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ON MUTUALLY ORTHOGONAL AND TOTALLY BALANCED SETS OF BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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Abstract: Families of balanced incomplete block designs with nested rows and columns are developed for multistage experimentation. In some designs the stages, or distinct treatment sets, are orthogonal to one another, and in others totally balanced. For given numbers of treatments, blocks, stages, and block size, the technique leads to a variety of totally balanced designs, from which the most efficient is selected.

Key words and phrases: Nested row-column design, mutually orthogonal designs, total balance, multistage experiments, method of differences.

1. Introduction

Multistage experiments, or experiments for multiple sets of treatments, are experiments for which more than one treatment set is applied to the same experimental material. These applications can be either simultaneous or successive, and the different treatment sets are referred to as stages. From the design viewpoint and assuming different treatment sets do not interact, one problem is to find the treatment assignments so that estimates of treatment effects from different stages are either orthogonal or totally balanced with respect to one another and to all blocking factors. This is the problem we shall attack in the nested row-column setting.

Nested row and column designs, introduced by Singh and Dey (1979), are designs for v treatments in b blocks of size $k = pq$, each block being a $p \times q$ array of p rows and q columns. Let λ_{ij}^R, λ_{ij}^C, and λ_{ij}^B be respectively the number of times treatments i and j appear together in rows, columns, and blocks. Then a balanced incomplete block design with nested rows and columns is a binary, equireplicate nested row-column design for which

$$p\lambda_{ij}^R + q\lambda_{ij}^C - \lambda_{ij}^B = \lambda;$$

it will be denoted by BIBRC(v, b, r, p, q, λ). Here r is the replication number and $\lambda = r(p-1)(q-1)/(v-1)$. Constructions for BIBRC's have been given by

Street (1981), Agrawal and Prasad (1982, 1983), Cheng (1986), Sreenath (1989), and Uddin and Morgan (1990, 1991). The main results of this paper show how additional sets of treatments may be applied to the series of BIBRC's given by Uddin and Morgan (1991), which include many parameter combinations found in the previous papers. Efficiencies relative to single stage designs are also derived.

For a list of references on multistage experimentation in a variety of situations the reader is referred to the review paper of Preece (1976). Row-column designs for more than two stages include sets of orthogonal Latin squares (see Raghavarao (1971) for a summary) and analogous Youden designs (e.g. Hedayat, Sieden and Federer (1972), Saha and Das (1988)). Generalizations include sets of mutually orthogonal Youden or F-hyperrectangles (Cheng (1980)) and mutually orthogonal F-hyperrectangles with variable numbers of symbols (Mandeli and Federer (1984)). The only result for multistage nested row-column designs we have found is that of Street (1981). Her Theorem 8, improved on below, gives BIBRC's for two sets of treatments.

2. Multistage BIBRC's

Suppose there are t sets of v treatments each to be applied in b blocks of size $p \times q$. Let $n = bpq$ be the total number of experimental units; one treatment from each of the sets is to be applied to each unit. The model under consideration is

$$Y = \mu I_n + B\beta + R\rho + G\gamma + A\tau + \epsilon$$

where Y is a $n \times 1$ vector of observations, ρ a $bp \times 1$ vector of row effects, γ a $bq \times 1$ vector of column effects, β a $b \times 1$ vector of block effects, $\tau = (\tau_1', \tau_2', \ldots, \tau_t')'$ and τ_i is a $v \times 1$ vector of treatment effects for the ith stage, and ϵ is a $n \times 1$ vector of experimental errors ($E(\epsilon) = 0$, $\mathrm{var}(\epsilon) = \sigma^2 I$). Using j to denote a vector of ones, then with appropriate ordering of the observations, $R = I_b \otimes j_q \otimes I_p$, $G = I_b \otimes I_q \otimes j_p$, and $B = I_b \otimes j_p \otimes j_q$ are the plot-row, plot-column, and plot-block incidence matrices. Finally, $A = (A_1, A_2, \ldots, A_t)$ where A_i is the $n \times v$ plot-treatment incidence matrix for treatment set i, $A_i j_v = j_n$. With fixed, unknown β, ρ, and γ, the appropriate (least squares) analysis is the within-rows-and-columns, or bottom stratum, analysis. Recovery of information from other strata is discussed in Section 3.

Least squares estimation of τ in the above model yields the reduced normal equations $C\hat{\tau} = Q$ where C is the partitioned matrix $C = (C_{ij})$ with

$$\begin{aligned} C_{ij} &= A_i'\Big[I - \frac{1}{q}RR' - \frac{1}{p}GG' + \frac{1}{pq}BB'\Big]A_j \\ &= A_i'A_j - \frac{1}{q}N_{R,i}N_{R,j}' - \frac{1}{p}N_{G,i}N_{G,j}' + \frac{1}{pq}N_{B,i}N_{B,j}', \end{aligned} \tag{2.1}$$

and $N_{R,i}$, $N_{G,i}$, and $N_{B,i}$ the treatment-row, treatment-column, and treatment-block incidence matrices for stage i.

Let J be a matrix of ones. We will say we have a set of t mutually orthogonal BIBRC's if each block is binary in treatments, if $C_{ii} = aI + bJ$ for some constants a, b and all $i = 1, \ldots, t$, and if $C_{ij} = 0$ for all $i \neq j = 1, \ldots, t$. We say we have a totally balanced set of BIBRC's if the condition on C_{ij} is replaced by $C_{ij} = yI + zJ$ for some constants $(y, z) \neq (0, 0)$ and all $i \neq j = 1, \ldots, t$. In either case the conditions on the C_{ii}'s say that each stage is a BIBRC. The latter condition on the C_{ij}'s generalizes to more than two stages the property of some two stage designs such as the BIBD's for two sets of treatments of Preece (1976), and is stricter than that of the multistage Youden designs of Hedayat, Seiden and Federer (1972). In the terminology of Preece (1976), this says that not only is each stage totally balanced with respect to each other stage, but that they possess total overall balance. The term "orthogonal" is used in the sense of Eccleston and Russell (1975). Their concept of orthogonality is now usually phrased "adjusted orthogonality", a terminology not used here as it has come to specifically refer to a property of row-column and of nested row and column designs that does not hold for the designs of this paper (cf. Eccleston and John (1986, 1988)). Simply put, the orthogonality condition $C_{ij} = 0$ implies that contrasts from different sets are orthogonally estimated in the bottom stratum.

