\mathbf{R}^{1/2}\left[\mathrm{diag}(\check{\mathbf{S}})\right]^{-1/2}$$

and $\check{W}(g)$ be the truncated version of $W(g)$. As have been proved in Karoui (2009), under the moment assumption, we shall select a sequence of $\eta_n = (\log n)^{-(1+\varepsilon)/2} \to 0$ as $n \to \infty$ which satisfies that

$$\mathrm{P}\left(\widehat{\mathbf{R}} \neq \check{\mathbf{R}}, i.o.\right) = 0.$$

Now define $\widetilde{\mathbf{X}} = (\widetilde{x}_{ij})$ with $\widetilde{x}_{ij} = (\check{x}_{ij} - \mathrm{E}\,\check{x}_{ij}) / \sqrt{\mathrm{E}\,(\check{x}_{ij} - \mathrm{E}\,\check{x}_{ij})^2}$. Also define $\widetilde{\mathbf{S}}$, $\widetilde{\mathbf{R}}$ and $\widetilde{W}(f)$ as the analogues of $\mathbf{S}$, $\mathbf{R}$ and $W(f)$ with $\mathbf{X}_n$ replaced by $\widetilde{\mathbf{X}}$. For large n and any $1 \le i \le p, 1 \le j \le n$, we have the following estimates

(1) $|\operatorname{E}\check{x}_{ij}| = \operatorname{E}|x_{ij} I(|x_{ij} \geq \eta_n \sqrt{n}|)| \leq \eta_n n^{-3/2} \operatorname{E}|x_{ij}|^4 = o(n^{-3/2})$,

(2)

$$\begin{aligned} \operatorname{E}(\check{x}_{ij} - \operatorname{E}\check{x}_{ij})^2 - 1 &= \operatorname{E} x_{ij}^2 I(|x_{ij} \geq \eta_n \sqrt{n}|) - \left(\operatorname{E} x_{ij} I(|x_{ij} \geq \eta_n \sqrt{n}|)\right)^2 \\ &\leq \operatorname{E} x_{ij}^2 I(|x_{ij} \geq \eta_n \sqrt{n}|) = o(n^{-1}), \end{aligned} \tag{4.2}$$

(3) $\frac{\sqrt{\operatorname{E}\left(\check{x}_{ij} - \operatorname{E}\check{x}_{ij}\right)^2} - 1}{\sqrt{\operatorname{E}\left(\check{x}_{ij} - \operatorname{E}\check{x}_{ij}\right)^2}} \leq 2\left(\operatorname{E}(\check{x}_{ij} - \operatorname{E}\check{x}_{ij})^2 - 1\right) = o(n^{-1})$.

It follows that as $n \to \infty$,

$$\begin{aligned} &\|n^{-1}\check{\mathbf{X}}\check{\mathbf{X}}^T - n^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T\| \leq \|n^{-1}\check{\mathbf{X}}\left(\check{\mathbf{X}} - \widetilde{\mathbf{X}}\right)^T\| + \|n^{-1}\left(\check{\mathbf{X}} - \widetilde{\mathbf{X}}\right)\widetilde{\mathbf{X}}^T\| \\ \leq &2\left((1+\sqrt{\rho_n})^2 \frac{\sqrt{\operatorname{E}(\check{x}_{ij} - \operatorname{E}\check{x}_{ij})^2} - 1}{\sqrt{\operatorname{E}(\check{x}_{ij} - \operatorname{E}\check{x}_{ij})^2}} + 2n^{-1/2} n |\operatorname{E}\check{x}_{ij}|\right) = o(n^{-1}). \end{aligned} \tag{4.3}$$

Let $\mathbf{g}_\ell$ to be the transpose of the ℓ-th row of the matrix $\mathbf{R}^{1/2}$, it can be verified that

$$\begin{aligned} &\max_{i \leq p} \|\mathbf{g}_i^T \left(\frac{1}{n}\check{\mathbf{X}}\check{\mathbf{X}}^T - \frac{1}{n}\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T\right)\mathbf{g}_i\| \\ \leq &\max_{i \leq p}\left(\|\frac{1}{n}\mathbf{g}_i^T \check{\mathbf{X}}\left(\check{\mathbf{X}} - \widetilde{\mathbf{X}}\right)^T \mathbf{g}_i\| + \|\frac{1}{n}\mathbf{g}_i^T\left(\check{\mathbf{X}} - \widetilde{\mathbf{X}}\right)\widetilde{\mathbf{X}}^T \mathbf{g}_i\|\right) = o_{a.s.}(n^{-1}), \end{aligned} \tag{4.4}$$

which implies $\|\operatorname{diag}(\check{\mathbf{S}} - \widetilde{\mathbf{S}})\| = o_{a.s.}(n^{-1})$. Then we get

$$\begin{aligned} &\|\check{\mathbf{R}}_n - \widetilde{\mathbf{R}}_n\| \\ \leq &\|\mathbf{R}^{-1}\left(\operatorname{diag}(\check{\mathbf{S}})^{-1/2} - \operatorname{diag}(\widetilde{\mathbf{S}})^{-1/2}\right) n^{-1}\mathbf{R}^{1/2}\check{\mathbf{X}}\check{\mathbf{X}}^T\mathbf{R}^{1/2}\operatorname{diag}(\check{\mathbf{S}})^{-1/2}\| \\ &+ \|\mathbf{R}^{-1}\operatorname{diag}(\widetilde{\mathbf{S}})^{-1/2}\mathbf{R}^{1/2}\left(n^{-1}\check{\mathbf{X}}\check{\mathbf{X}}^T - n^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T\right)\mathbf{R}^{1/2}\operatorname{diag}(\check{\mathbf{S}})^{-1/2}\| \\ &+ \|\mathbf{R}^{-1}\operatorname{diag}(\widetilde{\mathbf{S}})^{-1/2}\mathbf{R}^{1/2} n^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T\mathbf{R}^{1/2}\left(\operatorname{diag}(\check{\mathbf{S}})^{-1/2} - \operatorname{diag}(\widetilde{\mathbf{S}})^{-1/2}\right)\| = o_{a.s.}(n^{-1}). \end{aligned} \tag{4.5}$$

Combining the estimates above, we finally obtain for large n

$$\left|\check{W}(g) - \widetilde{W}(g)\right| \leq M \sum_{k=1}^{p}\left|\lambda_k\left(\check{\mathbf{R}}_n\right) - \lambda_k\left(\widetilde{\mathbf{R}}_n\right)\right| \leq Mp\|\check{\mathbf{R}}_n - \widetilde{\mathbf{R}}_n\| = o_{a.s.}(1), \tag{4.6}$$

where M is a bound on $|f'(z)|$. Therefore, we shall assume in the following that the underlying variables in the data matrix $\mathbf{X}$ are all truncated at $\eta_n\sqrt{n}$, centralized and rescalized to have unit variances.

Decomposing the Stieltjes transformation of the ESD of $\mathbf{R}^{-1}\widehat{\mathbf{R}}$

We now decompose the Stieltjes transformation of the ESD of $\mathbf{R}^{-1}\widehat{\mathbf{R}}$ into several terms. Denote

$$\mathbf{D} = \mathbf{R}^{-1/2}\left[\operatorname{diag}(\mathbf{S}) - \mathbf{I}_p\right]\mathbf{R}^{1/2}, \qquad \mathbf{D}_j = \mathbf{R}^{-1/2}\left[\operatorname{diag}\left(\frac{1}{n}\sum_{i\neq j}\mathbf{S}_j\right) - \frac{n-1}{n}\mathbf{I}_p\right]\mathbf{R}^{1/2},$$

$$\mathbf{L} = \mathbf{R}^{-1/2}\left[\operatorname{diag}(\mathbf{S})\right]^{1/2}\mathbf{R}\left[\operatorname{diag}(\mathbf{S})\right]^{1/2}\mathbf{R}^{-1/2}, \qquad \mathbf{H}(z) = \mathbf{R}^{-1/2}\mathbf{S}\mathbf{R}^{-1/2} - z\mathbf{L}.$$

Reorganize

$$ps_n(z) = \operatorname{tr}\left(\mathbf{R}^{-1/2}\widehat{\mathbf{R}}\mathbf{R}^{-1/2} - z\mathbf{I}_p\right)^{-1} = \operatorname{tr}\left(\mathbf{H}^{-1}(z)\mathbf{L}\right).$$

Moreover, by the identity

$$\mathbf{A}_0^{-1} - \mathbf{B}_0^{-1} = \mathbf{A}_0^{-1}(\mathbf{B}_0 - \mathbf{A}_0)\mathbf{B}_0^{-1} \tag{4.7}$$

for any invertible matrices $\mathbf{A}_0$ and $\mathbf{B}_0$, we have

$$\begin{aligned}\mathbf{H}^{-1}(z) =& \mathbf{A}^{-1}(z) + z\mathbf{H}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z)\\ =& \mathbf{A}^{-1}(z) + z\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z) + z^2\mathbf{H}^{-1}(z)\left[\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z)\right]^2.\end{aligned} \tag{4.8}$$

Hence, it follows that

$$\begin{aligned}ps_n(z) =& \operatorname{tr}\left[\mathbf{H}^{-1}(z)\right] + \operatorname{tr}\left[\mathbf{H}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]\\ =& \operatorname{tr}\left(\mathbf{A}^{-1}(z)\right) + \operatorname{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] + z\operatorname{tr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]\\ &+ z\operatorname{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]^2 + z^2\operatorname{tr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]\\ &+ z^2\operatorname{tr}\left\{\mathbf{H}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]^2\right\}\\ &+ z^3\operatorname{tr}\left\{\mathbf{H}^{-1}(z)\left[\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z)\right]^3\right\}.\end{aligned} \tag{4.9}$$

The following steps of proof will be focused on the analysis of the terms above.

Choosing the contour $\mathcal{C}$

Recall (4.1), an integral contour should be determined. We choose the contour $\mathcal{C}$ as

$$\begin{aligned}
&\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_u \cup \mathcal{C}_b \cup \mathcal{C}_r, \text{ where} \\
&\mathcal{C}_u = \{x + i\nu_0 : x \in [x_\ell, x_r]\}, \quad \mathcal{C}_\ell = \{x_\ell + i\nu : |\nu| \le \nu_0\} \\
&\mathcal{C}_b = \{x - i\nu_0 : x \in [x_\ell, x_r]\}, \quad \mathcal{C}_r = \{x_r + i\nu : |\nu| \le \nu_0\} \\
&(x_\ell, x_r) \supset \left((1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2\right),
\end{aligned}$$

and $\nu_0 > 0$ is to be determined. Let $\mathcal{C}_n = \mathcal{C} \cap \{z : |\Im z| > n^{-2}\}$. Moreover, for an $\varepsilon > 0$ sufficiently small so that

$$x_\ell + \varepsilon \le (1-\sqrt{\rho})^2 - \varepsilon \le (1+\sqrt{\rho})^2 + \varepsilon \le x_r - \varepsilon,$$

define

$$\mathcal{B}_n = \{(1-\sqrt{\rho})^2 - \varepsilon \le \lambda_{\min}(\mathbf{X}\mathbf{X}^T/n) < \lambda_{\max}(\mathbf{X}\mathbf{X}^T/n) < (1+\sqrt{\rho})^2 + \varepsilon\}.$$

By Theorem 5.9 in Bai and Silverstein (2010), we have that $P(\mathcal{B}_n) = o(n^{-t})$ for any given $t > 0$. Let

$$\widehat{M}_n(z) = \begin{cases} M_n(z), & \text{if } z \in \mathcal{C}_n \\ M_n(x_\ell + in^{-2}), & \text{if } \Re z = x_\ell, \Im z \in [0, n^{-2}] \\ M_n(x_\ell - in^{-2}), & \text{if } \Re z = x_\ell, \Im z \in [-n^{-2}, 0) \\ M_n(x_r + in^{-2}), & \text{if } \Re z = x_r, \Im z \in [0, n^{-2}] \\ M_n(x_r - in^{-2}), & \text{if } \Re z = x_r, \Im z \in [-n^{-2}, 0). \end{cases}$$

Observe that on event $\mathcal{B}_n$, when $\Re z$ equals either x_ℓ or x_r, we have $|M_n(z)| \le 1/\varepsilon$, hence

$$\begin{aligned}
&\left| p \oint_{\mathcal{C}} g(z)(M_n(z) - \widehat{M}_n(z))\, dz \right| = \left| p \oint_{\mathcal{C}\setminus\mathcal{C}_n} g(z)(M_n(z) - \widehat{M}_n(z))\, dz \right| \\
\le\ & K \frac{p}{n^2} \cdot 1/\varepsilon = o(1).
\end{aligned}$$

Therefore, in order to establish the limit for $p\oint\limits_{\mathcal{C}} g(z)M_n(z)\,dz$, it suffices to study $p\oint\limits_{\mathcal{C}} g(z)\widehat{M}_n(z)\,dz$. Furthermore, since $\Im(z)$ can be chosen to be arbitrarily small, the contribution from the segments C_ℓ and C_r can be made small as well. This allows us to focus only on $z \in \mathcal{C}_u \cup \mathcal{C}_b$ in the following.

CLT of LSS of the sample correlation matrix

We are here in a position to establish the CLT for the LSS of the sample correlation matrix. Indicated by (4.1), the limit of the process $M_n(z) = p(s_n(z) - s_{\rho_n}(z))$ should be derived. Recall that by (4.9), we have

$$\begin{aligned} M_n(z) =& \mathrm{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\left(\mathrm{tr}\mathbf{A}^{-1}(z)\right) + \mathrm{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] - \mathrm{Etr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] \\ &+ z\mathrm{tr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] - z\mathrm{Etr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] \\ &+ \mathrm{E}\left(\mathrm{tr}\mathbf{A}^{-1}(z)\right) - ps_{\rho_n}(z) + \mathrm{Etr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] + z\mathrm{Etr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] \\ &+ z \cdot I + z^2 \cdot II + z^2 \cdot III + z^3 \cdot IV. \end{aligned} \tag{4.10}$$

The analysis of $M_n(z)$ will be carried out by conducting the following steps.

$\mathcal{I}$**:** Derive the limit of $I = \mathrm{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]^2$ and $II = \mathrm{tr}\left[\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)^2\mathbf{A}^{-1}(z)\right]$;

$\mathcal{II}$**:** Show that

$$III = \mathrm{tr}\left\{\mathbf{H}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]^2\right\} = o_p(1), \tag{4.11}$$

and

$$IV = \mathrm{tr}\left\{\mathbf{H}^{-1}(z)\left[\left(\mathbf{L}-\mathbf{I}_p\right)\mathbf{A}^{-1}(z)\right]^3\right\} = o_p(1); \tag{4.12}$$

$\mathcal{III}$**:** Derive the limit of

$$\begin{aligned} V :=& \mathrm{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\left[\mathrm{tr}\mathbf{A}^{-1}(z)\right] + \mathrm{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] - \mathrm{Etr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] \\ &+ z\mathrm{tr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] - z\mathrm{Etr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]; \end{aligned} \tag{4.13}$$

$\mathcal{IV}$**:** Derive the limit of

$$\begin{aligned} VI :=& \mathrm{E}\mathrm{tr}\left[\mathbf{A}^{-1}(z)\right] - ps_{\rho_n}(z) + \mathrm{E}\mathrm{tr}\left[\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right] \\ &+ z\mathrm{E}\mathrm{tr}\left[\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right]; \end{aligned} \tag{4.14}$$

$\mathcal{V}$**:** Combining the steps above to derive the limit of $M_n(z)$ and consequently the CLT of the LSS of $\widetilde{\mathbf{R}}$.

We now process these steps one by one. It is worth pointing out that some repeating calculation will be omitted in order to reduce the length of this paper.

Step $\mathcal{I}$. In this step, we derive the limit of I and II, which is further divided into the following substeps.

$\mathcal{I}$**.1:** Decompose term I as

$$I = \frac{1+o(1)}{4}I_1 + \frac{1+o(1)}{4}I_2 + \frac{1+o(1)}{2}I_3 + \frac{1+o(1)}{4}I_4 + \frac{1+o(1)}{4}I_5 + \frac{1+o(1)}{16}I_6$$

to be specified below and show that $I_4 + I_5 + I_6 = o_p(1)$;

$\mathcal{I}$**.2:** Prove that

$$|I_1 - \mathrm{E}\left(I_1\right)| + |I_2 - \mathrm{E}\left(I_2\right)| + |I_3 - \mathrm{E}\left(I_3\right)| = o_p(1);$$

$\mathcal{I}$**.3:** Prove that $\mathrm{E}(I_1)$, $\mathrm{E}\left(I_2\right)$ converge to the same limit

$$\frac{1}{z^2(1+\underline{s}(z))^2}\left(2\rho + \beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right)$$

and $\mathrm{E}\left(I_3\right)$ converge to

$$\frac{1}{z^2(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\left(2r_{kl}^2 + \beta_x\sum_{h=1}^{p} g_{kh}^2 g_{\ell h}^2\right)\right];$$

$\mathcal{I}$.4: Combining the aforementioned results, we obtain that

$$z \cdot I \to \frac{1}{2z(1+\underline{s}(z))^2}\left(2\rho + \beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right) \\ + \frac{1}{2z(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\left(2r_{kl}^2 + \beta_x\sum_{h=1}^{p} g_{kh}^2 g_{\ell h}^2\right)\right]; \tag{4.15}$$

$\mathcal{I}$.5: Repeating the steps as done in I.1-I.4, we similarly get

$$z^2 II \xrightarrow{p} \frac{z^2}{2}\frac{\partial}{\partial z}\left(\frac{1}{z^2(1+\underline{s}(z))^2}\right)\cdot\left(2\rho + \beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right).$$

We now process these substeps one by one.

