Supplement to
"A nonparametric test for stationarity in functional time series"

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Abstract

This supplement contains the proof of Theorem 3.1 and additional technical material. Section S1 provides some properties of tensor products between Hilbert spaces and cumulant tensors. Section S2 provides the proof of Lemma 2.1. Section S3 contains the main steps of the proof of Theorem 3.1 while the bounds on cumulant tensors of local functional DFT are established in Section S4. Section S5 provides the derivation of the asymptotic first and second order structure of $\hat{m}_T$, the proof of Lemma 3.1 and the proof of Remark 3.1. Finally, we briefly mention some statistical applications in Section S6.

Keywords: time series, functional data, spectral analysis, local stationarity, measuring stationarity, relevant hypotheses

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S1. Some properties of tensor products of operators and cumulant tensors

Let $\mathscr{H}_i$ for each $i = 1,\dots,n$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_i$. The tensor of $\mathscr{H}_1,\dots,\mathscr{H}_n$ is denoted by

$$\mathscr{H} := \mathscr{H}_1 \otimes \dots \otimes \mathscr{H}_n = \bigotimes_{i=1}^{n} \mathscr{H}_i. \tag{S1.1}$$

If $\mathscr{H}_i = \mathscr{H}$ for each $i \in 1,\dots,n$, then (S1.1) is the n-th fold tensor product of $\mathscr{H}$. For $A_i \in \mathscr{H}_i, i = 1,\dots,n$ the object $\bigotimes_{i=1}^{n} A_i$ is a multi-antilinear functional that generates a linear manifold, the usual algebraic tensor product of vector spaces $\mathscr{H}_1,\dots,\mathscr{H}_n$, to which the scalar product

$$\langle \bigotimes_{i=1}^{n} A_i, \bigotimes_{i=1}^{n} B_i \rangle = \prod_{i=1}^{n} \langle A_i, B_i \rangle_i$$

can be extended to a pre-Hilbert space. The completion of the above algebraic tensor product is $\bigotimes_{i=1}^{n} \mathscr{H}_i$ (we refer to Section 2.6 of Kadison and Ringrose (1997) and Section 3.4 of Weidmann (1980) for more details about tensor products of Hilbert spaces).

We require some properties of operators on a separable Hilbert space. $\mathscr{L}(\mathscr{H})$ denotes the Banach space of bounded linear operators $A : \mathscr{H} \to \mathscr{H}$ with the operator norm given by $|||A|||_\infty = \sup_{\|x\|\le 1} \|Ax\|$. Each operator $A \in \mathscr{L}(\mathscr{H})$ has the adjoint operator $A^\dagger \in \mathscr{L}(\mathscr{H})$, which satisfies $\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$ for each $x, y \in \mathscr{H}$. $A \in \mathscr{L}(\mathscr{H})$ is called self-adjoint if $A = A^\dagger$ and non-negative definite if $\langle Ax, x\rangle \ge 0$ for each $x \in \mathscr{H}$. The conjugate of an operator $A \in \mathscr{L}(\mathscr{H})$, denoted by $\overline{A}$, is defined as $\overline{A}x = \overline{(A\overline{x})}$, where $\overline{x}$ denotes the complex conjugate of $x \in \mathscr{H}$. For $A, B, C \in \mathscr{L}(\mathscr{H})$ we define the following bounded linear mappings. The Kronecker product is defined as $(A \widetilde{\bigotimes} B)C = ACB^\dagger$, while the transpose Kronecker product is given by $(A \widetilde{\bigotimes}_\top B)C = (A \widetilde{\bigotimes} \overline{B})\overline{C}^\dagger$. For $A, B, C \in S_2(\mathscr{H})$, we shall denote, in analogy to elements $a, b \in \mathscr{H}$, the Hilbert tensor product as $A \bigotimes B$. We list the following useful properties:

Properties S1.1. *Let $\mathscr{H}_i = L^2_{\mathbb{C}}([0,1]^k)$ for $i = 1, \ldots, n$. Then for $a_i, b_i \in \mathscr{H}_i$ and $A_i, B_i \in S_2(\mathscr{H}_i)$, we have*

1. $\langle A, B \rangle_{HS} = \mathrm{Tr}(AB^\dagger)$
2. $\langle \bigotimes_{i=1}^n A_i, \bigotimes_{i=1}^n B_i \rangle_{HS} = \prod_{i=1}^n \langle A_i, B_i \rangle_{HS}$
3. $\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle_{HS} = \langle a_1 \otimes \overline{a}_2, b_1 \otimes \overline{b_2} \rangle_{\mathscr{H}_i \otimes \mathscr{H}_i} = \langle a_1, b_1 \rangle \overline{\langle a_2, b_2 \rangle}$
4. *If* $A_i \in S_1(H)$, *then* $\prod_{i=1}^n \mathrm{Tr}(A_i) = \mathrm{Tr}(\widetilde{\bigotimes}_{i=1}^n A_i)$
5. $\big((a_1 \otimes \overline{a}_2) \bigotimes (a_3 \otimes \overline{a}_4)\big) = \big((a_1 \otimes a_3) \widetilde{\bigotimes} (\overline{a}_2 \otimes \overline{a}_4)\big) = \big((a_1 \otimes a_4) \widetilde{\bigotimes}_\top a_2 \otimes a_3)\big)$

Let X be a random element on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ that takes values in a separable Hilbert space $\mathscr{H}$. More precisely, we endow $\mathscr{H}$ with the topology induced by the norm on $\mathscr{H}$ and assume that $X : \Omega \to \mathscr{H}$ is Borel-measurable. The *k-th order cumulant tensor* is defined by (van Delft and Eichler, 2018)

$$\mathrm{Cum}\big(X_1, \ldots, X_k\big) = \sum_{l_1, \ldots l_k \in \mathbb{N}} \mathrm{Cum}\big(\langle X_1, \psi_{l_1} \rangle, \ldots, \langle X_k, \psi_{l_k} \rangle\big)(\psi_{l_1} \otimes \cdots \otimes \psi_{l_k}), \quad \text{(S1.2)}$$

and the cumulants on the right hand side are as usual given by

$$\mathrm{Cum}\big(\langle X_1,\psi_{l_1}\rangle,\dots,\langle X_k,\psi_{l_k}\rangle\big) = \sum_{\nu=(\nu_1,\dots,\nu_p)} (-1)^{p-1}(p-1)! \prod_{r=1}^{p} \mathbb{E}\Big[\prod_{t\in\nu_r}\langle X_t,\psi_{l_t}\rangle\Big],$$

where the summation extends over all unordered partitions ν of $\{1,\dots,k\}$. The product theorem for cumulants (Brillinger, 1981, Theorem 2.3.2) can then be generalized (see e.g. Aue and van Delft, 2019, Theorem B.1) to simple tensors of random elements of $\mathscr{H}$, i.e., $X_t = \otimes_{j=1}^{J_t} X_{tj}$ with $j = 1,\dots,J_t$ and $t = 1,\dots,k$. The joint cumulant tensor is then be given by

$$\mathrm{Cum}(X_1,\dots,X_k) = \sum_{\nu=(\nu_1,\dots,\nu_p)} S_\nu\Big(\otimes_{n=1}^{p}\mathrm{Cum}\big(X_{tj}|(t,j)\in\nu_n\big)\Big), \tag{S1.3}$$

where S_ν is the permutation that maps the components of the tensor back into the original order, that is, $S_\nu(\otimes_{r=1}^{p}\otimes_{(t,j)\in\nu_r} X_{tj}) = X_{11}\otimes\cdots\otimes X_{kJ_t}$.

Suppose that $\{X_t^{(u)} : t\in\mathbb{Z}\}$ is a strictly stationary sequence of $\mathscr{H}$-valued random elements for each $u\in[0,1]$. The results of van Delft and Eichler (2018) imply that the *local k-th order cumulant spectral operator*

$$\mathscr{F}_{u,\omega_1,\dots,\omega_{k-1}} = \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\dots,t_{k-1}\in\mathbb{Z}} C_{u,t_1,\dots,t_{k-1}} e^{-\mathrm{i}\sum_{l=1}^{k-1}\omega_l t_l}$$

is well-defined, where $\omega_1,\dots,\omega_{k-1}\in[-\pi,\pi]$ and

$$C_{u,t_1,\dots,t_{k-1}} = \mathrm{Cum}\big(X_{t_1}^{(u)},\dots,X_{t_{k-1}}^{(u)},X_0^{(u)}\big) \tag{S1.4}$$

is the corresponding local cumulant operator of order k at time u. For $k =$

2, we obtain the time-varying spectral density operator $\mathscr{F}_{u,\omega}$ - the operator defined in (2.3) of the main paper - which is uniquely defined by the triangular array $\{X_{t,T} : 1 \le t \le T\}_{T\in\mathbb{N}}$ and twice-differentiable with respect to u and ω if (iv) of Assumption 3.1 of the main paper holds for $\ell = 2$ (see also Aue and van Delft (2019) for more details).

S2. Proof of Lemma 2.1

Proof of Lemma 2.1. Since $|||\cdot|||_2$ is induced by the Hilbert-Schmidt inner product, we have that

$$\begin{aligned}|||\mathscr{F}_{u,\omega} - \mathscr{G}_\omega|||_2^2 =& |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega|||_2^2 + \langle \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega\rangle_{\mathrm{HS}} \\ &+ \langle \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega\rangle_{\mathrm{HS}} + |||\widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega|||_2^2.\end{aligned}$$

By linearity and the definition of the Hilbert-Schmidt inner product,

$$\int_0^1 \langle \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega\rangle_{\mathrm{HS}} du = \left\langle \int_0^1 \mathscr{F}_{u,\omega} du - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega \right\rangle_{\mathrm{HS}} = 0.$$

A similar argument shows that $\int_0^1 \langle \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega\rangle_{\mathrm{HS}} du = 0$. Hence,

$$m^2 = \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega|||_2^2 du d\omega + \min_{\mathscr{G}} \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega|||_2^2 d\omega$$

and the infimum of the second term is achieved at $\mathscr{G}_\omega \equiv \widetilde{\mathscr{F}}_\omega$. The proof is complete. □

S3. Proof of Theorem 3.1

In this section we explain the main steps in the proof of Theorem 3.1. The proofs of various statements of this section are intricate and therefore postponed to Section S4 and Section S5. Let us recall that $T = NM$, where N defines the resolution in frequency of the local fDFT and M controls the number of nonoverlapping local fDFT's. To establish that $\sqrt{T}(\hat{m}_T - m^2) \xrightarrow{d} N(0, v^2)$ as $T \to \infty$ with v^2 given by (S5.19), we show that

$$\sqrt{T}[\mathbb{E}\,\hat{m}_T - m^2] \to 0, \tag{S3.5}$$

$$T \operatorname{Var} \hat{m}_T \to v^2, \tag{S3.6}$$

and

$$T^{n/2} \operatorname{cum}_n(\hat{m}_T) \to 0 \tag{S3.7}$$

for $n > 2$ as $T \to \infty$.

Recall first that $\hat{m}_T = 4\pi(\hat{F}_{1,T} - \hat{F}_{2,T} + \hat{B}_{N,T})$ and that Corollary S3.1, which is given below, implies the bias correction $\hat{B}_{N,T}$ does not affect the asymptotic distribution of $\hat{m}_T$. Therefore, the distributional properties of $\sqrt{T}(\hat{m}_T - m^2)$ will follow from the joint distributional structure of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$. In particular, multilinearity of cumulants implies that we have

$$\operatorname{cum}_n(\hat{m}_T) = (4\pi)^n \operatorname{cum}_n(\hat{F}_{1,T} - \hat{F}_{2,T}) = (4\pi)^n \sum_{x=0}^{n} (-1)^x \binom{n}{x} \operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}),$$

where $\text{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T})$ denotes the joint cumulant

$$\text{cum}(\underbrace{\hat{F}_{1,T}, \ldots, \hat{F}_{1,T}}_{x \text{ times}}, \underbrace{\hat{F}_{2,T}, \ldots, \hat{F}_{2,T}}_{n-x \text{ times}})$$

for $n, x \geq 0$.

