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CONTINGENTLY AND VIRTUALLY BALANCED INCOMPLETE BLOCK DESIGNS AND THEIR EFFICIENCIES UNDER VARIOUS OPTIMALITY CRITERIA

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Abstract. Even when a parameter set (v, k, λ) satisfies the necessary conditions for the existence of a Balanced Incomplete Block (BIB) design, the actual design may not exist or its existence may be unknown. We introduce two classes of designs, Contingently Balanced Incomplete Block (C-BIB) designs and Virtually Balanced Incomplete Block (V-BIB) designs, that may be considered in such cases. Both C-BIB and V-BIB designs are constructed from Unfinished Balanced Incomplete Block (U-BIB) designs, which can be constructed by a sequential search algorithm. Some V-BIB designs are shown to be highly efficient under A-, D-, and E-optimality criteria. Special attention is given to the parameter set (22, 8, 4) for which the existence of a BIB design is unknown. Highly efficient V-BIB designs exist for this parameter set. Also, C-BIB and V-BIB designs for (v, k, λ) may be used to construct BIB designs for parameter sets $(v, k, t\lambda)$, where $t > 1$ is an integer. This generalizes the well-known result that multiple copies of a BIB design form again a BIB design.

Key words and phrases: A-optimality, BIB(22, 33, 12, 8, 4), BIB designs, Block designs, D-optimality, E-optimality, Optimal designs, simple random sampling.

1. Introduction

An equireplicated binary design with v treatments in b blocks of size k, $k < v$, is called a *Balanced Incomplete Block* (BIB) design, denoted by BIB(v, b, r, k, λ), if it satisfies: (1) each treatment is replicated exactly r times; and (2) each pair of treatments appears within blocks exactly λ times. It is well known that the parameters of such a design must satisfy the following necessary conditions:

$$bk = rv, \quad r(k-1) = \lambda(v-1) \quad \text{and} \quad b \geq v. \tag{1.1}$$

Any parameter set (v, b, r, k, λ) that satisfies (1.1) may be denoted by (v, k, λ) for brevity.

A BIB design is known to be A-, D-, and E-optimal in the class of all connected block designs with the same values for v, b and k. However, even if the conditions in (1.1) are satisfied, a BIB design may not exist or its existence may be unknown. For such parameter sets, BIB designs are not available. For results on the existence and the nonexistence of BIB designs, see, among others, Hanani (1975), Hall (1986), and Mathon and Rosa (1990).

A prominent example of a parameter set for which the existence of a BIB design is unknown is (22, 8, 4). This is the smallest parameter set, in terms of b and k, that satisfies (1.1) and for which the existence of a BIB design is unknown. This parameter set has received considerable attention in the literature, such as in Hamada and Kobayashi (1978), Hall, Roth, van Rees and Vanstone (1988), Landgev and Tonchev (1989) and Hedayat, Stufken and Zhang (1995). It may be noted that the existence of a BIB$(22, 33t, 12t, 8, 4t)$ design is only unknown for $t = 1$. Hanani (1975) has shown that a BIB(22, 66, 24, 8, 8) design and a BIB(22, 99, 36, 8, 12) design exist, thereby establishing the existence of a BIB$(22, 33t, 12t, 8, 4t)$ design for any $t > 1$. (Note that $\lambda = 4t$ is a necessary condition based on (1.1) .)

When the necessary conditions for the existence of a BIB design are satisfied but the required BIB design is not available, an alternative design with "good" properties is needed. We propose a class of alternative designs, called Virtually Balanced Incomplete Block (V-BIB) designs, which possess some desirable statistical properties. A V-BIB design may be obtained by first constructing an intermediate design, an Unfinished Balanced Incomplete Block (U-BIB) design. If a U-BIB design cannot be completed to a V-BIB design, it may be possible to complete the U-BIB design to a design in a slightly larger class of designs, namely to a Contingently Balanced Incomplete Block (C-BIB) design. Precise definitions of these designs are given in Section 2.

A search algorithm for finding U-BIB designs is discussed in Section 3. A modified version of the algorithm is used to complete U-BIB designs to V-BIB designs, if possible.

It should be noted that the relationship between pairwise and variance balanced block designs also allows the algorithm to be employed for the construction of variance balanced block designs. For further discussion on this relationship see Hedayat and Federer (1974) and Hedayat and Stufken (1989).

Section 4 discusses properties of U-BIB designs, while Section 5 studies efficiencies of V-BIB designs with respect to some common optimality criteria.

Applications of C-BIB and V-BIB designs to construct BIB designs with larger values of b, r and λ are also considered. This generalizes the well known result that multiple copies of a BIB design form again a BIB design. This is also a useful tool to manipulate the support size of sampling plans that are, in terms of inclusion probabilities, equivalent to simple random sampling without replacement. Some results in this direction are discussed in Section 6.

2. Definition of U-BIB, C-BIB, and V-BIB Designs

By λ_{ij} we denote the number of blocks that contain both treatment i and j, $i \neq j$.

Definition 2.1. A binary block design with $v - w$ treatments in b blocks, each of size less than or equal to k, $k < v$, is called an *Unfinished Balanced Incomplete Block* (U-BIB) design with treatment deficiency w, denoted by U-BIB$(v, b, r, k, \lambda; w)$, if
(1) $bk = rv$, $r(k-1) = \lambda(v-1)$, $b \geq v$;
(2) all $v - w$ treatments are replicated r times; and
(3) $\lambda_{ij} = \lambda$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, v - w\}$.