Step $\mathcal{I}$.1 It is apparent that from Lemma 6.2

$$\|\mathbf{D}\| \xrightarrow{a.s.} 0 \quad \text{and} \quad \max_j \|\mathbf{D}_j\| \xrightarrow{a.s.} 0. \tag{4.16}$$

Using the Taylor formula as $x \to 1$

$$\sqrt{x} = 1 + \frac{1}{2}(x-1)(1+o(1)) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2(1+o(1)),$$

one finds that

$$\begin{aligned}\mathbf{L} =& \mathbf{I}_p + \mathbf{R}^{-1/2}\left\{[\mathrm{diag}(\mathbf{S})]^{1/2} - \mathbf{I}_p\right\}\mathbf{R}^{1/2} + \mathbf{R}^{1/2}\left\{[\mathrm{diag}(\mathbf{S})]^{1/2} - \mathbf{I}_p\right\}\mathbf{R}^{-1/2} \\ &+ \mathbf{R}^{-1/2}\left\{[\mathrm{diag}(\mathbf{S})]^{1/2} - \mathbf{I}_p\right\}\mathbf{R}\left\{[\mathrm{diag}(\mathbf{S})]^{1/2} - \mathbf{I}_p\right\}\mathbf{R}^{-1/2} \\ =& \mathbf{I}_p + \frac{1}{2}\mathbf{D} + \frac{1}{2}\mathbf{D}^T - \frac{1+o(1)}{8}\mathbf{D}^2 - \frac{1+o(1)}{8}\left(\mathbf{D}^T\right)^2 + \frac{1+o(1)}{4}\mathbf{D}\mathbf{D}^T. \end{aligned} \tag{4.17}$$

This implies

$$\begin{aligned} I =& \frac{1+o(1)}{4}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)^2 + \frac{1+o(1)}{4}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)^2 + \frac{1+o(1)}{2}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{A}^{-1}(z)\mathbf{D}^T\right) \\ &+ \frac{1+o(1)}{4}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right) + \frac{1+o(1)}{4}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right) \\ &+ \frac{1+o(1)}{16}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)^2 \end{aligned}$$

$$:=\frac{1+o(1)}{4}I_1+\frac{1+o(1)}{4}I_2+\frac{1+o(1)}{2}I_3+\frac{1+o(1)}{4}I_4+\frac{1+o(1)}{4}I_5+\frac{1+o(1)}{16}I_6.$$

In the following, we will show that $|I_4|+|I_5|+|I_6|=o_p(1)$. Define $\mathbf{g}_\ell$ to be the transpose of the ℓ-th row of the matrix $\mathbf{R}^{1/2}$ and $\mathbf{e}_\ell$ to be the ℓ-th column of the matrix $\mathbf{I}_p$. It is obvious from (4.2) that

$$\begin{aligned}
I_4 =&\frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\right]\\
=&\frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]\\
&-\frac{1}{n}\sum_{j=1}^{n}\beta_j(z)\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]\\
&-\frac{1}{n}\sum_{j=1}^{n}\beta_j(z)\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\right].
\end{aligned}$$

By (4.16) and Lemma 6.3, one finds that

$$\begin{aligned}
&\mathrm{E}\Bigg|\frac{1}{n}\sum_{j=1}^{n}\beta_j(z)\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]\\
&+\frac{1}{n}\sum_{j=1}^{n}\beta_j(z)\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\right]\Bigg|\\
\le&\frac{o(1)}{n}\sum_{j=1}^{n}\mathrm{E}\left[\mathbf{r}_j^T\mathbf{r}_j\left\|\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right\|\right]\\
\le&\frac{o(1)}{n}\sum_{j=1}^{n}\mathrm{E}^{1/2}\left(\mathbf{r}_j^T\mathbf{r}_j\right)^2\mathrm{E}^{1/2}\left\|\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right\|^2=o(1).
\end{aligned}$$

Here

$$\mathrm{E}\left\|\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right\|^2=\max_{k=1,\cdots,p}\mathrm{E}\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1\right)^2\le C. \tag{4.18}$$

Hence, we get from (4.16) and (4.18)

$$\begin{aligned}
I_4 =&\frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]+o_p(1)\\
=&\frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n^2}\sum_{j=1}^{n}\operatorname{tr}\left[\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{R}^{1/2}\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]\\
&+\frac{1}{n^2}\sum_{j=1}^{n}\operatorname{tr}\left[\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]\\
&+\frac{1}{n^2}\sum_{j=1}^{n}\operatorname{tr}\left[\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]+o_p(1)\\
=&\frac{1}{n}\sum_{j=1}^{n}\operatorname{tr}\left[\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right]+o_p(1).
\end{aligned}
$$

Define $\mathbf{M}_j(z)=\mathbf{R}^{1/2}\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{D}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}$. It follows from (4.16)

$$
\begin{aligned}
\mathrm{E}\left|I_4\right|^2=&\frac{1}{n^2}\sum_{j=1}^{n}\mathrm{E}\left|\operatorname{tr}\left[\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{M}_j(z)\right]\right|^2+o(1)\\
=&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1\right)\left(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1\right)\mathbf{e}_k^T\mathbf{M}_j(z)\mathbf{e}_k\mathbf{e}_\ell^T\overline{\mathbf{M}_j(z)}\mathbf{e}_\ell\right]+o(1)\\
\le&\frac{o(1)}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\left[2\left(\mathbf{g}_k^T\mathbf{g}_\ell\right)^2+\sum_{h=1}^{p}\left(\mathbf{e}_h^T\mathbf{g}_k\mathbf{g}_\ell^T\mathbf{e}_h\right)^2\right]=o(1).
\end{aligned}
$$

Consequently, we conclude

$$
I_4=o_p(1). \tag{4.19}
$$

Using the same method, it can be verified that

$$
I_5+I_6=o_p(1),
$$

which completes this substep.

Step $\mathcal{I}.2$ In this substep, three terms I_1, I_2 and I_3 will be shown to converge in probability to their means, say

$$
\left|I_1-\mathrm{E}\left(I_1\right)\right|+\left|I_2-\mathrm{E}\left(I_2\right)\right|+\left|I_3-\mathrm{E}\left(I_3\right)\right|=o_p(1).
$$

We start with $I_1-\mathrm{E}\left(I_1\right)=o_p(1)$. By (4.2), we obtain that

$$
I_1-\mathrm{E}\left(I_1\right)
$$

$$
\begin{aligned}
&=\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)^2-\mathrm{E}\left(\operatorname{tr}\mathbf{A}^{-1}(z)\mathbf{D}\right)^2\\
&=\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)^2-\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)^2\right)\\
&=\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\Bigg[-2\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\\
&\quad+\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)^2\\
&\quad+\frac{2}{n}\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\\
&\quad-\frac{4}{n}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\\
&\quad+\frac{2}{n}\left(\beta_j^2(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\\
&\quad+\frac{1}{n^2}\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right)^2\\
&\quad-\frac{2}{n^2}\left[\beta_j(z)\mathbf{r}_j^T\left[\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right]^2\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right]\\
&\quad+\frac{1}{n^2}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)^2\Bigg]\\
&=\sum_{j=1}^{n}(-2P_{j1}+P_{j2}+2P_{j3}-4P_{j4}+2P_{j5}+P_{j6}-2P_{j7}+P_{j8}).
\end{aligned}
$$

We now devote to investigating those terms. Firstly, by (4.3), one gets

$$
\begin{aligned}
\sum_{j=1}^{n}P_{j1}=&-\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j(z)\beta_j(z)\widehat{\gamma}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\\
&+\sum_{j=1}^{n}\mathrm{E}_j\left[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\operatorname{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right)\right].
\end{aligned}
$$

Denote $\mathrm{E}_{(j)}=\mathrm{E}\left(\cdot\mid\mathcal{F}_j\right)$ and

$$
\mathcal{F}_j=\sigma\{\mathbf{r}_1,\cdots,\mathbf{r}_{j-1},\mathbf{r}_{j+1},\cdots,\mathbf{r}_n\}.
$$

It follows that from (4.16) and Lemma 6.1

$$
\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j(z)\beta_j(z)\widehat{\gamma}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\right|^2\\
\le&C\sum_{j=1}^{n}\mathrm{E}\left|\widehat{\gamma}_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\operatorname{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right)\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left|\widehat{\boldsymbol{\gamma}}_j(z)\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right|^2\\
\leq & C\sum_{j=1}^{n}\mathrm{E}\left[\mathrm{E}_{(j)}^{1/2}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right|^4\right.\\
&\left.\times\mathrm{E}_{(j)}^{1/2}\left|\widehat{\boldsymbol{\gamma}}_j(z)\right|^4\right]+C\sum_{j=1}^{n}\mathrm{E}\left[\|\mathbf{D}_j\|^4\,\mathrm{E}_{(j)}\left|\widehat{\boldsymbol{\gamma}}_j(z)\right|^2\right]\leq\frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\,\|\mathbf{D}_j\|^4\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}\mathrm{E}_j\left[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right)\right]\right|^2\\
\leq & C\sum_{j=1}^{n}\mathrm{E}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right|^2\\
\leq & \frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\,\|\mathbf{D}_j\|^4\to 0.
\end{aligned}
$$

Hence, we have

$$
\sum_{j=1}^{n}P_{j1}=o_p(1). \tag{4.20}
$$

Secondly, we deal with $\sum_{j=1}^{n}P_{j2}$. By (4.3), we have

$$
\begin{aligned}
&\sum_{j=1}^{n}P_{j2}\\
= & \sum_{j=1}^{n}\left(\mathrm{E}_j-\mathrm{E}_{j-1}\right)\left[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right)\right]^2\\
&+\frac{2}{n}\sum_{j=1}^{n}\left(\mathrm{E}_j-\mathrm{E}_{j-1}\right)\left[b_j^2(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right)\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right]\\
&+2\sum_{j=1}^{n}\left(\mathrm{E}_j-\mathrm{E}_{j-1}\right)\left(b_j(z)\sqrt{\beta_j(z)}\sqrt{\widehat{\boldsymbol{\gamma}}_j(z)}\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)^2\\
&+\sum_{j=1}^{n}\left(\mathrm{E}_j-\mathrm{E}_{j-1}\right)\left(b_j(z)\beta_j(z)\widehat{\boldsymbol{\gamma}}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)^2.
\end{aligned}
$$

For the first term of the righthand side of the above equality, it yields that from Lemma 6.1

$$
\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}\left(\mathrm{E}_j-\mathrm{E}_{j-1}\right)\left[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right)\right]^2\right|^2\\
\leq\ & C\sum_{j=1}^{n}\mathrm{E}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right|^4\leq\frac{C\eta_n^4}{n}\sum_{j=1}^{n}\mathrm{E}\|\mathbf{D}_j\|^4\to 0.
\end{aligned}
$$

Other terms are similar. Hence, one gets

$$\sum_{j=1}^{n} P_{j2} = o_p(1). \tag{4.21}$$

Thirdly, we prove that $\sum_{j=1}^{n} P_{j3} = o_p(1)$. Define $\mathbf{g}_\ell$ to be the transpose of the l-th row of the matrix $\mathbf{G}$. It follows from (4.16) that

$$\begin{aligned}
\mathrm{E}\left|\sum_{j=1}^{n} P_{j3}\right|^2 \leq & C\sum_{j=1}^{n}\mathrm{E}\left|\sum_{\ell=1}^{p}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\left(\mathbf{r}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_j-\frac{1}{n}\right)\right|^2\\
\leq & C\sum_{j=1}^{n}\sum_{\ell_1=1}^{p}\sum_{\ell_2=1}^{p}\mathrm{E}\Bigg[\mathbf{e}_{\ell_1}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_1}\mathbf{e}_{\ell_2}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(\bar{z})\mathbf{D}_j\\
&\times\mathbf{A}_j^{-1}(\bar{z})\mathbf{R}^{-1/2}\mathbf{e}_{\ell_2}\left(\mathbf{r}_j^T\mathbf{g}_{\ell_1}\mathbf{g}_{\ell_1}^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\mathbf{g}_{\ell_2}\mathbf{g}_{\ell_2}^T\mathbf{r}_j-\frac{1}{n}\right)\Bigg]\\
\leq & \frac{C}{n^2}\sum_{j=1}^{n}\sum_{\ell_1=1}^{p}\sum_{\ell_2=1}^{p}\mathrm{E}\Bigg[\mathbf{e}_{\ell_1}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_1}\mathbf{e}_{\ell_2}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(\bar{z})\mathbf{D}_j\\
&\times\mathbf{A}_j^{-1}(\bar{z})\mathbf{R}^{-1/2}\mathbf{e}_{\ell_2}\left(2\left(\mathbf{g}_{\ell_1}^T\mathbf{g}_{\ell_2}\right)^2+\beta_x\sum_{h=1}^{p}\mathbf{e}_h^T\mathbf{g}_{\ell_1}\mathbf{g}_{\ell_1}^T\mathbf{e}_h\mathbf{e}_h^T\mathbf{g}_{\ell_2}\mathbf{g}_{\ell_2}^T\mathbf{e}_h\right)\Bigg]\\
\leq & \frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\left\|\mathbf{D}_j\right\|^2\to 0.
\end{aligned}$$

This shows that

$$\sum_{j=1}^{n} P_{j3} = o_p(1). \tag{4.22}$$

Next, combining (4.16), Lemma 6.1, and Lemma 6.3, we have

$$\begin{aligned}
\mathrm{E}\left|\sum_{j=1}^{n} P_{j4}\right| \leq & \frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right|\\
\leq & \frac{C}{n}\sum_{j=1}^{n}\max_{\ell=1,\cdots,p}\mathrm{E}\left[\|\mathbf{D}_j\|\mathbf{r}_j^T\mathbf{r}_j\left|\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1\right|\right]\\
\leq & \frac{C}{n}\sum_{j=1}^{n}\max_{\ell=1,\cdots,p}\mathrm{E}\left[\|\mathbf{D}_j\|\mathrm{E}_{(j)}^{1/2}\left|\mathbf{r}_j^T\mathbf{r}_j\right|^2\mathrm{E}_{(j)}^{1/2}\left|\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1\right|^2\right]\\
\leq & \frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\|\mathbf{D}_j\|\to 0.
\end{aligned}$$

Therefore, it yields that

$$\sum_{j=1}^{n} P_{j4} = o_p(1). \tag{4.23}$$

Using the same methods, we can get

$$\sum_{j=1}^{n}(P_{j5}+P_{j7}+P_{j8})=o_p(1). \tag{4.24}$$

Now we deal with P_{j6}. Denote

$$\mathbf{M}_q=\mathbf{g}_q\mathbf{g}_q^T=(m_{kj}^q), q=1,2,3,4,$$

then one finds that for any positive number t

$$\mathbf{e}_k^T\mathbf{M}_q\mathbf{e}_k\geq 0 \quad \text{and} \quad \mathrm{tr}\,(\mathbf{M}_q)^t=1.$$

By the proof of Lemma 6.1 [see Bai and Silverstein (2010)], we can get

$$\begin{aligned}
&\mathrm{E}\left[\prod_{q=1}^{4}\left(\mathbf{x}_j^T\mathbf{M}_q\mathbf{x}_j-\mathrm{tr}\,(\mathbf{M}_q)\right)\right]\\
=&\mathrm{E}\left[\prod_{q=1}^{4}\sum_{l_q=1}^{p}\left(x_{l_qj}^2-1\right)m_{l_ql_q}^q\right]+\mathrm{E}\left[\prod_{q=1}^{4}\sum_{l_q\neq k_q=1}^{p}x_{l_qj}x_{k_qj}m_{l_qk_q}^q\right]\\
\leq&C\left|\mathrm{E}\left[\sum_{l=1}^{p}\left(x_{lj}^2-1\right)^4m_{ll}^1m_{ll}^2m_{ll}^3m_{ll}^4\right]+\eta_n^2n\sum_{q_1\neq q_2=1}^{4}\sum_{l_{q_1}=1}^{p}\sum_{l_{q_2}=1}^{p}\left(m_{l_{q_1}l_{q_1}}^{q_1}\right)^3m_{l_{q_2}l_{q_2}}^{q_2}\right.\\
&\left.+\sum_{q_1\neq q_2=1}^{4}\sum_{l_{q_1}=1}^{p}\sum_{l_{q_2}=1}^{p}\left(m_{l_{q_1}l_{q_1}}^{q_1}\right)^2\left(m_{l_{q_2}l_{q_2}}^{q_2}\right)^2\right|+C\prod_{q=1}^{4}\mathrm{E}^{1/4}\left|\sum_{l_q\neq k_q=1}^{p}x_{l_qj}x_{k_qj}m_{l_qk_q}^q\right|^4\\
\leq&C\eta_n^4n^2\sum_{l=1}^{p}m_{ll}^1m_{ll}^2m_{ll}^3m_{ll}^4+C\eta_n^2n+\prod_{q=1}^{4}\left|\mathrm{tr}\left(\mathbf{M}_q\mathbf{M}_q^T\right)^2\right|^{1/4}\\
\leq&C\eta_n^4n^2\sum_{l=1}^{p}m_{ll}^1m_{ll}^2m_{ll}^3m_{ll}^4+C\eta_n^2n.
\end{aligned}$$