The first two moments (S3.5)-(S3.6) can be determined by the cumulant structure of order $n = 1$ and $n = 2$ of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$, respectively, while (S3.7) will follow from showing that

$$T^{n/2} \, \text{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}) \to 0$$

as $T \to \infty$ for each $n > 2$ and $0 \leq x \leq n$.

The main ingredient to our proof is the following result which allows us to re-express the cumulants of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$, which consists of Hilbert-Schmidt inner products of local periodogram tensors, into the trace of cumulants of simple tensors of the local functional DFT's. The proof of Theorem S3.1 is given in Section S4.

Theorem S3.1. *Let* $\mathbb{E}|||I_N^{u,\omega}|||_2^{2n} < \infty$ *for some* $n \in \mathbb{N}$ *uniformly in* u *and* ω*. Then*

$$\text{cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS}, \ldots, \langle I_N^{u_{j_{2n-1}},\omega_{k_{2n-1}}}, I_N^{u_{j_{2n}},\omega_{k_{2n}}}\rangle_{HS}\Big)$$
$$\text{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G} S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G} \text{cum}\big(D_N^{u_{j_p},\omega_{k_p}} | p \in P_g\big)\Big)\Big),$$

where the summation is over all indecomposable partitions $\boldsymbol{P} = P_1 \cup \ldots \cup P_G$ *of* *the array*

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ (n,1) & (n,2) & (n,3) & (n,4) \end{array} \tag{S3.8}$$

where $p = (l,m)$ *and* $k_p = (-1)^m k_{2l-\delta\{m\in\{1,2\}\}}$ *and* $j_p = j_{2l-\delta\{m\in\{1,2\}\}}$ *for* $l \in \{1,\dots,n\}$ *and* $m \in \{1,2,3,4\}$. *Here the function* $\delta_{\{A\}}$ *equals 1 if event* A *occurs and* 0 *otherwise.*

In order to establish (S3.6) and (S3.7), it is of importance to be able to determine which indecomposable partitions of the array (S3.8) are vanishing in a structured fashion. The following two results allow us to exploit the structure of the array. The next lemma provides a global bound on the cumulants that is implied by the behavior of the joint cumulants of the local fDFT's for different midpoints (Lemma S4.2). For a fixed partition $P = \{P_1,\dots,P_G\}$ of the array, denote the size of the partition by G.

Lemma S3.1. *If Assumption 3.1 is satisfied then for finite* n,

$$T^{n/2}\,\mathrm{cum}_{n-x,x}(\hat{F}_{1,T},\hat{F}_{2,T}) = O(T^{1-n/2}N^{G-n-1})$$

uniformly in $0 \le x \le n$.

Lemma S3.1 implies that for $n = 2$ partitions with $G \le 2$ vanish, while for $n > 2$ all partitions of size $G \le n+1$ will vanish asymptotically. Moreover, indecomposability of the array requires to stay on the frequency manifold (see (S4.11) of Corollary S4.1) and therefore imposes additional restrictions in frequency direction. In case $n = 2$, only those partitions of size $G \geq 2$ for which all sets are such that $\sum_{k \in P_g} \omega_k \equiv 0 \mod 2\pi$ will not vanish. For $n > 2$, the indecomposability of the partition and Corollary S4.1 also result in restrictions over frequencies $k_1, \ldots, k_n$. These restrictions are formalized in the following proposition.

Proposition S3.1. *For a partition of size* $G = n + r_1 + 1$ *with* $r_1 \geq 1$ *of the array* (S4.14) *with* $n > 2$*, only partitions with at least* r_1 *restrictions in frequency direction are indecomposable. For* $n = 2$, $G = n + r_1 + 1$ *with* $r_1 \geq 1$ *will have at least 1 restriction in frequency direction.*

Together, Lemma S3.1 and Proposition S3.1 allow to show that all higher order cumulants vanish asymptotically and therefore asymptotic normality can be established.

Theorem S3.2. *Under Assumption 3.1, we have for all* $x = 0, \ldots, n$ *and* $n > 2$,

$$T^{n/2} \operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}) \to 0 \quad \textit{as } T \to \infty.$$

Proof. By Lemma S3.1, it is direct that all partitions of size $G \leq n + 1$ vanish. We therefore only have to consider the case where $G = n + r_1 + 1$ with $r_1 \geq 1$. In this case, Proposition S3.1, yields an upperbound of the joint cumulant that is of order $O(T^{1-n/2} N^{n+r_1+1-n-1} N^{-r_1}) = O(T^{1-n/2})$. This establishes (S3.7). □

Combining Theorem S3.2 with (S5.16), we immediately obtain the following result for the bias correction

Corollary S3.1. *Under the conditions of Theorem 3.1,*

$$\sqrt{T}\big(\hat{B}_{N,T} - B_{N,T}\big) \xrightarrow{p} 0 \quad as \quad T \to \infty.$$

The derivation of the asymptotic first and second order structure of $\sqrt{T}\hat{F}_{1,T}$ and $\sqrt{T}\hat{F}_{2,T}$ can be found in Section S5. A straightforward calculation then yields the asymptotic variance ν^2 is simply given by

$$\nu^2 = \lim_{T\to\infty}\Big(16\pi^2 \mathrm{Var}(\hat{F}_{1,T}) + 16\pi^2 \mathrm{Var}(\hat{F}_{2,T}) - 32\pi^2 \mathrm{Cov}(\hat{F}_{1,T}, \hat{F}_{2,T})\Big).$$

We therefore obtain the following expression for the asymptotic variance

$$\begin{aligned}
\nu^2 =& 4\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1} \bigotimes \mathscr{F}_{u,\omega_2}\rangle_{HS}\, du\, d\omega_1\, d\omega_2 \\
&+ 4\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \langle \widetilde{\mathscr{F}}_{\omega_1,-\omega_1,-\omega_2}, \widetilde{\mathscr{F}}_{\omega_1} \bigotimes \widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}\, d\omega_1\, d\omega_2 \\
&- 8\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1} \bigotimes \widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}\, du\, d\omega_1\, d\omega_2 \\
&+ 4\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1} \bigotimes \mathscr{F}_{u,-\omega_2}\rangle_{HS}\, du\, d\omega_1\, d\omega_2 \\
&+ 4\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \langle \widetilde{\mathscr{F}}_{\omega_1,-\omega_1,\omega_2}, \widetilde{\mathscr{F}}_{\omega_1} \bigotimes \widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS}\, d\omega_1\, d\omega_2 \\
&- 8\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1} \bigotimes \widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS}\, du\, d\omega_1\, d\omega_2 \\
&+ 8\pi \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}^2_{u,\omega}|||_2^2\, du\, d\omega + 4\pi \int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}^2_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}\, du\, d\omega \\
&+ 4\pi \int_{-\pi}^{\pi}\int_0^1 \langle \widetilde{\mathscr{F}}_{\omega}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{u,\omega}\rangle_{HS}\, du\, d\omega \\
&- 16\pi \int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}\, du\, d\omega
\end{aligned}$$

$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}_{u,\omega}|||_2^4 du d\omega$$
$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega},\mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,\omega}\rangle_{HS} du d\omega$$
$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega},\mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,-\omega}\rangle_{HS} du d\omega$$
$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}}_{\omega}\bigotimes\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega$$
$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega},\widetilde{\mathscr{F}}_{\omega}\bigotimes\widetilde{\mathscr{F}}_{-\omega}\rangle_{HS} du d\omega$$
$$-8\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega},\mathscr{F}_{u,\omega}\bigotimes\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega$$
$$-8\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega},\mathscr{F}_{u,\omega}\bigotimes\widetilde{\mathscr{F}}_{-\omega}\rangle_{HS} du d\omega, \tag{S3.9}$$

Under H_0 this reduces to $\nu_{H_0}^2 = 4\pi\int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}}_{\omega}|||_2^4 d\omega$.

S4. Bounds on cumulant tensors of local functional DFT

The following lemma shows that the cumulant tensor of the local fDFT's evaluated at the same midpoint u_i and on the manifold $\sum_{j=1}^{k}\omega_j \equiv 0 \mod 2\pi$ can in turn be expressed in terms of higher order cumulant spectral operators.

Lemma S4.1. *If Assumption 3.1 is satisfied and* $\sum_{j=1}^{k}\omega_j \equiv 0 \mod 2\pi$ *then*

$$\left|\left|\left|Cum\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right)-\frac{(2\pi)^{1-k/2}}{N^{k/2-1}}\mathscr{F}_{u_i,\omega_1,\ldots,\omega_{k-1}}\right|\right|\right|_1 = O\left(N^{-k/2}\times\frac{N}{M^2}\right).$$

When evaluated off the manifold, i.e., $\sum_{j=1}^{k}\omega_j \neq 0 \mod 2\pi$, the above cumulant is of lower order (see Corollary S4.1). A direct consequence of the proof of Lemma S4.1 is the following corollary

Corollary S4.1. *We have for any* $p \geq 1$

$$\left|\left|\left|Cum\left(D_N^{u_1,\omega_1},\ldots,D_N^{u_k,\omega_k}\right)\right|\right|\right|_p = O\left(N^{1-k/2}\right) \tag{S4.10}$$

uniformly in $\omega_1,\ldots,\omega_k$ *and* $u_1,\ldots,u_k$*. Moreover, if* $\sum_{j=1}^k \omega_j \neq 0 \mod 2\pi$ *then*

$$\left|\left|\left|Cum\left(D_N^{u_1,\omega_1},\ldots,D_N^{u_k,\omega_k}\right)\right|\right|\right|_p = O\left(N^{-k/2}\right). \tag{S4.11}$$

Before we give the proofs, denote the function $\Delta^{(N)}(\omega) = \sum_{t=0}^{N-1} e^{-\mathrm{i}\omega t}$ for $\omega \in \mathbb{R}$. This function satisfies $|\Delta^{(N)}(\sum_{j=1}^k \omega_j)| = N$ for any $\omega_1,\ldots,\omega_k$ for which their sum lies on the manifold $\omega \equiv 0 \mod 2\pi$, while it is of reduced magnitude off the manifold. For the canonical frequencies $\omega_k = 2\pi k/N$ with $k \in \mathbb{Z}$, we moreover have

$$\Delta^{(N)}(\omega_k) = \begin{cases} N, & k \in N\mathbb{Z}; \\ 0, & k \in \mathbb{Z} \setminus N\mathbb{Z}. \end{cases} \tag{S4.12}$$