Definition 2.2. A connected binary block design with v treatments in b blocks, each of size k, $k < v$, is called a *Contingently Balanced Incomplete Block* (C-BIB) design with treatment deficiency w, denoted by C-BIB$(v, b, r, k, \lambda; w)$, if
(1) $bk = rv$, $r(k-1) = \lambda(v-1)$, $b \geq v$;
(2) all v treatments are replicated r times; and
(3) $\lambda_{ij} = \lambda$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, v - w\}$, and arbitrary otherwise.
A C-BIB design with treatment deficiency w in which all λ_{ij} differ by at most 1 from λ is called a *Virtually Balanced Incomplete Block* (V-BIB) design with treatment deficiency w.

In Definition 2.2, without loss of generality, we have assumed that the design is pairwise balanced with respect to the *first* $v - w$ treatments.

The following facts are clear: (1) The existence of a U-BIB$(v, b, r, k, \lambda; w)$ design implies the existence of a U-BIB$(v, b, r, k, \lambda; w')$ design for $w < w' < v$. (2) A C-BIB (V-BIB) design with treatment deficiency w is also a C-BIB (V-BIB) design with treatment deficiency w' for $w < w' < v$. (3) A BIB design is a special case of any one of these designs with treatment deficiency $w = 0$.

We will search for U-BIB designs with the smallest possible treatment deficiency. Unless such a design is already a BIB design, these will then be used to search for V-BIB designs. A V-BIB design with a small treatment deficiency should be highly efficient under all criteria for which a BIB design is highly efficient or optimal.

Note that deletion of w treatments from a BIB design results in a U-BIB design with treatment deficiency w. This is, however, of little interest to us, since our focus is on those parameters for which a BIB design does not exist or is unknown. The existence of a U-BIB design does of course not require the existence of the corresponding BIB design. The following example gives a U-BIB(15, 21, 7, 5, 2; 3) design although no corresponding BIB design exists. A complete computer search also showed that no U-BIB(15, 21, 7, 5, 2; 2) design exists.

Example 2.1. Treatments 1 through 12 in Table 2.1 form a U-BIB(15, 21, 7, 5, 2; 3) design. The 21 blocks in the design contain each of the 12 treatments 7 times and each pair of treatments 2 times. It can be extended to a V-BIB design by adding treatments 13, 14 and 15, while keeping the design binary and equireplicated. One way to do this is shown in Table 2.1. In Example 4.1 it will be seen that not every U-BIB design can be completed to a C-BIB design.

Table 2.1. A V-BIB(15, 21, 7, 5, 2; 3) design

1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5
2	2	3	4	5	7	8	3	4	5	6	10	4	5	6	7	5	7	8	6	6
3	6	9	6	9	10	11	11	7	8	9	12	6	7	8	8	8	11	9	7	9
4	7	10	12	13	12	13	12	9	10	11	13	10	13	14	9	11	14	10	10	12
5	8	11	15	14	14	14	13	14	15	13	15	14	15	15	12	12	15	13	11	15

3. Construction of U-BIB, C-BIB, and V-BIB Designs

In searching for U-BIB$(v, b, r, k, \lambda; w)$ designs we use an algorithm that attempts to fill an empty array consisting of bk cells (representing experimental units) in b columns (representing blocks) and k rows by assigning treatments to cells, one at a time, while ensuring that the eventual design will be a binary pairwise balanced design for the $v - w$ treatments that are actually included.

For the following discussion it is useful to introduce some notation. A U-BIB$(v, b, r, k, \lambda; w)$ design induces a $(v - w) \times r$ matrix $M = (m_{\text{in}})$ of block labels, where m_{in} denotes the nth block (starting to count from block 1) in which treatment i appears. By $\boldsymbol{m}_i$ we mean the ith row of M. Further, for two vectors $\boldsymbol{x} = (x_1, \ldots, x_r)$ and $\boldsymbol{y} = (y_1, \ldots, y_r)$ we write $\boldsymbol{x} \prec \boldsymbol{y}$ if there is a j, possibly equal to 1, such that $x_j < y_j$ and $x_h = y_h$ for all $h < j$. For any U-BIB design the following properties are observed:

(1) For all $i \in \{1, \ldots, v-w\}$, we have $1 \leq m_{i1} < \cdots < m_{ir} \leq b$.
(2) Each $m \in \{1, \ldots, b\}$ appears at most k times as an entry in M.
(3) For any two rows in M there are exactly λ block labels that appear in both rows.

There are additional properties that may be assumed without loss of generality, because for any U-BIB design with $v - w \geq 2$ one exists with the same parameters and these additional properties. These properties are:
(4) $\boldsymbol{m}_1 = (1, 2, \ldots, r)$ and $\boldsymbol{m}_2 = (1, 2, \ldots, \lambda, r+1, \ldots, 2r-\lambda)$.
(5) $\boldsymbol{m}_i \prec \boldsymbol{m}_{i'}$ if $i < i'$.
(6) If m', $m \in \{1, \ldots, b\}$, $m' < m$, have the property that for each of the first $i-1$ rows in M either both m' and m appear as an entry or neither one of them appears as an entry, then m can only appear as an entry in row i if m' does.

The following example provides a simple illustration of consequences of these properties.

Example 3.1. When trying to construct a U-BIB(22, 33, 8, 12, 4; 19) design there are essentially only four possibilities for the first three treatments. The first two rows of M are for all cases:

$$\begin{aligned}\boldsymbol{m}_1 &= (1,2,3,4,5,6,7,8,9,10,11,12),\\ \boldsymbol{m}_2 &= (1,2,3,4,13,14,15,16,17,18,19,20).\end{aligned}$$

The four different possibilities for the third row are:
Case 1: $\boldsymbol{m}_3 = (1,2,3,4,21,22,23,24,25,26,27,28)$,
Case 2: $\boldsymbol{m}_3 = (1,2,3,5,13,21,22,23,24,25,26,27)$,
Case 3: $\boldsymbol{m}_3 = (1,2,5,6,13,14,21,22,23,24,25,26)$,
Case 4: $\boldsymbol{m}_3 = (1,5,6,7,13,14,15,21,22,23,24,25)$.