By Cauchy-Schwarz inequality and the above inequality, it yields that

$$\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}P_{j6}\right|^2\\
=&\sum_{j=1}^{n}\mathrm{E}\left|\sum_{\ell_1=1}^{p}\sum_{\ell_2=1}^{p}\left(\mathbf{r}_j^T\mathbf{g}_{\ell_1}\mathbf{g}_{\ell_1}^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\mathbf{g}_{\ell_2}\mathbf{g}_{\ell_2}^T\mathbf{r}_j-\frac{1}{n}\right)\mathbf{e}_{\ell_1}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_2}\right.\\
&\left.\times\mathbf{e}_{\ell_2}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_1}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{j=1}^{n}\sum_{\ell_1,\ell_2,\ell_3,\ell_4=1}^{p}\mathrm{E}\Bigg[\left(\mathbf{r}_j^T\mathbf{g}_{\ell_1}\mathbf{g}_{\ell_1}^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\mathbf{g}_{\ell_2}\mathbf{g}_{\ell_2}^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\mathbf{g}_{\ell_3}\mathbf{g}_{\ell_3}^T\mathbf{r}_j-\frac{1}{n}\right)\\
&\quad\times\left(\mathbf{r}_j^T\mathbf{g}_{\ell_4}\mathbf{g}_{\ell_4}^T\mathbf{r}_j-\frac{1}{n}\right)\mathbf{e}_{\ell_1}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_2}\mathbf{e}_{\ell_2}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_1}\\
&\quad\times\mathbf{e}_{\ell_3}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(\bar{z})\mathbf{R}^{-1/2}\mathbf{e}_{\ell_4}\mathbf{e}_{\ell_4}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(\bar{z})\mathbf{R}^{-1/2}\mathbf{e}_{\ell_3}\Bigg]\\
&\le\frac{C\eta_n^4}{n^2}\sum_{j=1}^{n}\sum_{\ell_1,\ell_2,\ell_3,\ell_4=1}^{p}\sum_{k=1}^{p}m_{kk}^{\ell_1}m_{kk}^{\ell_2}m_{kk}^{\ell_3}m_{kk}^{\ell_4}\\
&\quad+\frac{C\eta_n^2}{n^3}\sum_{j=1}^{n}\mathrm{E}\left[\sum_{\ell_1,\ell_2=1}^{p}\mathbf{e}_{\ell_1}^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_{\ell_2}\mathbf{e}_{\ell_2}^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(\bar{z})\mathbf{R}^{1/2}\mathbf{e}_{\ell_1}\right]^2\\
&\le C\eta_n^2\to 0.
\end{aligned}
$$

Consequently, it follows that

$$\sum_{j=1}^{n}P_{j6}=o_p(1). \tag{4.25}$$

Together with (4.20)-(4.25), we conclude that

$$I_1-\mathrm{E}\left(I_1\right)=o_p(1). \tag{4.26}$$

Similar to the proof of (4.26), we shall prove $I_2-\mathrm{E}\left(I_2\right)=o_p(1)$ and $I_3-\mathrm{E}\left(I_3\right)=o_p(1)$.

Step $\mathcal{I}$.3 We now deduce the limit of the expectation of I_1. Rewrite

$$
\begin{aligned}
I_1 &= \mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)^2=\frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\right]\\
&= \frac{1}{n}\sum_{j=1}^{n}\mathrm{tr}\left[\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\right]\\
&\quad -\frac{2}{n}\sum_{j=1}^{n}\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\\
&\quad +\frac{1}{n}\sum_{j=1}^{n}\beta_j^2(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\\
&:= I_1.1-2\times I_1.2+I_1.3,
\end{aligned}
\tag{4.27}
$$

where the fourth equation holds due to the decomposition of $\mathbf{A}^{-1}(z)$ in (4.2).

To begin with, we prove that $\mathrm{E}(I_1.2) = o(1)$. Note that

$$\begin{aligned}\mathrm{E}\left(I_1.2\right) =& \mathrm{E}\left(\beta_1(z)\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{D}_1\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1\right) + \frac{1}{n}\mathrm{E}\Big(\beta_1(z)\\ & \times \mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1\Big)\\ :=& \mathrm{E}\left(I_1.2.1\right) + \mathrm{E}\left(I_1.2.2\right).\end{aligned}$$

Using (4.16) and Lemma 6.1, we get

$$\begin{aligned}&\left|\mathrm{E}\left(I_1.2.1\right)\right|\\ \le& Cn\sum_{\ell=1}^{p}\mathrm{E}\left|\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{D}_1\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1\left(\mathbf{r}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_1-\frac{1}{n}\right)\right|\\ \le& Cn\sum_{\ell=1}^{p}\mathrm{E}\Bigg|\Bigg(\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{D}_1\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1-\frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-2}(z)\mathbf{D}_1\\ &\times\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\Bigg)\left(\mathbf{r}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_1-\frac{1}{n}\right)\Bigg| + C\sum_{\ell=1}^{p}\mathrm{E}\left[\|\mathbf{D}_1\|\left|\mathbf{r}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_1-\frac{1}{n}\right|\right]\\ \le& Cn\sum_{\ell=1}^{p}\mathrm{E}\Bigg[\mathrm{E}_{(1)}^{1/2}\Bigg|\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{D}_1\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1-\frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-2}(z)\mathbf{D}_1\\ &\times\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\Bigg|^2\mathrm{E}_{(1)}^{1/2}\left|\mathbf{r}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_1-\frac{1}{n}\right|^2\Bigg] + C\sum_{\ell=1}^{p}\mathrm{E}\|\mathbf{D}_1\|\mathrm{E}\left|\mathbf{r}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_1-\frac{1}{n}\right|\\ \le& C\mathrm{E}\|\mathbf{D}_1\|\to 0.\end{aligned}$$

By Lemma 6.1 and Lemma 6.3, it follows that

$$\begin{aligned}\left|\mathrm{E}\left(I_1.2.2\right)\right| \le& \frac{C}{n}\mathrm{E}\left(\mathbf{r}_1^T\mathbf{r}_1\left\|\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p\right\|^2\right)\\ \le& \frac{C}{n}\mathrm{E}^{1/2}\left(\mathbf{r}_1^T\mathbf{r}_1\right)^2\mathrm{E}^{1/2}\left\|\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p\right\|^4\\ \le& \frac{C}{n}\max_{k=1,\cdots,p}\mathrm{E}^{1/2}\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1\right)^4 \le C\eta_n^2\to 0.\end{aligned}$$

These two inequalities show that

$$\mathrm{E}(I_1.2) = o(1).$$

Next we show that $\mathrm{E}(I_1.3) = o(1)$. Note that

$$\mathrm{E}\left(I_1.3\right)$$

$$
\begin{aligned}
=&\mathrm{E}\left(\beta_1^2(z)\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{D}_1\mathbf{A}_1^{-1}(z)\mathbf{r}_1\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1\right)\\
&+\frac{1}{n}\mathrm{E}\left(\beta_1(z)\mathbf{r}_1^T\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{r}_1\right)^2\\
:=&\mathrm{E}\left(I_1.3.1\right)+\mathrm{E}\left(I_1.3.2\right).
\end{aligned}
$$

Due to (4.16) and Lemma 6.1, it implies that

$$
\begin{aligned}
\left|\mathrm{E}\left(I_1.3.1\right)\right| \le& C\mathrm{E}\left[\left(\mathbf{r}_1^T\mathbf{r}_1\right)^2\|\mathbf{D}_1(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\|\right]\\
\le& C\max_{\ell=1,\cdots,p}\mathrm{E}\left[\|\mathbf{D}_1\|\mathrm{E}_{(1)}^{1/2}\left|\mathbf{r}_1^T\mathbf{r}_1\right|^4\mathrm{E}_{(1)}^{1/2}\left|\mathbf{X}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_1-1\right|^2\right]\\
\le& C\mathrm{E}\|\mathbf{D}_1\|\to 0
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\mathrm{E}\left(I_1.3.2\right)\right| \le& \frac{C}{n}\mathrm{E}\left[\left(\mathbf{r}_1^T\mathbf{r}_1\right)^2\|(\mathrm{diag}(\mathbf{S}_1)-\mathbf{I}_p)\|^2\right]\\
\le& \frac{C}{n}\max_{\ell=1,\cdots,p}\mathrm{E}\left[\mathrm{E}_{(1)}^{1/2}\left|\mathbf{r}_1^T\mathbf{r}_1\right|^4\mathrm{E}_{(1)}^{1/2}\left|\mathbf{X}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_1-1\right|^4\right]\\
\le& C\eta_n^2\to 0.
\end{aligned}
$$

Hence, we have $E(I_1.3)=o(1)$.

Now we analyze $\mathrm{E}(I_1.1)$. Noting that $g_k^Tg_k=1$, we have that

$$
\begin{aligned}
\mathrm{E}(I_1.1)=&\frac{1}{n^2}\sum_{j=1}^n\mathrm{Etr}\left((\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\right)^2\\
=&\frac{1}{n^2}\sum_{j=1}^n\sum_{k,\ell=1}^p\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\\
&\times(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1)(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\Big]\\
=&\frac{1}{n}\sum_{k,\ell=1}^p\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\\
&\times(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1)(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\Big].
\end{aligned}
$$

Further note that

$$
\mathrm{E}\left((\mathbf{X}_1^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_1-1)(\mathbf{X}_1^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_1-1)\right)=\beta_x\sum_{j=1}^p g_{\ell j}^2g_{kj}^2+2r_{k\ell}^2, \tag{4.28}
$$

where $\beta_x = \mathrm{E}\left(X_{11}^4\right) - 3$ and $r_{k\ell} = \mathbf{g}_k^T \mathbf{g}_\ell$. Therefore

$$\mathrm{E}(I_1.1) = \frac{1}{n}\sum_{k=1}^{p}\sum_{\ell=1}^{p}\left(\beta_x \sum_{j=1}^{p} g_{\ell j}^2 g_{kj}^2 + 2r_{k\ell}^2\right) \times \mathrm{E}\left(\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell \mathbf{e}_\ell^T \mathbf{R}^{1/2}\mathbf{A}_1^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right).$$

Using the decomposition of $\mathbf{A}_1^{-1}(z)$ in (4.5), we then get

$$\begin{aligned}\mathrm{E}(I_1.1) =& \frac{1}{n}\left(z - \frac{n-1}{n}b(z)\right)^{-2}\left(2p + \beta_x \sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right)\\ &- \frac{2}{n\left(z-(n-1)/n\cdot b(z)\right)}\sum_{k=1}^{p}\sum_{m=2}^{4}\left(2 + \beta_x \sum_{j=1}^{p} g_{kj}^4\right)\mathrm{E}\left(\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{B}_m(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right)\\ &+ \frac{2r_{k\ell}^2 + \beta_x \sum_{j=1}^{p} g_{kj}^2 g_{\ell j}^2}{n}\sum_{k,\ell=1}^{p}\sum_{u=2}^{4}\sum_{v=2}^{4}\mathrm{E}\left[\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{B}_u(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell \mathbf{e}_\ell^T \mathbf{R}^{1/2}\mathbf{B}_v(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right]\\ :=& \mathrm{E}(I_1.1.1) + \mathrm{E}(I_1.1.2) + \mathrm{E}(I_1.1.3).\end{aligned}$$

We first have that

$$\mathrm{E}(I_1.1.1) = \frac{2\rho_n + \beta_x n^{-1}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4}{z^2(1+\underline{s}(z))^2} + o(1)$$

by (2.17) of Bai and Silverstein (2004).

Next we show that both $\mathrm{E}(I_1.1.2)$ and $\mathrm{E}(I_1.1.3)$ are $o(1)$. By conditioning on $\mathcal{F}_1$ we get $\mathrm{E}(B_2(z)) = 0$. Furthermore, by (4.4) and Lemma 6.1, it implies that

$$\begin{aligned}&\left|\frac{1}{n}\mathrm{E}\sum_{k=1}^{p}\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{B}_3(z)\mathbf{R}^{-1/2}\mathbf{e}_k \cdot \left(2 + \beta_x \sum_{j=1}^{p} g_{kj}^4\right)\right|\\ \le\ & K\sum_{k=1}^{p}\left|\mathrm{E}\left[\left(\beta_{2(1)}(z) - b(z)\right)\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{r}_2\mathbf{r}_2^T \mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\ \le\ & K\sum_{k=1}^{p}\left|\mathrm{E}\left[\left(\beta_{2(1)}(z) - b_{21}(z)\right)\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{r}_2\mathbf{r}_2^T \mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\ & + K\sum_{k=1}^{p}\mathrm{E}\left[\left|b_{21}(z) - b(z)\right|\left|\mathrm{E}_{(2)}\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{r}_2\mathbf{r}_2^T \mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right|\right]\\ \le\ & K\sum_{k=1}^{p}\left|\mathrm{E}[-b_{21}^2(z)\widehat{\gamma}_{2(1)}(z) + \beta_{2(1)}(z)b_{21}^2(z)\widehat{\gamma}_{2(1)}^2(z)]\mathbf{e}_k^T \mathbf{R}^{1/2}\mathbf{r}_2\mathbf{r}_2^T \mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right| + o(1)\end{aligned}$$

$$
\begin{aligned}
&\le K\sum_{k=1}^{p}\left[\mathrm{E}\left|\mathrm{E}_{(2)}\widehat{\gamma}_{2(1)}(z)\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2\right|+\mathrm{E}\left|\widehat{\gamma}_{2(1)}^2(z)\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{e}_k\mathbf{e}_k^T\mathbf{r}_2\right|\right]+o(1)\\
&\le K\sum_{k=1}^{p}\mathrm{E}\left|\mathrm{E}_{(2)}\left[\widehat{\gamma}_{2(1)}(z)(\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2-n^{-1}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k)\right]\right|\\
&\quad +K\sum_{k=1}^{p}\mathrm{E}\left|\widehat{\gamma}_{2(1)}^2(z)(\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2-n^{-1}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k)\right|\\
&\quad +\frac{K}{n}\sum_{k=1}^{p}\mathrm{E}\left|[\widehat{\gamma}_{2(1)}(z)]^2\right|+o(1)\\
&= o(1),
\end{aligned}
$$

where K is a constant. Finally, by a similar decomposition to (4.2) we get that

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{k=1}^{p}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{B}_4(z)\mathbf{R}^{-1/2}\mathbf{e}_k\cdot\left(2+\beta_x\sum_{j=1}^{p}g_{kj}^4\right)\right]\right|\\
\le\ &\frac{K}{n^2}\sum_{k=1}^{p}\left|\sum_{i\ge 2}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{A}_{i1}^{-1}(z)-\mathbf{A}_1^{-1}(z))\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\
\le\ &\frac{K}{n^2}\sum_{k=1}^{p}\left|\sum_{i\ge 2}\mathrm{E}\left[\beta_{i(1)}(z)\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{i1}^{-1}(z)\mathbf{r}_i\mathbf{r}_i^T\mathbf{A}_{i1}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\
\le\ &\frac{K}{n}\sum_{k=1}^{p}\left|\mathrm{E}\left[\beta_{2(1)}(z)\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{r}_2\right]\right|\\
\le\ &\frac{K}{n}\sum_{k=1}^{p}\left|\mathrm{E}b_{2(1)}(z)\beta_{2(1)}(z)\widehat{\gamma}_{2(1)}(z)\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{r}_2\right|+o(1)\\
\le\ &\frac{K}{n}\sum_{k=1}^{p}\mathrm{E}\left|\widehat{\gamma}_{2(1)}(z)(\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{r}_2-n^{-1}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-2}(z)\mathbf{R}^{-1/2}\mathbf{e}_k)\right|\\
&+\frac{K}{n}\mathrm{E}\left|\widehat{\gamma}_{2(1)}(z)\right|+o(1)\\
\le\ &\frac{K}{n}\sum_{k=1}^{p}\mathrm{E}^{1/2}\left|\mathbf{r}_2^T\mathbf{A}_{21}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-1}(z)\mathbf{r}_2-\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{21}^{-2}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right|\\
&\quad\times\mathrm{E}^{1/2}\left|\widehat{\gamma}_{2(1)}(z)\right|^2+o(1)\\
=\ &o(1).
\end{aligned}
$$

To sum up, we proved that $E(I_1.1.2)=o(1)$.