Proof of Lemma S4.1. Let $p \in \{1,2\}$. Using linearity of cumulants we write

$$\begin{aligned} &\mathrm{Cum}\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right) \\ =&\frac{1}{(2\pi N)^{k/2}} \sum_{s_1,\ldots,s_k=0}^{N-1} \exp\left(-\mathrm{i}\sum_{j=1}^k s_j\omega_j\right) \mathrm{Cum}\left(X_{\lfloor u_i T\rfloor - N/2+s_1+1,T},\ldots,X_{\lfloor u_i T\rfloor - N/2+s_k+1,T}\right) \\ =&\frac{1}{(2\pi N)^{k/2}} \sum_{s_1,\ldots,s_k=0}^{N-1} \exp\left(-\mathrm{i}\sum_{j=1}^k s_j\omega_j\right) C_{u_i-\frac{N/2-s_1-1}{T},s_1-s_k,s_2-s_k,\ldots,s_{k-1}-s_k} + R^1_{k,M,N}, \end{aligned} \tag{S4.13}$$

where $C_{u,t_1,\ldots,t_{k-1}}$ denotes the local cumulant operator as defined in (S1.4). Using Lemma S.2.1 of Aue and van Delft (2019) and Assumption 3.1,

$$\left\|\left\|R^1_{k,M,N}\right\|\right\|_p \leq \frac{1}{(2\pi N)^{k/2}} \sum_{s_1,\ldots,s_k=0}^{N-1} \left(\frac{k}{T} + \sum_{j=1}^{k-1} \frac{|s_j - s_k|}{T}\right) |||\kappa_{k,s_1-s_k,\ldots,s_{k-1}-s_k}|||_p$$
$$\leq \frac{1}{(2\pi N)^{k/2}} \sum_{s_k=0}^{N-1} \frac{1}{T} \sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}} \left(1 + \sum_{j=1}^{k-1} |l_j|\right) |||\kappa_{k,l_1,\ldots,l_{k-1}}|||_p = O\left(N^{-k/2} M^{-1}\right).$$

In addition, we can write the first term of (S4.13) as

$$= \frac{1}{(2\pi N)^{k/2}} \sum_{s_1,\ldots,s_k=0}^{N-1} \exp\left(-\mathrm{i}s_k \sum_j \omega_j\right) \exp\left(-\mathrm{i} \sum_{j=1}^{k-1} \omega_j (s_j - s_k)\right) C_{u_i - \frac{N/2 - s_k - 1}{T}, s_1 - s_k, s_2 - s_k, \ldots, s_{k-1} - s_k}$$
$$= \frac{(2\pi)^{1-k/2}}{N^{k/2}} \sum_{s=0}^{N-1} e^{-\mathrm{i}s \sum_j \omega_j} \mathscr{F}_{u_i - \frac{N/2 - s - 1}{T}, \omega_1, \ldots, \omega_{k-1}} + R^2_{k,M,N},$$

where

$$\left\|\left\|R^2_{k,M,N}\right\|\right\|_p \leq \frac{1}{(2\pi N)^{k/2}} \sum_{s=0}^{N-1} \sum_{\substack{j:1,\ldots k-1: \\ |s_j - s| \geq N-1}} \left\|\left\| C_{u - \frac{N/2 - s - 1}{T}, s_1 - s, s_2 - s, \ldots, s_{k-1} - s} \right\|\right\|_p$$
$$\leq \frac{1}{(2\pi N)^{k/2}} \sum_{s=0}^{N-1} \sum_{\substack{j:1,\ldots k-1: \\ |s_j - s| \geq N-1}} |||\kappa_{k,s_1-s,s_2-s,\ldots,s_{k-1}-s}|||_p.$$
$$\leq \frac{1}{(2\pi N)^{k/2}} \sum_{s=0}^{N-1} \frac{1}{N^2} \sum_{\substack{j:1,\ldots k-1: \\ |l_j| > N}} N^2 |||\kappa_{k,l_1,l_2,\ldots,l_{k-1}}|||_p$$
$$\leq \frac{1}{(2\pi N)^{k/2}} \frac{1}{N} \sum_{\substack{j:1,\ldots k-1: \\ |l_j| > N}} |l_j|^2 |||\kappa_{k,l_1,l_2,\ldots,l_{k-1}}|||_p = o\left(N^{-k/2}\right).$$

Therefore, the cumulants satisfy

$$\mathrm{Cum}\left(D_N^{u_i,\omega_1}, \ldots, D_N^{u_i,\omega_k}\right) = \frac{(2\pi)^{1-k/2}}{N^{k/2}} \sum_{s=0}^{N-1} e^{-\mathrm{i}s \sum_j \omega_j} \mathscr{F}_{u_i - \frac{N/2 - s - 1}{T}, \omega_1, \ldots, \omega_{k-1}} + R^1_{k,M,N} + R^2_{k,M,N}.$$

On the manifold $\sum_{j=1}^{k} \omega_j \equiv 0 \mod 2\pi$, we have that $e^{-\mathrm{i}s\sum_j \omega_j} = 1$. Assumption 3.1(iv) and a Taylor expansion yield

$$\mathrm{Cum}\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right) = \frac{(2\pi)^{k/2-1}}{N^{k/2-1}} \mathscr{F}_{u_i,\omega_1,\ldots,\omega_{k-1}} + R^1_{k,M,N} + R^2_{k,M,N} + R^3_{k,M,N},$$

where

$$\begin{aligned}
|||R^3_{k,M,N}|||_p &= \left|\left|\left| \frac{(2\pi)^{k/2-1}}{N^{k/2}} \sum_{\ell=1}^{2} \sum_{s=0}^{N-1} \left(\frac{1-N/2+s}{T}\right)^{\ell} \frac{\partial^\ell}{\partial u^\ell} \mathscr{F}_{u,\omega_1,\ldots,\omega_k} \right|\right|\right|_p \\
&\leq \frac{(2\pi)^{k/2-1}}{N^{k/2}} O\left(\frac{N}{T} + \frac{N}{M^2}\right) \sum_{\ell=1}^{2} \sup_{u,\omega_1,\ldots,\omega_k} \left|\left|\left| \frac{\partial^\ell}{\partial u^\ell} \mathscr{F}_{u,\omega_1,\ldots,\omega_k} \right|\right|\right|_p = O\left(N^{-k/2}\left(\frac{N}{T} + \frac{N}{M^2}\right)\right),
\end{aligned}$$

which follows since $\sum_{s=0}^{N-1}\left(\frac{1-N/2+s}{T}\right) = \frac{(N-1)(1-N/2+N/2)}{T} = \frac{N}{2T}$ and similarly $\sum_{s=0}^{N-1}\left(\frac{1-N/2+s}{T}\right)^2 = O\left(\frac{N^3}{T^2}\right)$. We additionally note that, off manifold, the first term of (S4.13) can be bounded in norm by

$$\begin{aligned}
&= \left|\left|\left| \frac{1}{(2\pi N)^{k/2}} \sum_{s_1,\ldots,s_k=0}^{N-1} \exp\left(-\mathrm{i}s_k \sum_j \omega_j\right) \exp\left(-\mathrm{i}\sum_{j=1}^{k-1} \omega_j (s_j - s_k)\right) C_{u_i - \frac{N/2-s_k-1}{T}, s_1-s_k, s_2-s_k,\ldots,s_{k-1}-s_k} \right|\right|\right|_p \\
&\leq \frac{K}{(2\pi N)^{k/2}} \sum_{|l_1|\ldots,|l_{k-1}|<N} \left| \sum_{s_k=0}^{N-|l_j^*|} e^{-\mathrm{i}s_k \sum_j \omega_j} \right| \left|\left|\left| C_{u_i - \frac{N/2-s_k-1}{T}, l_1, l_2,\ldots,l_{k-1}} \right|\right|\right|_p \\
&\leq \frac{K}{(2\pi N)^{k/2}} \sum_{|l_1|\ldots,|l_{k-1}|<N} |l_j^*| \, |||\kappa_{k;t_1,\ldots,t_{l-1}}|||_p = O\left(N^{-k/2}\right),
\end{aligned}$$

for some constant K and where $j^* = \underset{j=1,\ldots,k-1}{\arg\max} |l_j|$. This finishes the proof of Lemma S4.1. □

Additionally, when the local fDFT's are evaluated on different midpoints then we have the following lemma.

Lemma S4.2. *If Assumption 3.1 is satisfied and* $|j_1 - j_2| > 1$ *for some midpoints* u_{j_1} *and* u_{j_2} *then*

$$\left\|\left\| Cum\left(D_N^{u_{j_1},\omega_1},\dots,D_N^{u_{j_k},\omega_k}\right)\right\|\right\|_1 = O\left(N^{-k/2}M^{-1}\right)$$

uniformly in $\omega_1,\dots,\omega_k$.

Proof of Lemma S4.2. Using again the linearity of cumulants we write

$$\begin{aligned}
&\mathrm{Cum}\left(D_N^{u_{j_1},\omega_1},\dots,D_N^{u_{j_k},\omega_k}\right)\\
&\quad= \frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\dots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{v=1}^{k}s_v\omega_v\right)\mathrm{Cum}\left(X_{\lfloor u_{j_1}T\rfloor-N/2+s_1+1,T},\dots,X_{\lfloor u_{j_k}T\rfloor-N/2+s_k+1,T}\right)\\
&\quad= \frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\dots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{v=1}^{k}s_v\omega_v\right)C_{u'_k,\lfloor u_{j_1}T\rfloor-\lfloor u_{j_k}T\rfloor+s_1-s_k,\dots,\lfloor u_{j_{k-1}}T\rfloor-\lfloor u_kT\rfloor+s_{k-1}-s_k}+R^1_{k,M,N}
\end{aligned}$$

where $u'_k = u_k - \frac{N/2-s_k-1}{T}$ and where $R^1_{k,M,N}$ is the error term derived in Lemma S4.1. Let

$$l_m = \lfloor u_{j_m}T\rfloor - \lfloor u_kT\rfloor + s_m - s_k \leftrightarrow s_m = t_{j_k} - t_{j_m} + l_m - s_k \qquad m = 1,\dots k-1$$

Similar to the proof of Lemma S4.1, we note that

$$\sum_{\substack{l_1,l_2,\dots l_{k-1},\\ |l_m|>N}} |||C_{u'_k,l_1,\dots,l_m,\dots,l_{k-1}}|||_p \le \sum_{\substack{l_1,l_2,\dots l_{k-1}\\ ,|l_m|>N}} \frac{|l_m|^2}{N^2}|||\kappa_{k,l_1,\dots,l_{k-1}}|||_p = O(N^{-2}).$$

From which it follows that if $|l_m| > N$, the term

$$\Big\|\Big\|\frac{1}{(2\pi N)^{k/2}}\sum_{s_k=0}^{N-1} e^{-\mathrm{i}s_k\sum_{v=1}^{k}\omega_v}\sum_{\substack{l_1,l_2,\ldots l_{k-1},\\ |l_m|>N}} e^{-\mathrm{i}\sum_{v=1}^{k-1}(t_{j_v}-t_{j_k}+l_v)\omega_v} C_{u'_k,l_1,\ldots,l_{k-1}}\Big\|\Big\|_p$$

is bounded by

$$\begin{aligned}
&\frac{1}{(2\pi N)^{k/2}}\sum_{s=0}^{N-1}\sum_{\substack{l_1,l_2,\ldots,l_{k-1}\\ |l_m|\geq N}} |||\kappa_{k,l_1,\ldots,l_m,\ldots,l_{k-1}}|||_p \\
&\leq \frac{1}{(2\pi N)^{k/2+2}}\sum_{s=0}^{N-1}\sum_{\substack{l_1,l_2,\ldots,l_{k-1}\\ |l_1|\geq N}} |l_m|^2|||\kappa_{k,l_1,\ldots,l_m,\ldots,l_{k-1}}|||_p = O\Big(N^{-k/2-1}\Big).
\end{aligned}$$

□

Proof of Theorem S3.1. First note that a sufficient condition for $\mathbb{E}|||I_N^{u,\omega}|||_2^p < \infty$ to exist is $\mathbb{E}\|D_N^{u,\omega}\|_2^{2p} < \infty$ or, in terms of moments of X, $\mathbb{E}\|X_{t,T}\|_2^{2p} < \infty$ for each $T \geq 1$, $1 \leq t \leq T$ and hence by Assumption 3.1

$$\mathrm{Cum}_n(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS}) \leq \prod_{l=1}^{n}\sum_{k_l=1}^{\lfloor N/2\rfloor}\sum_{j_l=1}^{M}\sqrt{\mathbb{E}|||I_N^{u_{j_l},\omega_{k_l}}|||_2^2}\sqrt{\mathbb{E}|||I_N^{u_{j_2},\omega_{k_2}}|||_2^2} < \infty.$$

