A ROBUST CALIBRATION-ASSISTED METHOD FOR LINEAR MIXED EFFECTS MODEL UNDER CLUSTER-SPECIFIC NONIGNORABLE MISSINGNESS

Yongchan Kwon[1], Jae Kwang Kim[2], Myunghee Cho Paik[1]

and Hongsoo Kim[1]

[1]*Seoul National University and* [2]*Iowa State University*

Supplementary Material

In this supplementary material, we prove the lemma 1, equations (3.7) and (3.8), and Theorem 1.

S1 Proof of Lemma 1

The key idea is to use the calibrated condition (3.3) and cluster-specific nonignorable assumptions (2.2). We first state the lemma again.

Lemma 1.

$$E\left\{\sum_{j=1}^{n_i} x_{ij}(y_{ij}-x_{ij}\beta)\right\} = E\left\{\sum_{j=1}^{n_i} \frac{\delta_{ij}}{\hat{\pi}_{ij}}(x_{ij}-\tilde{x}_i)(y_{ij}-x_{ij}\beta)\right\}, \qquad \text{(S1.1)}$$

where $\tilde{x}_i = {n_i}^{-1} \sum_{j=1}^{n_i} \left\{ \delta_{ij}/\hat{\pi}_{ij}(\gamma) - 1 \right\} x_{ij},$ *and*

$$E\left[\left\{\sum_{j=1}^{n_i}(y_{ij} - x_{ij}\beta)\right\}^2\right] = E\left[\left\{\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}(y_{ij} - x_{ij}\beta)\right\}^2 - C_i(\eta)\right], \tag{S1.2}$$

where $C_i(\eta) = \sum_{j=1}^{n_i} \left\{ \delta_{ij}/\hat{\pi}_{ij}^2(\gamma) - 1 \right\} \sigma^2.$

Proof. For (S1.1), it is enough to show

$$E\left[\sum_{j=1}^{n_i}\left\{\left(\frac{\delta_{ij}}{\hat{\pi}_{ij}} - 1\right)x_{ij} - \frac{\delta_{ij}}{\hat{\pi}_{ij}}\tilde{x}_i\right\}(y_{ij} - x_{ij}\beta)\right] = 0$$

$$\begin{aligned}
& E\left[\sum_{j=1}^{n_i}\left\{\left(\frac{\delta_{ij}}{\hat{\pi}_{ij}} - 1\right)x_{ij} - \frac{\delta_{ij}}{\hat{\pi}_{ij}}\tilde{x}_i\right\}(y_{ij} - x_{ij}\beta)\right] \\
&= E\left[\sum_{j=1}^{n_i}\left\{\left(\frac{\delta_{ij}}{\hat{\pi}_{ij}} - 1\right)x_{ij} - \frac{\delta_{ij}}{\hat{\pi}_{ij}}\tilde{x}_i\right\}(a_i + e_{ij})\right] \\
&= E\left[\sum_{j=1}^{n_i}\left\{\left(\frac{\delta_{ij}}{\hat{\pi}_{ij}} - 1\right)x_{ij} - \frac{\delta_{ij}}{\hat{\pi}_{ij}}\tilde{x}_i\right\}a_i\right] \\
&= E\left\{\sum_{j=1}^{n_i}\left(\frac{\delta_{ij}}{\hat{\pi}_{ij}} - 1\right)x_{ij}a_i\right\} - E\left(n_i\tilde{x}_i a_i\right) = 0,
\end{aligned}$$

where the last equality holds by definition of $\tilde{x}_i$ and the second equality holds because CSNI implies

$$E(e_{ij} \mid x_{ij}, a_i, \delta_{ij}) = E(e_{ij} \mid x_{ij}, a_i) = 0. \tag{S1.3}$$