Let x be a primitive element for the finite field GF_v of order v. Our constructions require the following two results due to Uddin and Morgan (1991):

Lemma 1. *If $v = mq + 1$ is a prime or prime power and $2 \leq p \leq m$, then the m blocks*

$$B_i = \begin{pmatrix} x^{i-1} & x^{m+i-1} & \cdots & x^{(q-1)m+i-1} \\ x^{i} & x^{m+i} & \cdots & x^{(q-1)m+i} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p+i-2} & x^{m+p+i-2} & \cdots & x^{(q-1)m+p+i-2} \end{pmatrix},$$

$i = 1, 2, \ldots, m$, are initial blocks for a BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1))$.

Lemma 2. *If $v = 2mq + 1$ is a prime or prime power, q odd and $2 \leq p \leq 2m$, then the m blocks*

$$B_i = \begin{pmatrix} x^{i-1} & x^{2m+i-1} & \cdots & x^{2(q-1)m+i-1} \\ x^{i} & x^{2m+i} & \cdots & x^{2(q-1)m+i} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p+i-2} & x^{2m+p+i-2} & \cdots & x^{2(q-1)m+p+i-2} \end{pmatrix},$$

$i = 1, 2, \ldots, m$, are initial blocks for a BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1)/2)$.

The construction technique is simple: treatment assignments for additional stages are defined by obtaining new sets of m initial blocks given by row permutations of the initial blocks of Lemma 1 or 2. The problems are in choosing the form of the permutations and demonstrating their existence. It will be seen that permutations yielding orthogonal sets are always to be preferred but do not always exist.

The result based on Lemma 1 will be derived first and in detail (it is less problematic than that for Lemma 2 and will allow us to jump to some of the results in that case). Let $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ be any permutation of $(1, \ldots, p)$ such that $\pi_h = h$ for exactly s values of $h \in \{1, \ldots, p\}$, and let $\pi(B_i)$ be the block derived by permuting the rows of B_i according to π; the same π will be used for each $i = 1, \ldots, m$. Then if $B_i + g$, $g \in GF_v$ is a block for treatment set 1, $\pi(B_i) + g$ is the treatment arrangement in that block for treatment set 2.

In Appendix I it is shown that for such a π,

$$C_{12} = m(s-1)(q-1)I - \frac{(s-1)(q-1)}{q}(J - I)$$

so that a totally balanced BIBRC for two sets of treatments has been constructed. If $s = 1$, i.e. π fixes exactly one row of the B_i's, then $C_{12} = 0$ and the result is an orthogonal pair of BIBRC's.

It is now easy to see how additional sets of treatments can be added: each additional set requires another row permutation, which for total balance must be such that each pair of permutations puts s rows in the same position. For t stages, t permutations are required, where the first may be taken as the identity permutation.

Example 1. Take $v = 13$, $m = 4$, $q = 3$, and $x = 2$ in Lemma 1. Using the permutations $(1,2,3,4)$, $(1,4,2,3)$, and $(1,3,4,2)$ of the ordered sequence $(1,2,3,4)$, initial blocks for a set of 3 orthogonal BIBRC$(13, 52, 48, 4, 3, 24)$'s are:

STAGE 1	1	3	9	2	6	5	4	12	10	8	11	7
	2	6	5	4	12	10	8	11	7	3	9	1
	4	12	10	8	11	7	3	9	1	6	5	2
	8	11	7	3	9	1	6	5	2	12	10	4
STAGE 2	1	3	9	2	6	5	4	12	10	8	11	7
	8	11	7	3	9	1	6	5	2	12	10	4
	2	6	5	4	12	10	8	11	7	3	9	1
	4	12	10	8	11	7	3	9	1	6	5	2
STAGE 3	1	3	9	2	6	5	4	12	10	8	11	7
	4	12	10	8	11	7	3	9	1	6	5	2
	8	11	7	3	9	1	6	5	2	12	10	4
	2	6	5	4	12	10	8	11	7	3	9	1

Which value of s is to be preferred can be determined from the C-matrix. For the design of Lemma 1

$$C = C_{11} = m(p-1)(q-1)I - \frac{(p-1)(q-1)}{q}(J-I).$$

So for a totally balanced t-stage design

$$\begin{aligned} C &= I_t \otimes C_{11} + (J_t - I_t) \otimes C_{12} \\ &= \frac{q-1}{q}[(p-s)I_t + (s-1)J_t] \otimes [(mq+1)I_v - J_v] \end{aligned}$$

by using $C_{12} = \frac{s-1}{p-1}C_{11}$ and rearranging terms. That C must be nonnegative definite with rank $t(v-1)$ (required for full estimability) implies that $s < p$, and for $s = 0$, $t < p$. Otherwise the only restriction is the obvious $s \geq 0$. A generalized inverse of C is

$$\begin{aligned} C^- &= \frac{q}{q-1}[(p-s)I_t + (s-1)J_t]^{-1} \otimes [(mq+1)I_v - J_v]^- \\ &= \frac{q}{v(q-1)(p-s)}\left[I_t - \frac{s-1}{p-s+t(s-1)}J_t\right] \otimes \left[I_v - \frac{1}{mq+1}J_v\right] \\ \Rightarrow \operatorname{tr}(C^-) &= \frac{tq^2m[p-s+(t-1)(s-1)]}{v(q-1)(p-s)[p-s+t(s-1)]}, \end{aligned} \tag{2.2}$$

which is a constant times the average variance of an elementary treatment contrast. The quantity s can now be chosen to minimize (2.2).

