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FRACTIONAL FACTORIAL SPLIT-PLOT DESIGNS WITH MINIMUM ABERRATION AND MAXIMUM ESTIMATION CAPACITY

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Abstract: Considering general prime or prime powered factorials, we give a finite projective geometric formulation for regular fractional factorial split-plot designs. This provides a unified framework for such designs and facilitates their systematic study under the criteria of minimum aberration and minimum secondary aberration; the latter criterion is considered to achieve finer discrimination. We investigate the role of complementary subsets in this context and observe that, unlike in classical fractional factorials, two such complementary subsets have to be handled simultaneously. Criteria based on estimation capacity are also studied to provide further statistical justification for our results. Finally, applications of the results to specific cases are summarized as tables.

Key words and phrases: Complementary subsets, finite projective geometry, minimum secondary aberration, two-phase randomization, sub plot, whole plot, wordlength pattern.

1. Introduction

Fractional factorial (FF) designs with minimum aberration (MA) have been the subject of much interest over the last two decades and have been used extensively in industrial and agricultural experiments. We refer to Chen, Sun and Wu (1993) for an excellent review, and to Suen, Chen and Wu (1997) and Cheng and Mukerjee (1998) for more recent results and further references.

While in a classical FF setting the experimental runs are completely randomized, in recent years attention has been focused on situations where it is impractical to perform the runs in such a completely random order. This happens, in particular, when the levels of some of the factors are very difficult or expensive to change. In situations of this kind, a fractional factorial split-plot (FFSP) design, which involves a two-phase randomization, can be conveniently used to reduce costs, and hence represents a practical design option. We refer to Box and Jones (1992) for an illuminating discussion of split-plot designs in industrial experiments. Further examples, one on a thin-film coating experiment and another on an experiment for improving the efficiency of a ball mill, were given in Huang, Chen and Voelkel (1998) and Bingham and Sitter (1999a), respectively.

As discussed later in more detail, the two-phase randomization in an FFSP design induces several novel features which distinguish it from a classical FF design. The most notable of these are:

(a) not all factors have the same status,

(b) inference is possible at two distinct levels of accuracy.

Because of the above, the study of FFSP designs with MA opens up significantly new issues. In particular, as a result of (a), reduction of the class of competing designs via isomorphism is possible to a much lesser extent in FFSP designs than in classical FF designs and this complicates the study of the former. Huang, Chen and Voelkel (1998) and Bingham and Sitter (1999a) investigated two-level FFSP designs with MA via primarily algorithmic approaches. Continuing with the two-level case, Bingham and Sitter (1999b) reported several theoretical results and highlighted some of the differences between classical FF and FFSP designs. Bingham and Sitter (2001) discussed design issues with reference to a real industrial FFSP experiment involving two-level factors.

In this paper, we consider general prime or prime-powered factorials and attempt to develop a unified framework for the study of FFSP designs. This is done through a finite projective geometric formulation which is given in Section 2 after presenting the necessary definitions and preliminaries. This formulation, used in the rest of the paper, is seen to be different from that arising in a classical FF setting. In Section 3, the criterion of minimum aberration is considered with reference to FFSP designs. It is noted that, because of (a) above, not infrequently there can be several nonisomorphic FFSP designs all having MA; cf. Bingham and Sitter (1999a, 2001). Therefore, taking cognizance of (b), we explore a criterion of minimum secondary aberration (MSA) which significantly narrows the class of completing nonisomorphic MA designs and very often yields a unique optimal design.

In Section 4, we investigate the role of complementary subsets for finding optimal FFSP designs under the twin criteria of MA and MSA. This provides a powerful tool for handling the practically important nearly saturated (i.e., highly fractionated) cases and, when specialized to the two-level case, supplements the findings in Bingham and Sitter (1999b). While this approach has been used by several authors (Chen and Hedayat (1996), Tang and Wu (1996), Suen, Chen and Wu (1997), Mukerjee and Wu (2001)) for obtaining MA designs in the classical setup, a novel feature that we encounter with FFSP designs is that two, rather than one, complementary subsets have to be handled simultaneously.

Notwithstanding the popularity of the criterion of MA among both theoreticians and practitioners, it is essentially a combinatorial criterion without a direct statistical meaning. Therefore, in Section 5, we study the issue of model robustness, via consideration of estimation capacity. This is done by suitably adapting the ideas in Cheng, Steinberg and Sun (1999), who considered classical FF designs, taking due recognition of (a) and (b) which are special to the FFSP setting. Quite pleasantly, it is found that the twin criteria of MA and MSA are excellent surrogates for the statistically more meaningful criteria based on estimation capacity. In Section 6, we apply the results to specific situations and summarize the findings as tables. For ease in presentation, the proofs are relegated to the appendix.

2. Description and a Projective Geometric Formulation

2.1. Description

Consider the setup of an s^n factorial experiment involving factors $Z_1, \ldots, Z_n$, each at s levels where $s(\geq 2)$ is a prime or prime power. A typical treatment combination will be denoted by an n-vector x over $GF(s)$, the finite field with s elements. A typical pencil belonging to a factorial effect is a nonnull n-vector b over $GF(s)$. For $\lambda(\neq 0) \in GF(s)$, b and λb represent the same pencil. A pencil b represents an i-factor interaction if it has exactly i nonzero elements. A 1-factor interaction is a main effect. See Bose (1947) or Dey and Mukerjee (1999, Ch.8) for more details on the representation of factorial effects via pencils.

Suppose among the n factors there are n_1, $1 \leq n_1 < n$, whose levels are very difficult or expensive to change. Without loss of generality, let these factors be $Z_1, \ldots, Z_{n_1}$. The levels of the remaining $n_2(= n - n_1)$ factors are not hard to change. In designing an FF plan in this situation, suppose the available resources allow experimentation with a total of $s^{n_1+n_2-p_1-p_2}$ treatment combinations which involve $s^{n_1-p_1}$ distinct factor level settings of $Z_1, \ldots, Z_{n_1}$; here $0 \leq p_1 < n_1$, $0 \leq p_2 < n_2$, and $p_1 + p_2 \geq 1$. Note that a restriction on the number of distinct settings of $Z_1, \ldots, Z_{n_1}$, in addition to that on the total number of treatment combinations to be included in the experiment, is natural since the levels of $Z_1, \ldots, Z_{n_1}$ are difficult or expensive to change. Because of the same reason, in order to save cost or time, instead of using a complete randomization, it is natural to adopt a two-phase randomization via the use of a split-plot structure, as follows.