The algorithm constructs a matrix M by adding one treatment at a time so that (1)-(6) are satisfied. If this is no longer possible a back tracking procedure is used to delete one or more treatments. If there is no BIB design, the algorithm will eventually search through all possibilities — though it can be stopped at any time and the largest U-BIB design obtained up to that time can be retrieved. For more details concerning the algorithm the reader is referred to Zhang (1994). Those who would like to obtain the algorithm may also contact this author.

Example 3.2. Following is the matrix M for a U-BIB(22, 33, 12, 8, 4; 3) design obtained by our algorithm. Note that it satisfies all of the properties (1)-(6).

$$M=\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12\\
1 & 2 & 3 & 4 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20\\
1 & 2 & 3 & 4 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28\\
1 & 2 & 5 & 6 & 13 & 14 & 21 & 22 & 29 & 30 & 31 & 32\\
1 & 3 & 7 & 8 & 15 & 16 & 23 & 24 & 29 & 30 & 31 & 33\\
1 & 4 & 9 & 10 & 17 & 18 & 25 & 26 & 29 & 30 & 32 & 33\\
1 & 5 & 6 & 7 & 15 & 19 & 20 & 25 & 27 & 28 & 32 & 33\\
1 & 8 & 11 & 12 & 13 & 14 & 17 & 26 & 27 & 28 & 31 & 33\\
2 & 5 & 7 & 11 & 17 & 18 & 19 & 23 & 24 & 26 & 31 & 32\\
2 & 5 & 9 & 12 & 16 & 17 & 20 & 21 & 23 & 27 & 29 & 33\\
2 & 8 & 9 & 11 & 15 & 16 & 20 & 22 & 26 & 28 & 30 & 32\\
2 & 9 & 10 & 12 & 13 & 15 & 19 & 22 & 24 & 25 & 31 & 33\\
3 & 6 & 9 & 11 & 13 & 18 & 20 & 23 & 25 & 28 & 29 & 31\\
3 & 6 & 10 & 12 & 13 & 16 & 19 & 23 & 26 & 27 & 30 & 32\\
3 & 7 & 9 & 11 & 14 & 18 & 19 & 21 & 22 & 27 & 30 & 33\\
3 & 7 & 10 & 12 & 14 & 17 & 20 & 22 & 24 & 28 & 29 & 32\\
4 & 5 & 8 & 10 & 16 & 18 & 19 & 22 & 27 & 28 & 29 & 31\\
4 & 6 & 10 & 11 & 15 & 17 & 20 & 21 & 24 & 27 & 30 & 31\\
4 & 8 & 11 & 12 & 14 & 15 & 19 & 21 & 23 & 25 & 29 & 32
\end{bmatrix}$$

The corresponding U-BIB design can, for this example, be completed to a C-BIB design by filling the blocks to which less than 8 treatments have been assigned with the remaining 3 treatments, replicating each of them 12 times and keeping the design binary.

Ideally we would like to complete a U-BIB design to a V-BIB design. An algorithm to search for such completions, based on minor modifications of our algorithm for the construction of U-BIB designs, has also been developed and is presented in Zhang (1994). The most conspicuous modification consists, of course, in a change of the requirement formulated by property (3). An interesting example of a V-BIB design obtained by this algorithm will be presented in Section 5.

We conclude this section with some results pertaining to the construction of V-BIB designs from given V-BIB and BIB designs. Proofs are simple and are omitted for brevity.

Theorem 3.1. *The complement* (*in the set of all k-subsets of* $\{1,2,\ldots,v\}$) *of a* V-BIB$(v,b,r,k,\lambda;w)$ *design without repeated blocks is a* V-BIB$(v,b',r',k,\lambda';w)$ *design, where*

$$b'=\binom{v}{k}-b \qquad r'=\binom{v-1}{k-1}-r \qquad \lambda'=\binom{v-2}{k-2}-\lambda.$$

Theorem 3.2. *Replacing each block in a* V-BIB$(v,b,r,k,\lambda;w)$ *design by its complement* (*in* $\{1,2,\ldots,v\}$) *results in a* V-BIB$(v,b,r',k',\lambda';w)$ *design, where* $r'=b-r$, $k'=v-k$, $\lambda'=b-2r+\lambda$.

Theorem 3.3. *Together the blocks of a* $\mathrm{BIB}(v, b_1, r_1, k, \lambda_1)$ *design and a* V-BIB $(v, b_2, r_2, k, \lambda_2; w)$ *design form a* $\text{V-BIB}(v, b_1 + b_2, r_1 + r_2, k, \lambda_1 + \lambda_2; w)$ *design.*

4. Properties of U-BIB Designs

Before focusing on V-BIB designs, we briefly consider some properties of the intermediate U-BIB designs, along the lines of Hedayat, Stufken and Zhang (1995). It is natural to ask whether a U-BIB design with a small enough treatment deficiency can always be completed to a BIB design. We show the answer to be affirmative if the treatment deficiency is 1 and provide examples to show that an unconditional statement is no longer true for U-BIB designs with treatment deficiency 2. U-BIB designs with treatment deficiency 2 can often, but not always, be completed to C-BIB designs or possibly to V-BIB designs.