The definition of scalar cumulants and a basis expansion yield

$$\begin{aligned}
&\mathrm{Cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS},\ldots,\langle I_N^{u_{j_{2n-1}},\omega_{k_{2n-1}}}, I_N^{u_{j_{2n}},\omega_{k_{2n}}}\rangle_{HS}\Big) \\
&\quad= \sum_{\nu=(\nu_1,\ldots,\nu_G)}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{(l,m)\in\nu_g}\langle I_N^{u_{j_l},\omega_{k_l}}, I_N^{u_{j_m},\omega_{k_m}}\rangle_{HS} \\
&\quad= \sum_{\nu=(\nu_1,\ldots,\nu_G)}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{(l,m)\in\nu_g}\mathrm{Tr}\Big((D_N^{u_{j_l},\omega_{k_l}}\otimes D_N^{u_{j_l},-\omega_{k_l}})\bigotimes(D_N^{u_{j_m},\omega_{k_m}}\otimes D_N^{u_{j_m},-\omega_{k_m}})\Big)
\end{aligned}$$

$$= \sum_{\nu=(\nu_1,\ldots,\nu_G)} (-1)^{G-1}(G-1)! \prod_{g=1}^{G} \mathbb{E} \prod_{(l,m)\in\nu_g} \mathrm{Tr}\Big((D_N^{u_{j_l},\omega_{k_l}} \otimes D_N^{u_{j_l},-\omega_{k_l}} \otimes D_N^{u_{j_m},-\omega_{k_m}} \otimes D_N^{u_{j_m},\omega_{k_m}}\Big),$$

where the summation extends over all unordered partitions $(\nu_1,\ldots,\nu_G), G = 1,\ldots,n$. The fact that the expectation operator commutes with the trace operator together with property S1.1.4 implies this can be written as

$$\mathrm{Tr}\Big(\sum_{\nu=(\nu_1,\ldots,\nu_G)} (-1)^{G-1}(G-1)! \widetilde{\bigotimes}_{g=1}^{G} \mathbb{E}\Big[\widetilde{\bigotimes}_{(l,m)\in\nu_g}((D_N^{u_{j_l},\omega_{k_l}} \otimes D_N^{u_{j_l},\omega_{k_l}}) \otimes (D_N^{u_{j_m},\omega_{k_m}} \otimes D_N^{u_{j_m},\omega_{k_m}}))\Big]\Big)$$

The product theorem for cumulant tensors (equation (S1.3)) then shows this equals

$$\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G} S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G} \mathrm{Cum}\big(D_N^{u_{j_p},\omega_{k_p}} | p \in \nu_g\big)\Big)\Big).$$

Here, the summation is over all indecomposable partitions $\boldsymbol{P} = P_1 \cup \ldots \cup P_G$ of the array

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ \vdots & & & \vdots \\ (n,1) & (n,2) & (n,3) & (n,4) \end{array} \tag{S4.14}$$

where $S_{\boldsymbol{P}}$ denote the permutation operator on $\otimes_{i=1}^{4n} L^2([0,1],\mathbb{C})$ that maps the components of the tensor according to the permutation $(1,\ldots,4n) \mapsto \boldsymbol{P}$ and where $p = (l,m)$, $k_p = (-1)^m k_{2l-\delta\{m\in\{1,2\}\}}$ and $j_p = j_{2l-\delta\{m\in\{1,2\}\}}$ for $l \in \{1,\ldots,n\}$ and $m \in \{1,2,3,4\}$. Here the function $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. $\square$

Lemma S4.3. *If Assumption 3.1 is satisfied then for finite n,*

$$T^{n/2}\,\mathrm{cum}_{n-x,x}(\hat{F}_{1,T},\hat{F}_{2,T}) =$$
$$\frac{1}{T^{n/2}M^{x}}\sum_{k_1,\dots,k_n=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,\dots,j_n\\ j_{n+1},\dots,j_{n+x}=1}}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\dots\cup P_G}S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G}\mathrm{cum}\big(D_N^{u_{j_p},\omega_{k_p}}|p\in\nu_g\big)\Big)\Big)$$
$$= O(T^{1-n/2}N^{G-n-1})$$

uniformly in $0\le x\le n$.

Proof of Lemma S4.3. For a fixed partition $P=\{P_1,\dots,P_G\}$, let the cardinality of set P_g be denoted by $|P_g|=\mathscr{C}_g$. By (S4.10) of Corollary S4.1 and Lemma S4.2 an upperbound of (S4.14) is given by

$$O\Big(T^{-n/2}M^{-x}\sum_{k_1,\dots,k_n=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,\dots,j_n\\ j_{n+1},\dots,j_{n+x}=1}}^{M}\prod_{g=1}^{G}\frac{1}{N^{\mathscr{C}_g/2-1}}M^{-\delta_{\{\exists p_1,p_2\in P_g:|j_{p_1}-j_{p_2}|>1\}}}\Big) \quad \text{(S4.15)}$$

Note that $|\mathscr{C}_g|\ge 2$ and that the partition must be indecomposable. We can therefore assume, without loss of generality, that row l hooks with row $l+1$ for $l=1,\dots,n-1$, i.e., within each partition there must be at least one set P_g that contains an element from both rows. For fixed j_l, there are only finitely many possibilities, say E, for j_{l+1} (Lemma S4.2). If the set does not cover another row, then the fact that j_l is fixed and j_{l+1} are fixed, another set must contain at least an element from row l or $l+1$. But since the sets must communicate there are only finitely many options for j_{l+2}. If, on other hand, the same set covers elements from yet another row then given a fixed j_l, there are again finitely many options for j_{l+1} and for j_{l+2}. This argument can be continued inductively to find (S4.15) is of order

$$O(N^{n/2} M^{-n/2-x} E^n M^{1+x} N^{-2n+G}) = O(T^{1-n/2} N^{G-n-1}). \qquad \square$$

Proposition S4.1. *For a partition of size $G = n + r_1 + 1$ with $r_1 \geq 1$ of the array* (S4.14) *with $n > 2$, only partitions with at least r_1 restrictions in frequency direction are indecomposable. For $n = 2$, $G = n + r_1 + 1$ with $r_1 \geq 1$ will have at least 1 restriction in frequency direction.*

Proof of Proposition S4.1. First consider the case $n > 2$. We note that a minimal amount of restrictions will be given by those partitions in which frequencies and their conjugates are always part of the same set, i.e., in which for fixed row l, the first two columns are in the same set and the last two columns are in the same set. Given we need that $G \geq n+2$ and $\mathscr{C}_g \geq 2$, indecomposability of the array means that the smallest number of restrictions is given by partitions that have one large set that covers the first two or last two columns and $n - r_1$ rows and for the rest has $\frac{4n-2(n-r_1)}{2} = n + r_1$ sets with $\mathscr{C}_g = 2$. This means there are no constraints in frequency in $n - r_1 - 1$ rows but for the array to hook there must be r_1 constraints in terms of frequencies in rows $n - r_1 - 1$ to row n.

Consider then the special case of $n = 2$, for which the above argument implies a partition of size $G = 3$ and thus $r_1 = 0$. For partitions are of size $G \geq 4$, indecomposability then requires the first row to hook with the second, which imposes at least one restriction in frequency direction since only those partitions for which $\sum_{k\in P_g}\omega_k \equiv 0 \mod 2\pi$ will not vanish. □

S5. Derivation of expectation and covariance

For (S3.5), Theorem S3.1 implies we can write

$$\mathbb{E}\hat{F}_{1,T} = \frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(\mathbb{E}\big[D_N^{u_{j_1},\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1-1}}\otimes D_N^{u_{j_1},\omega_{k_1-1}}\big]\Big).$$

Expressing this expectation in cumulant tensors, we get

$$\begin{aligned}\mathbb{E}\hat{F}_{1,T} = &\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1234}\Big(\mathrm{Cum}\big((D_N^{u_{j_1},\omega_{k_1}},D_N^{u_{j_1},-\omega_{k_1}},D_N^{u_{j_1},-\omega_{k_1-1}},D_N^{u_{j_1},\omega_{k_1-1}})\Big)\Big)\\ &+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1234}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}},D_N^{u_{j_1},-\omega_{k_1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1-1}},D_N^{u_{j_1},\omega_{k_1-1}})\Big)\Big)\\ &+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1324}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}},D_N^{u_{j_1},-\omega_{k_1-1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1}},D_N^{u_{j_1},\omega_{k_1-1}})\Big)\Big)\\ &+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1423}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}},D_N^{u_{j_1},\omega_{k_1-1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1}},D_N^{u_{j_1},-\omega_{k_1-1}})\Big)\Big),\end{aligned}$$

where S_{ijkl} denotes the permutation operator on $\otimes_{i=1}^{4}L^2([0,1],\mathbb{C})$ that maps the components of the tensor according to the permutation $(1,2,3,4)\mapsto(i,j,k,l)$. By Lemma S4.1 and Corollary S4.1, we thus find

$$\mathbb{E}\hat{F}_{1,T} = \frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\langle\mathscr{F}_{u_j,\omega_k},\mathscr{F}_{u_j,\omega_{k-1}}\rangle_{HS} + O(M^{-2}) + O(N^{-1}).$$

Similarly, for $\hat{F}_{2,T}$ we obtain

$$\begin{aligned}
\mathbb{E}\hat{F}_{2,T} =& \frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\mathrm{Cum}\big((D_N^{u_{j_1},\omega_k}, D_N^{u_{j_1},-\omega_k}, D_N^{u_{j_2},-\omega_k}, D_N^{u_{j_2},\omega_k})\big)\\
&+\frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(S_{1234}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_k}, D_N^{u_{j_1},-\omega_k})\otimes\mathrm{Cum}(D_N^{u_{j_2},-\omega_k}, D_N^{u_{j_2},\omega_k})\Big)\Big)\\
&+\frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(S_{1324}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_k}, D_N^{u_{j_2},-\omega_k})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_k}, D_N^{u_{j_2},\omega_k})\Big)\Big)\\
&+\frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(S_{1423}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_k}, D_N^{u_{j_2},\omega_k})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_k}, D_N^{u_{j_2},-\omega_k})\Big)\Big).
\end{aligned}$$

Corollary S4.1 and Lemma S4.2 then yield

$$\begin{aligned}
\mathbb{E}\hat{F}_{2,T} =& \frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\langle\mathscr{F}_{u_{j_1},\omega_k},\mathscr{F}_{u_{j_2}\omega_k}\rangle_{HS}+\frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1=1}^{M}\mathrm{Tr}\Big(S_{1324}\big(\mathscr{F}_{u_j,\omega_k}\bigotimes\mathscr{F}_{u_j,\omega_k}\big)\Big)\\
&+O(\frac{1}{T})+O(\frac{1}{M^2}).
\end{aligned}$$

Note that the permutation operator implies $\mathrm{Tr}\Big(S_{1324}\big(\mathscr{F}_{u_j,\omega_k}\otimes\mathscr{F}_{u_j,\omega_k}\big)\Big)=\mathrm{Tr}\big(\mathscr{F}_{u_j,\omega_k}\widetilde{\otimes}\mathscr{F}_{u_j,\omega_k}\big)$. Therefore, with slight abuse of notation,

$$\lim_{N,M\to\infty}\mathbb{E}\,\hat{F}_{2,T}=\frac{1}{4\pi}\int_{-\pi}^{\pi}\langle\int_0^1\mathscr{F}_{u,\omega}du,\int_0^1\mathscr{F}_{u,\omega}du\rangle_{HS}\mathrm{d}u\mathrm{d}\omega+\frac{N}{T}B_{N,T},$$

where the term $B_{N,T}$ is defined in (2.15). In complete analogy with the derivation of $\mathbb{E}\hat{F}_{1,T}$, we find that the estimator $\hat{B}_{N,T}$ defined in (2.17) is asymptotically unbiased, i.e.,

$$\lim_{T\to\infty}\mathbb{E}\hat{B}_{N,T}=B_{N,T}. \tag{S5.16}$$

Summarizing, we obtain

$$\sqrt{T}\Big[4\pi\mathbb{E}(\hat{F}_{1,T}-\hat{F}_{2,T}+\frac{N}{T}\hat{B}_{N,T})-\Big(\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}_{u,\omega}|||_2^2 du d\omega-\int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}_{\omega}}|||_2^2 d\omega\Big)\Big]\to 0$$

as $T\to\infty$, provided Assumption 2.1 is satisfied.