Select $s^{n_1-p_1}$ distinct factor level settings of $Z_1, \ldots, Z_{n_1}$. Randomly choose one of these and then run $s^{n_2-p_2}$ distinct combinations of $Z_{n_1+1}, \ldots, Z_n$ in a random order while holding $Z_1, \ldots, Z_{n_1}$ fixed. Repeat this for all the $s^{n_1-p_1}$ selected distinct settings of $Z_1, \ldots, Z_{n_1}$. The sets of $s^{n_2-p_2}$ combinations of $Z_{n_1+1}, \ldots, Z_n$ to be combined with the different settings of $Z_1, \ldots, Z_{n_1}$ are not required to be the same. This defines an $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP design. Under the kind of randomization just described, each of the $s^{n_1-p_1}$ selected factor level settings of $Z_1, \ldots, Z_{n_1}$ defines a whole plot (WP) consisting of $s^{n_2-p_2}$ individual runs, called subplots (SP), obtained through variation of $Z_{n_1+1}, \ldots, Z_n$. As such, $Z_1, \ldots, Z_{n_1}$ are called WP factors and $Z_{n_1+1}, \ldots, Z_n$ are called SP factors.

We consider regular $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP designs. These are given by a defining equation (see (2.1) below) and have an aliasing structure that facilitates analysis and interpretation. Such a design consists of treatment combinations x satisfying

$$Hx = 0, \tag{2.1}$$

where

$$H = \begin{bmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{bmatrix} \tag{2.2}$$

is a matrix over $GF(s)$, with H_{11}, H_{21} and H_{22} of orders $p_1 \times n_1, p_2 \times n_1$ and $p_2 \times n_2$ respectively, such that both H_{11} and H_{22} have full row rank. This rank condition on H_{11} and H_{22} ensures that the design involves, as required, $s^{n_1-p_1}$ distinct factor level settings of the WP factors, each of which appears in combination with $s^{n_2-p_2}$ distinct settings of the SP factors.

Any pencil b belongs to the defining equation of the FFSP design given by (2.1) if

$$b \in \mathcal{M}(H'), \tag{2.3}$$

where $\mathcal{M}(\cdot)$ denotes the column space of a matrix. Two distinct pencils $b^{(1)}$ and $b^{(2)}$, neither of which appears in the defining equation, are aliased with each other if for some $\lambda(\neq 0) \in GF(s)$,

$$b^{(1)} - \lambda b^{(2)} \in \mathcal{M}(H'). \tag{2.4}$$

The minimum number of nonzero elements in a pencil appearing in the defining equation is called the *resolution* of the design. As in the existing literature, hereafter, even without explicit mentioning, we consider only those regular FFSP designs which

(i) have resolution at least three, and
(ii) keep every pencil representing an SP factor main effect unaliased with pencils involving only the WP factors.

The requirement (i) ensures that no main effect pencil appears in the defining equation and that no two distinct main effect pencils are aliased with each other. The requirement (ii) ensures that no SP factor main effect is estimated at the WP level of the design. This is important since, because of two-phase randomization, FFSP designs have two sources of error, one at the WP and the other at the SP levels, typically the former is larger than the latter. The point just noted has further implications to be discussed later.

2.2. A projective geometric formulation

It is well-known that a systematic study of classical FF designs is greatly facilitated by consideration of a geometric approach where the set of factors is identified with a set of points in a finite projective geometry. A considerable modification of this theory is needed in the case of FFSP designs. In particular, as Theorem 1 below reveals, one needs to consider two sets of points, with distinctive properties, of a finite projective geometry: one for WP factors and the other for SP factors.

We first introduce some preliminaries. Let

$$t_1 = n_1 - p_1, \qquad t_2 = n_2 - p_2, \qquad t = t_1 + t_2, \tag{2.5}$$

and P denote the set of distinct points of the finite projective geometry $PG(t-1, s)$. The points in P are nonnull t-vectors over $GF(s)$ with mutually proportional vectors representing the same point. Then $\#P = L_t$, where $\#$ denotes cardinality, and

$$L_u = (s^u - 1)/(s-1), \quad u = 0, 1, 2, \ldots. \tag{2.6}$$

For $u = 1, 2, \ldots$, a $(u-1)$-flat of P is a subset of P, with cardinality L_u, which is closed, up to proportionality, under the formation of nonnull linear combinations. Clearly, such a flat is generated by u linearly independent points in P. Let e_1, $\ldots$, e_t be the t-dimensional unit vectors over $GF(s)$, and P_1 be a (t_1-1)-flat of P generated by the points e_i, $1 \le i \le t_1$. Define P_2 as the complement of P_1 in P. For any subset C of P, let $V(C)$ be a $t \times c$ matrix with columns given by the points in C, where $c = \#C$.

Definition 1. An ordered pair of subsets (C_1, C_2) of P is called an eligible (n_1, n_2)-pair if (a) $\#C_i = n_i$ $(i = 1, 2)$, (b) $C_i \subset P_i$ $(i = 1, 2)$, (c) $\text{rank}\{V(C_1)\} = t_1$, and (d) $\text{rank}\{V(C)\} = t$, where $C = C_1 \cup C_2$.

We are now in a position to present the main result of this section. The proof of the theorem is given in the appendix.

Theorem 1. *The existence of an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design is equivalent to the existence of an eligible (n_1, n_2)-pair of subsets (C_1, C_2) of P such that, with $C = C_1 \cup C_2$, and*

$$V(C) = [V(C_1) \vdots V(C_2)], \tag{2.7}$$

(i) *the treatment combinations included in the design are given by the vectors in $\mathcal{M}[V(C)']$,*
(ii) *a pencil b appears in the defining equation of the design if and only if $V(C)b = 0$,*
(iii) *two distinct pencils, $b^{(1)}$ and $b^{(2)}$, neither of which appears in the defining equation of the design, are aliased with each other if and only if $V(C)b^{(1)}$ and $V(C)b^{(2)}$ are proportional to the same point in P.*

In view of Theorem 1, while studying $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP designs, it will be enough to consider eligible (n_1,n_2)-pairs of subsets of P. The regular FFSP design corresponding to any such eligible pair of subsets (C_1,C_2) will be denoted by $d(C_1, C_2)$. The WP and SP factors correspond to the points in C_1 and C_2 respectively; vide (2.7) and Theorem 1(i). Considering the cardinalities of C_1, C_2, P_1 and P_2, it is evident that such a design exists if and only if

$$n_1 \leq L_{t_1} \quad \text{and} \quad n_2 \leq L_t - L_{t_1}, \tag{2.8}$$

where t_1, t, L_{t_1} and L_t are as in (2.5) and (2.6). If equality holds in both places in (2.8) then all designs are isomorphic. Therefore, hereafter, we assume that at least one of these inequalities is strict, i.e., $f > 0$, where

$$f = f_1 + f_2, \qquad f_1 = L_{t_1} - n_1, \qquad f_2 = L_t - L_{t_1} - n_2. \tag{2.9}$$

It is well known that there are $L_{p_1+p_2}$ distinct pencils appearing in the defining equation of an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design and that the remaining pencils are partitioned into L_t alias sets, each containing $s^{p_1+p_2}$ distinct pencils. Since in an FFSP design, inference at the WP level does not have the same status as that at the SP level, we distinguish between two types of alias sets: a *WP alias set* is one which contains some pencil involving only the WP factors; an *SP alias set* is one which contains no such pencil.