The first lemma tells us something about the structure of a U-BIB design with treatment deficiency 1. We reemphasize that all parameters are assumed to meet the conditions in (1.1).

Lemma 4.1. *A* $\text{U-BIB}(v, b, r, k, \lambda; 1)$ *design necessarily has* r *blocks of size* $k-1$ *and* $b - r$ *blocks of size* k.

Proof. Let n_s denote the number of blocks of size s in a $\text{U-BIB}(v, b, r, k, \lambda; 1)$ design, $0 \leq s \leq k$. Then $\sum n_s = b$, $\sum sn_s = r(v-1)$, and $\sum s(s-1)n_s = \lambda(v-1)(v-2)$. It follows that $\sum(k-s)(k-s-1)n_s = 0$. Since $(k-s)(k-s-1) \geq 0$ and $n_s \geq 0$ for $0 \leq s \leq k$ this implies that $n_s = 0$ unless $s = k$ or $s = k-1$. The number of blocks of size $k-1$ must then be given by $n_{k-1} = vr - (v-1)r = r$.

Theorem 4.1. *A* $\text{U-BIB}(v, b, r, k, \lambda; 1)$ *design can always be completed to a* $\mathrm{BIB}(v, b, r, k, \lambda)$ *design.*

Proof. Based on Lemma 4.1, without loss of generality, it can be assumed that the first $b - r$ blocks in a $\text{U-BIB}(v, b, r, k, \lambda; 1)$ design are of size k and the last r blocks are of size $k-1$. Let r_{1i} and r_{2i} denote the replication of treatment i, $i = 1, 2, \ldots, v-1$, in the first $b - r$ and the last r blocks, respectively. Then $r_{1i} + r_{2i} = r$ and $r_{1i}(k-1) + r_{2i}(k-2) = \lambda(v-2)$, which leads to $r_{1i} = r - \lambda$ and $r_{2i} = \lambda$. Thus, each of the $v-1$ treatments in the U-BIB design appears exactly λ times in the last r blocks, and adding treatment v to these results in a BIB design.

Hence, if a $\mathrm{BIB}(v, b, r, k, \lambda)$ design does not exist then the smallest possible treatment deficiency for a U-BIB design is 2. The following example illustrates that such U-BIB designs can indeed not always be extended to a BIB design.

Example 4.1. A U-BIB(13, 26, 12, 6, 5; 2) design in which one block contains only 3 treatments does exist. Such a design can, of course, not be completed to a C-BIB design, and consequently not to a BIB(13, 26, 12, 6, 5) design. Perhaps more interesting, a U-BIB(13, 26, 12, 6, 5; 2) design that cannot be completed to a BIB(13, 26, 12, 6, 5) design, even though each block in the U-BIB design contains at least 4 treatments, does also exist. An example of such a U-BIB design is given in Table 4.1. What makes this example especially intriguing is that a BIB(13, 26, 12, 6, 5) design does exist.

Table 4.1. A U-BIB(13, 26, 12, 6, 5; 2) design

1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	4
2	2	2	2	2	3	4	4	4	5	6	6	3	4	4	4	5	5	5	4	4	4	5	5	5	5
3	3	3	3	6	6	5	5	7	9	7	7	7	6	6	6	7	7	8	7	7	8	6	6	6	6
4	4	8	8	7	9	8	10	9	10	8	8	11	8	10	10	8	9	9	8	9	9	7	8	9	7
5	5	9	11	9	10		11	11	11	10	11		9	11		10	10		10	10	11	11	10		8
6	7	10											11				11						11		9

In Section 5 we present a V-BIB(22, 33, 12, 8, 4; 2) design, obtained by completing a U-BIB(22, 33, 12, 8, 4; 2) design. Without solving the existence problem of a BIB(22, 33, 12, 8, 4) design this is the best result we can hope for. In terms of block sizes, there are at most two types of such U-BIB designs. To describe these, we use again n_s to denote the number of blocks of size s in the design.

Proposition 4.1. *In a* U-BIB(22, 33, 12, 8, 4; 2) *design one of the following possibilities must prevail:*

(1) $n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = 0$, $n_6 = 4$, $n_7 = 16$, $n_8 = 13$

or (2) $n_0 = n_1 = n_2 = n_3 = n_4 = 0$, $n_5 = n_6 = 1$, $n_7 = 19$, $n_8 = 12$.

Proof. For any U-BIB$(v, b, r, k, \lambda; w)$ design the equalities $\sum n_s = b$, $\sum sn_s = r(v-w)$, and $\sum s(s-1)n_s = \lambda(v-w)(v-w-1)$ must hold. For $v = 22$, $b = 33$, $r = 12$, $k = 8$, $\lambda = 4$, and $w = 2$, the only solutions in nonnegative integers are those in the statement of the proposition.

The solution in (2) cannot be completed to a C-BIB design (and therefore also not to a BIB design) since the one block of size 5 cannot be completed to a binary block. We do not have an example of a design corresponding to solution (2), and in view of the above do not intend to pursue this any further. The V-BIB(22, 33, 12, 8, 4; 2) design in Section 5 is based on a U-BIB design with the structure described by (1).

5. Efficiencies of V-BIB Designs

We now consider the efficiency of V-BIB designs under various optimality criteria. Since designs with a small treatment deficiency are in a sense closer to BIB designs, we concentrate our discussion on V-BIB designs with treatment deficiency 2. A brief comment concerning more general results is included in Remark 5.2.

We write $\Lambda_i(s)$ to denote the number of treatments j, $j \neq i$, that appear in exactly s blocks with treatment i.