For the equation (S1.2),

$$E\left[\left\{\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}(y_{ij}-x_{ij}\beta)\right\}^2-C_i\right]=E\left[\left\{\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}(a_i+e_{ij})\right\}^2-C_i\right]$$
$$=E\left[\left\{n_ia_i+\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}e_{ij}\right\}^2-\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)^2}-1\right\}\sigma^2\right]$$
$$=E\left[(n_ia_i)^2+\left\{\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}e_{ij}\right\}^2-\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}^2(\gamma)}-1\right\}\sigma^2\right]$$
$$=E\left[(n_ia_i)^2+n_i\sigma^2+\sum_{j=1}^{n_i}\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)^2}(e_{ij}^2-\sigma^2)\right]=n_i^2D+n_i\sigma^2,$$

and

$$E\left[\left\{\sum_{j=1}^{n_i}(y_{ij}-x_{ij}\beta)\right\}^2\right]=E\left[\left\{\sum_{j=1}^{n_i}(a_i+e_{ij})\right\}^2\right]=E\left\{(n_ia_i+\sum_{j=1}^{n_i}e_{ij})^2\right\},$$
$$=E\left\{(n_ia_i)^2+\left(\sum_{j=1}^{n_i}e_{ij}\right)^2\right\}=n_i^2D+n_i\sigma^2.$$

Thus, we have (S1.2). □

Additionally, for the equation (3.7),

$$E\left[\bar{x}_i\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(y_{ij}-x_{ij}\beta)\right]$$
$$=E\left[\bar{x}_i\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i+e_{ij})\right]$$
$$=E\left[\bar{x}_i\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}a_i\right]=0,$$

where the second equality follows from (S1.3). Similarly, for the equation

(3.8) in the paper, it is enough to show that

$$E\left[\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(y_{ij}-x_{ij}\beta)^2\right]=0. \qquad \text{(S1.4)}$$

Note that

$$\begin{aligned}
&E\left[\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(y_{ij}-x_{ij}\beta)^2\right]\\
&=E\left[\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i+e_{ij})^2\right]\\
&=\sum_{j=1}^{n_i}E\left[E\left\{\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i+e_{ij})^2 \mid x_{ij},a_i,\delta_{ij}\right\}\right]\\
&=\sum_{j=1}^{n_i}E\left[E\left\{\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i^2+2a_ie_{ij}+e_{ij}^2) \mid x_{ij},a_i,\delta_{ij}\right\}\right]\\
&=\sum_{j=1}^{n_i}E\left[E\left\{\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i^2+e_{ij}^2) \mid x_{ij},a_i,\delta_{ij}\right\}\right]\\
&=E\left[\sum_{j=1}^{n_i}\left\{\frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}-1\right\}(a_i^2+\sigma^2)\right]=0,
\end{aligned}$$

where the fourth equality follows from (S1.3) and the last equality follows from (3.3). Therefore, (S1.4) is established.

S2 Proof of Theorem 1

We first check the conditions for the asymptotic normality of $U_1(\eta) = \sum_{i=1}^{K} U_{1i}(\eta)$ for $p = 1$. Since

$$\begin{aligned} U_{1i}(\eta) &= \sum_{j=1}^{n_i} \left\{ \frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}(x_{ij} - \tilde{x}_i)(y_{ij} - x_{ij}\beta) - \bar{x}_i \tau_i \frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}(y_{ij} - x_{ij}\beta) \right\} \\ &= \sum_{j=1}^{n_i} c_{ij}(\gamma)\left(a_i + e_{ij}\right), \end{aligned}$$

where

$$c_{ij}(\gamma) = \frac{\delta_{ij}}{\hat{\pi}_{ij}(\gamma)}\left(x_{ij} - \tilde{x}_i - \bar{x}_i \tau_i\right).$$

Define $B_K^2 = \sum_{i=1}^{K} V\{U_{1i}(\eta)\}$. Since $\sum_{j=1}^{n_i} \delta_{ij} > 0$, $\hat{\pi}_{ij}(\gamma)$ is bounded away from zero. Under conditions of bounded moments, we have

$$\sum_{i=1}^{K} n_i^2 \cdot C_1 \leq B_K^2 \leq \sum_{i=1}^{K} n_i^2 \cdot C_2$$

for some constants $C_2 > C_1 > 0$.