Consider now an FFSP design $d(C_1, C_2)$. By Theorem 1(iii), there is a one-one correspondence between the L_t alias sets of the design and the L_t distinct points in P. From Definition 1, the points in C_1 span every point in P_1. Since the points in C_1 correspond to the WP factors, by (2.7) and Theorem 1(iii), given any point in P_1 it follows that there exists a pencil, involving only the WP factors, which belongs to the corresponding alias set. Thus the points in P_1 correspond to WP alias sets. Similarly, noting that the points in C_1 do not span any point in P_2, it is clear that the points in P_2 correspond to SP alias sets. As $C_2 \subset P_2$, no SP factor main effect pencil belongs to a WP alias set.

3. Minimum and Minimum Secondary Aberration

3.1. The criteria

Consider an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design $d = d(C_1, C_2)$. For $i = 1,\ldots,n$, let $A_i(d)$ be the number of distinct i-factor interaction pencils appearing in the defining equation of d. Since d has resolution three or more, we have $A_1(d) = A_2(d) = 0$. The sequence $W(d) = \{A_3(d), \ldots, A_n(d)\}$ is called the wordlength pattern of d. Given two such designs d_1 and d_2, we say that d_1 has less aberration than d_2 if there exists a positive integer i_0 such that $A_i(d_1) = A_i(d_2)$ for $i < i_0$ and $A_{i_0}(d_1) < A_{i_0}(d_2)$. A design has *minimum aberration* (MA) if no design has less aberration.

Given n_1,n_2,p_1 and p_2, there can, however, be several nonisomorphic FFSP designs all having MA. These designs have identical wordlength pattern. In fact, this problem is much more pronounced with FFSP designs than with classical FF designs, since in the former the roles of a WP factor and an SP factor are not interchangeable. Taking cognizance of the distinction between the WP and SP level errors in an FFSP design, we now explore a criterion for discriminating among rival nonisomorphic MA designs.

It is well known that in a full factorial split-plot design, all pencils involving only the WP factors are tested against the WP level error while all pencils involving at least one SP factor are tested against the SP level error. Pencils of these two kinds will be called WP-type and SP-type pencils respectively. Since the WP level error is typically larger than the SP level error, a good FFSP design should try to avoid assignment of SP-type pencils, especially those representing lower order factorial effects, to WP alias sets. Considering a regular FFSP design d, let $B_i(d)$ be the number of distinct i factor interaction pencils of SP-type that appear in the WP alias sets of d. As stipulated earlier, no pencil representing the main effect of an SP factor appears in a WP alias set. Thus, we always have $B_1(d) = 0$. Bingham and Sitter (2001) essentially suggested a smaller value of $B_2(d)$ as a criterion for discriminating among nonisomorphic MA designs. In the same spirit, we consider a secondary wordlength pattern $W^*(d) = \{B_2(d), \ldots, B_n(d)\}$ and propose sequential minimization of $B_2(d)$, $B_3(d)$, etc.

More formally, given two nonisomorphic MA $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP designs d_1 and d_2, we say that d_1 has less secondary aberration than d_2 if there exists a positive integer i_0 such that $B_i(d_1) = B_i(d_2)$ for $i < i_0$ and $B_{i_0}(d_1) < B_{i_0}(d_2)$. An MA design has *minimum secondary aberration* (MSA) if no MA design has less secndary aberration.

Remark 1. A classical FF in a block design also involves two wordlength patterns, one arising from the defining equation and the other arising from confounding with blocks. One might wonder about a possible similarity between the latter and our secondary wordlength pattern. Such similarity is, however, superficial and our approach differs from those for block designs both conceptually and operationally. Conceptually, this is so because interactions between block versus treatment factors are normally ignored in block designs, whereas our secondary wordlength pattern relates to SP-type pencils including those which involve both WP and SP factors. Operationally, the approach is different because in block designs the two wordlength patterns are interpenetrated (Sitter, Chen and Feder (1997)), or linearly combined (Chen and Cheng (1999)), or treated separately with admissibility considerations (Sun, Wu and Chen (1997), Mukerjee and Wu (1999)), whereas we consider the $A_i(d)$'s and $B_i(d)$'s sequentially, the former being given precedence over the latter.

Remark 2. While the twin criteria of MA and MSA minimize the $A_i(d)$'s and the $B_i(d)$'s sequentially in the order $\{A_3(d),\ldots, A_n(d), B_2(d),\ldots, B_n(d)\}$, other criteria can be considered. For example, if the relegation of an SP-type pencil representing a lower order interaction to a WP alias set is treated as very serious, then a rearrangement with some $B_i(d)$'s preceding some $A_j(d)$'s may be considered. We do not discuss these possibilities here since then, unlike in our approach, the resulting optimal designs are not guaranteed to have MA. Given the widespread popularity of the MA criterion, this may be undesirable. It is, however, reassuring to note that the tools developed here are capable of handling such modified criteria. As an illustration, under sequential minimization in the order $\{A_3(d), B_2(d), A_4(d), B_3(d), \ldots\}$, the formulae in Theorem 3 of Section 4 are readily applicable. One can also check that the optimal designs reported in the examples of Section 4 continue to remain so under such a modified criterion.

Remark 3. Returning to the present twin criteria of MA and MSA, one might have reservations about the use of the $B_i(d)$ for higher values of i in discriminating among rival nonisomorphic MA designs, on the ground that typically SP-type pencils, representing higher order interactions, will be aliased with pencils representing lower order factorial effects. Even then Theorem 3 below, which pertains to the $B_i(d)$ for smaller values of i, should be useful. It is satisfying to note in this context that for none of the unique MSA designs reported here, either in the examples of Section 4 or in the tables of Section 6, does one have to go beyond considering $B_2(d)$ to achieve uniqueness. Thus the criterion of MSA can often become equivalent to what was proposed in Bingham and Sitter (2001). On the other hand, even for two-level factorials, the tools developed here yield optimal designs in many new cases.

3.2. A result

We introduce some more notation. Let Q be a nonempty subset of the finite projective geometry P consisting of q distinct points of the latter. For $i \geq 1$, let $\Omega(i, q)$ be the set of q-vectors over $GF(s)$ having exactly i nonzero elements. Also, let P^* be any flat of P such that P^* and Q are disjoint. For $i \geq 1$, define

$$A_i(Q) = (s-1)^{-1}\#\{\beta : \beta \in \Omega(i, q),\ V(Q)\ \beta = 0\}, \tag{3.1}$$

$$M_i(P^*, Q) = (s-1)^{-1}\#\{\beta : \beta \in \Omega(i, q),\ V(Q)\beta \text{ is nonnull but proportional to some point in } P^*\}. \tag{3.2}$$

Clearly,

$$A_1(Q) = A_2(Q) = M_1(P^*, Q) = 0, \tag{3.3}$$

$$A_i(Q) = M_i(P^*, Q) = 0 \text{ for } i > q. \tag{3.4}$$

Consider now an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design $d = d(C_1, C_2)$. Let $C = C_1 \cup C_2$. Since pencils with proportional elements are identical, by Theorem 1(ii) and (3.1),

$$A_i(d) = A_i(C), \quad 3 \le i \le n. \tag{3.5}$$

The following result, proved in the appendix, will play a crucial role in the study of MSA.