Theorem 5.1. *For a* V-BIB$(v, b, r, k, \lambda; 2)$ *design we have*
(1) *for a fixed* $i \in \{1, 2, \ldots, v-2\}$, $\lambda_{i(v-1)} + \lambda_{iv} = 2\lambda$;
(2) $\Lambda_{v-1}(\lambda - 1) = \Lambda_{v-1}(\lambda + 1) = \Lambda_v(\lambda - 1) = \Lambda_v(\lambda + 1)$;
(3) $\lambda_{(v-1)v} = \lambda$.

Proof. By counting the number of pairs involving treatment 1 we obtain

$$\lambda(v-1) = r(k-1) = \sum_{i=2}^{v} \lambda_{1i} = \lambda(v-3) + \lambda_{1(v-1)} + \lambda_{1v}.$$

This implies (1), since the role of treatment 1 in this argument can be played by any treatment in $\{1, 2, \ldots, v-2\}$.

This also implies that $\Lambda_{v-1}(\lambda - 1) = \Lambda_v(\lambda + 1)$ and $\Lambda_{v-1}(\lambda + 1) = \Lambda_v(\lambda - 1)$. Further, by counting pairs that involve treatment v, we have

$$\lambda(v-1) = r(k-1) = \lambda\Lambda_v(\lambda) + (\lambda+1)\Lambda_v(\lambda+1) + (\lambda-1)\Lambda_v(\lambda-1) = \lambda(v-1) + \Lambda_v(\lambda+1) - \Lambda_v(\lambda-1).$$

Hence $\Lambda_v(\lambda+1) = \Lambda_v(\lambda-1)$, which, in conjunction with the previous observation, implies (2).

Finally, by counting pairs that involve treatments $v-1$ or v and by using (1), we obtain

$$2\lambda(v-1) = 2\lambda_{(v-1)v} + \sum_{i=1}^{v-2} \sum_{j=v-1}^{v} \lambda_{ij} = 2\lambda_{(v-1)v} + 2\lambda(v-2)$$

which implies (3).

It should be observed, for later use, that results (1) and (3) hold for any C-BIB design with treatment deficiency 2.

In view of Theorem 5.1, we introduce the following concept.

Definition 5.1. For a V-BIB$(v, b, r, k, \lambda; 2)$ design, the *pairwise deficiency index* of the design is defined as the common value of $\Lambda_{v-1}(\lambda - 1)$, $\Lambda_{v-1}(\lambda+1)$, $\Lambda_v(\lambda-1)$ and $\Lambda_v(\lambda + 1)$.

To obtain bounds for the efficiencies of V-BIB designs with treatment deficiency 2 and pairwise deficiency index p we need to know eigenvalues of the information matrix for such designs. As usual, the information matrix for a proper block design is written as

$$C_d = \mathrm{diag}(r_1, r_2, \ldots, r_v) - NN'/k,$$

where r_i is the replication of the ith treatment, $i = 1, 2, \ldots, v$, and N is the $v \times b$ incidence matrix of the design. By $0 = \mu_0 < \mu_1 \leq \ldots \leq \mu_{v-1}$ we will denote the eigenvalues of C_d.

Theorem 5.2. *The nonzero eigenvalues of the C-matrix of a* V-BIB$(v, b, r, k, \lambda; 2)$ *design with pairwise deficiency index p are $\mu_1 = (\lambda v - 2\sqrt{p})/k$, $\mu_i = \lambda v/k$, for $i = 2, 3, \ldots, v-2$, and $\mu_{v-1} = (\lambda v + 2\sqrt{p})/k$.*

Proof. Possibly after a permutation of the treatments, we can conclude from Theorem 5.1 that the $v \times v$ concordance matrix NN' of the V-BIB design is equal to the matrix

$$NN' = \left[\begin{array}{cccccc|cc}
r & & & & & & & \\
& r & & & & & & \\
& & \ddots & & \lambda & & & \\ \cline{7-8}
& & & & & & \lambda+1 & \lambda-1 \\
& & & \ddots & & & \vdots & \vdots \\
& & & & & & \lambda+1 & \lambda-1 \\ \cline{7-8}
& \lambda & & & \ddots & & \lambda-1 & \lambda+1 \\
& & & & r & & \vdots & \vdots \\
& & & & & r & \lambda-1 & \lambda+1 \\ \hline
\lambda+1 & \cdots & \lambda+1 & \lambda-1 & \cdots & \lambda-1 & r & \lambda \\
\lambda-1 & \cdots & \lambda-1 & \lambda+1 & \cdots & \lambda+1 & \lambda & r
\end{array}\right],$$

where the number of occurrences of both $\lambda - 1$ and $\lambda + 1$ in either of the last two rows and columns is p. Defining the $(v-2) \times 1$ vector $\boldsymbol{y}$ as $(0, \ldots 0, -1, \ldots, -1, 1, \ldots 1)'$, where the number of entries equal to -1 and to 1 are both equal to p, it is easily seen that there are $v - 4$ linearly independent eigenvectors of NN' with eigenvalue $r - \lambda$ that are of the form $(\boldsymbol{x}', 0, 0)'$, where $\boldsymbol{x}$ is a $(v-2) \times 1$ vector such that $\boldsymbol{x}'\mathbf{1}_{v-2} = \boldsymbol{x}'\boldsymbol{y} = 0$. Other eigenvectors are $\mathbf{1}_v$, $(\boldsymbol{y}', \sqrt{p}, -\sqrt{p})'$, $(\boldsymbol{y}', -\sqrt{p}, \sqrt{p})'$, and $(1, \ldots, 1, -(v-2)/2, -(v-2)/2)'$; the corresponding eigenvalues of NN' are rk, $r - \lambda - 2\sqrt{p}$, $r - \lambda + 2\sqrt{p}$, and $r - \lambda$, respectively. The result now follows easily.