S5.1 Covariance structure of $\sqrt{T}\hat{F}_{1,T}$

$$T\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{1,T})=T\mathrm{Cum}\Big(\frac{1}{T}\sum_{k_1=1}^{\lfloor N/2\rfloor}\sum_{j_1=1}^{M}\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS},\frac{1}{T}\sum_{k_2=1}^{\lfloor N/2\rfloor}\sum_{j_2=1}^{M}\overline{\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}}\Big).$$

Using again Theorem S3.1

$$\mathrm{Cum}_2(\sqrt{T}\hat{F}_{1,T})=\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G}\mathrm{cum}\big(D_N^{u_{j_p},\omega_{k_p}}|p\in\nu_g\big)\Big)\Big),$$

where $p=(l,m)$ with $k_p=(-1)^{l-m}k_l-\delta_{\{m\in\{3,4\}\}}$ and $j_p=j_l$ for $l\in\{1,2\}$ and $m\in\{1,2,3,4\}$ and where $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. In particular, we are interested in all indecomposable partitions of the array

$$\begin{array}{cccc}\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{1} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{2} & \underbrace{D_N^{u_{j_1},-\omega_{k_1-1}}}_{3} & \underbrace{D_N^{u_{j_1},\omega_{k_1-1}}}_{4}\\ \underbrace{D_N^{u_{j_2},-\omega_{k_2}}}_{5} & \underbrace{D_N^{u_{j_2},\omega_{k_2}}}_{6} & \underbrace{D_N^{u_{j_2},\omega_{k_2-1}}}_{7} & \underbrace{D_N^{u_{j_2},-\omega_{k_2-1}}}_{8}\end{array}$$

By Lemma S4.3, all partitions of size $G<3$, will be of lower order. By Proposition S4.1), the only partitions that remain are those that contain either one fourth order cumulant and two second order cumulants or those consisting only of second order cumulants. Additionally, Corollary S4.1 and Lemma S4.2

imply that, in order for the partitions with structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$ to be indecomposable, there must be at least one restriction in time. More restrictions in terms of frequency would mean the partition term is of lower order.

For the structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are therefore

$$\mathrm{Tr}\Big(S_{(1256)(34)(78)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(1278)(34)(56)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2-1}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3456)(12)(78)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3478)(12)(56)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2-1}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big),$$

where $\mathscr{E}_k$ denotes an operator on $L^2_{\mathbb{C}}([0,1]^{\lfloor k/2\rfloor})$ that satisfies $|||\mathscr{E}_k|||_1 = O(N^{-k/2}\times\frac{N}{M^2})$. For the partitions with structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, there must be at least one restriction in terms of time and frequency for the partition to be indecomposable. Those with more than the minimum restrictions are of lower order. For the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the significant indecomposable partitions are

$$\mathrm{Tr}\Big(S_{(12)(37)(56)(48)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(12)(36)(78)(45)}\Big(\delta_{j_1,j_2}\delta_{k_1-1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2-1}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(15)(26)(37)(48)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(15)(26)(34)(78)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(18)(27)(34)(56)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2-1}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\epsilon_2\big]\Big).$$

Using Remark S5.1 and in particular equation (S5.17) and (S5.18) below, the corresponding terms of the covariance, $T\mathrm{Cov}(\hat{F}_{1,T}, \hat{F}_{1,T})$ equal

$$
\begin{aligned}
&\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\Big[\Big\langle \frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}, \mathscr{F}_{u_{j_1},\omega_{k_1-1}}\bigotimes \mathscr{F}_{u_{j_2},\omega_{k_2-1}}\Big\rangle_{HS} + O(\frac{1}{T})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\Big[\Big\langle \frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2-1}}, \mathscr{F}_{u_{j_1},\omega_{k_1-1}}\bigotimes \mathscr{F}_{u_{j_2},-\omega_{k_2}}\Big\rangle_{HS} + O(\frac{1}{T})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\Big[\Big\langle \frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}}, \mathscr{F}_{u_{j_1},-\omega_{k_1}}\bigotimes \mathscr{F}_{u_{j_2},\omega_{k_2-1}}\Big\rangle_{HS} + O(\frac{1}{T})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\Big[\Big\langle (\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2-1}}, \mathscr{F}_{u_{j_1},-\omega_{k_1}}\bigotimes \mathscr{F}_{u_{j_2},-\omega_{k_2}}\Big\rangle_{HS} + O(\frac{1}{T})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\delta_{k_1,k_2}\Big[\langle \mathscr{F}^{\dagger}_{u_{j_1},-\omega_{k_1}}\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}, \mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\mathscr{F}^{\dagger}_{u_{j_2},-\omega_{k_2}}\rangle + O(\frac{1}{M^2})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\Big[\langle \mathscr{F}^{\dagger}_{u_{j_1},-\omega_{k_1}}\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}, \mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\mathscr{F}_{u_{j_2},-\omega_{k_2-1}}\rangle_{HS} + O(\frac{1}{M^2})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\delta_{k_1,k_2}\Big[\langle \mathscr{F}_{u_{j_1},\omega_{k_1}}, \mathscr{F}_{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle \mathscr{F}_{u_{j_1},-\omega_{k_1}}, \mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\rangle_{HS} + O(\frac{1}{M^2})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\delta_{k_1,k_2}\Big[\langle \mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}\mathscr{F}_{u_{j_1},\omega_{k_1}}, \mathscr{F}_{u_{j_1},\omega_{k_1-1}}\bigotimes \mathscr{F}_{u_{j_2},\omega_{k_2-1}}\rangle_{HS} + O(\frac{1}{M^2})\Big]\\
&+\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\delta_{j_1,j_2}\delta_{k_1,k_2-1}\Big[\langle \mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u_{j_1},-\omega_{k_1}}, \mathscr{F}_{u_{j_1},\omega_{k_1-1}}\bigotimes \mathscr{F}_{u_{j_2},-\omega_{k_2}}\rangle_{HS} + O(\frac{1}{M^2})\Big].
\end{aligned}
$$

Then, using self-adjointness of the spectral density operator and that, for any function $g:\mathbb{R}\to\mathbb{K}$, we have $\int_{-\pi}^{\pi} g(\omega)d\omega = \int_{-\pi}^{\pi} g(-\omega)d\omega$, it follows that the first and fourth term, the second and third term, the fifth and sixth term of the above expression are, respectively, equal in the limit. Hence, as $N, M\to\infty$,

$$
\begin{aligned}
NM\mathrm{Cov}(\hat{F}_{1,T}, \hat{F}_{1,T}) \to &\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \Big\langle \mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \mathscr{F}_{u,\omega_2}\Big\rangle_{HS} du d\omega_1 d\omega_2\\
&+\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \Big\langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \mathscr{F}_{u,-\omega_2}\Big\rangle_{HS} du d\omega_1 d\omega_2
\end{aligned}
$$

$$+\frac{2}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}^{2}_{u,\omega}|||_2^2 du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}_{u,\omega}|||_2^4 du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,\omega}\rangle_{HS} du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,-\omega}\rangle_{HS} du d\omega.$$

Remark S5.1 (**A note on the permutation for the 2nd order cumulants**). In order to give meaning to the covariance structure, we need to investigate how it 'operates' as a result of the permutation that occurs due to the cumulant operation. For the second order cumulant structure, Theorem S3.1] implies that the original order of the simple tensors has structure $\mathrm{Tr}(S_{1234}\cdot\otimes\cdot\otimes\cdot\otimes\cdot\widetilde{\bigotimes}S_{5678}\cdot\otimes\cdot\otimes\cdot\otimes\cdot)$ which leads to the following correspondence of simple tensors

$$\begin{aligned}1 &\leftrightarrow 3\\ 2 &\leftrightarrow 4\\ 5 &\leftrightarrow 7\\ 6 &\leftrightarrow 8\end{aligned}$$

Let $X\in\mathscr{H}^{\otimes^4}$ and $Y,Z,X\in\mathscr{H}^{\otimes^2}$, then using properties S1.1 we find

$$\begin{aligned}
\mathrm{Tr}\Big(S_{(1256)(12)(56)}X\otimes Y\otimes Z) &= \mathrm{Tr}\big(X(\overline{Y}\bigotimes Z)^{\dagger}\big) = \langle X,\overline{Y}\bigotimes Z\rangle_{HS}\\
\mathrm{Tr}\Big(S_{(1256)(12)(56)}X\otimes Y\otimes Z) &= \mathrm{Tr}\big(X(\overline{Y}\bigotimes Z)^{\dagger}\big) = \langle X,\overline{Y}\bigotimes Z\rangle_{HS}\\
\mathrm{Tr}\Big(S_{(1256)(12)(56)}X\otimes Y\otimes Z) &= \mathrm{Tr}\big(X(\overline{Y}\bigotimes Z)^{\dagger}\big) = \langle X,\overline{Y}\bigotimes Z\rangle_{HS}\\
\mathrm{Tr}\Big(S_{(1256)(12)(56)}X\otimes Y\otimes Z) &= \mathrm{Tr}\big(X(\overline{Y}\bigotimes Z)^{\dagger}\big) = \langle X,\overline{Y}\bigotimes Z\rangle_{HS} \qquad \text{(S5.17)}
\end{aligned}$$

and for $W, X, Y, Z \in X \in \mathscr{H}^{\otimes^2}$

$$\mathrm{Tr}\Big(S_{(12)(15)(56)(26)} W \otimes X \otimes Y \otimes Z\Big) = \langle \overline{W}^{\dagger} X, \overline{Z} Y^{\dagger} \rangle_{HS}$$
$$\mathrm{Tr}\Big(S_{(12)(16)(56)(25)} W \otimes X \otimes Y \otimes Z\Big) = \langle \overline{W}^{\dagger} X, \overline{ZY} \rangle_{HS}$$
$$\mathrm{Tr}\Big(S_{(15)(26)(15)(26)} W \otimes X \otimes Y \otimes Z\Big) = \langle W, \overline{Y} \rangle_{HS} \langle X, \overline{Z} \rangle_{HS}$$
$$\mathrm{Tr}\Big(S_{(15)(26)(12)(56)} W \otimes X \otimes Y \otimes Z\Big) = \langle W \widetilde{\bigotimes} \overline{X}, (\overline{Y} \bigotimes Z) \rangle_{HS}$$
$$\mathrm{Tr}\Big(S_{(16)(25)(12)(56)} W \otimes X \otimes Y \otimes Z\Big) = \langle W \widetilde{\bigotimes}_{\top} X, (\overline{Y} \bigotimes Z) \rangle_{HS}. \tag{S5.18}$$

S5.2 Covariance structure of $\sqrt{T}\hat{F}_{2,T}$

We have that $T\mathrm{Cov}(\hat{F}_{2,T}, \hat{F}_{2,T})$ is given by

$$T\mathrm{Cum}\Big(\frac{1}{NM^2} \sum_{k_1=1}^{\lfloor N/2 \rfloor} \sum_{j_1, j_2=1}^{M} \langle I_N^{u_{j_1}, \omega_{k_1}}, I_N^{u_{j_2}, \omega_{k_1}} \rangle_{HS}, \frac{1}{NM^2} \sum_{k_2=1}^{\lfloor N/2 \rfloor} \sum_{j_3, j_4=1}^{M} \overline{\langle I_N^{u_{j_3}, \omega_{k_2}}, I_N^{u_{j_4}, \omega_{k_2}} \rangle}_{HS}\Big).$$