Theorem 2. *For* $2 \le i \le n$,

$$B_i(d) = A_i(C_1) - A_i(C) + \sum_{r=2}^{i} \binom{n_1}{i-r} (s-1)^{i-r} \{A_r(C_2) + M_r(P_1, C_2)\}, \tag{3.6}$$

where $\binom{n_1}{i-r}$ *is interpreted as zero if* $i - r > n_1$.

The terms on the right side of (3.6) have meaningful interpretation. As noted in (3.5), $A_i(C) = A_i(d)$. Similarly, $A_i(C_1)$ $(A_r(C_2))$ is the number of distinct i-factor (r-factor) interaction pencils involving only the WP (SP) factors and appearing in the defining equation of d. Also, by Theorem 1(iii) and (3.2), $M_r(P_1, C_2)$ is the number of distinct r-factor interaction pencils, involving only the SP factors, that appear in the WP alias sets of d.

4. Complementary Subsets

We now show how the study of optimal designs under the twin criteria of MA and MSA is facilitated through consideration of complementary subsets. Lemma 1 below is helpful for this purpose. We refer to Mukerjee and Wu (1999, 2001) for proofs of parts (i)−(iv) of this lemma. Part (v) of Lemma 1 is new but has a similar proof, omitted here.

Lemma 1. *Let* P *denote the set of points in the finite projective geometry* $PG(t-1, s)$, P^* *be any* $(u-1)$*-flat of* P, *and* Q *be any nonempty subset of* P *such that* P^* *and* Q *are disjoint. Let* $q = \#Q, \hat{Q} = P - (P^* \cup Q)$. *Then*
(i) $A_3(P^* \cup Q) =$ *constant* $+ A_3(Q) + M_2(P^*, Q)$,
(ii) $A_4(P^* \cup Q) =$ *constant* $+ A_4(Q) + M_3(P^*, Q) + \gamma_1 M_2(P^*, Q)$,
(iii) $M_2(P^*, Q) =$ *constant* $+ M_2(P^*, \hat{Q})$,
(iv) $M_3(P^*, Q) =$ *constant* $- M_3(P^*, \hat{Q}) - \gamma_2 M_2(P^*, \hat{Q})$,
(v) $M_4(P^*, Q) =$ *constant* $+ (s^u - 1) A_3(\hat{Q}) + M_4(P^*, \hat{Q}) + (s^u + 3s - 7) M_3(P^*, \hat{Q}) + \gamma_3 M_2(P^*, \hat{Q})$,
where $\gamma_1, \gamma_2, \gamma_3$ *and the constants appearing on the right sides of (i)−(v) are constants which may depend on* t, q, u *or* s, *but not on the particular choice of* P^* *and* Q (*explicit constants are not required for our purpose*).

Consider now an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design $d = d(C_1, C_2)$. Define the two complementary subsets $F_i = P_i - C_i, i = 1, 2$. Let $F = F_1 \cup F_2 = P - C$, where $C = C_1 \cup C_2$. Following (2.9), the cardinalities of F_1, F_2 and F are f_1, f_2 and f respectively. The next theorem expresses the $A_i(d)$ and $B_i(d)$, for small values of i, in terms of F_1, F_2 and F. While parts (i)−(iii) of this theorem are proved in Suen, Chen and Wu (1997), parts (iv)−(vi), relating to the secondary wordlength pattern, are new. The proofs of these latter parts involve the use of (3.3), (3.5), Theorem 2, Lemma 1, and also the results in Suen, Chen and Wu (1997). Proofs are omitted.

Theorem 3.
(i) $A_3(d) =$ *constant* $-A_3(F)$,
(ii) $A_4(d) =$ *constant* $+(3s-5)A_3(F) + A_4(F)$,
(iii) $A_5(d) =$ *constant* $-\mu A_3(F) - (4s-7)A_4(F) - A_5(F)$,
(iv) $B_2(d) =$ *constant* $+M_2(P_1, F_2)$,
(v) $B_3(d) =$ *constant* $+A_3(F) - \mu_{32} M_2(P_1, F_2) - h_3(F_1, F_2)$,
(vi) $B_4(d) =$ *constant* $-(3s-5)A_3(F) - A_4(F) + \mu_{42} M_2(P_1, F_2) + \mu_{43} h_3(F_1, F_2) + h_4(F_1, F_2)$,
where

$$h_3(F_1, F_2) = A_3(F_1) + A_3(F_2) + M_3(P_1, F_2), \tag{4.1}$$

$$h_4(F_1, F_2) = A_4(F_1) + A_4(F_2) + M_4(P_1, F_2) - (s-1) f_1 A_3(F_1), \tag{4.2}$$

and μ,μ_{32},μ_{42},μ_{43} *and the constants appearing on the right sides of* (i)−(vi), *being constants which may depend on* n_1, n_2, p_1, p_2 *or* s, *but not on the particular choice of* C_1 *or* C_2 (*explicit constants are not needed in the sequel*).

At the expense of heavier algebra, one can extend Theorem 3 to get expressions for $B_i(d)$ in terms of F_1, F_2 and F for still higher values of i. Our Theorem 2, which is quite general, can be combined with the results in Suen, Chen and Wu (1997) and Mukerjee and Wu (1999) for this purpose. The resulting expressions for higher values of i will, however, be complicated. Nevertheless, in most applications, the present version of Theorem 3 will suffice for completely characterizing optimal designs under the twin criteria of MA and MSA. The examples below, as well as the findings reported in Section 6, illustrate this point.

Though the consideration of complementary subsets is in the spirit of what one does in the classical FF setup (Suen, Chen and Wu (1997)), there is one major difference. Here an arbitrary set of cardinality f cannot be a candidate for F. We need that F should be decomposable as $F = F_1 \cup F_2$, where $\#F_i = f_i$ and $F_i \subset P_i$, $i = 1, 2$. This special feature of the split-plot setting is highlighted in Example 1 below. Another specialty of the present setup is that one may have to consider the criterion of MSA, which does not arise in classical FF designs – see Example 2 below.