To formulate efficiency results, we consider the A-, D- and E-optimality criteria. Taking a possibly nonexisting BIB design as the bench mark, known to be optimal under each of these criteria if it exists, we define the efficiencies (or, strictly speaking, lower bounds for the efficiencies) as follows. The A-efficiency is defined as $k(v-1)/(\lambda v \sum(1/\mu_i))$; the D-efficiency is defined as $(k/(\lambda v))/(\prod(1/\mu_i)^{1/(v-1)})$; and the E-efficiency is defined as $k\mu_1/(\lambda v)$.

Theorem 5.3. *A* V-BIB$(v, b, r, k, \lambda; 2)$ *design with pairwise deficiency index* p *has an A-efficiency of* $(v-1)/(v-3+2/(1-4p/(\lambda v)^2))$, *a D-efficiency of* $(1-4p/(\lambda v)^2)^{1/(v-1)}$, *and an E-efficiency of* $1-2\sqrt{p}/(\lambda v)$.

Proof. This follows immediately from Theorem 5.2 and the definitions of efficiency.

Returning to the parameter set (22, 8, 4), the following example provides a V-BIB design for this parameter set with treatment deficiency 2 and pairwise deficiency index 2.

Example 5.1. Table 5.1 provides a V-BIB(22, 33, 12, 8, 4; 2) design with pairwise deficiency index 2. The efficiencies for this design under the A-, D- and E-optimality criteria are 99.99%, 99.99% and 96.78%, respectively.

Table 5.1. A V-BIB(22, 33, 12, 8, 4; 2) design with pairwise deficiency index 2

1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4	5
2	2	2	2	3	3	5	6	7	7	8	8	3	3	4	4	5	6	7	7	5	5	5	6	6	6	5	5	5	6	6	6	6
3	3	5	5	4	4	7	9	9	11	10	11	7	10	8	10	9	11	8	8	7	7	9	8	8	8	8	8	10	7	7	7	9
4	4	6	6	9	14	8	10	14	12	13	12	11	14	12	11	12	12	9	9	10	12	11	9	10	12	9	11	13	9	10	11	10
5	9	13	17	13	15	10	12	16	14	15	13	13	16	15	15	13	14	10	15	12	17	13	11	14	13	12	14	16	15	12	13	11
6	10	14	18	17	19	11	16	17	15	18	16	16	17	16	17	19	18	13	17	15	19	14	16	19	15	14	16	18	16	13	18	15
7	11	15	19	18	21	21	20	20	18	20	17	20	18	20	19	21	21	14	18	16	20	15	18	20	17	18	17	19	19	14	20	17
8	12	16	20	21	22	22	22	21	19	22	19	21	22	21	20	22	22	19	21	18	22	20	19	21	22	20	22	21	22	17	22	21

Remark 5.1. For a V-BIB$(v, b, r, k, \lambda; 2)$ design, the possible values for the pairwise deficiency index p are given by $1, 2, \ldots, [v/2]-1$, where $[\cdot]$ denotes the largest integer function. For the parameter set (22, 8, 4), a V-BIB design with treatment deficiency 2 can thus have a pairwise deficiency index between 1 and 10. We have found examples for all these values, except for $p = 1$ and for $p = 10$.

Remark 5.2. Since all of the parameters in this paper satisfy the conditions in (1.1), all binary equireplicated designs have the property that for every treatment i the average value of the λ_{ij}, $j \neq i$, is equal to λ. It is therefore not too surprising that highly efficient designs can be found among those with most of the λ_{ij} equal to λ and the remaining values differing from λ by 1. A V-BIB design with treatment deficiency 2 and minimal pairwise deficiency index should therefore be highly efficient, and possibly optimal if a BIB design does not exist. Based on examples in Zhang (1994), it appears that V-BIB designs are indeed most efficient if few of the λ_{ij} differ from λ. Thus, in principal the best V-BIB design could be one with a treatment deficiency that exceeds 2, but if there exists a V-BIB design with treatment deficiency 2 and a low pairwise deficiency index then, based on our experience, it is likely that the most efficient design is among those with treatment deficiency 2.

6. Use of C-BIB and V-BIB Designs in Constructing BIB Designs

C-BIB and V-BIB designs can be used to construct certain BIB designs. While these tend to have larger values for λ, there are at least two reasons why they are nevertheless of interest. First, if λ is the minimal possible value for the parameter set (v,k,λ) based on the necessary conditions for the existence of a BIB design, and if the existence of such a BIB design is unknown or if such a design is known not to exist, it is from a combinatorial point of view natural to ask for which values of $t \geq 2$ a BIB design for $(v,k,t\lambda)$ is known to exist. Second, if BIB designs are used in sampling for a sampling design with inclusion probabilities that match those for simple random sampling without replacement, the number of blocks (or, equivalently, the value of λ) is rather immaterial. In such a context, methods to find different BIB designs for given values of v and k can be useful to manipulate the support size and the support of these sampling designs. The reader is referred to Chakrabarti (1963), Hedayat (1979), and Hedayat and Sinha (1991) for further explanation.

The basic idea for the constructions in this section is to join collections of blocks of various C-BIB designs, all with the same treatment deficiency. Our approach will be to start with one initial C-BIB design and to obtain others by permuting the last w treatments, or more, in the initial design. The first result follows if we combine the blocks of all $w!$ C-BIB designs that can be obtained from an initial C-BIB design by using all possible permutations of the last w treatments.