Lemma 3. *The minimum of* (2.2) *over* $0 \leq s \leq p-1$, *with* $t < p$ *for* $s = 0$, *occurs at* $s = 1$.

Proof. Ignoring $tq^2m/(q-1)v$, (2.2) at $s = 1$ is $1/(p-1)$. Then

$$\frac{p-s+(t-1)(s-1)}{(p-s)[p-s+t(s-1)]} - \frac{1}{p-1} = \frac{(s-1)^2(t-1)}{(p-s)[p-s+t(s-1)](p-1)} > 0$$

for $s \neq 1$.

Lemma 3 says that in terms of efficiency an orthogonal set is always preferred. The analysis is also simplest in this case, as C^- has diagonal block form and the treatment sum of squares will orthogonally decompose into components for each stage.

How many stages can be accommodated? Consider a $p \times t$ array A on the symbols $1, \ldots, p$ and say that A has s-pair balance if
(i) each column contains each symbol once

and
(ii) each pair of columns has exactly s like pairs in rows.
An array A with s-pair balance will be denoted by $A(p,t,s)$. The columns of an $A(p,t,s)$ will be the permutations needed to define the stages of the BIBRC. Summarizing the above results:

Theorem 1. *Let $v = mq+1$ be a prime or prime power with $m \geq p \geq 2$. The existence of $A(p,t,1)$ implies the existence of a set of t mutually orthogonal* BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1))$*'s.*

Corollary 1. *There exists a set of $p-1$ mutually orthogonal* BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1))$*'s for $v = mq+1$ a prime or prime power and $m \geq p \geq 3$.*

Corollary 1 follows since $A(p,p-1,1)$ can always be constructed by adjoining the row $(p,p,\ldots,p)$ to a Latin square on the first $p-1$ symbols. No more columns can be added to this array, but there do exist arrays $A(p,t,1)$ with $t \geq p$. Some trial and error constructions appear in Table 1. These show that, in addition to the arrays of Corollary 1, 5 stages can be accommodated for $p = 5$, 8 stages for $p = 6$, and 10 stages for $p = 7$. For $p = 3, 4$, and 5 enumeration shows the maximum t has been obtained.

Table 1. Examples of $A(p,t,1)$ for $3 \leq p \leq 7$

1	1
2	3
3	2

1	1	3
2	4	1
3	2	2
4	3	4

1	1	4	2	2
2	3	3	1	3
3	4	1	4	5
4	5	2	3	4
5	2	5	5	1

1	1	6	3	2	5	2	5
2	3	2	4	1	1	3	3
3	4	4	5	5	4	1	6
4	2	5	2	4	3	5	4
5	6	1	1	6	2	4	1
6	5	3	6	3	6	6	2

1	1	7	6	5	1	6	4	1	1
2	5	6	1	6	6	5	6	3	7
3	2	2	5	4	5	1	3	7	4
4	3	1	4	3	7	2	2	6	2
5	4	5	2	2	3	3	1	2	6
6	7	4	7	1	2	4	7	4	5
7	6	3	3	7	4	7	5	5	3

Now consider the lemma 2 designs. Again additional stages will be defined by permutations, but here another condition must be imposed. Say an array $A^\sigma(p,t,s)$ has symmetric s-pair balance if it has s-pair balance with the further property that if a pair (a,b) is formed by the rows of any two given columns, so is the pair (b,a). For $A^\sigma(p,t,s)$, s is odd if and only if p is odd, which will be an important consideration with regard to efficiency and orthogonal sets. Defining a t-stage BIBRC by permuting the rows of the B_i's of Lemma 2 according to the columns of an $A^\sigma(p,t,s)$, the result of Appendix II gives

$$\begin{aligned} C &= I_t \otimes C_{11} + (J_t - I_t) \otimes C_{12} \\ &= \frac{q-1}{2q}[(p-s)I_t + (s-1)J_t] \otimes [(2mq+1)I_v - J_v]. \end{aligned}$$

Mimicking the g-inverse preceding (2.2) gives

$$\operatorname{tr}(C^-) = \frac{2q^2mt}{v(q-1)(p-s)}\left[\frac{p-s+(t-1)(s-1)}{p-s+t(s-1)}\right]. \tag{2.3}$$

It follows from Lemma 3 that the average variance is again minimized for $s=1$, which is the orthogonal case. However if p is even then s must be even, so other values must be considered.

Lemma 4. *The minimum of* (2.3) *over even s for $0 \le s \le p-1$, with $t < p$ for $s=0$, is at $s=2$.*

Proof. Write $W_s = \frac{p-s+(t-1)(s-1)}{(p-s)(p-s+t(s-1))}$. Then after some manipulation,

$$W_s - W_2 = \frac{(s-2)(t-1)[s(p+t-3)-(t-2)]}{(p-2)(p+t-2)(p-s)[p-s+t(s-1)]} \ge 0 \quad \text{for} \quad s \ne 1.$$

Hence we use $A^\sigma(p,t,1)$ if p is odd, and $A^\sigma(p,t,2)$ if p is even. These arrays are constructed by building up from $A^\sigma(p,t,0)$'s with smaller p.