Using again Theorem S3.1,

$$\mathrm{Cum}_2(\hat{F}_{2,T}) = \frac{1}{N^2 M^4} \sum_{k_1, k_2=1}^{\lfloor N/2 \rfloor} \sum_{\substack{j_1, j_2, \\ j_3, j_4=1}}^{M} \mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1 \cup \ldots \cup P_G} S_{\boldsymbol{P}} \Big(\otimes_{g=1}^{G} \mathrm{cum}\big(D_N^{u_{j_p}, \omega_{k_p}} | p \in \nu_g\big)\Big)\Big),$$

where $p = (l, m)$ with $k_p = (-1)^{l-m} k_l$ and $j_p = j_{2l - \delta_{\{m \in \{1,2\}\}}}$ for $l \in \{1, 2\}$ and $m \in \{1, 2, 3, 4\}$ and where $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. That is, we are interested in all indecomposable partitions of the array

$$\begin{array}{cccc}
\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{1} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{2} & \underbrace{D_N^{u_{j_2},-\omega_{k_1}}}_{3} & \underbrace{D_N^{u_{j_2},\omega_{k_1}}}_{4} \\
\underbrace{D_N^{u_{j_3},-\omega_{k_2}}}_{5} & \underbrace{D_N^{u_{j_3},\omega_{k_2}}}_{6} & \underbrace{D_N^{u_{j_4},\omega_{k_2}}}_{7} & \underbrace{D_N^{u_{j_4},-\omega_{k_2}}}_{8}
\end{array}$$

For the same reason as above, we only have to consider the structures $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$ and $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$. For the structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are again

$$\mathrm{Tr}\Big(S_{(1256)(34)(78)}\Big(\delta_{j_1,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_4},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(1278)(34)(56)}\Big(\delta_{j_1,j_4}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3456)(12)(78)}\Big(\delta_{j_2,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_4},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3478)(12)(56)}\Big(\delta_{j_2,j_4}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},-\omega_{k_2}}+\mathscr{E}_2\Big]\Big).$$

For the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are in this case

$$\mathrm{Tr}\Big(S_{(3478)(12)(56)}\big(\delta_{j_2,j_4}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_1}}+\mathscr{E}_2\big]\big)\Big)$$
$$\mathrm{Tr}\Big(S_{(12)(36)(78)(45)}\big(\delta_{j_2,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_4},\omega_{k_2}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_1}}+\mathscr{E}_2\big]\big)\Big)$$
$$\mathrm{Tr}\Big(S_{(15)(26)(34)(78)}\big(\delta_{j_1,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_4},\omega_{k_2}}+\mathscr{E}_2\big]\big)\Big)$$
$$\mathrm{Tr}\Big(S_{(18)(27)(34)(56)}\big(\delta_{j_1,j_4}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}+\mathscr{E}_2\big]\big)\Big).$$

Using Remark S5.1, we find that $\mathrm{Cov}(\sqrt{T}\hat{F}_{2,T})$ equals

$$\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_1,j_3}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}},\overline{\mathscr{F}_{u_{j_2},-\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_4},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_1,j_4}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}},\overline{\mathscr{F}_{u_{j_2},-\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_2,j_3}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},-\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_4},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_2,j_4}\Big[\langle(\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_2,j_4}\delta_{k_1,k_2}\Big[\langle\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}^{\dagger}\mathscr{F}_{u_{j_2},-\omega_{k_1}},\overline{\mathscr{F}_{u_{j_2},\omega_{k_1}}}\mathscr{F}^{\dagger}_{u_{j_3},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_2,j_3}\delta_{k_1,k_2}\Big[\langle\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}^{\dagger}\mathscr{F}_{u_{j_2},-\omega_{k_1}},\overline{\mathscr{F}_{u_{j_2},\omega_{k_1}}\mathscr{F}_{u_{j_4},\omega_{k_2}}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_1,j_3}\delta_{k_1,k_2}\Big[\langle\mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1}}},\overline{\mathscr{F}_{u_{j_2},-\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_4},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M}\delta_{j_1,j_4}\delta_{k_1,k_2}\Big[\langle\mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u_{j_1},-\omega_{k_1}},\overline{\mathscr{F}_{u_{j_2},-\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{M^2})\Big].$$

So that, as $N,M\to\infty$,

$$\begin{aligned}NM\mathrm{Cov}(\hat{F}_{2,T},\hat{F}_{2,T})\to&\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle\widetilde{\mathscr{F}}_{\omega_1,-\omega_1,-\omega_2},\widetilde{\mathscr{F}}_{\omega_1}\bigotimes\widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}d\omega_1d\omega_2\\&+\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle\widetilde{\mathscr{F}}_{\omega_1,-\omega_1,\omega_2},\widetilde{\mathscr{F}}_{\omega_1}\bigotimes\widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS}d\omega_1d\omega_2\\&+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}^2_{u,\omega},\mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}dud\omega\\&+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle\widetilde{\mathscr{F}}_{\omega}\mathscr{F}_{u,\omega},\mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{u,\omega}\rangle_{HS}dud\omega\end{aligned}$$

$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \widetilde{\mathscr{F}_\omega}\bigotimes\widetilde{\mathscr{F}_\omega}\rangle_{HS} du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \widetilde{\mathscr{F}_\omega}\bigotimes\widetilde{\mathscr{F}_{-\omega}}\rangle_{HS} du d\omega.$$

S5.3 Cross-covariance $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$

Using again Theorem S3.1

$$T\mathrm{Cum}_2(\hat{F}_{1,T},\hat{F}_{2,T}) = \frac{T}{N^2M^3}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G} S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G}\mathrm{cum}\big(D_N^{u_{j_p},\omega_{k_p}}|p\in\nu_g\big)\Big)\Big),$$

where this time we are interested in all indecomposable partitions of the array

$$\begin{matrix} \underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{1} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{2} & \underbrace{D_N^{u_{j_1},-\omega_{k_1-1}}}_{3} & \underbrace{D_N^{u_{j_1},\omega_{k_1-1}}}_{4} \\ \underbrace{D_N^{u_{j_2},-\omega_{k_2}}}_{5} & \underbrace{D_N^{u_{j_2},\omega_{k_2}}}_{6} & \underbrace{D_N^{u_{j_3},\omega_{k_2}}}_{7} & \underbrace{D_N^{u_{j_3},-\omega_{k_2}}}_{8} \end{matrix}$$

By Lemma S4.3 and Proposition S4.1), we only have to consider partitions of the form $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$ and $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$. The only significant terms of first form are again

$$\mathrm{Tr}\Big(S_{(1256)(34)(78)}\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(1278)(34)(56)}\delta_{j_1,j_3}\Big[\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3456)(12)(78)}\delta_{j_1,j_2}\Big[\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(3478)(12)(56)}\delta_{j_1,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$

and for the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are

$$\mathrm{Tr}\Big(S_{(12)(37)(56)(48)}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\big(\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2]\Big)$$
$$\mathrm{Tr}\Big(S_{(12)(36)(78)(45)}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\mathscr{F}_{u_{j_3},\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2]\Big)$$
$$\mathrm{Tr}\Big(S_{(15)(26)(34)(78)}\delta_{j_1,j_2}\delta_{k_1,k_2})^2\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2}}+\mathscr{E}_2]\Big)$$
$$\mathrm{Tr}\Big(S_{(18)(27)(34)(56)}\delta_{j_1,j_3}\delta_{k_1,k_2})^2\delta_{j_1,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2]\Big),$$

which implies by Remark S5.1 that $T\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{2,T})$ equals

$$\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_2}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}}\bigotimes\mathscr{F}_{u_{j_3},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_3}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}}\bigotimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_2}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_3},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_3}\Big[\langle\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}\bigotimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{T})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\Big[\langle\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}^{\dagger}\mathscr{F}_{u_{j_1},-\omega_{k_1-1}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1-1}}\mathscr{F}^{\dagger}_{u_{j_2},-\omega_{k_2}}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\Big[\langle\overline{\mathscr{F}_{u_{j_1},\omega_{k_1}}}^{\dagger}\mathscr{F}_{u_{j_1},-\omega_{k_1-1}},\overline{\mathscr{F}_{u_{j_1},\omega_{k_1-1}}\mathscr{F}_{u_{j_3},\omega_{k_2}}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_2}\delta_{k_1,k_2}\delta_{j_1,j_2}\delta_{k_1,k_2}\Big[\langle\mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1}}},\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}}\bigotimes\mathscr{F}_{u_{j_2},\omega_{k_2}}\rangle_{HS}+O(\frac{1}{M^2})\Big]$$
$$+\frac{1}{NM^2}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\delta_{j_1,j_3}\delta_{k_1,k_2}\delta_{j_1,j_3}\delta_{k_1,k_2}\Big[\langle\mathscr{F}_{u_{j_1},\omega_{k_1}}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u_{j_1},-\omega_{k_1}},\overline{\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}}\bigotimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\rangle_{HS}+O(\frac{1}{M^2})\Big].$$

Hence, as $N,M\to\infty$,

$$NM\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{2,T})\to\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2},\mathscr{F}_{u,\omega_1}\bigotimes\widetilde{\mathscr{F}_{\omega_2}}\rangle_{HS}du\,d\omega_1 d\omega_2$$

$$\frac{2}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS} du d\omega_1 d\omega_2$$
$$+\frac{2}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\bigotimes \widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \mathscr{F}_{u,\omega}\bigotimes \widetilde{\mathscr{F}}_{-\omega}\rangle_{HS} du d\omega.$$

S5.4 Limiting Variance of $\widehat{m}_T$

The limiting variance of $\widehat{m}_T$ is given by

$$\nu^2 = \lim_{T\to\infty}\Big(16\pi^2\mathrm{Var}(\hat{F}_{1,T}) + 16\pi^2\mathrm{Var}(\hat{F}_{2,T}) - 32\pi^2\mathrm{Cov}(\hat{F}_{1,T}, \hat{F}_{2,T})\Big).$$

The above therefore yields the following expression for the asymptotic variance

$$\nu^2 = 4\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \mathscr{F}_{u,\omega_2}\rangle_{HS} du d\omega_1 d\omega_2$$
$$+4\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle \widetilde{\mathscr{F}}_{\omega_1,-\omega_1,-\omega_2}, \widetilde{\mathscr{F}}_{\omega_1}\bigotimes \widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS} d\omega_1 d\omega_2$$
$$-8\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS} du d\omega_1 d\omega_2$$
$$+4\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \mathscr{F}_{u,-\omega_2}\rangle_{HS} du d\omega_1 d\omega_2$$
$$+4\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle \widetilde{\mathscr{F}}_{\omega_1,-\omega_1,\omega_2}, \widetilde{\mathscr{F}}_{\omega_1}\bigotimes \widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS} d\omega_1 d\omega_2$$
$$-8\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}_{u,\omega_1,-\omega_1,\omega_2}, \mathscr{F}_{u,\omega_1}\bigotimes \widetilde{\mathscr{F}}_{-\omega_2}\rangle_{HS} du d\omega_1 d\omega_2$$
$$+8\pi\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}^2_{u,\omega}|||_2^2 du d\omega + 4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle \mathscr{F}^2_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega$$
$$+4\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle \widetilde{\mathscr{F}}_{\omega}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{u,\omega}\rangle_{HS} du d\omega$$

$$
\begin{aligned}
&-16\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega + 4\pi \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u,\omega}|||_2^4 du d\omega\\
&+4\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,\omega}\rangle_{HS} du d\omega\\
&+4\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \mathscr{F}_{u,\omega}\bigotimes\mathscr{F}_{u,-\omega}\rangle_{HS} du d\omega\\
&+4\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \widetilde{\mathscr{F}}_{\omega}\bigotimes\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega\\
&+4\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \widetilde{\mathscr{F}}_{\omega}\bigotimes\widetilde{\mathscr{F}}_{-\omega}\rangle_{HS} du d\omega\\
&-8\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}\mathscr{F}_{u,\omega}, \mathscr{F}_{u,\omega}\bigotimes\widetilde{\mathscr{F}}_{\omega}\rangle_{HS} du d\omega\\
&-8\pi\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\widetilde{\bigotimes}_{\top}\mathscr{F}_{u,-\omega}, \mathscr{F}_{u,\omega}\bigotimes\widetilde{\mathscr{F}}_{-\omega}\rangle_{HS} du d\omega. \qquad \text{(S5.19)}
\end{aligned}
$$

Under H_0 this reduces to $\nu^2_{H_0} = 4\pi\int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}}_{\omega}|||_2^4 d\omega$.