We now show that $E(I_1.1.3)=o(1)$. Denoting $d_{kl}=2r_{k\ell}^2+\beta_x\sum_{h=1}^{p}g_{kh}^2g_{\ell h}^2$, we have

$$
\left|\frac{1}{n}\sum_{k,\ell=1}^{p}d_{kl}\mathrm{E}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{B}_2(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{B}_2(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right|
$$

$$\begin{aligned}
&\le \frac{K}{n}\left|\sum_{k,\ell=1}^{p}\sum_{i,j=2}^{n} d_{kl}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_i\mathbf{r}_i^T-n^{-1}\mathbf{I})\mathbf{A}_{i1}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_j\mathbf{r}_j^T-n^{-1}\mathbf{I})\mathbf{A}_{j1}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\
&\le K\left|\sum_{k,\ell=1}^{p} d_{kl}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{21}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{21}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\
&\quad +Kn\left|\sum_{k,\ell=1}^{p} d_{kl}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{21}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I})\mathbf{A}_{31}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right]\right|\\
&\le \frac{K}{n^2}\sum_{k,\ell=1}^{p} d_{kl}+Kn\left|\sum_{k,\ell=1}^{p} d_{kl}\mathrm{E}\Big[\beta_{2(31)}\beta_{3(21)}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\right.\\
&\qquad\qquad\left.\times(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big]\right|\\
&\le Kn\left|\sum_{k,\ell=1}^{p} d_{kl}\mathrm{E}\Big[\beta_{2(31)}\beta_{3(21)}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\right.\\
&\qquad\qquad\left.\times(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big]\right|+o(1).
\end{aligned}$$

Note that

$$\beta_{2(31)}(z)=b_{321}(z)-\beta_{2(31)}(z)b_{321}(z)\widehat{\gamma}_{2(31)}(z)$$

and

$$\beta_{3(21)}(z)=b_{321}(z)-\beta_{3(21)}(z)b_{321}(z)\widehat{\gamma}_{3(21)}(z).$$

Using (1.15) of Bai and Silverstein (2004) many times, we have for each $k,l=1,\cdots,p$,

$$\begin{aligned}
&\mathrm{E}\left[b_{321}^2(z)\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right]\\
=\;&\mathrm{E}\left[b_{321}^2(z)\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{r}_3\right]\\
&-\frac{1}{n^2}\mathrm{E}\left[b_{321}^2(z)\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{r}_2(\mathbf{r}_2^T\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2-n^{-1}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell)\right]\\
=\;&\mathrm{E}\Big[b_{321}^2(z)\Big(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3-\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\\
&\qquad\cdot\mathbf{A}_{321}^{-2}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big)\cdot\left(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{r}_3-\frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)\Big]\\
&+\frac{1}{n^2}\mathrm{E}\left[b_{321}^2(z)\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-2}\mathbf{r}_2-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_{321}^{-2}\right)\right)\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2-\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right)\right.\\
&\left.\cdot\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right]+\frac{1}{n^4}\mathrm{E}\left[b_{321}^2(z)\mathrm{tr}\left(\mathbf{A}_{321}^{-2}\right)\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right]\\
&-\frac{1}{n^4}\mathrm{E}\left[b_{321}^2(z)\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-3}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right]
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{n^2}\mathrm{E}\left[b_{321}^2(z)\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2 - n^{-1}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)^2\right] \\
= \; & \mathrm{E}\Big[b_{321}^2(z)\Big(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3 - \frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I}) \\
& \cdot\mathbf{A}_{321}^{-2}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big)\cdot\left(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{r}_3 - \frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)\Big] \\
& +O(n^{-3}) \\
= \; & \frac{1}{n^2}\mathrm{E}\Big[b_{321}^2(z)\Big(\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I})\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{r}_2 \\
& +\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-2}\mathbf{r}_2 \\
& +\beta_x\sum_{h=1}^p\mathbf{e}_h^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{e}_h\mathbf{e}_h^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{e}_h\Big)\Big] \\
& +O(n^{-3}) \\
= \; & \frac{1}{n^2}\mathrm{E}\Big[b_{321}^2(z)\Big(2\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2 - \frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right) \\
& \cdot\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{r}_2 - \frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-3}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right) \\
& +\beta_x\sum_{h=1}^p\mathbf{e}_h^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{e}_h\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{r}_2 - \frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\right) \\
& \cdot\left(\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{e}_h\mathbf{e}_h^T\mathbf{A}_{321}^{-1}\mathbf{r}_2 - \frac{1}{n}\mathbf{e}_h^T\mathbf{A}_{321}^{-2}\mathbf{e}_h\right)\Big)\Big] + O(n^{-3}) \\
= \; & O(n^{-3}).
\end{aligned}
$$

By Lemma 6.1, it follows that for each $k, l = 1, \cdots, p$,

$$
\begin{aligned}
& \mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_3\mathbf{r}_3^T - n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_2 \\
& \qquad \cdot\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\beta_{231}(z)b_{321}(z)\widehat{\boldsymbol{\gamma}}_{2(31)}(z)\Big] \\
\le \; & K\mathrm{E}\Big|\widehat{\boldsymbol{\gamma}}_{2(31)}(z)\left(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{r}_3 - \frac{1}{n}\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\right) \\
& \cdot\Big(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - \frac{1}{n}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_3 \\
& \quad -\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - \frac{1}{n}\mathbf{I}_p)\mathbf{A}_{321}^{-2}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big)\Big| \\
& +\frac{K}{n}\mathrm{E}\Big|\widehat{\boldsymbol{\gamma}}_{2(31)}(z)\Big(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - \frac{1}{n}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_3 \\
& \qquad -\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - \frac{1}{n}\mathbf{I}_p)\mathbf{A}_{321}^{-2}\mathbf{r}_2\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\Big)\Big| \\
& +\frac{K}{n^2}\mathrm{E}\left|\widehat{\boldsymbol{\gamma}}_{2(31)}(z)\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T - n^{-1}\mathbf{I})\mathbf{A}_{321}^{-2}\mathbf{r}_2\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{K}{n}\mathrm{E}\Big|\widehat{\gamma}_{2(31)}(z)\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{r}_2\Big(\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell \\
&\quad\cdot\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{r}_3-\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-1}\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\Big)\Big| \\
&+\frac{K}{n^2}\mathrm{E}\Big|\widehat{\gamma}_{2(31)}(z)\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_{321}^{-1}\mathbf{r}_2\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})\mathbf{A}_{321}^{-2}\mathbf{R}^{-1/2}\mathbf{e}_\ell\Big| \\
\le\ &\frac{K}{n^2}\mathrm{E}\Big|\widehat{\gamma}_{2(31)}(z)\mathbf{r}_2^T\mathbf{r}_2\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-\frac{1}{n}\mathbf{I}_p)^2\mathbf{R}^{-1/2}\mathbf{e}_k\Big| \\
&+\frac{K}{n^2}\mathrm{E}\left|\widehat{\gamma}_{2(31)}(z)\mathbf{r}_2^T\mathbf{r}_2\sqrt{\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I})^2\mathbf{R}^{-1/2}\mathbf{e}_k}\right| \\
\le\ &\frac{K}{n^2}\mathrm{E}^{1/2}\left|\widehat{\gamma}_{2(31)}(z)\right|^2=O(n^{-5/2}).
\end{aligned}
$$

Furthermore, for each $k,l=1,\cdots,p$,

$$
\begin{aligned}
&\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_2 \\
&\qquad\cdot\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\beta_{321}(z)b_{321}(z)\widehat{\gamma}_{3(21)}(z)\Big]=O(n^{-5/2})
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{1/2}(\mathbf{r}_2\mathbf{r}_2^T-n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_3\mathbf{r}_3^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}(\mathbf{r}_3\mathbf{r}_3^T-n^{-1}\mathbf{I}_p)\mathbf{A}_{321}^{-1}\mathbf{r}_2 \\
&\qquad\cdot\mathbf{r}_2^T\mathbf{A}_{321}^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\beta_{321}(z)\beta_{231}(z)b_{321}^2(z)\widehat{\gamma}_{3(21)}(z)\widehat{\gamma}_{2(31)}(z)\Big]=O(n^{-5/2}).
\end{aligned}
$$

Hence, we conclude that

$$
\left|\frac{1}{n}\sum_{k,\ell=1}^{p}d_{kl}\mathrm{E}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{B}_2(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{B}_2(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right|\le Kn\sum_{k,\ell=1}^{p}d_{kl}+o(1)=o(1).
$$

Similarly, we have

$$
\frac{1}{n}\sum_{k,\ell=1}^{p}d_{kl}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{B}_u(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{B}_v(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right]=o(1),\quad\text{for}\quad u,v=2,3,4.
$$

Combining the estimates above we get the desired convergence that

$$
\mathrm{E}\left(I_1\right)\ =\ \frac{2\rho_n+\beta_x\frac{1}{n}\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{p}g_{kj}^4}{z^2(1+\underline{s}(z))^2}+o(1) \tag{4.29}
$$

$$\to \frac{1}{z^2(1+\underline{s}(z))^2}\left(2\rho+\beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right).$$

Repeating the proof of (4.29), it implies that

$$\mathrm{E}\,(I_2) \qquad \to \frac{1}{z^2(1+\underline{s}(z))^2}\left(2\rho+\beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right)$$

and

$$\begin{aligned}
&\mathrm{E}\,(I_3)\\
= &\ \frac{1}{n}\sum_{j=1}^{n}\mathrm{Etr}\left[\mathbf{A}_j^{-1}(z)\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p)\mathbf{R}^{1/2}\right]+o(1)\\
= &\ \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{1/2}\mathbf{e}_k\\
&\quad\cdot\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1\right)\left(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1\right)\Big]+o(1)\\
= &\ \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{1/2}\mathbf{e}_k\\
&\quad\cdot\left(2r_{kl}^2+\beta_x\sum_{h=1}^{p}g_{kh}^2g_{\ell h}^2\right)\Big]+o(1)\\
\to &\ \frac{1}{z^2(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathrm{E}\Big[\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\left(2r_{kl}^2+\beta_x\sum_{h=1}^{p}g_{kh}^2g_{\ell h}^2\right)\Big].
\end{aligned}$$

Step $\mathcal{I}$.4 By Step $\mathcal{I}$.1-$\mathcal{I}$.3, we conclude that

$$\begin{aligned}
z\cdot I \to &\frac{1}{2z(1+\underline{s}(z))^2}\left(2\rho+\beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right)\\
&+\frac{1}{2z(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\left(2r_{kl}^2+\beta_x\sum_{h=1}^{p}g_{kh}^2g_{\ell h}^2\right)\right].
\end{aligned}\tag{4.30}$$

Step $\mathcal{I}$.5 Since

$$II=\frac{1}{2}\cdot\frac{\partial I}{\partial z},$$

it is apparent that

$$\begin{aligned} & z^2 \cdot II \\ \xrightarrow{p} & \frac{z^2}{4}\frac{\partial}{\partial z}\left(\frac{1}{z^2(1+\underline{s}(z))^2}\right)\cdot\left(2\rho+\beta_x \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{p} g_{kj}^4\right) \\ & +\frac{z^2}{4}\frac{\partial}{\partial z}\left(\frac{1}{z^2(1+\underline{s}(z))^2}\right)\cdot\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\left(2r_{kl}^2+\beta_x\sum_{h=1}^{p} g_{kh}^2 g_{\ell h}^2\right)\right], \end{aligned} \tag{4.31}$$

which is similar to the proof of (4.30).

Step $\mathcal{II}$ In this step, we are going to showing that

$$III + IV = o_p(1).$$

Firstly, it can be verified that from Lemma 6.2

$$\|\mathbf{H}^{-1}(z)\| \le \frac{1}{\Im z} \quad \text{a.s.}$$

Using (4.15) and Lemma 6.4, one has

$$\begin{aligned} |III+IV| \le & C\left\|\mathbf{H}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right\|\operatorname{tr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)^2 \\ \le & C\left\|\mathbf{L}-\mathbf{I}_p\right\| \le C\left\|\mathbf{D}\right\| \to 0 \quad \text{in probability.} \end{aligned}$$

Therefore, we conclude that

$$III + IV = o_p(1). \tag{4.32}$$

Step $\mathcal{III}$. In this step we derive the CLT of

$$\begin{aligned} V = & \left[\operatorname{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\left(\operatorname{tr}\mathbf{A}^{-1}(z)\right)\right] + \left[\operatorname{tr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right) - \mathrm{E}\operatorname{tr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)\right] \\ & + z\left[\operatorname{tr}\left(\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right) - \mathrm{E}\operatorname{tr}\left(\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)\right] \\ := & V.1 + V.2 + V.3. \end{aligned}$$

The outline of this step is as follows:

$\mathcal{III}$**.1:** Prove $V.1 = \sum_{j=1}^{n} W_j(z) + o_p(1)$ where $W_j(z) = -\mathrm{E}_j\left(\frac{d}{dz} b_j(z)\widehat{\gamma}_j(z)\right)$.

$\mathcal{III}$**.2:** Show that

$$V.2 = \sum_{j=1}^{n} Q_{j1}(z) + o_p(1)$$

and

$$V.3 = z \cdot \frac{\partial}{\partial z} \sum_{j=1}^{n} Q_{j1}(z) + o_p(1)$$

where $Q_{j1}(z) = \frac{1}{n}\mathrm{E}_j \mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j) - \mathbf{I}_p\right)\mathbf{R}^{1/2}\right)$.

$\mathcal{III}$**.3:** Show that the process indexed by z

$$\left(\sum_{j=1}^{n}\left(W_j(z) + Q_{j1}(z) + z \cdot \frac{\partial}{\partial z} Q_{j1}(z)\right)\right)$$

converges weakly to a Gaussian process.

Step $\mathcal{III}$.1 According to Bai and Silverstein (2004) (Page 562), we obtain

$$V.1 = \mathrm{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\right) = \sum_{j=1}^{n} W_j(z) + o_p(1), \tag{4.33}$$

where $W_j(z) = -\mathrm{E}_j\left(\frac{d}{dz} b_j(z)\widehat{\gamma}_j(z)\right)$. We note here that Zheng, Bai and Yao (2015) also obtain that $\mathrm{tr}\mathbf{A}^{-1}(z) - ps_{\rho_n}(z)$ converges to a Gaussian process with the following mean function

$$\frac{\rho \underline{s}^3(z)(1+\underline{s}(z))^{-3}}{(1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2})^2} + \frac{\beta_x \cdot \rho \underline{s}^3(z)(1+\underline{s}(z))^{-3}}{1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}}$$

and the covariance function

$$2\left[\frac{\frac{\partial \underline{s}(z_1)}{\partial z_1}\frac{\partial \underline{s}(z_2)}{\partial z_2}}{(\underline{s}(z_2)-\underline{s}(z_1))^2} - \frac{1}{(z_1-z_2)^2}\right] + \frac{\beta_x \rho \frac{\partial \underline{s}(z_1)}{\partial z_1}\frac{\partial \underline{s}(z_2)}{\partial z_2}}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2}.$$

Step $\mathcal{III}$.2 It follows that from (4.17) that

$$\begin{aligned} V.2 =& \frac{1+o(1)}{2}\left[\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right) - \mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)\right] \\ &+ \frac{1+o(1)}{2}\left[\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right) - \mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)\right] \end{aligned}$$

$$+\frac{1+o(1)}{4}\left[\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)-\operatorname{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)\right].$$

Similar to the proof of (4.26), we see that

$$\frac{1+o(1)}{4}\left[\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)-\operatorname{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)\right]=o_p(1).$$

Hence, one finds that

$$\begin{aligned}V.2=&\frac{1+o(1)}{2}\left[\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)-\operatorname{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)\right]\\&+\frac{1+o(1)}{2}\left[\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)-\operatorname{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)\right]+o_p(1)\\:=&\frac{1+o(1)}{2}V.2.1+\frac{1+o(1)}{2}V.2.2+o_p(1).\end{aligned}$$

Using (4.2), we obtain that

$$\begin{aligned}V.2.1=&\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)-\mathrm{E}\left(\operatorname{tr}\mathbf{A}^{-1}(z)\mathbf{D}\right)\\=&\sum_{j=1}^n(\mathrm{E}_j-\mathrm{E}_{j-1})\left(\operatorname{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)-\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{D}_j\right)\right)\\=&\frac{1}{n}\sum_{j=1}^n(\mathrm{E}_j-\mathrm{E}_{j-1})\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}\left(\mathbf{S}_j\right)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right)\\&-\sum_{j=1}^n(\mathrm{E}_j-\mathrm{E}_{j-1})\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\\&-\frac{1}{n}\sum_{j=1}^n(\mathrm{E}_j-\mathrm{E}_{j-1})\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\operatorname{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\\=&\sum_{j=1}^n\left(Q_{j1}(z)-Q_{j2}(z)-Q_{j3}(z)\right).\end{aligned}\tag{4.34}$$

We shall show that both $\sum_{j=1}^n Q_{j2}(z)$ and $\sum_{j=1}^n Q_{j3}(z)$ are $o_p(1)$.

We start with $\sum_{j=1}^n Q_{j2}(z)$. By (4.3), we have

$$\begin{aligned}\sum_{j=1}^n Q_{j2}(z)\quad=\quad&\sum_{j=1}^n\mathrm{E}_j\left[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\operatorname{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\right)\right)\right]\\&-\sum_{j=1}^n(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j^2(z)\widehat{\gamma}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\end{aligned}$$

$$+\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j^2(z)\beta_j(z)\widehat{\boldsymbol{\gamma}}_j^2(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right).$$

Using (4.16) and Lemma 6.1, it follows that

$$\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j^2(z)\widehat{\boldsymbol{\gamma}}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\right|^2\\
\le& C\sum_{j=1}^{n}\mathrm{E}\left|\widehat{\boldsymbol{\gamma}}_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right|^2\\
\le& C\sum_{j=1}^{n}\mathrm{E}\left|\widehat{\boldsymbol{\gamma}}_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right)\right|^2\\
&+\frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left|\widehat{\boldsymbol{\gamma}}_j(z)\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right|^2\\
\le& C\sum_{j=1}^{n}\mathrm{E}\left[\mathrm{E}_{(j)}^{1/2}\left|\widehat{\boldsymbol{\gamma}}_j(z)\right|^4\mathrm{E}_{(j)}^{1/2}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-2}(z)\mathbf{D}_j\right)\right|^4\right]\\
&+\frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left\|\mathbf{D}_j\right\|^2\le\frac{C\eta_n^4}{n}\sum_{j=1}^{n}\mathrm{E}\left\|\mathbf{D}_j\right\|^2+\frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left\|\mathbf{D}_j\right\|^2\to 0,
\end{aligned}$$

and

$$\mathrm{E}\left|\sum_{j=1}^{n}(\mathrm{E}_j-\mathrm{E}_{j-1})\left(b_j^2(z)\beta_j(z)\widehat{\boldsymbol{\gamma}}_j^2(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\right|\to 0.$$

Furthermore,

$$\begin{aligned}
&\mathrm{E}\left|\sum_{j=1}^{n}\mathrm{E}_j\left[b_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\right)\right]\right|^2\\
\le\quad& C\sum_{j=1}^{n}\mathrm{E}\left|\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{D}_j\mathbf{A}_j^{-1}(z)\right)\right|^2\\
\le\quad& \frac{C}{n}\sum_{j=1}^{n}\mathrm{E}\left\|\mathbf{D}_j\right\|^2\to 0.
\end{aligned}$$

That is, $\sum_{j=1}^{n}Q_{j2}(z)=o_p(1)$.