Lemma 5. *Let p be even, that is, $p = 2^\omega d$ where d is odd and $\omega \ge 1$. Then for $t \le 2^\omega$ there exists $A^\sigma(p,t,0)$, $A^\sigma(p+1,t,1)$, and $A^\sigma(p+2,t,2)$.*

Proof. Given $A^\sigma(p,t,0)$ on the symbols $1,2,\ldots,p$, $A^\sigma(p+1,t,1)$ is obtained by adding a row $(p+1,p+1,\ldots,p+1)$, and $A^\sigma(p+2,t,2)$ by also adding the row $(p+2,p+2,\ldots,p+2)$. So $A^\sigma(p,2^\omega,0)$ must be constructed.

Let $d=1$ and write $A_\omega = A^\sigma(2^\omega,2^\omega,0)$. Then

$$A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Continuing this sequence,

$$A_{\omega+1} = \begin{pmatrix} A_\omega & A_\omega + 2^\omega \\ A_\omega + 2^\omega & A_\omega \end{pmatrix}.$$

Now for $p = 2^\omega d$ with $d > 1$

$$A^\sigma(p, 2^\omega, 0) = \begin{pmatrix} A_\omega \\ A_\omega + 2^\omega \\ \vdots \\ A_\omega + (d-1)2^\omega \end{pmatrix}.$$

Lemma 5 becomes much more interesting when it is stated that the maximum value of t in $A^\sigma(p+s, t, s)$ has been achieved for each $s = 0, 1, 2$. The proof of this fact may be found in Appendix III. We can now state our final construction results.

Theorem 2. *Let $v = 2mq + 1$ be a prime or prime power where q is odd, and let odd p be such that $3 \leq p \leq 2m-1$. Then there exists a set of t mutually orthogonal* BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1)/2)$*'s where $t \leq 2^\omega$ for $p = 2^\omega d + 1$ and d is odd.*

Theorem 3. *Let $v = 2mq + 1$ be a prime or prime power where q is odd, and let even p be such that $4 \leq p \leq 2m$. Then there exists a set of t* BIBRC$(v, mv, mpq, p, q, p(p-1)(q-1)/2)$*'s totally balanced for t sets of treatments, where $t \leq 2^\omega$ for $p = 2^\omega d + 2$ and d is odd.*

For Theorem 2 the design of Lemma 2 is permuted according to the columns of $A^\sigma(p, t, 1)$, and in Theorem 3 according to those of $A^\sigma(p, t, 2)$. Although there are other combinatorial possibilities here (and in Theorem 1) for totally balanced designs, efficiency arguments have driven the choices of permuting arrays. Theorems 2 and 3 improve Theorem 8 of Street (1981), which is restricted to even $p \leq m$ and $t = 2$.

Example 2. Take $v = 13, m = 2, q = 3,$ and $x = 2$ in Lemma 2. Initial blocks for a pair of totally balanced BIBRC(13, 26, 24, 4, 3, 12)'s are

STAGE 1	1	3	9	2	6	5
	2	6	5	4	12	10
	4	12	10	8	11	7
	8	11	7	3	9	1
STAGE 2	2	6	5	4	12	10
	1	3	9	2	6	5
	4	12	10	8	11	7
	8	11	7	3	9	1

3. The Information in Other Strata

In this section the questions of the information in higher strata, and that in the t-stage versus the single stage design, are addressed. The reader is referred to Speed (1982) and Houtman and Speed (1983) for definitions and notations pertaining to strata efficiencies, treatment decompositions, etc. The four strata, 1–4 respectively, for a nested row and column design are the block, row, column, and within-rows-and-columns strata. Explicit expressions for the strata projectors may be found in e.g. Cheng (1986). In the following, any normalized contrast comparing effects within the same treatment set will be referred to as a standard contrast. All references to t-stage designs assume $t > 1$.

The BIBRC's of Lemmas 1 and 2 have the desirable property that the block component, row component, and column component designs are each balanced incomplete block designs. It follows easily that these designs are generally balanced with respect to the treatment decomposition $T_0^* + T_1^*$ where $T_0^* = \frac{1}{v}J_v$ and $T_1^* = I_v - \frac{1}{v}J_v$. The strata efficiencies for contrasts in T_1^* are $\lambda_1^* = (v-pq)/[pq(v-1)]$, $\lambda_2^* = v(p-1)/[pq(v-1)]$, $\lambda_3^* = v(q-1)/[pq(v-1)]$, and $\lambda_4^* = v(p-1)(q-1)/[pq(v-1)]$. Interestingly, the t-stage designs are not generally balanced with respect to the analogous decomposition for t treatment sets. The coarsest decomposition with respect to which the t-stage designs are generally balanced is given by $T_0 = \frac{1}{vt}J_{vt}$, $T_1 = \frac{1}{t}J_t \otimes (I_v - \frac{1}{v}J_v)$, and $T_2 = (I_t - \frac{1}{t}J_t) \otimes (I_v - \frac{1}{v}J_v)$. The corresponding strata efficiencies are $\lambda_{11} = (v-pq)t/qf_1$, $\lambda_{21} = v[(p-1)+(t-1)(s-1)]/qf_1$, $\lambda_{31} = v(q-1)t/qf_1$, $\lambda_{41} = v(q-1)[(p-1)+(s-1)(t-1)]/qf_1$ for T_1, and $\lambda_{12} = \lambda_{32} = 0$, $\lambda_{22} = 1/q$, $\lambda_{42} = (q-1)/q$ for T_2, with $f_1 = p(v-1)+(t-1)(sv-p)$. Since they lie in $T_1 + T_2$, standard contrasts are not estimable in the block or column stratum, as they would be in a single stage design. It is then of interest to ask what proportion of information is lost in the t-stage design relative to a single stage.