Example 1. Let $s = 2, f_1 = 4, f_2 = 1$. Since $f_1 = 4$, by (2.9), we have $t_1 \geq 3$. Up to isomorphism, there is a unique design which maximizes $A_3(F)$. This corresponds to

$$F_1 = \{e_1, e_2, e_3, e_1 + e_2\}, \qquad F_2 = \{e_{t_1+1}\}, \tag{4.3}$$

where as before e_1,e_2,... are the unit t-vectors over $GF(s)$. By Theorem 3(i), this is the unique MA design up to isomorphism. Here $f = 5$ and, following Tang and Wu (1996), for $s = 2, f = 5$, the classical FF design with MA is given by

$$F = \{e_i, e_j, e_u, e_i + e_j, e_i + e_u\} \tag{4.4}$$

where i, j, u are distinct. Clearly, (4.4) is not isomorphic to $F_1 \cup F_2$, with F_1 and F_2 as in (4.3). In fact, with $f_1 = 4, f_2 = 1$, F as in (4.4) is not decomposable as $F = F_1 \cup F_2$, where $\#F_i = f_i$ and $F_i \subset P_i$, $i = 1, 2$, and hence cannot arise in the present split-plot setup.

Example 2. Let $s = 3, f_1 = 1, f_2 = 4$. Since $f_1 = 1$, by (2.9), $t_1 \geq 2$.

(a) If $t_2 = 1$, then, as in the last example, the unique MA design up to isomorphism is given by

$$F_1 = \{e_1\}, \qquad F_2 = \{e_{t_1+1}, e_1 + e_{t_1+1}, e_1 + 2e_{t_1+1}, e_2 + e_{t_1+1}\}. \tag{4.5}$$

(b) If $t_2 \geq 2$ then, up to isomorphism, there are three distinct designs, say d_1, d_2, d_3, which maximize $A_3(F)$. These correspond to (4.5),

$$F_1 = \{e_1\}, \qquad F_2 = \{e_{t_1+1}, e_1 + e_{t_1+1}, e_1 + 2e_{t_1+1}, e_{t_1+2}\}, \tag{4.6}$$

$$F_1 = \{e_1\}, \qquad F_2 = \{e_{t_1+1}, e_{t_1+2}, e_{t_1+1} + e_{t_1+2}, e_{t_1+1} + 2e_{t_1+2}\}, \tag{4.7}$$

respectively. Here $f = 5$ and all these designs have the same $A_4(F)$ and $A_5(F)$. Hence, they all have MA; cf. Suen, Chen and Wu (1997). By (3.2) and (4.5)$-$(4.7), for d_1, d_2, d_3, the quantity $M_2(P_1, F_2)$ equals 6, 3 and 0 respectively. Hence by Theorem 3(iv), d_3 is the unique MSA design up to isomorphism.

It is not hard to see that if $C_i = P_i - F_i, i = 1, 2$, where F_1, F_2 are as in (4.3), (4.5) or (4.7), then the pair (C_1, C_2) satisfies the rank conditions of Definition 1. In particular, taking $t_1 = 3$ and $t_2 = 2$, or $t_1 = 4$ and $t_2 = 1$ in Example 1, and recalling (2.5), (2.6) and (2.9), we get the $2^{(3+23)-(0+21)}$ or $2^{(11+15)-(7+14)}$ FFSP designs with MA. These optimal 32-run FFSP designs have been hitherto unreported. Similarly the case $s = 3$ considered in Example 2 is new in the context of FFSP designs. More applications of Theorem 3 are summarized in Section 6.

5. Estimation Capacity

We now consider the issue of estimation capacity. The objective here is to choose a design retaining full information on the main effects and as much information as possible on the two-factor interactions, in the sense of entertaining the maximum possible model diversity.

Consider an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design $d = d(C_1, C_2)$. For $u = 1, 2, \ldots$, let $E_u(d)$ be the number of models, containing all the main effects and u distinct two-factor interaction pencils, that are estimable by d under the absence of interactions involving three or more factors. If a design maximizing $E_u(d)$ for every u exists, then it is said to have *maximum estimation capacity* (MEC). This concept is due to Cheng, Steinberg and Sun (1999) who considered classical FF designs. In the context of FFSP designs, however, the problem of non-uniqueness persists also with the criterion of MEC because of lack of interchangeability between the WP and SP factors. Thus, as will be seen later in the setup of Example 2(b), all three designs given by (4.5)−(4.7) have MEC.

The distinction between inference at the WP and SP levels again helps in achieving further discrimination. Since the WP level error is typically larger than the SP level error, given two nonisomorphic designs with MEC, it is natural to prefer the one having a better performance at the SP level. From this point of view, let $E_u^*(d)$ be the number of models, containing all the main effects and u distinct two-factor interaction pencils, that d can estimate under the absence of all three-factor or higher order interactions, such that the u two-factor interaction pencils are estimated at the SP level of d. If there are rival nonisomorphic designs with MEC, and if among them there is one which maximizes $E_u^*(d)$ for every u, then we say such a design has *maximum split-plot level estimation capacity* (MSPEC).

The quantities $E_u^*(d)$ ignore estimation at the WP level except for the main effects of the WP factors which, at any rate, have to be estimated at that level. This, however, does not mean that our approach to estimation capacity ignores estimation at the WP level altogether. Just as the criterion of MSA was proposed as a supplement to that of MA, we are now proposing the criterion of MSPEC as a follow-up of MEC for finer discrimination. The criterion of MEC based on the $E_u(d)$ does take care of WP level estimation as well.

Theorem 4 below summarizes the main tools needed for the study of FFSP designs with MEC or MSPEC. Its proof follows along the lines of Cheng, Steinberg and Sun (1999) and Cheng and Mukerjee (1998) and is hence omitted; see the former paper for the definition of upper weak majorization used in Theorem 4(b). As in these papers, the aliasing pattern has to be considered explicitly to reach Theorem 4. In particular following Cheng and Mukerjee (1998), the quantities $m_i(d)$ defined in (5.1) below can be seen to represent the numbers of distinct two-factor interaction pencils in the f alias sets of d which do not contain any main effect pencil — of these f alias sets, the first f_1 (last f_2) are of the WP (SP) type; vide (2.9) and the last paragraph of Section 2.

With reference to a regular FFSP design $d = d(C_1, C_2)$, define the complementary subsets F_1 and F_2 as in Section 4. Let $F_1 = \{\alpha_1, \ldots, \alpha_{f_1}\}, F_2 = \{\alpha_{f_1+1}, \ldots, \alpha_f\}$, where $f = f_1 + f_2$. For $1 \leq i \leq f$, let

$$m_i(d) = \frac{1}{2}(s-1)(L_t - 2f + 1) + \phi_i(d), \tag{5.1}$$

where $\phi_i(d)$ is the number of linearly dependent triplets $\{\alpha_i,\alpha_j,\alpha_u\}$ such that $\alpha_i,\alpha_j,\alpha_u$ are distinct members of $F = F_1 \cup F_2$ and $j<u$. Let $\phi(d) = (\phi_1(d), \ldots, \phi_f(d))'$ and $\phi^*(d) = (\phi_{f_1+1}(d), \ldots, \phi_f(d))'$.