S5.5 Proof of Lemma 3.1 (consistency of variance estimate)

Proof of Lemma 3.1: We write

$$
\begin{aligned}
\mathbb{E}(\hat{\nu}^2_{H_0}) =& \frac{16\pi^2}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\mathbb{E}\Big[\frac{1}{M}\sum_{j=1}^{M}\langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}}\rangle_{HS}\Big]^2\\
=& \frac{16\pi^2}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\mathrm{Var}\Big[\frac{1}{M}\sum_{j=1}^{M}\langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}}\rangle_{HS}\Big] + \frac{16\pi^2}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\Big(\mathbb{E}\Big[\frac{1}{M}\sum_{j=1}^{M}\langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}}\rangle_{HS}\Big]\Big)^2\\
=& \frac{16\pi^2}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Cov}\Big[\langle I_N^{u_{j_1},\omega_k}, I_N^{u_{j_1},\omega_{k-1}}\rangle_{HS}, \langle I_N^{u_{j_2},\omega_k}, I_N^{u_{j_2},\omega_{k-1}}\rangle_{HS}\Big]\\
&+ \frac{16\pi^2}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\Big(\frac{1}{M}\sum_{j=1}^{M}\mathbb{E}\Big[\langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}}\rangle_{HS}\Big]\Big)^2.
\end{aligned}
$$

Using Theorem S3.1, we can write the first term as

$$\frac{16\pi^2}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G} S_{\boldsymbol{P}}\Big(\otimes_{g=1}^{G}\mathrm{cum}\big(D_N^{u_{j_p},\omega_{k_p}}|p\in\nu_g\big)\Big)\Big), \qquad \text{(S5.20)}$$

where $p=(l,m)$ and $k_p=(-1)^m k_{2l-\delta\{m\in\{1,2\}\}}$ and $j_p=j_{2l-\delta\{m\in\{1,2\}\}}$ for $l\in\{1,\ldots,n\}$ and $m\in\{1,2,3,4\}$. In this case, we are interested in all indecomposible partitions of the array

$$\begin{array}{cccc}
\underbrace{D_N^{u_{j_1},\omega_k}}_{1} & \underbrace{D_N^{u_{j_1},-\omega_k}}_{2} & \underbrace{D_N^{u_{j_1},-\omega_{k-1}}}_{3} & \underbrace{D_N^{u_{j_1},\omega_{k-1}}}_{7} \\
\underbrace{D_N^{u_{j_2},-\omega_{k-1}}}_{5} & \underbrace{D_N^{u_{j_2},\omega_{k-1}}}_{6} & \underbrace{D_N^{u_{j_2},\omega_k}}_{7} & \underbrace{D_N^{u_{j_2},-\omega_k}}_{8}.
\end{array}$$

Indecomposability immediately implies that there must be one restriction in time. Using the results in Section S3 and a similar argument as in Section S5 will show that all of these are at most order $O(\frac{1}{M})$ and hence will vanish as $M\to\infty$. For example,

$$\begin{aligned}
&\mathrm{Tr}\Big(S_{(18)(27)(36)(45)}\delta_{j_1,j_2}\big(\mathscr{F}_{u_{j_1},\omega_k}\otimes\mathscr{F}_{u_{j_1},-\omega_{k-1}}\otimes\mathscr{F}_{u_{j_1},-\omega_k}\otimes\mathscr{F}_{u_{j_1},\omega_{k-1}}+\mathscr{E}_2\big]\Big)\\
&\mathrm{Tr}\Big(S_{(12)(78)(36)(45)}\delta_{j_1,j_2}\big(\mathscr{F}_{u_{j_1},\omega_k}\otimes\mathscr{F}_{u_{j_2},\omega_k}\otimes\mathscr{F}_{u_{j_1},-\omega_k}\otimes\mathscr{F}_{u_{j_1},\omega_{k-1}}+\mathscr{E}_2\big]\Big)\\
&\vdots
\end{aligned}$$

Using again Theorem Theorem S3.1, we can prove similar to the proof of $\mathbb{E}\hat{F}_{1,T}$ that the second term converges to $4\pi\int_{-\pi}^{\pi}\Big(\int_0^1 |||\mathscr{F}_{u,\omega}|||_2^2 du\Big)^2 d\omega$. Under H_0, we have $\mathscr{F}_{\omega,u}\equiv\mathscr{F}_{\omega}$ and it follows therefore that the second term converges to

$4\pi\int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}}_\omega|||_2^4 d\omega$ if the null is satisfied. For the variance of the estimator, we write

$$\mathrm{Var}(\hat{\upsilon}^2_{H_0}) = \mathbb{E}[\hat{\upsilon}^2_{H_0}]^2 - \big(\mathbb{E}[\hat{\upsilon}^2_{H_0}]\big)^2. \tag{S5.21}$$

Under H_0, the above derivation yields that the second term of (S5.21) converges to $\big(4\pi\int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}}_\omega|||_2^4 d\omega)^2$. Consider then decomposing the first term of (S5.21) as

$$\begin{aligned}\mathbb{E}[\hat{\upsilon}^2_{H_0}]^2 =& \frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\mathbb{E}\Big[\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big]^2\\ =&\frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\mathrm{Var}\Big[\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big]\end{aligned} \tag{S5.22}$$

$$+\frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\Big[\mathbb{E}\Big(\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big)\Big]^2 \tag{S5.23}$$

We consider(S5.22) and (S5.23) separately. Using the product theorem for cumulants, (S5.22) equals

$$\begin{aligned}&\frac{2^8\pi^4}{N^2M^4}\sum_{k_1,k_2}\sum_{j_1,j_2,j_3,j_4}\mathrm{Cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}, \overline{\langle I_N^{u_{j_3},\omega_{k_1}}, I_N^{u_{j_3},\omega_{k_1-1}}\rangle}_{HS}\Big)\\ &\qquad\times\mathrm{Cum}\Big(\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}, \overline{\langle I_N^{u_{j_4},\omega_{k_2}}, I_N^{u_{j_4},\omega_{k_2-1}}\rangle}_{HS}\Big)\\ &+\frac{2^8\pi^4}{N^2M^4}\sum_{k_1,k_2}\sum_{j_1,j_2,j_3,j_4}\mathrm{Cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}, \overline{\langle I_N^{u_{j_4},\omega_{k_2}}, I_N^{u_{j_4},\omega_{k_2-1}}\rangle}_{HS}\Big)\\ &\qquad\times\mathrm{Cum}\Big(\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}, \overline{\langle I_N^{u_{j_3},\omega_{k_1}}, I_N^{u_{j_3},\omega_{k_1-1}}\rangle}_{HS}\Big).\end{aligned}$$

Theorem S3.1 then shows that indecomposability of the first of these terms implies the restrictions $k_1 = k_2$ and $\{j_1, j_2\} \cap \{j_3, j_4\} = \emptyset$, while indecomposability of the second implies the constraints $\{j_1, j_2\} \cap \{j_3, j_4\} = \emptyset$ only. Therefore, (S5.22) is of order $O(\frac{1}{NM^2} + \frac{1}{M^2})$ and hence converges to zero as $N, M \to \infty$. Finally, it is straightforward to show using a similar argument that (S5.23) equals

$$\frac{2^8\pi^4}{N^2} \sum_{k_1,k_2} \Big[\frac{1}{M^2} \sum_{j_1,j_2} \mathrm{Cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}, \overline{\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}}\Big)$$
$$+\mathbb{E}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\mathbb{E}\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big)\Big]^2$$
$$= \frac{2^8\pi^4}{N^2} \sum_{k_1,k_2} \Big[O(\frac{1}{M}) + \frac{1}{M^2} \sum_{j_1,j_2} \mathbb{E}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\mathbb{E}\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big)\Big]^2$$
$$= \frac{2^8\pi^4}{N^2} \sum_{k_1,k_2} \Big[\frac{1}{M}\sum_{j_1} |||\mathscr{F}_{u_{j_1},\omega_{k_1}}|||_2^2 \frac{1}{M}\sum_{j_2} |||\mathscr{F}_{u_{j_2},\omega_{k_2}}|||_2^2 + O(\frac{1}{M})\Big]^2.$$

Under H_0, the latter converges to $\big(4\pi \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega|||_2^4 d\omega\big)^2$. Altogether, the above derivation shows that $\mathbb{E}[\hat{v}_{H_0}^2]^2 \to \big(4\pi \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega|||_2^4 d\omega\big)^2$ as $N, M \to \infty$. Since the second term of (S5.21) converges to the same limit, we thus find $\mathrm{Var}(\hat{v}_{H_0}^2) \to 0$ and consequently $\hat{v}_{H_0}^2 \xrightarrow{p} v_{H_0}^2$.

S5.6 Proof of Remark 3.1

We remark that, in case of nonzero constant mean, the local fDFT used in (2.16) is implicitly defined as

$$D_N^{u,\omega} = (2\pi N)^{-1/2} \sum_{s=0}^{N-1} (X_{\lfloor uT\rfloor - N/2+s+1,T} - \mu) e^{-i\omega s} = D_N^{u,\omega} - (2\pi N)^{-1/2} \mu \sum_{s=0}^{N-1} e^{-i\omega s},$$

where $\mu = \mathbb{E}X_{t,T}$ for all t, T. If $\mu \neq 0$, then we can center the data using the sample mean from which we obtain as a raw estimator

$$\tilde{D}_N^{u,\omega} := (2\pi N)^{-1/2} \sum_{s=0}^{N-1} (X_{\lfloor uT \rfloor - N/2+s+1,T} - \hat{\mu}_T) e^{-i\omega s}.$$

Let $\tilde{I}_N^{u,\omega} := \tilde{D}_N^{u,\omega} \otimes \tilde{D}_N^{u,\omega}$ and correspondingly denote $\widetilde{\widehat{m}}_T$ the statistic as given in (2.16) but where $\tilde{I}_N^{u,\omega}$ replaces $I_N^{u,\omega}$ in (2.13), (2.14) and (2.15), respectively. We shall prove that $\widetilde{\widehat{m}}_T$ has the same limiting distribution as $\widehat{m}_T$. Observe first that

$$\tilde{I}_N^{u,\omega} = I_N^{u,\omega} + (2\pi N)^{-1/2} \Big(D_N^{u,\omega} \otimes (\mu - \hat{\mu}_T) \sum_{s=0}^{N-1} e^{-\mathrm{i}\omega s} + (\mu - \hat{\mu}_T) \sum_{s=0}^{N-1} e^{-\mathrm{i}\omega s} \otimes D_N^{u,\omega} \Big)$$
$$+ (\mu - \hat{\mu}_T) \otimes (\mu - \hat{\mu}_T) \frac{1}{2\pi N} \sum_{s,t=0}^{N-1} e^{-\mathrm{i}\omega(s-t)} := I_N^{u,\omega} + R_{1,N}^{\omega} + R_{2,N}^{\omega}.$$