First, the variance of a standard contrast $l'(T_1+T_2)\tau$ estimated in the bottom stratum of any t-stage design of Section 2 is, aside from the stratum variance,

$$V_{4t} = l'\left(\frac{T_1}{\lambda_{41}f_1} + \frac{T_2}{\lambda_{42}f_2}\right)l = \frac{q\{(p-s)+(t-1)[(p-s)+t(s-1)]\}}{vt(q-1)(p-s)[(p-s)+t(s-1)]}$$

($f_2 = v(p-s)$ and f_1 are eigenvalues of $A'A$). That of a single stage design is $V_{41} = \frac{q}{v(p-1)(q-1)}$. Thus the t-stage versus 1-stage relative efficiency for stratum 4 is

$$\frac{V_{41}}{V_{4t}} = \frac{t}{p-1}\left[\frac{1}{(p-s)+t(s-1)} + \frac{t-1}{p-s}\right]^{-1}.$$

No information is lost in the bottom stratum when this is 1, which occurs if and only if $s = 1$, the orthogonal case. For totally balanced designs with $s = 2$ and $t = 2$ (e.g. Example 2) the efficiency is $\frac{p(p-2)}{(p-1)^2}$.

Looking now at just a t-stage design, and assuming for simplicity in comparing across strata that the strata variances are all equal ($=\sigma^2$, say), the variance of the overall estimator of a standard contrast obtained by combining stratum 2 and stratum 4 information is

$$V_t = \frac{(p-s)+(t-1)(s-1)}{v(p-s)[(p-1)+(t-1)(s-1)]}\sigma^2.$$

Hence the proportion of available information on standard contrasts that occurs in the bottom stratum is

$$\frac{V_t}{V_{4t}} = \frac{q-1}{q}.$$

For moderate to large q the bottom stratum analysis will usually be satisfactory, but may not be so for q small. This ratio additionally shows that maximizing with respect to s the information in the bottom stratum, as was done in Section 2, simultaneously maximizes the information in the row stratum. It can also be shown that $s=1$ gives orthogonality in stratum 2.

In a single stage design all strata may be used and the combined estimator has variance $V_1 = \frac{\sigma^2}{r}$, so

$$\frac{V_1}{V_t} = \frac{v(p-s)[(p-1)+(t-1)(s-1)]}{p(v-1)[(p-s)+(t-1)(s-1)]}$$

is the overall t-stage versus single stage relative efficiency. When $s=1$ this is $\frac{v(p-1)}{p(v-1)}$ and is independent of t.

Given the great savings of experimental material afforded by a t-stage design and the fact that the stratum 4 variance will usually be smallest, these results appear to be satisfactory.

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Appendix I: C_{12} for a Lemma 1 Design

From (2.1) it is clear that C_{12} is determined by the plot, row, column, and block differences between B_i and $\pi(B_i)$ for $i=1,2,\ldots,m$. Of these only the plot and row differences are affected by the choice of π, and then only by $s=\#\{h:\pi_h=h\}$. Denoting the hth row of B_i by

$$B_{ih} = \left(x^{i+h-2}, x^{i+h+m-2}, \ldots, x^{i+h+(q-1)m-2}\right),$$

if $\pi_h = l$ then the plot differences for the hth rows of the m blocks are

$$\begin{aligned}(B_{ih} - B_{il}) &= x^{i+h-2} - x^{i+l-2}, x^{i+h+m-2} - x^{i+l+m-2}, \ldots, \\ &\quad x^{i+h+(q-1)m-2} - x^{i+l+(q-1)m-2}, \quad i = 1, 2, \ldots, m \\ &= (x^0, x^1, \ldots, x^{m-1}) \otimes (x^{h-1} - x^{l-1}) \otimes (x^0, x^m, \ldots, x^{(q-1)m})\end{aligned}$$

which are every non-zero element of GF_v exactly once for $h \neq l$, and $v-1$ copies of 0 for $h = l$. Likewise the row differences for the hth rows are

$$B_{ih} - B_{il}, B_{ih} - x^m B_{il}, \ldots, B_{ih} - x^{(q-1)m} B_{il} =$$
$$x^{i-1}(x^{h-1} - x^{l-1}, x^{h-1} - x^{m+l-1}, \ldots, x^{h-1} - x^{(q-1)m+l-1}) \otimes (x^0, x^m, \ldots, x^{(q-1)m})$$

for $i = 1, 2, \ldots, m$, which are each non-zero element q times if $h \neq l$, or each non-zero $q-1$ times and $v-1$ copies of 0 if $h = l$. The block and column differences are unaffected by row permutations, so are the same as in the proof of Theorem 1 of Uddin and Morgan (1991), except that here plot differences are included. So with the proof there and letting $h = 1, 2, \ldots, p$ in the above lists, we have

differences	frequencies of nonzeros	frequencies of 0
plots	$p-s$	smq
rows	$pq-s$	smq
columns	$p(p-1)$	mpq
blocks	$p(pq-1)$	mpq

$$\begin{aligned}\text{Hence}\quad A_1'A_2 &= smqI + (p-s)(J-I) \\ N_{R,1}N_{R,2}' &= smqI + (pq-s)(J-I) \\ N_{G,1}N_{G,2}' &= mpqI + p(p-1)(J-I) \\ N_{B,1}N_{B,2}' &= mpqI + p(pq-1)(J-I)\end{aligned}$$

$$\Rightarrow C_{12} = m(s-1)(q-1)I - \frac{1}{q}(s-1)(q-1)(J-I).$$

Appendix II: C_{12} for a Lemma 2 Design

Again we look at the differences between B_i and $\pi(B_i)$, where now $\#\{h : \pi_h = h\} = s$, $\pi_h = l \Rightarrow \pi_l = h$, and $p-s$ is even. Write $B_{ih} = (x^{i+h-2}, x^{i+h+2m-2}, \ldots, x^{i+h+2(q-1)m-2})$ for row h of B_i.