Theorem 4. (a) *For any regular FFSP design d,*

$$E_u(d) = \begin{cases} \boldsymbol{\Sigma} \prod_{j=1}^{u} m_{i_j}(d) & \text{if } u \leq f, \\ 0 & \text{otherwise}, \end{cases}$$

$$E_u^*(d) = \begin{cases} \boldsymbol{\Sigma}^* \prod_{j=1}^{u} m_{i_j}(d) & \text{if } u \leq f_2, \\ 0 & \text{otherwise}, \end{cases}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^*$ are sums over $1 \leq i_1 < \cdots < i_u \leq f$ and $f_1 + 1 \leq i_1 < \cdots < i_u \leq f$, respectively.

(b) *Let d_1 and d_2 be $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP designs. If $\phi(d_1)$ $[\phi^*(d_1)]$ is upper weakly majorized by $\phi(d_2)$ $[\phi^*(d_2)]$ and not obtainable from $\phi(d_2)$ $[\phi^*(d_2)]$ by permuting its elements then $E_u(d_1) \geq E_u(d_2)$ $[E_u^*(d_1) \geq E_u^*(d_2)]$ for all u, with strict inequality for some u.*

Theorem 4 and (5.1) greatly simplify the study of estimation capacity in FFSP designs, especially in the practically important nearly saturated cases where F is of small size, and hence the $\phi_i(d)$ are not hard to obtain. For illustration, we now revisit Examples 1 and 2.

Example 1. (*continued*) Here $s = 2, f_1 = 4, f_2 = 1$. As $s = 2$, the quantity $\phi_i(d)$ in (5.1) has a geometric interpretation as the number of lines in F that pass through α_i. Since $f_2 = 1$ and no point in $F_2(\subset P_2)$ can be spanned exclusively by the points in $F_1(\subset P_1)$, clearly no line of F can pass through the single point in F_2. Hence by Theorem 4(b), a design has MEC if and only if F_1 consists of three collinear points and one isolated point. Up to isomorphism, the design in (4.3) is the unique one having this property and hence MEC. Recall that this is the unique MA design too.

Here $f = 5$. Following Cheng and Mukerjee (1998), for $s = 2, f = 5$, the classical FF design with MEC corresponds to (4.4). As discussed earlier, such a choice F is, however, not allowed in our context.

Example 2. (*continued*) Here $s = 3, f_1 = 1, f_2 = 4, f = 5$. Since $f_1 = 1$, by (2.9), $t_1 \geq 2$ and hence by (2.5), $t \geq 3$. Therefore, by (2.6),

$$\frac{1}{2}(s-1)(L_t - 2f + 1) = L_t - 9 \geq 4. \tag{5.2}$$

Write $\phi^{(1)} = (0, 3, 3, 3, 3)'$ and $\phi^{(2)} = (1, 1, 1, 1, 2)'$. It can be seen that for every design d, either $\phi^{(1)}$ or $\phi^{(2)}$ is upper weakly majorized by $\phi(d)$. Furthermore, if $d^{(1)}$ and $d^{(2)}$ be two designs such that $\phi(d^{(i)})$ is a permutation of $\phi^{(i)}$ $i = 1, 2$, then by (5.1), (5.2) and Theorem 4(a), after some algebra one gets $E_u(d^{(1)}) \geq E_u(d^{(2)})$ for every u, with strict inequality for $1 \leq u \leq 5$. Hence recalling Theorem 4(b), a design has MEC if and only if $\phi(d)$ is a permutation of $\phi^{(1)}$ which happens if and only if F consists of the four points of a 1-flat and an isolated point.

(a) If $t_2 = 1$ then the above argument shows that the design given by (4.5) uniquely has MEC, up to isomorphism. Recall that this is also the unique MA design.

(b) If $t_2 \geq 2$ then in a similar manner there are three distinct designs, up to isomorphism, which have MEC. These are d_1, d_2 and d_3 as described by (4.5)$-$(4.7). Since $\phi^*(d_1) = \phi^*(d_2) = (3, 3, 3, 0)'$ and $\phi^*(d_3) = (3, 3, 3, 3)'$, by Theorem 4(b), up to isomorphism, d_3 is the unique design having MSPEC. Recall that d_3 is the unique MSA design as well.

The twin criteria of MA and MSA are in perfect agreement with the twin criteria of MEC and MSPEC in the last two examples. The findings reported in Section 6 below reveal that this happens quite commonly. All the designs reported in the tables of Section 6 to have MA or MSA also have MEC or MSPEC. In fact, in a vast majority of cases, these are the unique optimal designs under both sets of criteria. This leads to the satisfying conclusion that the criteria of MA and MSA are excellent surrogates for the statistically more meaningful criteria of MEC and MSPEC.

6. Tables

In Table 1 we summarize optimal designs, under the criteria discussed above, for nearly saturated cases given by $f(= f_1 + f_2) \leq 5$. This is done using Theorem 3 and 4 and (5.1). Examples 1 and 2 are illustrative of the details underlying this table. Because of the lack of interchangeability between the WP and SP factors, and also as we intend to cover all prime or prime powered values of s, a large number of cases are in Table 1. Incidentally, by Theorem 4(a), the criterion of MSPEC does not arise if $f_2 = 0$ and it reduces to that of MEC if $f_1 = 0$.