Lemma 6.1. *A* C-BIB$(v,b,r,k,\lambda;w)$ *design can be used to construct a* BIB$(v,tb,tr,k,t\lambda)$ *design, where* $t=w!$.

While the result of Lemma 6.1 is of interest if $w=2$, for increasing values of w the value of $w!$ becomes rapidly large. Certainly from a combinatorial point of view this diminishes the value of the result. If the C-BIB design possesses some additional structure much better results are possible. In order to formulate results in that direction we need to introduce some terminology.

Definition 6.1. By a *regular* C-BIB *design with treatment deficiency* w we mean a C-BIB design with treatment deficiency w that has the property that

$\lambda_{ij} = \lambda$ for all $i, j \in \{v-w+1, \dots, v\}$, $i \neq j$.

Definition 6.2. By the ith *diagonal of an* $m \times m$ *matrix*, $i = 1, 2, \dots, m$, we mean the set of m cells of the matrix defined by $\{(j, j+i-1) : j = 1, 2, \dots, m\}$. The 1st diagonal will also be called the main diagonal.

Here and elsewhere, labels for rows or columns in an $m \times m$ matrix that exceed m should be reduced modulo m.

Definition 6.3. By a *constant diagonal* C-BIB$(v, b, r, k, \lambda; w)$ *design* of order m, $m \geq w$, we mean a C-BIB$(v, b, r, k, \lambda; w)$ design such that, possibly after a permutation of the first $v - w$ treatments or the last w treatments, the following equality holds for all $i \in \{2, 3, \dots, m\}$:

$$\sum_{j=1}^{m} \lambda_{(v-m+j)(v-m+j+i-1)} = \lambda m.$$

Thus a constant diagonal C-BIB design of order m has an $m \times m$ submatrix of its concordance matrix, induced by m treatments that include the last w treatments, such that the entries on each diagonal other than the main diagonal sum to λm.

Since a V-BIB design is also a C-BIB design, Definitions 6.1 and 6.3 make the expressions a regular and a constant diagonal V- BIB design also meaningful. The next lemma provides a relationship between the two concepts.

Lemma 6.2. *A regular* V-BIB$(v, b, r, k, \lambda; w)$ *design is a constant diagonal* V-BIB *design of both order* w *and order* $w + 1$.

Proof. The validity of the statement for order w is obvious, since all off-diagonal elements of the relevant $w \times w$ submatrix of the concordance matrix are λ. For order $w + 1$, note that after a permutation of the last w treatments we may take the lower $(w + 1) \times (w + 1)$ submatrix along the diagonal of the concordance matrix without loss of generality as:

$$\left[\begin{array}{c|ccccccccc}
r & \lambda+1 & \cdots & \lambda+1 & \lambda & \cdots & \lambda & \lambda-1 & \cdots & \lambda-1 \\
\hline
\lambda+1 & r & & & & & & & & \\
\vdots & & r & & & & & & & \\
\lambda+1 & & & r & & & & \lambda & & \\
\lambda & & & & \cdot & & & & & \\
\vdots & & & & & \cdot & & & & \\
\lambda & & & & & & \cdot & & & \\
\lambda-1 & & & \lambda & & & & \cdot & & \\
\vdots & & & & & & & & r & \\
\lambda-1 & & & & & & & & & r
\end{array}\right],$$

where the number of times that $\lambda+1$ appears in the first row is equal to that for $\lambda-1$. It is obvious that the entries on a diagonal other than the main diagonal add to $(w+1)\lambda$.

The next result addresses the use of the concepts defined in this section for the construction of BIB designs.

Theorem 6.1. *A constant diagonal* C-BIB$(v,b,r,k,\lambda;w)$ *design of order m can be used to construct a* BIB *design for the parameter set* $(v,k,\lambda m)$.

Proof. We will identify m C-BIB designs for the parameter set (v,k,λ) whose blocks can be combined to form the desired BIB design. If needed, we permute the first $v-w$ or the last w treatments so that the last m treatments have the property required for a constant diagonal C-BIB design of order m. Let d_j denote the C-BIB design obtained from this initial C-BIB design after permuting treatment $v-m+i$, $i=1,\ldots,m$, to treatment $v-m+i+j-1$, $j=1,\ldots,m$. (We reduce $i+j-1$ modulo m if it exceeds m.) Together the blocks of these d_j's then form the desired BIB design. To see this, the number of times that two treatments, say $v-m+i_1$ and $v-m+i_2$, appear simultaneously in a block of this design is equal to

$$\sum_{j=1}^{m}\lambda_{(v-m+i_1-j+1)(v-m+i_2-j+1)}=\sum_{j=1}^{m}\lambda_{(v-m+j)(v-m+j+i_2-i_1)}=\lambda m$$

Hence the result.

The previous results have some immediate consequences.

Theorem 6.2. *A* C-BIB$(v,b,r,k,\lambda;2)$ *design can be used to construct* BIB *designs for the parameter sets* $(v,k,t\lambda)$ *for any* $t\geq 2$.

Proof. From the observation after Theorem 5.1, we know that a C-BIB design with treatment deficiency 2 is a regular C-BIB design. Hence, the 3×3 submatrix of the concordance matrix generated by the last three treatments is of the form:

$$\begin{bmatrix} r & \alpha & 2\lambda-\alpha \\ \alpha & r & \lambda \\ 2\lambda-\alpha & \lambda & r \end{bmatrix}.$$

The design is therefore a constant diagonal C-BIB design of order 2 and 3. By Theorem 6.1 this means that a BIB design as required exists for $t=2$ and for $t=3$. But since any $t\geq 4$ can be written as a sum of multiples of 2 and 3, these two BIB designs can be used to form the required BIB designs for any value of $t\geq 4$.