Consider rows h and l where $\pi_h = l$ and $\pi_l = h$. The plot differences for these two rows over all squares are

$$\begin{aligned}&(B_{ih} - B_{il}, B_{il} - B_{ih}) \\ &= \pm(x^{i+h-2} - x^{i+l-2}) \otimes (x^0, x^{2m}, \ldots, x^{2(q-1)m}), \quad i = 1, 2, \ldots, m \\ &= (x^{h-2} - x^{l-2})(x^0, x^1, \ldots, x^{2mq-1})\end{aligned}$$

since $-1 = x^{mq}$ and q is odd. Since there are $(p-s)/2$ such pairs of rows defined by π, the plot differences are each non-zero element $(p-s)/2$ times and 0 with frequency smq.

The row differences for rows h and l are

$$\pm(B_{ih} - B_{il}), \pm(B_{ih} - x^{2m}B_{il}), \ldots, \pm(B_{ih} - x^{2(q-1)m}B_{il}) \quad i = 1, 2, \ldots, m$$

which simplifies to

$$(x^{h-1} - x^{l-1}, x^{h-1} - x^{2m+l-1}, \ldots, x^{h-1} - x^{2(q-1)m+l-1}) \otimes (x^0, x^1, \ldots, x^{2mq} - 1)$$

i.e. each non-zero element q times. If $h = l$, the row differences for row h are

$$(B_{ih} - B_{ih}), (B_{ih} - x^{2m}B_{ih}), \ldots, (B_{ih} - x^{2(q-1)m}B_{ih}) \quad i = 1, 2, \ldots, m.$$

Since the B_{ih}'s for $i = 1, 2, \ldots, m$ form the difference set of Sprott (1954, Theorem 2.1), these are 0 mq times and each non-zero $(q-1)/2$ times. So the row differences for all rows are 0 smq times each non-zero $(pq-s)/2$ times.

Taking the column and block differences from the proof of Theorem 2 of Uddin and Morgan (1991) (and adding 0 differences for plots),

$$\begin{aligned}
A_1'A_2 &= smqI + \frac{p-s}{2}(J-I) \\
N_{R,1}N_{R,2}' &= smqI + \frac{pq-s}{2}(J-I) \\
N_{G,1}N_{G,2}' &= mpqI + \frac{p(p-1)}{2}(J-I) \\
N_{B,1}N_{B,2}' &= mpqI + \frac{p(pq-1)}{2}(J-I)
\end{aligned}$$

$$\Rightarrow C_{12} = m(s-1)(q-1)\left[I - \frac{1}{2mq}(J-I)\right] = \frac{s-1}{p-1}C_{11}.$$

Appendix III: Maximum t in $A^\sigma(p,t,s), s = 0, 1, 2$

It is simple to prove that every $A^\sigma(p,t,1)$ has a constant row, so that analogous to the construction of Lemma 5, its existence is equivalent to that of $A^\sigma(p-1,t,0)$. Likewise one may easily prove that for $t \neq 3$ or 4, $A^\sigma(p,t,2)$ has two constant rows, so that its existence is equivalent to that of $A^\sigma(p-2,t,0)$. The exception for $t = 4$ is that an $A^\sigma(p,4,2)$ that does not have two constant rows must be of the form

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 4 & 3 & 3 & 4 \\ 3 & 4 & 4 & 3 \\ 5 & 6 & 5 & 6 \\ 6 & 5 & 6 & 5 \end{pmatrix}$$

and A_2 is an $A^\sigma(p-6,4,0)$ on the symbols $7, 8, \ldots, p$. For $t = 3$ the noncompliant $A^\sigma(p,3,2)$ must be three columns of the $A^\sigma(p,4,2)$ just described. Since the work below implies that an $A^\sigma(p-6,3,0)$ or $A^\sigma(p-6,4,0)$ must have $p-6$ a multiple of 4, these exceptions are of no consequence for the existence of $A^\sigma(p,t,2)$ of maximum t : they do not give larger t than already obtained in Lemma 5.

Thus the conjecture following Lemma 5 amounts to proving

Theorem A1. *Write $p = 2^\omega d$ where d is odd and $\omega \geq 0$. $A^\sigma(p,t,0)$ exists if and only if $t \leq 2^\omega$.*

Given Lemma 5, only the nonexistence for $t > 2^\omega$ remains to be demonstrated, and the case $\omega = 0$ is trivial, so here $\omega \geq 1$ and p is even. This will be done via a series of steps, the key to which is showing that every $A^\sigma(p,t,0)$ can be put in standard form. Before defining this notion some notation is needed. The elements of the matrix array $A = A^\sigma(p,t,0)$ are denoted by a_{ij}. The columns of A are $a_1, \ldots, a_t$ where $a_j = (a_{1j}, a_{2j}, \ldots, a_{pj})'$. The partition of a_j into successive 2-tuples is written $a_j = (a_{1j}^{*\prime}, a_{2j}^{*\prime}, \ldots, a_{\frac{p}{2}j}^{*\prime})'$ where $a_{ij}^* = (a_{2i-1,j}, a_{2i,j})'$. The a_{ij}^*'s are the *elementary 2-tuples* of column j. The reversal of an elementary 2-tuple is $a_{ij}^{*R} = (a_{2i,j}, a_{2i-1,j})'$. The reversal of column j is found by reversing all of its elementary 2-tuples, $a_j^R = (a_{1j}^{*R\prime}, a_{2j}^{*R\prime}, \ldots, a_{\frac{p}{2}j}^{*R\prime})'$.

Now define $A = A^\sigma(p,t,0)$ to be in *standard form* if for every odd $j \leq t-1$, $a_{j+1} = a_j^R$.