Recall that $\Delta_N(\omega) = \sum_{s=0}^{N-1} e^{-i\omega s}$ satisfies $\Delta_N(\omega) = N$, for all $\omega \equiv 0 (\mathrm{mod}\, 2\pi)$ and that $\Delta_N(\omega_k) = 0$ for any $\omega_k = 2\pi k/N$ with $k \in \mathbb{Z}, k \not\equiv 0 (\mathrm{mod}\, N)$. Hence, $|||\tilde{I}_N^{u,\omega_k} - I_N^{u,\omega_k}|||_2^2 = 0$ for all $k = 1, \ldots, N$. Therefore, using $\tilde{I}_N^{u,\omega}$ rather than $I_N^{u,\omega}$ only affects the estimator via $\hat{F}_{1,T}$ and $\hat{B}_{N,T}$ because the zero frequency arises in these terms due to the lag when the summand is evaluated at $k = 1$, i.e.,

$$\begin{aligned}
|\widehat{m}_T - \widetilde{\widehat{m}}_T| &= \Big| \frac{1}{T} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_1}, I_N^{u_j,\omega_0} \rangle_{HS} + \frac{1}{NM} \sum_{j=1}^{M} \mathrm{Tr}(I_N^{u_j,\omega_1}) \,\mathrm{Tr}(I_N^{u_j,\omega_0}) \\
&\quad - \frac{1}{T} \sum_{j=1}^{M} \langle \tilde{I}_N^{u_j,\omega_1}, \tilde{I}_N^{u_j,\omega_0} \rangle_{HS} - \frac{1}{NM} \sum_{j=1}^{M} \mathrm{Tr}(\tilde{I}_N^{u_j,\omega_1}) \,\mathrm{Tr}(\tilde{I}_N^{u_j,\omega_0}) \Big| \\
&\leq \frac{1}{T} \sum_{j=1}^{M} \Big| \langle I_N^{u_j,\omega_1}, R_{1,N}^{\omega_0} + R_{2,N}^{\omega_0} \rangle + \mathrm{Tr}(I_N^{u_j,\omega_1}) \,\mathrm{Tr}(R_{1,N}^{\omega_0} + R_{2,N}^{\omega_0}) \Big|. \quad \text{(S5.24)}
\end{aligned}$$

By the Cauchy Schwarz inequality, the properties of the tensor product and applying the Cauchy Schwarz inequality again,

$$\begin{aligned}
&\frac{1}{T}\sum_{j=1}^{M}\mathbb{E}\big|\langle I_N^{u_j,\omega_1}, R_{1,N}^{\omega_0}\rangle\big| \\
&\quad\le \frac{N^{1/2}}{(2\pi)^{1/2}T}\sum_{j=1}^{M}\mathbb{E}\Big[|||I_N^{u_j,\omega_1}|||_2\Big(|||D_N^{u,\omega_0}\otimes(\mu-\hat{\mu}_T)|||_2+|||(\mu-\hat{\mu}_T)\otimes D_N^{u,\omega_0}|||_2\Big)\Big] \\
&\quad\le \frac{2N^{1/2}}{(2\pi)^{1/2}T}\sum_{j=1}^{M}\sqrt{\mathbb{E}(\|D_N^{u,\omega_1}\|_2^2\|D_N^{u,\omega_0}\|_2)^2}\sqrt{E\|\mu-\hat{\mu}_T\|_2^2}. \qquad \text{(S5.25)}
\end{aligned}$$

Similarly, by the Cauchy Schwarz inequality

$$\frac{1}{T}\sum_{j=1}^{M}\mathbb{E}\big|\langle I_N^{u_j,\omega_1}, R_{2,N}^{\omega_0}\rangle\big| \le \frac{N}{(2\pi)^{1/2}T}\sum_{j=1}^{M}\sqrt{\mathbb{E}(\|D_N^{u,\omega_1}\|_2^4}\sqrt{E\|\mu-\hat{\mu}_T\|_2^4}. \qquad \text{(S5.26)}$$

Let $Y_T = \hat{\mu}_T - \mu$ and note that $\mathbb{E}\|Y_T\|_2^2 < \infty$, whereas $\mathbb{E}Y_T = 0$. We thus have $\mathbb{E}\|Y_T\|_2^2 = |||\mathrm{Var}(Y_T)|||_1 = |||\mathrm{Var}(T^{-1}\sum_{t=1}^{T}X_{t,T}-\mu)|||_1$ and Minkowski's inequality yields

$$\begin{aligned}
\mathbb{E}\|\hat{\mu}_T-\mu\|_2^2 &= \frac{1}{T^2}\Big|\Big|\Big|\sum_{t,s=1}^{T}(\mathbb{E}\big[(X_{t,T}-\mu)\otimes(X_{s,T}-\mu)\big]-C_{t/T,t-s}+C_{t/T,t-s})\Big|\Big|\Big|_1 \\
&\le \frac{1}{T^2}\sum_{t,s=1}^{T}|||\mathbb{E}\big[(X_{t,T}-\mu)\otimes(X_{s,T}-\mu)\big]-C_{t/T,t-s}|||_1+\frac{1}{T^2}\sum_{t,s=1}^{T}|||C_{t/T,t-s}|||_1,
\end{aligned}$$

where $C_{t/T,t-s} = \mathbb{E}(X_t^{(t/T)}-\mu)\otimes(X_{t-s}^{(t/T)}-\mu)$. Then, by Lemma S.2.1 of Aue and van Delft (2019) and Assumption 3.1,

$$\mathbb{E}\|\mu-\hat{\mu}_T\|_2^2 \le \frac{1}{T}\sup_u\sum_{|h|\le T}|||C_{u,h}|||_1+O(\frac{1}{T}) \le \frac{1}{T}\sum_{h\in\mathbb{Z}}|||\kappa_{2;h}|||_1+O(\frac{1}{T}) = O(\frac{1}{T}).$$

This shows that (S5.25) is of order $O((NT)^{-1/2})$ under Assumption 3.1. To derive (S5.26), observe that by the properties of the trace, simple tensor and the product theorem for cumulant tensors (S1.3)

$$\mathbb{E}\| Y_T\|_2^4 = \mathrm{Tr}(\mathbb{E}(Y_T \otimes Y_T \otimes Y_T \otimes Y_T)) = |||\mathrm{Cum}(Y_T, Y_T, Y_T, Y_T)|||_1 + 3|||\mathrm{Var}(Y_T)|||_1^2. \tag{S5.27}$$

Since we already showed that $|||\mathrm{Var}(Y_T)|||_1^2 = O(T^{-2})$, it remains to derive a bound for the first term of (S5.27). Again, by Assumption 3.1 and Lemma S.2.1 of Aue and van Delft (2019),

$$\begin{aligned}
|||\mathrm{Cum}(Y_T, Y_T, Y_T, Y_T)|||_1 &\leq \frac{1}{T^4}\sum_{t_1,\ldots,t_4} |||C_{t_1/T;t_1-t_2,t_1-t_3,t_1-t_4}|||_1 \\
&\quad + \frac{1}{T^4}\sum_{t_1,\ldots,t_4} \Big|\Big|\Big|\mathrm{Cum}\big[(X_{t_1,T}-\mu,\ldots,(X_{t_4,T}-\mu)\big] - C_{t_1/T;t_1-t_2,t_1-t_3,t_1-t_4}\Big|\Big|\Big|_1 \\
&\leq \frac{1}{T^3}\sum_{h_1,h_2,h_3\in\mathbb{Z}} (1+|h^\star|)|||\kappa_{4;h_1,h_2,h_3}|||_1 \\
&\quad + \frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4\in\mathbb{Z}} \Big(\frac{1}{T}\big(4+\sum_{j=1}^{3}|t_1-t_j|\big)|||\kappa_{4;t_1-t_2,t_1-t_3,t_1-t_4}|||_1 \\
&= O(T^{-3}) + O(T^{-4}) = O(T^{-3}),
\end{aligned}$$

where $h^\star = \min(h_1, h_2, h_3)$. Hence, we obtain for (S5.27) an order of $O(T^{-3} + T^{-2})$ and therefore for (S5.26) an order of $O(T^{-1})$. Using that $\mathrm{Tr}(x\otimes y) = \langle x, y\rangle$, $x, y \in \mathscr{H}$, a similar derivation shows the same bound for the second term of (S5.24), from which it now follows that $|\widehat{m}_T - \widetilde{\widehat{m}}_T| = O_p(T^{-1})$.

S6. Statistical applications

As Theorem 3.1 provides the asymptotic distribution at any point of the alternative it has several important applications, ,such as confidence intervals for the measure of stationarity and tests for precise hypotheses (see Berger and Delampady, 1987). These are briefly mentioned here.

(a) The probability of a type II of the test (3.4) can be calculated approximately by the formula

$$\mathbb{P}\big(\widehat{m}_T \le \widehat{v}_{H_0} u_{1-\alpha}/\sqrt{T}\big) \approx \Phi\Big(\frac{v_{H_0}}{v}\, u_{1-\alpha} - \sqrt{T}\,\frac{m^2}{v}\Big), \qquad \text{(S6.28)}$$

where $v_{H_0}^2$ and v^2 are defined in Theorem 3.1 and 3.1, respectively, and Φ is the distribution function of the standard normal distribution.

(b) An asymptotic confidence interval for the measure of stationarity m^2 is given by

$$\Big[\max\Big\{0, \widehat{m}_T - \frac{\hat{v}_{H_1}}{\sqrt{T}} u_{1-\alpha/2}\Big\}, \widehat{m}_T + \frac{\hat{v}_{H_1}}{\sqrt{T}} u_{1-\alpha/2}\Big], \qquad \text{(S6.29)}$$

where $\hat{v}_{H_1}^2$ denotes an estimator of the variance in Theorem 3.1.

(c) Similarly, one can use Theorem 3.1 to test for *similarity to stationarity* considering the hypotheses

$$H_\Delta : m^2 \ge \Delta \qquad \text{vs.} \qquad K_\Delta : m^2 < \Delta\,, \qquad \text{(S6.30)}$$

where Δ is a pre-specified constant such that for a value of m^2 smaller than Δ the experimenter defines the second order properties to be similar to stationarity. For example, if the functional time series deviates only slightly from second order stationarity, it is often reasonable to work under the assumption of stationarity as many procedures are robust against small deviations from this assumption and procedures specifically adapted to non-stationarity usually have a larger variability.
An asymptotic level α test for these hypotheses is obtained by rejecting the null hypothesis, whenever

$$\widehat{m}_T - \Delta < \frac{\hat{v}_{H_1}}{\sqrt{T}} u_\alpha \,. \tag{S6.31}$$

Note that this test allows to decide for "approximate second order stationarity" with a controlled type I error. It follows from Theorem 3.1 and a straightforward calculation that

$$\lim_{T\to\infty} \mathbb{P}\Big(\widehat{m}_T - \Delta < \frac{\hat{v}_{H_1}}{\sqrt{T}} u_\alpha\Big) = \begin{cases} 0 & \text{if } m^2 > \Delta \\ \alpha & \text{if } m^2 = \Delta \\ 1 & \text{if } m^2 < \Delta \end{cases}, \tag{S6.32}$$

which means that the test (S6.31) is a consistent and asymptotic level α test for the hypotheses (S6.30). For the hypotheses of a *relevant difference* $H: m^2 \le \Delta$ vs. $K :> \Delta$ a corresponding asymptotic level α test can be constructed similarly and the details are omitted.

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