Now, to achieve the asymptotic normality of $B_K^{-1} U_1(\eta)$, we can use Liapounov condition:

$$\lim_{K \to \infty} \frac{\sum_{i=1}^{K} E\{|U_{1i}(\eta)|^{2+\delta}\}}{B_K^{2+\delta}} = 0. \quad \text{(S2.5)}$$

Now, there exists $C_3 = O(1)$ such that

$$E\{|U_{1i}(\eta)|^{2+\delta}\} \leq \sum_{i=1}^{K} n_i^{2+\delta} \cdot C_3,$$

and we have

$$\frac{\sum_{i=1}^{K} E\{|U_{1i}(\eta)|^{2+\delta}\}}{B_K^{2+\delta}} \le \frac{\sum_{i=1}^{K} n_i^{2+\delta}}{\left(\sum_{i=1}^{K} n_i^2\right)^{(2+\delta)/2}} \cdot C_4,$$

for some $C_4 = O(1)$. Thus, (3.13) implies (S2.5) and the asymptotic normality of $U_1(\eta)$ can be established. Asymptotic normality of $U_1(\eta)$ when $p > 1$ and that of $\Psi(\eta) = \{U_1(\eta), U_2(\eta), U_3(\eta), \psi(\gamma)\}$ can be established similarly, using Cramer-Wold device.

To establish the asymptotic normality of the solution $\hat{\eta}_K$ to $\Psi_K(\eta) = 0$, where $\Psi_K(\eta) = \Psi(\eta)$, we apply the first-order Taylor expansion to get

$$0 = B_{2K}^{-1}\Psi_K(\eta^*) + \mathbf{\Gamma}(\tilde{\eta}_K)(\hat{\eta}_K - \eta^*), \tag{S2.6}$$

where $B_{2K}^2 = \sum_{i=1}^{K} V\{\Psi_i(\eta)\}$, $\mathbf{\Gamma}(\cdot) = \partial(B_{2K}^{-1}\Psi_K)(\cdot)/\partial\eta^T$, and $\tilde{\eta}_K$ lies on the line segment between $\hat{\eta}_K$ and η^*. Now, define

$$J_K^2 = \frac{\sum_{i=1}^{K} n_i^2}{(\sum_{i=1}^{K} n_i)^2}.$$

Note that $J_K^2 = O(K^{-1})$ by condition (3.12). Then,

$$J_K^2\mathbf{\Gamma}(\tilde{\eta}_K) = J_K^2\mathbf{\Gamma}(\eta^*) + o_p(1).$$

Since we can obtain $J_K^2\mathbf{\Gamma}(\eta^*)$ converges in probability to its mean

$$M_1(\eta^*) = \lim_{K\to\infty} E\left\{J_K^2\mathbf{\Gamma}(\eta^*)\right\},$$

and $B_{2K}^{-1}\Psi_K(\eta^*) = O_p(J_K^{-1})$ by central limit theorem. Therefore, if $M_1(\eta^*)$

is nonsigular,

$$J_K^{-1}(\hat{\eta}_K - \eta^*) = -\{M_1(\eta^*)\}^{-1} B_{2K}^{-1} \Psi_K(\eta^*) + o_p(1)$$

which establishes the asymptotic normality of $J_K^{-1}(\hat{\eta}_K - \eta^*)$. Since $K^{1/2}(\hat{\eta}_K - \eta^*) = (KJ_K^2)^{1/2} J_K^{-1}(\hat{\eta}_K - \eta^*) = CJ_K^{-1}(\hat{\eta}_K - \eta^*) + o_p(1)$, where $C^2 = \lim_{K\to\infty}(KJ_K^2)$, the asymptotic normality of $K^{1/2}(\hat{\eta}_K - \eta^*)$ also follows. See Chapter 6.2.1 of Bickel and Doksum (1977) for more details about regularity conditions.

References

Bickel, P.J. and Doksum, K.A. (1977). *Mathematical statistics : basic ideas and selected topics, Vol 1.* Prentice Hall.