Next we prove that $\sum_{j=1}^{n}Q_{j3}(z)=o_p(1)$. Applying Lemma 6.1 and Lemma 6.3, one finds that

$$\mathrm{E}\left|\sum_{j=1}^{n}Q_{j3}(z)\right|^2\le\frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left[\mathbf{r}_j^T\mathbf{r}_j\left|\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right\|\right]^2$$

$$\leq \frac{C}{n^2}\sum_{j=1}^{n} \mathrm{E}^{1/2}\left|\mathbf{r}_j^T\mathbf{r}_j\right|^4 \mathrm{E}^{1/2}\left|\mathrm{diag}(\mathbf{S}_j) - \mathbf{I}_p\right\|^4$$
$$\leq C\eta_n^2 \to 0.$$

Consequently, it follows that

$$\sum_{j=1}^{n} Q_{j3}(z) \quad = \quad o_p(1).$$

Combining the two estimates with (4.34), we then obtain that

$$V.2.1 = \sum_{j=1}^{n} Q_{j1}(z) + o_p(1).$$

Using the same method of the proof of the above inequality, it yields that

$$\begin{aligned} V.2.2 =& \frac{1}{n}\sum_{j=1}^{n}(\mathrm{E}_j - \mathrm{E}_{j-1})\mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{1/2}\left(\mathrm{diag}\left(\mathbf{S}_j\right) - \mathbf{I}_p\right)\mathbf{R}^{-1/2}\right) + o_p(1) \\ =& \sum_{j=1}^{n} Q_{j1}(z) + o_p(1). \end{aligned}$$

Consequently, it follows that

$$V.2 = (1 + o(1))\sum_{j=1}^{n} Q_{j1}(z) + o_p(1).$$

Note that from (4.28)

$$\begin{aligned} \mathrm{E}\left|\sum_{j=1}^{n} Q_{j1}(z)\right|^2 \leq& \frac{C}{n^2}\sum_{j=1}^{n}\mathrm{E}\left|\sum_{k=1}^{p}\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j - 1\right)\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right|^2 \\ =& \frac{C}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left[\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j - 1\right)\left(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j - 1\right)\right] \\ & \cdot \mathrm{E}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\right] \\ \leq& \frac{C}{n^2}\sum_{j=1}^{n}\sum_{k,l=1}^{p}\mathrm{E}\left[\left(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j - 1\right)\left(\mathbf{X}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j - 1\right)\right] \\ =& \frac{C}{n^2}\sum_{j=1}^{n}\sum_{k,l=1}^{p}\mathrm{E}\left[2r_{k\ell}^2 + \beta_x\sum_{h=1}^{p} g_{\ell h}^2 g_{kh}^2\right] \leq C. \end{aligned}$$

Hence, we conclude that

$$V.2 = \sum_{j=1}^{n} Q_{j1}(z) + o_p(1). \tag{4.35}$$

Because $V.3 = z\frac{\partial}{\partial z}V.2$, we have

$$V.3 = z\sum_{j=1}^{n} \frac{\partial}{\partial z} Q_{j1}(z) + o_p(1)$$

similar to the proof of (4.35). Combining (4.33) and (4.35) yields

$$V = \sum_{j=1}^{n}\left[W_j(z) + Q_{j1}(z) + z\cdot\frac{\partial}{\partial z}Q_{j1}(z)\right] + o_p(1).$$

Step $\mathcal{III}$.3 Next we show that

$$\sum_{j=1}^{n}\left[W_j(z) + Q_{j1}(z) + z\cdot\frac{\partial}{\partial z}Q_{j1}(z)\right]$$

converges weakly to a Gaussian process in z.

For any $z_1,\ldots,z_r \in \mathbf{C}_+$, $\alpha_1,\ldots,\alpha_r \in \mathbb{R}$ and any $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{j=1}^{n} E\left(\left|\sum_{\ell=1}^{r}\alpha_l Q_{j1}(z_\ell)\right|^2 I\left(\left|\sum_{\ell=1}^{r}\alpha_l Q_{j1}(z_\ell)\right| \geq \varepsilon\right)\right)\\
\leq\ & \frac{1}{\varepsilon^2}\sum_{j=1}^{n} E\left|\sum_{\ell=1}^{r}\alpha_l Q_{j1}(z_\ell)\right|^4\\
=\ & \frac{1}{n^4\varepsilon^2}\sum_{j=1}^{n} E\left|\sum_{\ell=1}^{r}\alpha_\ell \mathrm{E}_j \mathrm{tr}\left(\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\left(\mathrm{diag}\left(\mathbf{S}_j\right) - \mathbf{I}\right)\mathbf{R}^{1/2}\right)\right|^4\\
=\ & \frac{1}{n^4\varepsilon^2}\sum_{j=1}^{n} E\left|\sum_{\ell=1}^{r}\alpha_\ell E_j\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k\cdot(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j - 1)\right|^4\\
\leq\ & \frac{C}{n^4\varepsilon^2}\sum_{j=1}^{n}\sum_{\ell=1}^{r}\alpha_\ell^4\cdot E\left|E_j\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k\cdot(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j - 1)\right|^4\\
\leq\ & \frac{C}{n^4\varepsilon^2}\sum_{j=1}^{n}\sum_{\ell=1}^{r}\alpha_\ell^4\cdot E\left|\mathbf{X}_j^T\left(\sum_{k=1}^{p}\mathbf{g}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{g}_k^T\right)\mathbf{X}_j - \mathrm{tr}(\mathbf{A}_j^{-1}(z_\ell))\right|^4\\
\leq\ & \frac{C}{\varepsilon^2}\sum_{\ell=1}^{r}\alpha_\ell^4\cdot\eta_n^4 = o(1),
\end{aligned}$$

where the second to the last inequality hold due to Jensen's inequality, and in the last inequality we used the bound (9.9.6) on Page 271 of Bai and Silverstein (2010) applied with $q = 4$ and the fact that

$$\operatorname{tr}\left(\sum_{k=1}^{p}\mathbf{g}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{g}_k^T\right)=\sum_{k=1}^{p}\mathbf{g}_k^T\mathbf{g}_k\cdot\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k$$
$$=\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\mathbf{e}_k=\operatorname{tr}\left(\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_\ell)\mathbf{R}^{-1/2}\right)=\operatorname{tr}(\mathbf{A}_j^{-1}(z_\ell)).$$

The other terms have the similar results.

Next we derive the limit of the quadratic variation process, which is a sum involving the following six processes:

$$\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(W_j(z_1)W_j(z_2)\right),\quad\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)Q_{j1}(z_2)\right),\quad\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(\frac{\partial Q_{j1}(z_1)}{\partial z_1}\frac{\partial Q_{j1}(z_2)}{\partial z_2}\right),$$

and

$$\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(W_j(z_1)Q_{j1}(z_2)\right),\quad\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(W_j(z_1)\frac{\partial Q_{j1}(z_2)}{\partial z_2}\right),\quad\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)\frac{\partial Q_{j1}(z_2)}{\partial z_2}\right).$$

In fact we only need to derive the limits of the following terms

$$\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)W_j(z_2)\right),\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)Q_{j1}(z_2)\right).$$

First we have

$$\begin{aligned}
&\sum_{j=1}^{n}E_{j-1}\left(Q_{j1}(z_1)Q_{j1}(z_2)\right)\\
=&\sum_{j=1}^{n}\mathrm{E}_{j-1}\Bigg[\sum_{k=1}^{p}\mathrm{E}_j\left(\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\left(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n}\right)\right)\\
&\qquad\cdot\sum_{\ell=1}^{p}\mathrm{E}_j\left(\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_2)\mathbf{R}^{-1/2}\mathbf{e}_\ell\left(\mathbf{r}_j^T\mathbf{g}_\ell\mathbf{g}_\ell\mathbf{r}_j-\frac{1}{n}\right)\right)\Bigg]\\
=&\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}_j\left(\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\right)\mathrm{E}_j\left(\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_2)\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)\\
&\qquad\cdot\mathrm{E}_{j-1}\left[\left(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_j-\frac{1}{n}\right)\right]\\
=&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}_j\left(\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\right)\mathrm{E}_j\left(\mathbf{e}_\ell^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_2)\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)\\
&\qquad\cdot\left(\beta_x\sum_{h=1}^{p}\left(\mathbf{e}_h^T\mathbf{R}^{1/2}\mathbf{e}_k\mathbf{e}_l^T\mathbf{R}^{1/2}\mathbf{e}_h\right)^2+2r_{k\ell}^2\right)
\end{aligned}$$

$$\xrightarrow{p} \quad \frac{1}{z_1 z_2(1+\underline{s}(z_1))(1+\underline{s}(z_2))} \lim_{n\to+\infty}\left(\frac{2}{n}\mathrm{tr}\mathbf{R}^2 + \frac{\beta_x}{n}\sum_{h=1}^{p}\left(\mathbf{e}_h^T\mathbf{R}\mathbf{e}_h\right)^2\right) \tag{4.36}$$

where in the last equation we used (4.28), and the last convergence is due to Lemma 6.2. of Zheng (2012).

Secondly,

$$\begin{aligned}
&\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)W_j(z_2)\right)\\
=\ & -\frac{\partial}{\partial z_2}\left\{\sum_{j=1}^{n}\mathrm{E}_{j-1}\left[\mathrm{E}_j\left(\mathrm{tr}\mathbf{A}_j^{-1}(z_1)\frac{1}{n}\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right)\right.\right.\\
&\left.\left.\cdot\mathrm{E}_j\left(b_j(z_2)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}(z_2)\mathbf{r}_j-\frac{\mathrm{tr}\mathbf{A}_j^{-1}(z_2)}{n}\right)\right)\right]\right\}\\
=\ & -\frac{\partial}{\partial z_2}\left\{\sum_{j=1}^{n}\sum_{k=1}^{p}\mathrm{E}_{j-1}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\left(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n}\right)\right.\right.\\
&\left.\left.\cdot\breve{b}_j(z_2)\left(\mathbf{r}_j^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\breve{\mathbf{A}}_j^{-1}(z_2)\right)\right]\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&E\left(\left(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n}\right)\left(\mathbf{r}_j^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{r}_j-\frac{1}{n}\mathrm{tr}\breve{\mathbf{A}}_j^{-1}(z_2)\right)\middle|\,\mathbf{r}_1,\ldots,\mathbf{r}_{j-1},\breve{\mathbf{r}}_{j+1},\ldots,\breve{\mathbf{r}}_n\right)\\
=&\frac{2}{n^2}\mathbf{g}_k^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{g}_k+\frac{\beta_x}{n^2}\sum_{h=1}^{p}\left(\mathbf{e}_h^T\mathbf{R}^{1/2}\mathbf{e}_k\right)^2\mathbf{e}_h^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{e}_h.
\end{aligned}$$

Then we see that

$$\begin{aligned}
&\sum_{j=1}^{n}\mathrm{E}_{j-1}\left(Q_{j1}(z_1)W_j(z_2)\right)\\
=\ & -\frac{1}{n^2}\cdot\frac{\partial}{\partial z_2}\left\{b(z_2)\sum_{j=1}^{n}\sum_{k=1}^{p}\mathrm{E}_{j-1}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\cdot\left(2\mathbf{g}_k^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{g}_k\right.\right.\right.\\
&\left.\left.\left.+\beta_x\sum_{h=1}^{p}\left(\mathbf{e}_h^T\mathbf{R}^{1/2}\mathbf{e}_k\right)^2\mathbf{e}_h^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{e}_h\right)\right]\right\}+o(1)\\
=\ & \frac{1}{n^2}\cdot\frac{\partial}{\partial z_2}\left\{z_2\underline{s}(z_2)\sum_{j=1}^{n}\sum_{k=1}^{p}\mathrm{E}_{j-1}\left[\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z_1)\mathbf{R}^{-1/2}\mathbf{e}_k\cdot\left(2\mathbf{g}_k^T\breve{\mathbf{A}}_j^{-1}(z_2)\mathbf{g}_k\right.\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad +\beta_x \sum_{h=1}^{p} \left(\mathbf{e}_h^T \mathbf{R}^{1/2} \mathbf{e}_k \right)^2 \mathbf{e}_h^T \breve{\mathbf{A}}_j^{-1}(z_2) \mathbf{e}_h \Big) \Big] \Big\} + o(1) \\
\xrightarrow{p} \quad & \frac{\partial}{\partial z_2} \left(\frac{z_2 \underline{s}(z_2)}{z_1 z_2 (1+\underline{s}(z_1))(1+\underline{s}(z_2))} \right) \\
&\qquad \cdot \left(2\rho + \lim_{n\to\infty} \frac{\beta_x}{n} \sum_{h=1}^{p} \mathbf{e}_h^T \mathbf{R} \mathbf{e}_h \right) \\
= \quad & -\frac{1}{z_1(1+\underline{s}(z_1))} \frac{\partial}{\partial z_2} \left(\frac{1}{1+\underline{s}(z_2)} \right) \cdot (2+\beta_x)\rho,
\end{aligned}
$$

where by (4.3) and some computation, we have $|b_j(z) - b(z)| \le K n^{-1}$ and $|b(z) - z\underline{s}(z)| \le K n^{-1}$. Moreover, $s(z)$ is the Stieltjes transformation of the Mčenko-Pastur law, $\underline{s}(z) = -(1-\rho)z^{-1} + \rho s(z)$ and $z = -\underline{s}^{-1}(z) + \rho(1+\underline{s}(z))^{-1}$.

Combining the results above and by the martingale CLT and Lemma 2.14 of Bai and Silverstein (2010) we then obtain that the process

$$
\sum_{j=1}^{n} \left[W_j(z) + Q_{j1}(z) + z \cdot \frac{\partial}{\partial z} Q_{j1}(z) \right]
$$

converges weakly to a Gaussian process with mean zero and covariance functions

$$
\begin{aligned}
& v(z_1, z_2) \\
= \quad & 2\left(\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2) - \underline{s}(z_1))^2} - \frac{1}{(z_2 - z_1)^2} \right) + \rho\beta_x \cdot \frac{\partial}{\partial z_1} \left(\frac{1}{(1+\underline{s}(z_1))} \right) \cdot \frac{\partial}{\partial z_2} \left(\frac{1}{1+\underline{s}(z_2)} \right) \\
& + \frac{\left(\rho\beta_x t_{\mathbf{G}} + \lim_{n\to\infty} \frac{2}{n} \mathrm{tr}\mathbf{R}^2 \right)}{z_1 z_2 (1+\underline{s}(z_1))(1+\underline{s}(z_2))} \\
& + z_1 z_2 \left(\rho\beta_x t_{\mathbf{G}} + \lim_{n\to\infty} \frac{2}{n} \mathrm{tr}\mathbf{R}^2 \right) \frac{\partial}{\partial z_1} \left(\frac{1}{z_1(1+\underline{s}(z_1))} \right) \cdot \frac{\partial}{\partial z_2} \left(\frac{1}{z_2(1+\underline{s}(z_2))} \right) \\
& - (\rho\beta_x + 2\rho) \frac{1}{z_1(1+\underline{s}(z_1))} \cdot \frac{\partial}{\partial z_2} \left(\frac{1}{1+\underline{s}(z_2)} \right) \\
& - (\rho\beta_x + 2\rho) \frac{1}{z_2(1+\underline{s}(z_2))} \cdot \frac{\partial}{\partial z_1} \left(\frac{1}{1+\underline{s}(z_1)} \right) \\
& - (\rho\beta_x + 2\rho) z_1 \cdot \frac{\partial}{\partial z_1} \left(\frac{1}{z_1(1+\underline{s}(z_1))} \right) \cdot \frac{\partial}{\partial z_2} \left(\frac{1}{1+\underline{s}(z_2)} \right) \\
& - (\rho\beta_x + 2\rho) z_2 \cdot \frac{\partial}{\partial z_1} \left(\frac{1}{1+\underline{s}(z_1)} \right) \cdot \frac{\partial}{\partial z_2} \left(\frac{1}{z_2(1+\underline{s}(z_2))} \right) \\
& + (\rho\beta_x t_{\mathbf{G}} + \lim_{n\to\infty} \frac{2}{n} \mathrm{tr}\mathbf{R}^2) z_1 \cdot \frac{\partial}{\partial z_1} \left(\frac{1}{z_1(1+\underline{s}(z_1))} \right) \cdot \frac{1}{z_2(1+\underline{s}(z_2))}
\end{aligned}
$$

$$
\begin{aligned}
&+(\rho\beta_x t_{\mathbf{G}} + \lim_{n\to\infty}\frac{2}{n}\mathrm{tr}\mathbf{R}^2)z_2 \cdot \frac{1}{z_1(1+\underline{s}(z_1))}\cdot\frac{\partial}{\partial z_2}\left(\frac{1}{z_2(1+\underline{s}(z_2))}\right)\\
= \quad & 2\left(\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2)-\underline{s}(z_1))^2}-\frac{1}{(z_2-z_1)^2}\right)+\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2}\\
&\cdot\left(\rho\beta_x t_{\mathbf{G}}+\lim_{n\to\infty}\frac{2}{n}\mathrm{tr}\mathbf{R}^2-\rho\beta_x-4\rho\right). \qquad (4.37)
\end{aligned}
$$