Table 1. Optimal nearly saturated FFSP designs

f_1	f_2	s	t_2	Optimal design and optimality criteria
1	0	≥ 2	≥ 1	All isomorphic
0	1	≥ 2	≥ 1	All isomorphic
2	0	≥ 2	≥ 1	All isomorphic
1	1	≥ 2	≥ 1	All isomorphic
0	2	≥ 2	1	All isomorphic
0	2	≥ 2	≥ 2	$F_2 = \{\theta_1, \theta_2\}$; unique MSA, MEC.
3	0	≥ 2	≥ 1	$F_1 = \{e_1, e_2, e_1 + e_2\}$; unique MA, unique MEC.
2	1	≥ 2	≥ 1	All isomorphic
1	2	≥ 2	≥ 1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \theta_1\}$; unique MA, unique MEC.
0	3	2	1	All isomorphic
0	3	2	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2\}$; unique MA, unique MEC.
0	3	≥ 3	1	$F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1\}$; unique MA, unique MEC.
0	3	≥ 3	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2\}$; unique MSA, MEC.
4	0	2	≥ 1	$F_1 = \{e_1, e_2, e_1 + e_2, e_3\}$; unique MA, unique MEC.
4	0	≥ 3	≥ 1	$F_1 = \{e_1, e_2, e_1 + \lambda_1 e_2, e_1 + \lambda_2 e_2\}$; unique MA, unique MEC.
3	1	≥ 2	≥ 1	$F_1 = \{e_1, e_2, e_1 + e_2\}, F_2 = \{\theta_1\}$; unique MA, unique MEC.
2	2	≥ 2	≥ 1	$F_1 = \{e_1, e_2\}, F_2 = \{\theta_1, e_1 + \theta_1\}$; unique MA, unique MEC.
1	3	2	1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \theta_1, e_2 + \theta_1\}$; unique MA, unique MEC.
1	3	2	≥ 2	$F_1 = \{e_1\}, F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2\}$; unique MSA, unique MSPEC.
1	3	≥ 3	≥ 1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1\}$; unique MA, unique MEC.
0	4	2	1	$F_2 = \{\theta_1, e_1 + \theta_1, e_2 + \theta_1, e_3 + \theta_1\}$; unique MA, MEC.
0	4	2	2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, e_1 + \theta_1\}$; unique MA, unique MEC.
0	4	2	≥ 3	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_3\}$; unique MSA, MEC.
0	4	3	1	$F_2 = \{\theta_1, e_1 + \theta_1, e_1 + 2\theta_1, e_2 + \theta_1\}$; unique MA, unique MEC.
0	4	3	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_1 + 2\theta_2\}$; unique MA, unique MEC.
0	4	≥ 4	1	$F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1, e_1 + \lambda_3\theta_1\}$; unique MA, unique MEC.
0	4	≥ 4	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \lambda_1\theta_2, \theta_1 + \lambda_2\theta_2\}$; unique MSA, MEC.
5	0	2	≥ 1	$F_1 = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3\}$; unique MA, unique MEC.
5	0	3	≥ 1	$F_1 = \{e_1, e_2, e_1 + e_2, e_1 + 2e_2, e_3\}$; unique MA, unique MEC.
5	0	≥ 4	≥ 1	$F_1 = \{e_1, e_2, e_1 + \lambda_1 e_2, e_1 + \lambda_2 e_2, e_1 + \lambda_3 e_2\}$; unique MA, unique MEC.
4	1	2	≥ 1	$F_1 = \{e_1, e_2, e_3, e_1 + e_2\}, F_2 = \{\theta_1\}$; unique MA, unique MEC.
4	1	≥ 3	≥ 1	$F_1 = \{e_1, e_2, e_1 + \lambda_1 e_2, e_1 + \lambda_2 e_2\}, F_2 = \{\theta_1\}$; unique MA, unique MEC.
3	2	≥ 2	≥ 1	$F_1 = \{e_1, e_2, e_1 + e_2\}, F_2 = \{\theta_1, e_1 + \theta_1\}$; unique MA, unique MEC.
2	3	2	≥ 1	$F_1 = \{e_1, e_2\}, F_2 = \{\theta_1, e_1 + \theta_1, e_2 + \theta_1\}$; unique MA, unique MEC.
2	3	≥ 3	≥ 1	$F_1 = \{e_1, e_2\}, F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1\}$; unique MA, unique MEC.
1	4	2	1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \theta_1, e_2 + \theta_1, e_1 + e_2 + \theta_1\}$; unique MA, unique MEC.
1	4	2	≥ 2	$F_1 = \{e_1\}, F_2 = \{\theta_1, \theta_2, e_1 + \theta_1, \theta_1 + \theta_2\}$; unique MSA, unique MSPEC.
1	4	3	1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \theta_1, e_1 + 2\theta_1, e_2 + \theta_1\}$; unique MA, unique MEC.
1	4	3	≥ 2	$F_1 = \{e_1\}, F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_1 + 2\theta_2\}$; unique MSA, unique MSPEC.
1	4	≥ 4	≥ 1	$F_1 = \{e_1\}, F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1, e_1 + \lambda_3\theta_1\}$; unique MA, unique MEC.
0	5	2	1	For $t_1 = 3$, all isomorphic,
				For $t_1 \geq 4$, take $F_2 = \{\theta_1, e_1 + \theta_1, e_2 + \theta_1, e_3 + \theta_1, e_4 + \theta_1\}$; unique MA, MEC.
0	5	2	2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, e_1 + \theta_1, e_1 + \theta_2\}$; unique MA, unique MEC.
0	5	2	≥ 3	$F_2 = \{\theta_1, \theta_2, \theta_3, \theta_1 + \theta_2, \theta_1 + \theta_3\}$; unique MSA, MEC.
0	5	3	1	$F_2 = \{\theta_1, e_1 + \theta_1, e_1 + 2\theta_1, e_2 + \theta_1, e_2 + 2\theta_1\}$; unique MA, unique MEC.
0	5	3	2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_1 + 2\theta_2, e_1 + \theta_1\}$; unique MA, unique MEC.
0	5	3	≥ 3	$F_2 = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_1 + 2\theta_2, \theta_3\}$; unique MSA, MEC.
0	5	4	1	$F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1, e_1 + \lambda_3\theta_1, e_2 + \theta_1\}$; unique MA, unique MEC.
0	5	4	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \lambda_1\theta_2, \theta_1 + \lambda_2\theta_2, \theta_1 + \lambda_3\theta_2\}$; unique MA, unique MEC.
0	5	≥ 5	1	$F_2 = \{\theta_1, e_1 + \lambda_1\theta_1, e_1 + \lambda_2\theta_1, e_1 + \lambda_3\theta_1, e_1 + \lambda_4\theta_1\}$; unique MA, unique MEC.
0	5	≥ 5	≥ 2	$F_2 = \{\theta_1, \theta_2, \theta_1 + \lambda_1\theta_2, \theta_1 + \lambda_2\theta_2, \theta_1 + \lambda_3\theta_2\}$; unique MSA, MEC.

Table 2. Optimal three-level FFSP designs in 27 runs for $n_1 + n_2 \leq 7$

n_1	n_2	p_1	p_2	Optimal design and optimality criteria
1	3	0	1	$C_1 = \{1\}, C_2 = \{2, 3, 123\}$; unique MA, unique MEC.
2	2	0	1	$C_1 = \{1, 2\}, C_2 = \{3, 123\}$; unique MA, unique MEC.
3	1	1	0	All isomorphic.
1	4	0	2	$C_1 = \{1\}, C_2 = \{2, 3, 23, 12^23\}$; unique MSA, MEC.
2	3	0	2	$C_1 = \{1, 2\}, C_2 = \{3, 13, 123^2\}$; MSA, unique MSPEC.
3	2	1	1	$C_1 = \{1, 2, 12\}, C_2 = \{3, 12^23\}$; unique MA, unique MEC.
4	1	2	0	All isomorphic.
1	5	0	3	$C_1 = \{1\}, C_2 = \{2, 3, 12, 12^23, 12^23^2\}$; unique MA, unique MEC.
2	4	0	3	$C_1 = \{1, 2\}, C_2 = \{3, 13, 123^2, 12^23^2\}$; unique MA, unique MEC.
3	3	1	2	$C_1 = \{1, 2, 12\}, C_2 = \{3, 12^23, 12^23^2\}$; unique MA, unique MEC.
4	2	2	1	All isomorphic.
1	6	0	4	$C_1 = \{1\}, C_2 = \{2, 3, 12, 13^2, 23^2, 12^23^2\}$; unique MSA, MEC.
2	5	0	4	$C_1 = \{1, 2\}, C_2 = \{3, 13^2, 23, 123, 123^2\}$; unique MA, unique MEC.
3	4	1	3	$C_1 = \{1, 2, 12\}, C_2 = \{3, 13^2, 23^2, 12^23^2\}$; MSA, unique MSPEC.
4	3	2	2	$C_1 = \{1, 2, 12, 12^2\}, C_2 = \{3, 13, 23\}$; unique MA, unique MEC.