Note that A retains the properties of an $A^\sigma(p,t,0)$ under permutations of rows and of columns, and that standard form is unaffected by permuting pairs of rows corresponding to elementary 2-tuples. It follows that the first two columns can always be put into standard form, i.e. $a_2 = a_1^R$. Assuming that this has been done, it is straightforward to prove that

Result 1. *For fixed $j \geq 3$, $a_{lj}^* = a_{l'1}^*$ or $a_{l'1}^{*R}$ for some $l' \neq l$.*

Simply put, Result 1 says that every column of A contains the same set of unordered elementary 2-tuples, though in possibly different sequences. Generalizing this,

Result 2. *If for some l and some $j, j' (j \neq j')$, $a^*_{lj} = a^{*R}_{lj'}$, then for k such that $k \neq j$ and $k \neq j'$ there exists $m \neq l$ such that either $a^*_{mj} = a^*_{lk}$, $a^*_{mj'} = a^{*R}_{lk}$, $a^*_{mk} = a^*_{lj}$ or $a^*_{mj} = a^{*R}_{lk}$, $a^*_{mj'} = a^*_{lk}$, $a^*_{mk} = a^{*R}_{lj}$.*

Result 2, easily proven using Result 1, says that if three columns contain the 2×3 subarray $\begin{pmatrix} a & b & c \\ b & a & d \end{pmatrix}$ then the same three columns also contain one of the 2×3 subarrays $\begin{pmatrix} c & d & a \\ d & c & b \end{pmatrix}$ or $\begin{pmatrix} d & c & b \\ c & d & a \end{pmatrix}$.

Result 3. *The existence of $A^\sigma(p, t, 0)$ implies the existence of $A^\sigma(p, t, 0)$ in standard form.*

Proof. Write $A = A^\sigma(p, t, 0)$. Permute the rows and columns of A so that for some $s > 0$ the first $2s$ columns of A are in standard form, and for all $j, j' > 2s$, $a_{j'} \neq a^R_j$; for a given A, s is a unique number. It will be shown that A can be modified so that the first $2s + 2$ columns are in standard form $(2s + 2 \leq t)$. Consider a_{2s+1}.

Case 1. Suppose for all $j > 2s+1$ and all l, $a^{*R}_{l,2s+1} \neq a^*_{l,j}$. Then replacing a_{2s+2} by a^R_{2s+1} gives an $A^\sigma(p, t, 0)$ with the first $2s+2$ columns in standard form. To show this, a stronger claim will be proved, namely that (A, a^R_{2s+1}) is an $A^\sigma(p, t + 1, 0)$ for which the first $2s + 2$ columns can be put in standard form. Sufficient is that $B = (a_{2s+1}, a_j, a^R_{2s+1})$ is an $A^\sigma(p, 3, 0)$ for each $j > 2s + 1$. Consider the 2×2 subarray $(a^*_{i,2s+1}, a^*_{ij})$ in the first two columns of B. Since A is an $A^\sigma(p, t, 0)$ containing the first two columns of B there is also a 2×2 subarray $(a^*_{i',2s+1}, a^*_{i'l})$ with $\{a^*_{i',2s+1} = a^*_{il}$ and $a^*_{i'l} = a^*_{i,2s+1}\}$ or $\{a^*_{i',2s+1} = a^{*R}_{il}$ and $a^*_{i'l} = a^{*R}_{i,2s+1}\}$. Hence in the last two columns of B, for any 2×2 subarray $(a^*_{il}, a^{*R}_{i,2s+1})$ there is a 2×2 subarray $(a^*_{i'l}, a^{*R}_{i',2s+1}) = (a^*_{i,2s+1}, a^{*R}_{il})$ or $(a^{*R}_{i,2s+1}, a^*_{il})$, which is what needed to be shown.

Case 2. Suppose for some fixed $j > 2s + 1$ and some but not all l, $a^{*R}_{l,2s+1} = a^*_{l,j}$. Switch columns $2s + 2$ and j so that now $a^{*R}_{l,2s+1} = a^*_{l,2s+2}$ for some but not all l. Then permute pairs of rows (meaning that elementary 2-tuples are permuted but not changed) so that in the resulting A, $a^{*R}_{l,2s+1} = a^*_{l,2s+2}$ for $l = 1, \ldots, q$, say and for no larger l. This allows the resulting A bo be partitioned as $\binom{A_1}{A_2}$ where A_1 is $2q \times t$.

It will now be shown that the symbols in A_1 are disjoint from those in A_2. The first $2q$ rows of (a_{2s+1}, a_{2s+2}) are clearly disjoint from the last $p - 2q$, since the first $2q$ rows are a standard form $A^\sigma(2q, 2, 0)$. Now consider any a^*_{il} with $i \leq q$ and $l \neq 2s + 1, 2s + 2$, i.e. any $a^*_{il} \in A_1$ but not in (a_{2s+1}, a_{2s+2}). By Result 2 it must occur in standard form in columns $(2s + 1, 2s + 2)$ and therefore only in the first $2q$ rows of those columns. It follows that every column of A_1 contains, in some order, the same q unordered elementary 2-tuples. Since each unordered elementary 2-tuple occurs exactly once in a given column, A_1 and A_2 are disjoint.

Since A_1 and A_2 are symbol disjoint, A_2 must be an $A^\sigma(p-2q,t,0)$. Furthermore, permuting the columns of A_2 will not change the fact that A is an $A^\sigma(p,t,0)$. Also A_2 has its first $2s$ columns in standard form and A_1 has its first $2s+2$ columns in standard form. The problem is now to modify A_2 so that its first $2s+2$ columns are in standard form. This is the problem that the proof started with, except that now the number of rows (symbols) has been reduced. The procedure is to treat A_2 as if it were the A at the start of the proof and go through the same arguments. Since p is finite this process must reach an end.