Step $\mathcal{IV}$ Now we derive the limit of the term VI defined in (4.14). Firstly,Zheng, Bai and Yao (2015) proved that

$$
\mathrm{E}\mathrm{tr}\mathbf{A}^{-1}(z)-ps_{\rho_n}(z) \quad \to \quad \frac{\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{(1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2})^2}+\frac{\beta_x\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}}. \qquad (4.38)
$$

Then it follows that from (4.17)

$$
\begin{aligned}
\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)=&\frac{1}{2}\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)+\frac{1}{2}\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)\\
&+\frac{1+o(1)}{4}\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)-\frac{1+o(1)}{8}\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^2\right) \qquad (4.39)\\
&-\frac{1+o(1)}{8}\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{D}^T\right)^2\right).
\end{aligned}
$$

We begin with the third term of the righthand side of (4.39). It is apparent that

$$
\begin{aligned}
\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)=&\frac{1}{n}\sum_{j=1}^n\mathrm{E}\mathrm{tr}\left(\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\right)\\
=&\frac{1}{n^2}\sum_{j=1}^n\mathrm{E}\mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{-1/2}\right)\\
&+\frac{1}{n}\sum_{j=1}^n\mathrm{E}\mathrm{tr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}_j^T\right)\\
&-\frac{1}{n}\sum_{j=1}^n\mathrm{E}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right).
\end{aligned}
$$

By conditioning on $\mathcal{F}_j$ one sees that the second term equals 0. Applying (4.16) and (4.18), we get

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{j=1}^n\mathrm{E}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\mathbf{D}^T\mathbf{A}_j^{-1}(z)\mathbf{r}_j\right)\right|\\
\le&\frac{o(1)}{n}\sum_{j=1}^n\mathrm{E}\left(\mathbf{r}_j^T\mathbf{r}_j\left\|\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right\|\right)\le\frac{o(1)}{n}\sum_{j=1}^n\mathrm{E}^{1/2}\left(\mathbf{r}_j^T\mathbf{r}_j\right)^2=o(1).
\end{aligned}
$$

Therefore, we see that from (4.28)

$$\begin{aligned}
&\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\mathbf{D}^T\right)\\
=&\frac{1}{n^2}\sum_{j=1}^{n}\mathrm{Etr}\left(\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{-1/2}\right)+o(1)\\
=&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\mathrm{E}\left(\left(\mathbf{X}_j\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{X}_j-1\right)\left(\mathbf{X}_j\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1\right)\mathbf{e}_\ell^T\mathbf{R}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)+o(1)\\
=&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}\left(\beta_x\sum_{h=1}^{p}g_{\ell h}^2g_{kh}^2+2r_{k\ell}^2\right)\mathrm{E}\left(\mathbf{e}_\ell^T\mathbf{R}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_\ell\right)+o(1)\\
\to&-\frac{1}{z(1+\underline{s}(z))}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\left(\beta_x\sum_{h=1}^{p}g_{\ell h}^2g_{kh}^2+2r_{k\ell}^2\right)\left(\mathbf{e}_\ell^T\mathbf{R}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell\right), \qquad (4.40)
\end{aligned}$$

where the proof of (4.40) is similar to those of (4.36). It can also be verified that

$$\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^2\right)\to-\frac{1}{z(1+\underline{s}(z))}\left(\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^4+2\rho\right) \qquad (4.41)$$

and

$$\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{D}^T\right)^2\right)\to-\frac{1}{z(1+\underline{s}(z))}\left(\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^4+2\rho\right). \qquad (4.42)$$

Next, using (4.2) we have

$$\begin{aligned}
\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right) &= \frac{1}{n}\sum_{j=1}^{n}\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right)\\
&= \frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right)\\
&= \frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right)\\
&\quad -\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right)\\
&= -\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(\beta_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right).
\end{aligned}$$

By the decomposition (4.3) of $\beta_j(z)$ and (9.9.6) of Bai and Silverstein (2010), we have

$$\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}\right)$$

$$
\begin{aligned}
&= -\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(b_j(z)\mathbf{r}_j^T\mathbf{A}_j^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right)+o(1)\\
&= -\frac{1}{n}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\Big[b_j(z)\left(\mathbf{r}_j^T\mathbf{A}_j^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{r}_j-\frac{1}{n}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-2}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\right)(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\Big]\\
&\quad -\frac{1}{n^2}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\left(b_j(z)\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-2}(z)\mathbf{R}^{-1/2}\mathbf{e}_k(\mathbf{X}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{X}_j-1)\right)+o(1).\\
&:= VI.2.1+VI.2.2+o(1). \qquad (4.43)
\end{aligned}
$$

By conditioning on $\mathcal{F}_j$ one sees that the second term equals 0. Furthermore, one has

$$
\begin{aligned}
VI.2.1 &= -\frac{1}{n^2}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\Big[b_j(z)\Big(2\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{g}_k\mathbf{g}_k^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{-1/2}\mathbf{e}_k\\
&\quad +\beta_x\sum_{h=1}^{p}g_{kh}^2\mathbf{e}_h^T\mathbf{A}_j^{-1}\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{1/2}\mathbf{A}_j^{-1}(z)\mathbf{e}_h\Big)\Big]+o(1)\\
&\to \frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\left(2\mathbf{e}_k^T\mathbf{R}\mathbf{e}_k+\beta_x\sum_{h=1}^{p}g_{kh}^3\mathbf{e}_h^T\mathbf{R}^{-1/2}\mathbf{e}_k\right), \qquad (4.44)
\end{aligned}
$$

where the proof of (4.44) is similar to those of (4.36). Similarly, we get

$$
\begin{aligned}
&\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\mathbf{D}^T\right)\\
&= -\frac{1}{n^2}\sum_{k=1}^{p}\sum_{j=1}^{n}\mathrm{E}\Big[b_j(z)\Big(2\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{g}_k\mathbf{g}_k^T\mathbf{A}_j^{-1}(z)\mathbf{R}^{1/2}\mathbf{e}_k\\
&\quad +\beta_x\sum_{h=1}^{p}g_{kh}^2\mathbf{e}_h^T\mathbf{A}_j^{-1}\mathbf{R}^{1/2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}^{-1/2}\mathbf{A}_j^{-1}(z)\mathbf{e}_h\Big)\Big]+o(1)\\
&\to \frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\left(2\mathbf{e}_k^T\mathbf{R}\mathbf{e}_k+\beta_x\sum_{h=1}^{p}g_{kh}^3\mathbf{e}_h^T\mathbf{R}^{-1/2}\mathbf{e}_k\right). \qquad (4.45)
\end{aligned}
$$

Combining (4.39)-(4.45), we see that

$$
\begin{aligned}
&\mathrm{Etr}\left(\mathbf{A}^{-1}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right)\\
&\to \frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}\left(2\rho+\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^3\mathbf{e}_h^T\mathbf{R}^{-1/2}\mathbf{e}_k\right)\\
&\quad -\frac{1}{4z(1+\underline{s}(z))}\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\cdot\left(\beta_x\sum_{h=1}^{p}g_{\ell h}^2g_{kh}^2+2r_{k\ell}^2\right)\\
&\quad +\frac{1}{4z(1+\underline{s}(z))}\left(\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^4+2\rho\right). \qquad (4.46)
\end{aligned}
$$

It follows that the limit of the third term of the term VI defined in (4.14)

$$\begin{aligned}
&z\mathrm{Etr}\left(\mathbf{A}^{-2}(z)\left(\mathbf{L}-\mathbf{I}_p\right)\right) \\
\to &z\frac{\partial}{\partial z}\left(\frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}\right)\left(2\rho+\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^3\mathbf{e}_h^T\mathbf{R}^{-1/2}\mathbf{e}_k\right) \\
&-z\frac{\partial}{\partial z}\left(\frac{1}{4z(1+\underline{s}(z))}\right)\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\cdot\left(\beta_x\sum_{h=1}^{p}g_{\ell h}^2g_{kh}^2+2r_{k\ell}^2\right) \\
&+z\frac{\partial}{\partial z}\left(\frac{1}{4z(1+\underline{s}(z))}\right)\left(\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^4+2\rho\right). \qquad (4.47)
\end{aligned}$$

Combining (4.13), (4.38), (4.46) and(4.47), the limit of the term VI defined in (4.14) is

$$\begin{aligned}
&\frac{\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{(1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2})^2}+\frac{\beta_x\cdot\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}} \\
&+\left[\frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}+z\frac{\partial}{\partial z}\left(\frac{\underline{s}(z)}{z(1+\underline{s}(z))^2}\right)\right]\left(2\rho+\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^3\mathbf{e}_h^T\mathbf{R}^{-1/2}\mathbf{e}_k\right) \\
&-\left[\frac{1}{4z(1+\underline{s}(z))}+z\frac{\partial}{\partial z}\left(\frac{1}{4z(1+\underline{s}(z))}\right)\right]\left[\lim_{n\to\infty}\frac{1}{n}\sum_{k,\ell=1}^{p}\mathbf{e}_k^T\mathbf{R}^{-1}\mathbf{e}_\ell r_{kl}\right. \\
&\qquad\left.\cdot\left(\beta_x\sum_{h=1}^{p}g_{\ell h}^2g_{kh}^2+2r_{k\ell}^2\right)-\left(\beta_x\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{p}\sum_{h=1}^{p}g_{kh}^4+2\rho\right)\right]. \qquad (4.48)
\end{aligned}$$

Step $\mathcal{IV}$ We are finally ready to establish the CLT of LSS of the matrix $\mathbf{R}^{-1}\widehat{\mathbf{R}}$.

Combining (4.9), (4.30), (4.31), (4.11), (4.12), (4.37) and (4.48), we see that $M_n(z)$ converges to a Gaussian process $M(z)$ with the mean function and covariance function as follows

$$\begin{aligned}
\mathrm{E}M(z)=&\frac{\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{(1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2})^2}+\frac{\beta_x\cdot\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}} \\
&+\left(-\frac{\underline{s}'(z)}{(1+\underline{s}(z))^2}+\frac{2\underline{s}'(z)}{(1+\underline{s}(z))^3}\right)\frac{[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})]\rho}{4} \qquad (4.49) \\
&-\left(\frac{\underline{s}'(z)}{(1+\underline{s}(z))^2}+\frac{2\underline{s}'(z)}{(1+\underline{s}(z))^3}\right)\cdot\frac{(\beta_xc_g+2)\,\rho}{4}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Cov}(M(z_1),M(z_2)) \\
=&2\left(\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2)-\underline{s}(z_1))^2}-\frac{1}{(z_2-z_1)^2}\right)+\rho\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2} \qquad (4.50) \\
&\cdot\left(\beta_x+2d_{\mathbf{R}}-\beta_x-4\right).
\end{aligned}$$

The remaining is to prove the tightness, which should have been proved in Step III. We need to obtain the tightness of $\mathrm{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\left(\mathrm{tr}\mathbf{A}^{-1}(z)\right)$ and $\sum_{j=1}^{n} Q_{j1}(z)$. Notice that the tightness of $\mathrm{tr}\mathbf{A}^{-1}(z) - \mathrm{E}\left(\mathrm{tr}\mathbf{A}^{-1}(z)\right)$ has been proved by Bai and Silverstein (2004), now we will devote to prove the tightness of $\sum_{j=1}^{n} Q_{j1}(z)$.

We want to prove, as a sufficient condition,

$$\sup_{n,z_1,z_2\in\mathcal{C}^+} \frac{\mathrm{E}\left|\sum_{j=1}^{n}(Q_{j1}(z_1)-Q_{j1}(z_2))\right|^2}{|z_1-z_2|^2} \le K.$$

To this end, we only need to prove

$$\sup_{n,z_1,z_2\in\mathcal{C}_n} \frac{\mathrm{E}\left|\sum_{j=1}^{n}(Q_{j1}(z_1)-Q_{j1}(z_2))\right|^2}{|z_1-z_2|^2} \le K.$$

Apparently, we have

$$\begin{aligned}
&\frac{\sum_{j=1}^{n}(Q_{j1}(z_1)-Q_{j1}(z_2))}{z_1-z_2}\\
=\ & \frac{1}{n}\sum_{j=1}^{n}\mathrm{E}_j\left[\mathrm{tr}\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\mathbf{R}^{-1/2}\left(\mathrm{diag}(\mathbf{S}_j)-\mathbf{I}_p\right)\mathbf{R}^{1/2}\right]\\
=\ & \sum_{j=1}^{n}\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathrm{E}_j\left[\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\right]\mathbf{R}^{-1/2}\mathbf{e}_k\cdot(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n}).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{\mathrm{E}|\sum_{j=1}^{n}(Q_{j1}(z_1)-Q_{j1}(z_2))|^2}{|z_1-z_2|^2}\\
=\ & \mathrm{E}\left|\sum_{j=1}^{n}\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathrm{E}_j\left[\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\right]\mathbf{R}^{-1/2}\mathbf{e}_k\cdot(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n})\right|^2\\
=\ & \sum_{j=1}^{n}\mathrm{E}\left|\sum_{k=1}^{p}\mathbf{e}_k^T\mathbf{R}^{1/2}\mathrm{E}_j\left[\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\right]\mathbf{R}^{-1/2}\mathbf{e}_k\cdot(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n})\right|^2\\
=\ & \sum_{j=1}^{n}\sum_{k,\ell=1}^{p}E\Big\{\mathbf{e}_k^T\mathbf{R}^{1/2}\mathrm{E}_j\left[\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\right]\mathbf{R}^{-1/2}\mathbf{e}_k\mathbf{e}_\ell^T\mathbf{R}^{1/2}\\
&\qquad\cdot\mathrm{E}_j\left[\mathbf{A}_j^{-1}(z_1)\mathbf{A}_j^{-1}(z_2)\right]\mathbf{R}^{-1/2}\mathbf{e}_\ell\Big\}\cdot\mathrm{E}\left[(\mathbf{r}_j^T\mathbf{g}_k\mathbf{g}_k^T\mathbf{r}_j-\frac{1}{n})(\mathbf{r}_j^T\mathbf{g}_\ell\mathbf{g}_\ell^T\mathbf{r}_j-\frac{1}{n})\right]
\end{aligned}$$

$$\leq \quad \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{p}(2r_{k\ell}^2+\beta_x\sum_{h=1}^{p}g_{\ell h}^2 g_{kh}^2)\leq K,$$

where K is a constant and $\mathrm{E}|\mathbf{A}_j^{-1}(z)|^p \leq K$. So the tightness is proved.

Following the representation (4.1), the LSS of sample correlation matrix $\mathbf{R}^{-1}\widehat{\mathbf{R}}$ is as follows

$$\sum_{k=1}^{p} g_\ell(\lambda_k) - p\int g_\ell(x)\,dF_{\rho_n}(x) \quad = \quad -\frac{1}{2\pi\mathbf{i}}\int g_\ell(z)\left((\mathbf{R}^{-1}\widehat{\mathbf{R}}-z\mathbf{I})^{-1}-ps_{\rho_n}(z)\right)dz$$

for $\ell = 1,\ldots,m$, where $F_\rho(\cdot)$ is the LSD of $\mathbf{R}^{-1}\widehat{\mathbf{R}}$ and F_{ρ_n} is obtained by replacing ρ by ρ_n. Thus the random vector

$$\left(\sum_{k=1}^{p} g_1(\lambda_k) - p\int g_1(x)\,dF_{\rho_n}(x),\ldots,\sum_{k=1}^{p} g_m(\lambda_k) - p\int g_m(x)\,dF_{\rho_n}(x)\right) \tag{4.51}$$

converges to an m-dimensional Gaussian random vector $X_{g_1},\ldots,X_{g_m}$ with mean functions and covariance functions as follows

$$\mathrm{E}X_{g_\ell} = -\frac{1}{2\pi\mathbf{i}}\oint g_\ell(z)\cdot \mathrm{E}M(z)\,dz$$

and

$$\mathrm{Cov}(X_{g_\ell},X_{g_j}) \quad = \quad -\frac{1}{4\pi^2}\oint\oint g_\ell(z_1)g_j(z_2)\cdot\mathrm{Cov}(M(z_1),M(z_2))\,dz_1\,dz_2.$$

We finally complete the proof of Theorem 3.1.