Theorem 6.3. *A regular* V-BIB$(v,b,r,k,\lambda;3)$ *design can be used to construct* BIB *designs for the parameter sets* $(v,k,t\lambda)$ *for any* $t \geq 3$.

Proof. It suffices to show that the given V-BIB design is a constant diagonal design for any of the orders 3, 4 and 5. The proof may then be completed by arguments similar to those in the proof of Theorem 6.2.

As a first step, by Lemma 6.2, it is a constant diagonal V-BIB design of orders 3 and 4. It remains to be shown that it is also a constant diagonal V-BIB design of order 5. For any treatment $i \in \{1,2,\ldots,v-3\}$ we have one of the following:

(1) $\lambda_{i(v-2)} = \lambda_{i(v-1)} = \lambda_{iv} = \lambda$, or

(2) $\{\lambda_{i(v-2)},\, \lambda_{i(v-1)},\, \lambda_{iv}\} = \{\lambda-1, \lambda, \lambda+1\}$.

If (1) holds for some i, we may take this to be $i = v-3$. By an appropriate permutation of the last three treatments we can then assume that the 5×5 submatrix of the concordance matrix, induced by the last 5 treatments, looks like:

$$\begin{bmatrix} r & \lambda & \alpha & 2\lambda-\alpha & \lambda \\ \lambda & r & \lambda & \lambda & \lambda \\ \alpha & \lambda & r & \lambda & \lambda \\ 2\lambda-\alpha & \lambda & \lambda & r & \lambda \\ \lambda & \lambda & \lambda & \lambda & r \end{bmatrix},$$

where α is equal to $\lambda-1, \lambda$, or $\lambda+1$. Hence, the design is a constant diagonal V-BIB design of order 5.

Assume next, for each treatment among the first $v-3$ that (2) holds. Without loss of generality we may assume that $\lambda_{(v-3)(v-2)} = \lambda$. If $\lambda_{i(v-2)} = \lambda$ for all $i \in \{1,2,\ldots,v-3\}$, then the design actually has treatment deficiency 2, and the claimed result can be strengthened by virtue of Theorem 6.2. Hence, we may assume, without loss of generality, that $\lambda_{(v-4)(v-2)} = \lambda+1$. Possibly, after permuting treatments $v-1$ and v, we then obtain the following 5×5 submatrix of the concordance matrix, induced by the last 5 treatments:

$$\begin{bmatrix} r & \lambda & \lambda+1 & \lambda-1 & \lambda \\ \lambda & r & \lambda & \alpha & 2\lambda-\alpha \\ \lambda+1 & \lambda & r & \lambda & \lambda \\ \lambda-1 & \alpha & \lambda & r & \lambda \\ \lambda & 2\lambda-\alpha & \lambda & \lambda & r \end{bmatrix},$$

where α is equal to $\lambda-1$ or $\lambda+1$. Again it is clear that the design is a constant diagonal V-BIB design of order 5. This concludes the proof.

Example 6.1. Consider the V-BIB(22, 33, 12, 8, 4; 2) design in Example 5.1. Call it d_1. Permute treatments 21 and 22 in d_1 and call the resulting design d_2. Together the blocks of d_1 and d_2 form a BIB(22, 66, 24, 8, 8) design.

Example 6.2. Consider again the V-BIB(22, 33, 12, 8, 4; 2) design given in Example 5.1. Call it d_1. Cyclically permute treatments 20, 21 and 22 and call the resulting design d_2. Repeat the same permutation on d_2 and call the resulting design d_3. Together the blocks of d_1, d_2 and d_3 form a BIB(22, 99, 36, 8, 12) design. With the BIB design from Example 6.1 this design can be used to form a BIB$(22, 33t, 12t, 8, 4t)$ for any $t \geq 2$.

Example 6.3. The V-BIB(15, 21, 7, 5, 2; 3) design in Table 2.1 is regular. Hence, we know from Theorem 6.3 that a BIB$(15, 21t, 7t, 5, 2t)$ design exists for any $t \geq 3$. In this case however, this technique does not provide the best known result, which is that a BIB design exists for any $t \geq 2$.

It is of course not hard to count the support size of the designs constructed here. It is also easily seen that a small treatment deficiency for the original design means that a large number of blocks will be unaltered in a permuted version of the design, implying that the constructed BIB designs will contain a large number of repeated blocks and a relatively small support size. For example, the BIB(22, 66, 24, 8, 8) design constructed in Example 6.1 is nonisomorphic to the BIB design in Hanani (1975), which has 66 distinct blocks.

7. Summary

When a BIB(v, b, r, k, λ) design does not exist or its existence is unknown, a U-BIB$(v, b, r, k, \lambda; w)$ design with treatment deficiency w may be found by a sequential search algorithm. The U-BIB design can possibly be used to construct a V-BIB design which, provided that the treatment deficiency w is small, has high efficiencies with respect to A-, D-, and E-optimality criteria. In particular, when $w = 2$ a V-BIB$(v, b, r, k, \lambda; 2)$ design has high efficiencies that improve with decreasing values for the pairwise deficiency index. An example of a V-BIB(22, 33, 12, 8, 4; 2) design is given.

C-BIB designs can also be used to construct BIB designs by taking multiple permuted copies of it. This generalizes the well-known result of constructing a BIB design by taking multiple copies of a smaller BIB design. It also has a direct application in survey sampling to reduce the support size and manipulate the support for sampling designs that are equivalent to simple random sampling without replacement.

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