A consequence of Case 1 in the proof of Result 3 is that $A^\sigma(p,t,0)$ in standard form for which t is odd can be extended by addition of another column to give $A^\sigma(p,t+1,0)$ in standard form.

Result 4. *The existence of $A^\sigma(p,t,0)$ is equivalent to the existence of $A^\sigma(2p,2t,0)$.*

Proof. Given $A = A^\sigma(p,t,0)$ on the symbols $1,\ldots,p$, $B = A^\sigma(2p,2t,0)$ can be constructed by

$$B = \begin{pmatrix} A & A+p \\ A+p & A \end{pmatrix}.$$

Now let $B = A^\sigma(2p,2t,0)$ be given. By Result 3 it can be taken in standard form, and so without loss of generality its first two columns are $(B_1,B_2,\ldots,B_p)'$ where B_i is the 2×2 matrix $B_i = \binom{2i-1 \quad 2i}{2i \quad 2i-1}$. Furthermore if for the purposes of this proof we take $\binom{2i \quad 2i-1}{2i-1 \quad 2i}$ to be the "same" matrix as B_i, then columns $(j,j+1)$ are an arrangement of the B_i's for each odd j. Using Result 2, the pattern of the B_i's is that of an $A^\sigma(p,t,0)$. That is, replacing the 2×2 matrix B_i in B everywhere by the scalar $i, i=1,\ldots,p$ gives an $A^\sigma(p,t,0)$.

We can now prove Theorem A1. Let $A = A^\sigma(p,t^*,0)$ with maximun $t=t^*$ be given, and remember that $p = 2^\omega d$ for some $\omega \geq 1$ and odd d. By Result 3, A can be taken in standard form and t^* is even. Applying Result 4, $A_1 = A^\sigma(\frac{p}{2},\frac{t^*}{2},0)$ can be constructed. By Result 3, A_1 can be taken in standard form and $\frac{t^*}{2}$ is even, for if not an additional column could be added to A_1 and then Result 4 used to construct $A^\sigma(p,t^*+2,0)$, a contradiction. Continuing for $i=2,\ldots,\omega$, construct $A_i = A^\sigma(\frac{p}{2^i},\frac{t^*}{2^i},0)$ by applying Result 4 to A_{i-1}. Now $A_\omega = A^\sigma(d,\frac{t^*}{2^\omega},0)$ where d is odd. Such an array with an odd number of symbols can have only 1 column, and the result is proved.

References

Agrawal, H. L. and Prasad, J. (1982). Some methods of construction of balanced incomplete block designs with nested rows and columns. *Biometrika* **69**, 481–483.

Agrawal, H. L. and Prasad, J. (1983). On construction of balanced incomplete block designs with nested rows and columns. *Sankhyā Ser.B* **30**, 345–350.

Cheng, C.-S. (1980). Orthogonal arrays with variable numbers of symbols. *Ann. Statist.* **8**, 447–453.

Cheng, C.-S. (1986). A method for constructing balanced incomplete-block designs with nested rows and columns. *Biometrika* **73**, 695–700.

Eccleston, J. A. and John, J. A. (1986). Recovery of row and column information in row-column designs with adjusted orthogonality. *J. Roy. Statist. Soc. Ser.B* **48**, 238–243.

Eccleston, J. A. and John, J. A. (1988). Adjusted orthogonal nested row-column designs. *Austral. J. Statist.* **30**, 78–84.

Eccleston, J. A. and Russell, K. G. (1975). Connectedness and orthogonality in multi-factor designs. *Biometrika* **62**, 341–345.

Houtman, A. M. and Speed, T. P. (1983). Balance in designed experiments with orthogonal block structure. *Ann. Statist.* **11**, 1069–1085.

Hedayat, A., Seiden, E. and Federer, W. T. (1972). Some families of designs for multi-stage experiments: mutually balanced Youden designs where the number of treatments is prime power or twin primes. I. *Ann. Math. Statist.* **43**, 1517–1527.

Ipinyomi, R. A. and John, J. A. (1985). Nested generalized cyclic row-column designs. *Biometrika* **72**, 403–419.

Mandeli, J. P. and Federer, W. T. (1984). On the construction of mutually orthogonal F-hyperrectangles. *Util. Math.* **25**, 315–324.

Preece, D. A. (1966). Some balanced incomplete block designs for two sets of treatments. *Biometrika* **53**, 497–506.

Preece, D. A. (1976). Non-orthogonal Graeco-Latin designs. In *Combin. Math.* **4**, 7–26. *Lecture Notes in Mathematics 560.* Springer-Verlag, Berlin.

Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* John Wiley, New York.

Saha, G. M. and Das, A. D. (1988). A note on construction of mutually balanced Youden designs. *Util. Math.* **33**, 5–8.

Singh, M. and Dey, A. (1979). Block designs with nested rows and columns. *Biometrika* **66**, 321–326.

Speed, T. P. (1982). General balance. In *Encyclopedia of Statistical Sciences* **3** (Edited by S. Kotz and N. L. Johnson), 320–326. John Wiley, New York.

Sprott, D. A. (1954). A note on balanced incomplete block designs. *Canad. J. Math.* **6**, 341–346.

Sreenath, P. R. (1989). Construction of some balanced incomplete designs with nested rows and columns. *Biometrika* **76**, 399–402.

Street, D. J. (1981). Graeco-Latin and nested row and column designs. In *Combin. Math.* **8** (Edited by K. L. McAvaney), 304–313. *Lecture Notes in Mathematics 884.* Springer-Verlag, Berlin.

Uddin, N. and Morgan, J. P. (1990). Some constructions for balanced incomplete block designs with nested rows and columns. *Biometrika* **77**, 193–202.

Uddin, N. and Morgan, J. P. (1991). Two constructions for balanced incomplete block designs with nested rows and columns. *Statistica Sinica* **1**, 229–232.

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