5. Proof of Corollary 3.1 and Example 3.2

5.1 Proof of Corollary 3.1

Let $\underline{s}(z) = -\frac{1}{1+\sqrt{\rho}\xi}$, then

$$dz = h(1-\xi^{-2})d\xi \quad \text{and} \quad z = 1+h\xi+h\xi^{-1}+h^2 = |1+h\xi|^2$$

where $|\xi| = 1$ and $h=\sqrt{\rho}$, because $z = -\underline{s}^{-1}(z)+\rho(1+\underline{s}(z))^{-1}$ (e.g. Wang and Yao (2013)). When ξ runs counterclockwisely the unit circle, z will run counterclockwisely a contour that encloses the support interval $[a,b]=[(1-h)^2,(1+h)^2]$. So now we regard z as a function of ξ. Moreover, we have

$$\underline{s}'(z)=\frac{\underline{s}^2(z)[1+\underline{s}(z)]^2}{[1+\underline{s}(z)]^2-\rho\underline{s}^2(z)},$$

since

$$z=-\frac{1}{\underline{s}(z)}+\frac{\rho}{1+\underline{s}(z)}=-\frac{-[1+(1-\rho)\underline{s}(z)]}{\underline{s}(z)[1+\underline{s}(z)]}=-(1-\rho)\frac{[\underline{s}(z)+(1-\rho)^{-1}]}{\underline{s}(z)[1+\underline{s}(z)]}.$$

Substitute the above expression into Theorem 3.1, we obtain Corollary 3.1.

5.2 Proof of Example 3.2

Let $g_1(x)=x$, $g_2(x)=x^2$, $g_3(x)=x^3$, $g_4(x)=x^4$, and $g_5(x)=\log(x)$. It is well known that the moments of the standard M-P distribution with index ρ take values

$$m_k(\rho)=\sum_{r=0}^{k-1}\frac{1}{r+1}\binom{r}{k}\binom{k-1}{r}\rho^k.$$

From this, it is easy to calculate the centering terms

$$\int g_1(x)f_{\rho_{n-1}}(x)dx=m_1(\rho_{n-1})=1,\quad \int g_2(x)f_{\rho_{n-1}}(x)dx=m_2(\rho_{n-1})=1+\rho_{n-1},$$
$$\int g_3(x)f_{\rho_{n-1}}(x)dx=m_3(\rho_{n-1})=1+3\rho_{n-1}+\rho_{n-1}^2,$$
$$\int g_4(x)f_{\rho_{n-1}}(x)dx=m_4(\rho_{n-1})=1+6\rho_{n-1}+6\rho_{n-1}^2+\rho_{n-1}^3.$$

While the centering term $\int g_5(x)f_{\rho_{n-1}}(x)dx=\frac{\rho_{n-1}-1}{\rho_{n-1}}\log(1-y_{n-1})-1, \rho_{n-1}<1$, has been proved in Section 9.12.3 in Bai and Silverstein (2010).

Using Theorem 3.1, now we focus on the expressions of the expectation parameters $\mathrm{E}\,X_{g_i}$ of LSS for $j=1,2,3,4,5$ and (co)variance parameters $\mathrm{Cov}(X_{g_j},X_{g_k})$ for $j,k=1,2,3,4,5$. Firstly, we compute $\mathrm{E}X_{g_j}$ for $j=1,2,3,4$. Denoting

$$\mu_1(g)=-\frac{1}{2\pi i}\oint_{\mathcal{C}} g(z)\left(-\frac{1}{(1+\underline{s}(z))^2}+\frac{2}{(1+\underline{s}(z))^3}\right)\underline{s}'(z)\,dz,$$

$$\mu_2(g) = -\frac{1}{2\pi i}\oint_{\mathcal{C}} g(z)\left(\frac{1}{(1+\underline{s}(z))^2} + \frac{2}{(1+\underline{s}(z))^3}\right)\underline{s}'(z)\,dz,$$
$$\mu_3(g) = -\frac{1}{2\pi i}\oint g(z)\frac{\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{(1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2})^2}\,dz,$$
$$\mu_4(g) = -\frac{1}{2\pi i}\oint g(z)\frac{\beta_x\cdot\rho\underline{s}^3(z)(1+\underline{s}(z))^{-3}}{1-\rho\underline{s}^2(z)(1+\underline{s}(z))^{-2}}\,dz,$$

it follows that

$$\mathrm{E}X_g = \mu_3(g) + \beta_x\mu_4(g) + \frac{[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})]\rho}{4}\mu_1(g) - \frac{(\beta_x c_g+2)\,\rho}{4}\mu_2(g).$$

By Wang and Yao (2013), one gets

$$\begin{aligned}
\mu_3(g) &= \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi|=1} g\left(|1+h\xi|^2\right)\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right]d\xi,\\
\mu_4(g) &= \frac{1}{2\pi i}\oint_{|\xi|=1} g\left(|1+h\xi|^2\right)\frac{1}{\xi^3}d\xi.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\mu_1(g) &= -\frac{1}{2\pi i}\oint_{|\xi|=1} g\left(|1+h\xi|^2\right)\left(-\left(1+\frac{1}{h\xi}\right)^2 + 2\left(1+\frac{1}{h\xi}\right)^3\right)\frac{h}{(1+h\xi)^2}\,d\xi\\
&= -\frac{1}{2\pi i}\oint_{|\xi|=1} g\left(|1+h\xi|^2\right)\left(\frac{1}{h\xi^2} + \frac{2}{h^2\xi^3}\right)d\xi,\\
\mu_2(g) &= -\frac{1}{2\pi i}\oint_{|\xi|=1} g\left(|1+h\xi|^2\right)\left(\frac{3}{h\xi^2} + \frac{2}{h^2\xi^3}\right)d\xi.
\end{aligned}$$

Note that

$$\begin{aligned}
|1+h\xi|^8 &= (1+h^2)^4 + 12(1+h^2)^2h^2 + 6h^4 + h^4\left(\xi^{-4}+\xi^4\right)\\
&\quad +4(1+h^2)h^3\left(\xi^{-3}+\xi^3\right) + \left(4h^4+6(1+h^2)^2h^2\right)\left(\xi^{-2}+\xi^2\right)\\
&\quad +\left(12(1+h^2)h^3 + 4(1+h^2)^3h\right)\left(\xi^{-1}+\xi\right),\\
|1+h\xi|^6 &= (1+h^2)^3 + 6h^2(1+h^2) + \left(3h(1+h^2)^2+3h^3\right)\left(\xi+\xi^{-1}\right)\\
&\quad +3h^2(1+h^2)\left(\xi^2+\xi^{-2}\right) + h^3(\xi^3+\xi^{-3}),\\
|1+h\xi|^4 &= (1+h^2)^2 + 2h^2 + 2h(1+h^2)\left(\xi+\xi^{-1}\right) + h^2(\xi^2+\xi^{-2}).
\end{aligned}$$

Then we have

$$\begin{aligned}
\mu_1(g_4) =& -\frac{1}{2\pi i}\oint_{|\xi|=1} |1+h\xi|^8 \left(\frac{1}{h\xi^2}+\frac{2}{h^2\xi^3}\right) d\xi,\\
=& -\frac{1}{2\pi i}\oint_{|\xi|=1} \frac{12(1+h^2)h^2+4(1+h^2)^3}{\xi} d\xi - \frac{1}{\pi i}\oint_{|\xi|=1} \frac{4h^2+6(1+h^2)^2}{\xi} d\xi,\\
=& -4\left(3(1+h^2)h^2+(1+h^2)^3\right) - 4\left(2h^2+3(1+h^2)^2\right)\\
=& -4\left(3(1+\rho)\rho+(1+\rho)^3+2\rho+3(1+\rho)^2\right),\\
\mu_2(g_4) =& -4\left(9(1+\rho)\rho+3(1+\rho)^3+2\rho+3(1+\rho)^2\right).
\end{aligned}$$

$$\begin{aligned}
\mu_3(g_4) =& \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi|=1} |1+h\xi|^8 \left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right] d\xi\\
=& \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\left[(1+h^2)\xi+h\xi^2+h\right]^4}{\xi^3(\xi^2-r^{-2})} d\xi\\
&- \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{(1+h^2)^4+12(1+h^2)^2h^2+6h^4}{\xi} d\xi\\
=& \lim_{r\downarrow 1}\left.\frac{\left[(1+h^2)\xi+h\xi^2+h\right]^4}{\xi^3(\xi+r^{-1})}\right|_{\xi=r^{-1}} + \lim_{r\downarrow 1}\left.\frac{\left[(1+h^2)\xi+h\xi^2+h\right]^4}{\xi^3(\xi-r^{-1})}\right|_{\xi=-r^{-1}}\\
&+ \lim_{r\downarrow 1}\left.\frac{4h^4+6h^2(1+h^2)^2}{(\xi^2-r^{-2})}\right|_{\xi=0} - (1+h^2)^4 - 12(1+h^2)^2h^2 - 6h^4\\
=& \frac{(1+h)^8}{2}+\frac{(1-h)^8}{2} - 4h^4 - 6h^2(1+h^2)^2 - (1+h^2)^4 - 12(1+h^2)^2h^2 - 6h^4\\
=& 6\rho^2+6(1+\rho)^2\rho,\\
\mu_4(g_4) =& \frac{1}{2\pi i}\oint_{|\xi|=1} |1+h\xi|^8 \frac{1}{\xi^3} dz = 4\rho^2+6(1+\rho)^2\rho.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathrm{E}X_{g_4} =& 6\rho^2+6(1+\rho)^2\rho+2\beta_x\rho\left[2\rho+3(1+\rho)^2\right]\\
&- \rho[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})]\left[3(1+\rho)\rho+(1+\rho)^3+2\rho+3(1+\rho)^2\right]\\
&+ \rho(\beta_x c_g+2)\left[9(1+\rho)\rho+3(1+\rho)^3+2\rho+3(1+\rho)^2\right].
\end{aligned}$$

Moreover, we have

$$\mu_1(g_3) \quad = \quad -\frac{1}{2\pi i}\oint_{|\xi|=1} |1+h\xi|^6 \left(\frac{1}{h\xi^2}+\frac{2}{h^2\xi^3}\right) d\xi$$

$$
\begin{aligned}
&= -\left(3(1+h^2)^2+3h^2\right)-6(1+h^2) = -3\left((1+\rho)^2+3\rho+2\right),\\
\mu_2(g_3) &= -3\left(3(1+\rho)^2+5\rho+2\right),
\end{aligned}
$$

$$
\begin{aligned}
\mu_3(g_3) &= \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi|=1}|1+h\xi|^6\left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right]d\xi\\
&= \frac{(1+h)^6}{2}+\frac{(1-h)^6}{2}-3h^2(1+h^2)-(1+h^2)^3-6h^2(1+h^2) = 3\rho(1+\rho),\\
\mu_4(g_3) &= \frac{1}{2\pi i}\oint_{|\xi|=1}|1+h\xi|^6\frac{1}{\xi^3}d\xi = 3\rho(1+\rho).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\mathrm{E}X_{g_3} =& 3\rho(1+\rho)+3\beta_x\rho(1+\rho)+\frac{3\rho(\beta_x c_g+2)}{4}\left(3(1+\rho)^2+5\rho+2\right)\\
&-\frac{3\rho\left[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})\right]}{4}\left((1+\rho)^2+3\rho+2\right).
\end{aligned}
$$

By Wang and Yao (2013), it yields that

$$
\begin{aligned}
&\mu_3(g_1)+\beta_x\mu_4(g_1) = 0,\\
&\mu_3(g_2)+\beta_x\mu_4(g_2) = \rho+\beta_x\rho,\\
&\mu_3(g_5)+\beta_x\mu_4(g_5) = \frac{\log(1-\rho)}{2}-\frac{\beta_x\rho}{2}.
\end{aligned}
\tag{5.1}
$$

Furthermore, we have

$$
\begin{aligned}
\mu_1(g_1) &= -\frac{1}{2\pi i}\oint_{|\xi|=1}|1+h\xi|^2\left(\frac{1}{h\xi^2}+\frac{2}{h^2\xi^3}\right)d\xi = -1, \quad \mu_2(g_1) = -3,\\
\mu_1(g_2) &= -\frac{1}{2\pi i}\oint_{|\xi|=1}|1+h\xi|^4\left(\frac{1}{h\xi^2}+\frac{2}{h^2\xi^3}\right)d\xi = -2(1+h^2)-2 = -2(2+\rho),\\
\mu_2(g_2) &= -2(4+3\rho).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathrm{E}X_{g_1} &= -\frac{[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})]\rho}{4}+\frac{3\left(\beta_x c_g+2\right)\rho}{4}\\
\mathrm{E}X_{g_2} &= \rho+\beta_x\rho-\frac{[8-2a_{\mathbf{R}}+\beta_x(4a_g-h_{\mathbf{R}})]\rho}{2}(2+\rho)+\frac{\left(\beta_x c_g+2\right)\rho}{2}(4+3\rho).
\end{aligned}
$$

Note that

$$\oint_{|\xi|=1} \log(1+h\xi)\tfrac{1}{\xi}\,d\xi = 0, \qquad \oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi}\,d\xi = 0,$$
$$\oint_{|\xi|=1} \log(1+h\xi)\tfrac{1}{\xi^2}\,d\xi = 2\pi i\cdot h, \qquad \oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi^2}\,d\xi = 0,$$
$$\oint_{|\xi|=1} \log(1+h\xi)\tfrac{1}{\xi^3}\,d\xi = -2\pi i\cdot \tfrac{1}{2}h^2, \qquad \oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi^3}\,d\xi = 0,$$
$$\oint_{|\xi|=1} \log(1+h\xi)\tfrac{1}{\xi^4}\,d\xi = 2\pi i\cdot \tfrac{1}{3}h^3, \qquad \oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi^4}\,d\xi = 0,$$
$$\oint_{|\xi|=1} \log(1+h\xi)\tfrac{1}{\xi^5}\,d\xi = -2\pi i\cdot \tfrac{1}{4}h^4, \qquad \oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi^5}\,d\xi = 0.$$

This yields that

$$\mu_1(g_5) = -\frac{1}{2\pi i}\oint_{|\xi|=1} \log(|1+h\xi|^2)\left(\frac{1}{h\xi^2}+\frac{2}{h^2\xi^3}\right)d\xi = 0, \mu_2(g_5) = -2.$$

Together with (5.1), we have

$$\mathrm{E}X_{g_5} = \frac{\log(1-\rho)}{2} - \frac{\beta_x\rho}{2} + \frac{(\beta_x c_g+2)\,\rho}{2}.$$

Next, we will show the (co)variances of X_{g_j} and X_{g_k}, $j,k=1,2,3,4$. By Wang and Yao (2013), it implies that

$$\begin{aligned}\mathrm{Cov}(X_{g_j},X_{g_k}) = &-\frac{1}{2\pi^2}\lim_{r\downarrow 1}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1}\frac{g_j(|1+h\xi_1|^2)g_k(|1+h\xi_2|^2)}{(\xi_1-r\xi_2)^2}\,d\xi_1\,d\xi_2\\ &-\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\frac{g_j(|1+h\xi_1|^2))}{\xi_1^2}d\xi_1\oint_{|\xi_2|=1}\frac{g_k(|1+h\xi_2|^2))}{\xi_2^2}d\xi_2\\ &\cdot(2\beta_x+2d_{\mathbf{R}}-2\beta_x q_{\mathbf{G}}-4).\end{aligned}$$

In the following, we will compute for $j,k=1,2,3,4$

$$\begin{aligned}a_1(g_k,\xi_1) &= \frac{1}{2\pi i}\lim_{r\downarrow 1}\oint_{|\xi_2|=1}\frac{g_k(|1+h\xi_2|^2)}{(\xi_1-r\xi_2)^2}\,d\xi_2,\\ a_2(g_j,g_k) &= \frac{1}{2\pi i}\oint_{|\xi_1|=1} a_1(g_k,\xi_1)g_j(|1+h\xi_1|^2)\,d\xi_1,\end{aligned}$$

$$a_3(g_j) = \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{g_j(|1+h\xi|^2)}{\xi^2} d\xi.$$

To begin with, we will compute some integrals. By the residue theorem, it follows that

$$\begin{aligned} a_3(g_4) =& \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{12(1+h^2)h^3+4(1+h^2)^3h}{\xi} d\xi = 12(1+h^2)h^3+4(1+h^2)^3h, \\ a_3(g_3) =& \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{3h(1+h^2)^2+3h^3}{\xi} d\xi = 3h(1+h^2)^2+3h^3, \\ a_3(g_2) =& \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{2h(1+h^2)}{\xi} d\xi = 2h(1+h^2), \\ a_3(g_1) =& \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{h}{\xi} d\xi = h, \\ a_3(g_5) =& \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\log(1+h\xi)}{\xi^2} d\xi = h. \end{aligned} \tag{5.2}$$