For notational simplicity in Table 1, we write $e_{t_1+i} = \theta_i$, $i = 1, 2, \ldots$. Also whenever $\lambda_1, \lambda_2, \ldots$ are mentioned in this table, these have to be interpreted as distinct nonzero elements of $GF(s)$.

The general framework of this paper allows us to consider the case $s = 3$ in some detail. Table 2 shows optimal three-level FFSP designs in 27 runs for $n_1 + n_2 \leq 7$. Here $s = 3, t = 3$ and, for notational simplicity, the points of $PG(t-1, s)$ are denoted by $e_1 = 1, e_2 + 2e_3 = 23^2, e_1 + 2e_2 + e_3 = 12^23$, etc. The cases $n_1 + n_2 \geq 8$ are not shown in Table 2 since they correspond to $f \leq 5$.

The word "unique", as used in Tables 1 and 2, is always up to isomorphism. Also in these tables, "all isomorphic" means "all designs are isomorphic".

We also considered optimal two-level FFSP designs in 16 runs with a view to strengthening the corresponding table in Bingham and Sitter (1999a) through consideration of MSA and criteria based on estimation capacity. These details are omitted here to save space but are available on request.

Appendix

Proof of Theorem 1. First consider a regular $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP design as specified by (2.1) with H as in (2.2). Then it is not hard to see that there exists a matrix

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} \tag{A.1}$$

over $GF(s)$, with G_{11}, G_{12} and G_{22} of orders $t_1 \times n_1$, $t_1 \times n_2$ and $t_2 \times n_2$ respectively, (see (2.5)) such that

$$rank(G_{ii}) = t_i, \qquad i = 1, 2, \tag{A.2}$$

$$HG' = 0. \tag{A.3}$$

Now by the definition of H, it has full row rank. Similarly, by (A.1) and (A.2), G has full row rank. Also, by (2.5), $[H' \ G']'$ is a square matrix. Hence by (A.3), for any n-vector ξ over $GF(s)$,

$$\xi \in \mathcal{M}(H') \Leftrightarrow G\xi = 0, \qquad \xi \in \mathcal{M}(G') \Leftrightarrow H\xi = 0. \tag{A.4}$$

Recall that the design has resolution at least three and that it keeps every pencil representing an SP factor main effect unaliased with pencils involving only the WP factors. Hence by (2.3), (2.4), (A.1) and (A.2) one can deduce from (A.4) that (I) no column of G is null and no two distinct columns of G are proportional to each other, (II) no column of G_{22} is null.

The fact (II) is special to the present split-plot setting and does not arise in classical FF designs. Because of (I), the columns of G represent distinct points in $PG(t-1, s)$. Let C_1 and C_2 be the sets of points given by the first n_1 and last n_2 columns respectively, of G. Then, with $C = C_1 \cup C_2$,

$$V(C_1) = \begin{bmatrix} G_{11} \\ 0 \end{bmatrix}, \qquad V(C_2) = \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix}, \qquad V(C) = G. \tag{A.5}$$

As noted above, G has full row rank. Hence from (A.2), (A.5) and (II) above, (C_1, C_2) is an eligible (n_1, n_2)-pair of subsets of P. The validity of (i)$-$(iii) in the statement of the theorem now follows from (2.1), (2.3), (2.4), (A.4) and (A.5).

Conversely, essentially reversing the above steps, one can show that given an eligible (n_1, n_2)-pair of subsets (C_1, C_2) of P, it is possible to construct an $s^{(n_1+n_2)-(p_1+p_2)}$ regular FFSP design such that (i)$-$(iii) in the statement of the theorem hold.

Proof of Theorem 2. Let $\mathcal{P}$ be the set of all vectors over $GF(s)$ which are nonnull but proportional to some point in P_1, and $\mathcal{P}_0 = \mathcal{P} \cup \{0\}$. Also, for $2 \le i \le n$, let $\Omega^*(i, n) = \{\beta : \beta \in \Omega(i, n)$, the last n_2 elements of β are not all zeros$\}$. Since pencils with proportional elements are identical, recalling the definition of $B_i(d)$, by Theorem 1(iii), we have $B_i(d) = (s-1)^{-1}\#\{\beta : \beta \in \Omega^*(i, n), V(C)\beta \in \mathcal{P}\}$. Similarly, by (3.1), $A_i(C) - A_i(C_1) = (s-1)^{-1}\#\{\beta : \beta \in \Omega^*(i, n), V(C)\beta = 0\}$. Hence

$$\begin{aligned} B_i(d) + A_i(C) - A_i(C_1) &= (s-1)^{-1}\#\{\beta : \beta \in \Omega^*(i, n), \ V(C)\beta \in \mathcal{P}_0\} \\ &= (s-1)^{-1} \sum_{r=1}^{i} \#\triangle(i, r, n_1, n_2), \end{aligned} \tag{A.6}$$

$$\begin{aligned}\triangle(i,r,n_1,n_2) &= \{\beta=(\beta_1',\beta_2')' : \beta_1\in\Omega(i-r,n_1), \beta_2\in\Omega(r,n_2),\\ &\qquad\qquad\qquad\qquad V(C_1)\beta_1+V(C_2)\beta_2\in\mathcal{P}_0\}\\ &= \{\beta=(\beta_1',\beta_2')' : \beta_1\in\Omega(i-r,n_1), \beta_2\in\Omega(r,n_2), V(C_2)\beta_2\in\mathcal{P}_0\}.\end{aligned} \tag{A.7}$$

The last step follows as $\mathcal{P}_0$ is closed under addition and $V(C_1)\beta_1 \in \mathcal{P}_0$ for every $\beta_1 \in \Omega(i-r, n_1)$. By (3.1), (3.2), (A.7) and the definition of $\mathcal{P}_0$

$$\begin{aligned}\triangle(i,r,n_1,n_2) &= \{\#\Omega(i-r,n_1)\}[\#\{\beta_2 : \beta_2\in\Omega(r,n_2), V(C_2)\beta_2=0\}\\ &\qquad\qquad\qquad\qquad +\#\{\beta_2 : \beta_2\in\Omega(r,n_2), V(C_2)\beta_2\in\mathcal{P}\}]\\ &= \binom{n_1}{i-r}(s-1)^{i-r}[(s-1)\{A_r(C_2)+M_r(P_1,C_2)\}].\end{aligned} \tag{A.8}$$

If one substitutes (A.8) in (A.6) and employs (3.3), then the result follows.

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