Furthermore, we obtain

$$\begin{aligned} a_1(g_4,\xi_1) =& \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{h^4}{\xi_2^4(\xi_1-r\xi_2)^2}\, d\xi_2 + \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{4(1+h^2)h^3}{\xi_2^3(\xi_1-r\xi_2)^2}\, d\xi_2 \\ &+ \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{4h^4+6(1+h^2)^2h^2}{\xi_2^2(\xi_1-r\xi_2)^2}\, d\xi_2 \\ &+ \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{12(1+h^2)h^3+4(1+h^2)^3h}{\xi_2(\xi_1-r\xi_2)^2}\, d\xi_2 \\ =& \frac{4h^4}{\xi_1^5} + \frac{12(1+h^2)h^3}{\xi_1^4} + \frac{8h^4+12(1+h^2)^2h^2}{\xi_1^3} \\ &+ \frac{12(1+h^2)h^3+4(1+h^2)^3h}{\xi_1^2} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} a_1(g_3,\xi_1) =& \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{h^3}{\xi_2^3(\xi_1-r\xi_2)^2}\, d\xi_2 + \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{3h^2(1+h^2)}{\xi_2^2(\xi_1-r\xi_2)^2}\, d\xi_2 \\ &+ \lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1} \frac{3h(1+h^2)^2+3h^3}{\xi_2(\xi_1-r\xi_2)^2}\, d\xi_2 \\ =& \frac{3h^3}{\xi_1^4} + \frac{6h^2(1+h^2)}{\xi_1^3} + \frac{3h(1+h^2)^2+3h^3}{\xi_1^2}, \end{aligned} \tag{5.4}$$

where the second equality holds because $\frac{1}{r}\xi_1$ is not a pole as $r>1$. By Taylor formula

$$\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k},$$

it follows that

$$\begin{aligned}a_1(g_5,\xi_1) =&\lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2)}{(\xi_1-r\xi_2)^2}\,d\xi_2+\lim_{r\downarrow 1}\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2^{-1})}{(\xi_1-r\xi_2)^2}\,d\xi_2 \\ =&\lim_{r\downarrow 1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h^k}{k}\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{1}{\xi_2^k(\xi_1-r\xi_2)^2}\,d\xi_2 \\ =&\lim_{r\downarrow 1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h^k}{k}\frac{1}{(k-1)!}\frac{d}{d\xi_2^{k-1}}(\xi_1-r\xi_2)^{-2}\Big|_{\xi_2=0} \\ =&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h^k}{\xi_1^{k+1}}=\frac{h}{\xi_1(\xi_1+h)}.\end{aligned} \tag{5.5}$$

Next, we will show the covariances of X_{g_j} and $X_{g_5}, j=1,2,3,4,5$. By Wang and Yao (2013), one has

$$\mathrm{Cov}(X_{g_5},X_{g_5})=-2\log(1-\rho)+\rho\,(2d_{\mathbf{R}}-4)$$

$$\mathrm{Cov}(X_{g_1},X_{g_5})=2\rho+\rho\,(2d_{\mathbf{R}}-4)\,.$$

Using (5.3), we compute that

$$\begin{aligned}a_2(g_4,g_5)=&\frac{1}{2\pi i}\oint_{|\xi_1|=1}a_1(g_4,\xi_1)\log(|1+h\xi_1|^2)\,d\xi_1 \\ =&\frac{4h^4}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^5}\,d\xi_1+\frac{12(1+h^2)h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^4}\,d\xi_1 \\ &+\frac{8h^4+12(1+h^2)^2h^2}{2\pi i}\lim_{r\downarrow 1}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^3}\,d\xi_1 \\ &+\frac{12(1+h^2)h^3+4(1+h^2)^3h}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^2}\,d\xi_1 \\ =&-h^8+4(1+h^2)^2h^6-2h^4\left(2h^2+3(1+h^2)^2\right) \\ &+4h^2(1+h^2)\left(3h^2+(1+h^2)^2\right).\end{aligned}$$

Moreover, it follows from (5.4) that

$$\begin{aligned}a_2(g_3,g_5)=&\frac{1}{2\pi i}\oint_{|\xi_1|=1}a_1(g_3,\xi_1)\log(|1+h\xi_1|^2)\,d\xi_1 \\ =&\frac{3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^4}\,d\xi_1+\frac{6h^2(1+h^2)}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^3}\,d\xi_1\end{aligned}$$

$$
\begin{aligned}
&+\frac{3h(1+h^2)^2+3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)}{\xi_1^2}\,d\xi_1\\
=&h^6-3h^4(1+h^2)+3h^2\left((1+h^2)^2+h^2\right).
\end{aligned}
$$

Applying (5.5), we see that

$$
\begin{aligned}
a_2(g_2,g_5)=&\frac{1}{2\pi i}\oint_{|\xi_1|=1}a_1(g_5,\xi_1)|1+h\xi_1|^4\,d\xi_1\\
=&\frac{(1+h^2)^2h+2h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{1}{\xi_1(\xi_1+h)}\,d\xi_1+\frac{2h^2(1+h^2)}{2\pi i}\oint_{|\xi_1|=1}\frac{1}{\xi_1+h}\,d\xi_1\\
&+\frac{2h^2(1+h^2)}{2\pi i}\oint_{|\xi_1|=1}\frac{1}{\xi_1^2(\xi_1+h)}\,d\xi_1+\frac{h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{\xi_1}{\xi_1+h}\,d\xi_1\\
&+\frac{h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{1}{\xi_1^3(\xi_1+h)}\,d\xi_1=2h^2(1+h^2)-h^4.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathrm{Cov}(X_{g_4},X_{g_5})=&2a_2(g_4,g_5)+a_3(g_4)a_3(g_5)(2d_{\mathbf{R}}-4)\\
=&-2\rho^4+8(1+\rho)^2\rho^3-4\rho^2[2\rho+3(1+\rho)^2]\\
&+4\rho(1+\rho)[3\rho+(1+\rho)^2](2d_{\mathbf{R}}-2),\\
\mathrm{Cov}(X_{g_3},X_{g_5})=&2a_2(g_3,g_5)+a_3(g_3)a_3(g_5)(2d_{\mathbf{R}}-\beta_x-4)\\
=&2\rho^3-6\rho^2(1+\rho)+3\rho[(1+\rho)^2+\rho](2d_{\mathbf{R}}-2),\\
\mathrm{Cov}(X_{g_2},X_{g_5})=&2a_2(g_2,g_5)+a_3(g_2)a_3(g_5)(2d_{\mathbf{R}}-\beta_x-4)\\
=&-2\rho^2+2\rho(1+\rho)(2d_{\mathbf{R}}-2).
\end{aligned}
$$

Subsequently, we deduce the covariance of X_{g_j}, $j=1,2,3,4$. From (5.3), we see that

$$
\begin{aligned}
a_2(g_4,g_4)=&\frac{1}{2\pi i}\oint_{|\xi_1|=1}a_1(g_4,\xi_1)|1+h\xi_1|^8\,d\xi_1\\
=&\frac{4h^4}{2\pi i}\oint_{|\xi_1|=1}\frac{|1+h\xi_1|^8}{\xi_1^5}\,d\xi_1+\frac{12(1+h^2)h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{|1+h\xi_1|^8}{\xi_1^4}\,d\xi_1\\
&+\frac{8h^4+12(1+h^2)^2h^2}{2\pi i}\lim_{r\downarrow 1}\oint_{|\xi_1|=1}\frac{|1+h\xi_1|^8}{\xi_1^3}\,d\xi_1
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{12(1+h^2)h^3 + 4(1+h^2)^3 h}{2\pi i} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^8}{\xi_1^2}\, d\xi_1 \\
=&4h^8 + 48(1+h^2)^2 h^6 + 8h^4[2h^2 + 3(1+h^2)^2]^2 \\
&+ 16h^2(1+h^2)^2[3h^2 + (1+h^2)^2]^2,
\end{aligned}
$$

$$
\begin{aligned}
a_2(g_3, g_4) =& \frac{12(1+h^2)h^3}{2\pi i} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^6}{\xi_1^4}\, d\xi_1 \\
&+ \frac{8h^4 + 12(1+h^2)^2 h^2}{2\pi i} \lim_{r\downarrow 1} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^6}{\xi_1^3}\, d\xi_1 \\
&+ \frac{12(1+h^2)h^3 + 4(1+h^2)^3 h}{2\pi i} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^6}{\xi_1^2}\, d\xi_1 \\
=&12(1+h^2)h^6 + 12h^4(1+h^2)[2h^2 + 3(1+h^2)^2] \\
&+ 12h^2[3(1+h^2)h^2 + (1+h^2)^3][(1+h^2)^2 + h^2], \\
a_2(g_2, g_4) =& \frac{8h^4 + 12(1+h^2)^2 h^2}{2\pi i} \lim_{r\downarrow 1} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^4}{\xi_1^3}\, d\xi_1 \\
&+ \frac{12(1+h^2)h^3 + 4(1+h^2)^3 h}{2\pi i} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^4}{\xi_1^2}\, d\xi_1 \\
=&4h^4[2h^2 + 3(1+h^2)^2] + 8h^2(1+h^2)[3(1+h^2)h^2 + (1+h^2)^3], \\
a_2(g_1, g_4) =& \frac{12(1+h^2)h^3 + 4(1+h^2)^3 h}{2\pi i} \oint_{|\xi_1|=1} \frac{|1+h\xi_1|^2}{\xi_1^2}\, d\xi_1 \\
=&4h^2[3(1+h^2)h^2 + (1+h^2)^3].
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\mathrm{Cov}(X_{g_4}, X_{g_4}) =& 2a_2(g_4, g_4) + a_3^2(g_4)\left(2d_{\mathbf{R}} - 4\right) \\
=& 8\rho^4 + 96\rho^3(1+\rho)^2 + 16\rho^2\left(2\rho + 3(1+\rho)^2\right)^2 \\
&+ 16\rho(1+\rho)^2\left(3\rho + (1+\rho)^2\right)^2\left(2d_{\mathbf{R}} - 2\right), \\
\mathrm{Cov}(X_{g_3}, X_{g_4}) =& 2a_2(g_3, g_4) + a_3(g_3)a_3(g_4)\left(\beta_x + 2d_{\mathbf{R}} - \beta_x - 4\right) \\
=& 24(1+\rho)\rho^3 + 24\rho^2(1+\rho)\left(2\rho + 3(1+\rho)^2\right) + 12\rho\left(3(1+\rho)\rho + (1+\rho)^3\right) \\
&\cdot \left((1+\rho)^2 + \rho\right)\left(2d_{\mathbf{R}} - 2\right),
\end{aligned}
$$

$$\begin{aligned}\mathrm{Cov}(X_{g_2}, X_{g_4}) =& 2a_2(g_2, g_4) + a_3(g_2)a_3(g_4)\left(\beta_x + 2d_{\mathbf{R}} - \beta_x - 4\right)\\ =& 8\rho^2\left(2\rho + 3(1+\rho)^2\right) + 8\rho(1+\rho)\left(3(1+\rho)\rho + (1+\rho)^3\right)\left(2d_{\mathbf{R}} - 2\right),\\ \mathrm{Cov}(X_{g_1}, X_{g_4}) =& 2a_2(g_1, g_4) + a_3(g_1)a_3(g_4)\left(2d_{\mathbf{R}} - 4\right)\\ =& 4\rho\left(3(1+\rho)\rho + (1+\rho)^3\right)\left(2d_{\mathbf{R}} - 2\right).\end{aligned}$$

By calculating and (5.4), one has

$$\begin{aligned}a_2(g_3, g_3) =& \frac{1}{2\pi i}\oint_{|\xi_1|=1} a_1(g_3, \xi_1)|1 + h\xi_1|^6\, d\xi_1\\ =& \frac{3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^6}{\xi_1^4}\, d\xi_1 + \frac{6h^2(1+h^2)}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^6}{\xi_1^3}\, d\xi_1\\ &+ \frac{3h(1+h^2)^2 + 3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^6}{\xi_1^2}\, d\xi_1\\ =& 3h^6 + 18h^4(1+h^2)^2 + 9h^2\left((1+h^2)^2 + h^2\right)^2,\end{aligned}$$

$$\begin{aligned}a_2(g_2, g_3) =& \frac{6h^2(1+h^2)}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^4}{\xi_1^3}\, d\xi_1\\ &+ \frac{3h(1+h^2)^2 + 3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^4}{\xi_1^2}\, d\xi_1\\ =& 6h^4(1+h^2) + 6h^2\left(1+h^2\right)\left((1+h^2)^2 + h^2\right),\\ a_2(g_1, g_3) =& \frac{3h(1+h^2)^2 + 3h^3}{2\pi i}\oint_{|\xi_1|=1}\frac{|1 + h\xi_1|^2}{\xi_1^2}\, d\xi_1 = 3h^2\left((1+h^2)^2 + h^2\right).\end{aligned}$$

Consequently, we acquire

$$\begin{aligned}\mathrm{Cov}(X_{g_3}, X_{g_3}) =& 2a_2(g_3, g_3) + a_3^2(f_3)\left(\beta_x + 2d_{\mathbf{R}} - \beta_x - 4\right)\\ =& 6\rho^3 + 36\rho^2(1+\rho)^2 + 9\rho\left((1+\rho)^2 + \rho\right)^2\cdot\left(2d_{\mathbf{R}} - 2\right),\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(X_{g_2}, X_{g_3}) =& 2a_2(g_2, g_3) + a_3(g_2)a_3(g_3)\left(\beta_x + 2d_{\mathbf{R}} - \beta_x - 4\right)\\ =& 12\rho^2(1+\rho) + 6\rho\left(1+\rho\right)\left((1+\rho)^2 + \rho\right)\cdot\left(2d_{\mathbf{R}} - 2\right),\\ \mathrm{Cov}(X_{g_1}, X_{g_3}) =& 2a_2(g_1, g_3) + a_3(g_1)a_3(g_3)\left(2d_{\mathbf{R}} - 4\right) = 3\rho\left((1+\rho)^2 + \rho\right)\left(2d_{\mathbf{R}} - 2\right).\end{aligned}$$

Finally, from Wang and Yao (2013) it can be proved that

$$\mathrm{Cov}(X_{g_1}, X_{g_1}) = \rho\,(2d_{\mathbf{R}} - 2)\,, \quad \mathrm{Cov}(X_{g_1}, X_{g_2}) = 2\rho(1+\rho)\,(2d_{\mathbf{R}} - 2)\,,$$

$$\mathrm{Cov}(X_{g_2}, X_{g_2}) = 4\rho^2 + 4\rho\,(1+\rho)^2\,(2d_{\mathbf{R}} - 2)\,.$$

Thus the proof of Example 3.2 is complete.

6. Some Auxiliary Lemmas

Lemma 6.1 (Lemma B.26 in Bai and Silverstein (2010))**.** Let $\mathbf{A} = (a_{jk})$ be an $n\times n$ nonrandom matrix and $\mathbf{X} = (x_1, \cdots, x_n)'$ be a random vector of independent entries. Assume that $\mathrm{E}x_j = 0$, $\mathrm{E}|x_j|^2 = 1$ and $\mathrm{E}|x_j|^l \le \nu_l$. Then for $p \ge 1$,

$$\mathrm{E}\,|\mathbf{X}^*\mathbf{A}\mathbf{X} - \mathrm{tr}\mathbf{A}|^p \le C_p \left[(\nu_4 \mathrm{tr}\mathbf{A}\mathbf{A}^*)^{p/2} + \nu_{2p}\mathrm{tr}\,(\mathbf{A}\mathbf{A}^*)^{p/2}\right]$$

where C_p is a constant depending on p only.

Lemma 6.2 (Lemma 4 in Karoui (2009))**.** Let us focus on $\mathbf{S} = \frac{1}{n}\mathbf{G}\mathbf{X}\mathbf{X}^T\mathbf{G}^T$. When $p/n \to c$, we have $\max_{j=1,\cdots,n}\left|\sqrt{\mathbf{e}_j^T\mathbf{S}\mathbf{e}_j} - 1\right| \xrightarrow{a.s.} 0$.

Lemma 6.3. For rectangular matrix $\mathbf{A}$, complex vectors $\mathbf{a}$ and $\mathbf{b}$, we have $|\mathbf{a}^*\mathbf{A}\mathbf{b}| \le \|\mathbf{A}\|\sqrt{\mathbf{a}^*\mathbf{a}}\sqrt{\mathbf{b}^*\mathbf{b}}$.

Lemma 6.4. For rectangular matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, $|\mathrm{tr}\,(\mathbf{ABCD})| \le \|\mathbf{A}\|\|\mathbf{C}\|\,[\mathrm{tr}\,(\mathbf{B}\mathbf{B}^*)]^{1/2}\,[\mathrm{tr}\,(\mathbf{D}\mathbf{D}^*)]^{1/2}$.

Bibliography

Bai, Z. D. & Silverstein, J. W. (1998). No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.* **26(1)**, 316-345.

Bai, Z. D. & Silverstein, J. W. (2004). CLT for linear spectral statistics of large dimensional sample covariance matrices. *Ann. Probab.* **32(1)**, 553-605.

Bai, Z. D. & Silverstein, J. W. (2010). *Spectral Analysis of Large Dimensional Ransom Matrices*. Science Press: Beijing.

Karoui, N. E. (2009). Concentration of measure and spectra of random matrices: with applications to correlation matrices, elliptical distributions and beyond. *Ann. Appl. Probab.*, **19**(6), 2362-2405.

Wang, Q. W. and Yao, J. F. (2013). On the sphericity test with large-dimensional observations. *Electronic J. Statistics.* **7**, 2164-2192.

Zheng, S. R(2012). Central limit theorems for linear spectral statistics of large dimensional F-matrices. *Annales De L Institut Henri Poincar Probabilits Et Statistiques.* **48(2)**, 444-476.

Zheng, S. R, Bai, Z. D. and Yao, J. F. (2015). Substitution principle for clt of linear spectral statistics of high-dimensional sample covariance matrices with applications to hypothesis testing. *Ann. Statist.*. **43(2)**, 546-591.