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ON ESTIMATION OF MEAN SQUARED ERRORS OF BENCHMARKED EMPIRICAL BAYES ESTIMATORS

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Abstract: We consider benchmarked empirical Bayes (EB) estimators under the basic area-level model of Fay and Herriot while requiring the standard benchmarking constraint. In this paper we determine the excess mean squared error (MSE) from constraining the estimates through benchmarking. We show that the increase due to benchmarking is $O(m^{-1})$, where m is the number of small areas. Furthermore, we find an asymptotically unbiased estimator of this MSE and compare it to the second-order approximation of the MSE of the EB estimator or, equivalently, of the MSE of the empirical best linear unbiased predictor (EBLUP), that was derived by Prasad and Rao (1990). Morever, using methods similar to those of Butar and Lahiri (2003), we compute a parametric bootstrap estimator of the MSE of the benchmarked EB estimator under the Fay-Herriot model and compare it to the MSE of the benchmarked EB estimator found by a second-order approximation. Finally, we illustrate our methods using SAIPE data from the U.S. Census Bureau, and in a simulation study.

Key words and phrases: Benchmarking, empirical bayes, Fay-Herriot, mean squared error, parametric bootstrap, small-area.

1. Introduction

Small area estimation has become increasingly popular recently due to a growing demand for such statistics. It is well known that direct small-area estimators usually have large standard errors and coefficients of variation. In order to produce estimates for these small areas, it is necessary to borrow strength from other related areas. Accordingly, model-based estimates often differ widely from the direct estimates, especially for areas with small sample sizes. One problem that arises in practice is that the model-based estimates do not aggregate to the more reliable direct survey estimates. Agreement with the direct estimates is often a political necessity to convince legislators of the utility of small area estimates. The process of adjusting model-based estimates to correct this problem is known as benchmarking. Another key benefit of benchmarking is protection against model misspecification as pointed out by You, Rao, and Dick (2004) and Datta et al. (2011).

In recent years, the literature on benchmarking has grown. Among others, Pfeffermann and Barnard (1991); You and Rao (2003); You, Rao, and Dick (2004); Pfeffermann and Tiller (2006); and Ugarte, Militino, and Goicoa (2009) have made an impact on the continuing development of this field. Specifically, Wang, Fuller, and Qu (2008) provided a frequentist method wherein an augmented model was used to construct a best linear unbiased predictor (BLUP) that automatically satisfies the benchmarking constraint. In addition, Datta et al. (2011) developed very general benchmarked Bayes estimators, that covered most of the earlier estimators that were motivated from either a frequentist or Bayesian perspective. Specifically, they found benchmarked Bayes estimators under the Fay and Herriot (1979) model.

Due to the fact that they borrow strength, model-based estimates typically show a substantial improvement over direct estimates in terms of mean squared error (MSE). It is of particular interest to determine how much of this advantage is lost by constraining the estimates through benchmarking. The aforementioned work of Wang, Fuller, and Qu (2008) and Ugarte, Militino, and Goicoa (2009) examined this question through simulation studies but did not derive any probabilistic results. They showed that the MSE of the benchmarked EB estimator was slightly larger than the MSE of the EB estimator for their simulation studies. In Section 3, we derive a second-order approximation of the MSE of the benchmarked Bayes EB estimator to show that the increase due to benchmarking is $O(m^{-1})$, where m is the number of small areas.

In this paper, we are concerned with the basic area-level model of Fay and Herriot (1979). We propose benchmarked EB estimators in Section 2. In Section 3, we derive a second-order asymptotic expansion of the MSE of the benchmarked EB estimator. In Section 4, we find an estimator of this MSE and compare it to the second-order approximation of the MSE of the EB estimator or, equivalently, the MSE of the EBLUP, that was derived by Prasad and Rao (1990). Finally, in Section 5, using methods similar to those of Butar and Lahiri (2003), we compute a parametric bootstrap estimator of the mean squared error of the benchmarked EB estimator under the Fay and Herriot (1979) model and compare it to our estimators from Section 2. Section 6 contains an application based on Small Area Income and Poverty Estimation Data (SAIPE) from the U.S. Census Bureau as well as a simulation study. Some concluding remarks are made in Section 7.

2. Benchmarked Empirical Bayes Estimators

Consider the area-level random effects model

$$\hat{\theta}_i = \theta_i + e_i, \quad \theta_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + u_i, \quad i = 1, \ldots, m, \tag{2.1}$$

where e_i and u_i are mutually independent with $e_i \overset{ind.}{\sim} N(0, D_i)$ and $u_i \overset{iid}{\sim} N(0, \sigma_u^2)$. This model was first considered in the context of estimating income for small areas (population less than 1,000) by Fay and Herriot (1979). In (2.1), the D_i are known as are the $p \times 1$ design vectors $\boldsymbol{x}_i$. However, the vector of regression coefficients $\boldsymbol{\beta}_{p\times 1}$ is unknown.

When the variance component σ_u^2 is known and $\boldsymbol{\beta}$ has a uniform prior on $\mathbb{R}^p$, then the Bayes estimator of θ_i is given by $\hat{\theta}_i^B = (1 - B_i)\hat{\theta}_i + B_i \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}$ where $B_i = D_i(\sigma_u^2 + D_i)^{-1}$, $\tilde{\boldsymbol{\beta}} \equiv \tilde{\boldsymbol{\beta}}(\sigma_u^2) = (X'V^{-1}X)^{-1}X'V^{-1}\hat{\boldsymbol{\theta}}$, and $V = \text{Diag}(\sigma_u^2 + D_1, \ldots, \sigma_u^2 + D_m)$. Suppose now we want to match the weighted average of some estimates δ_i to the weighted average of the direct estimates, which we denote by t. We assume for our calculations that $t = \sum_i w_i \hat{\theta}_i =: \bar{\hat{\theta}}_w$. We denote the normalized weights by w_i, so that $\sum_i w_i = 1$. Under the loss $L(\theta, \delta) = \sum_i w_i(\theta_i - \delta_i)^2$, and subject to $\sum_i w_i \delta_i = \sum_i w_i \hat{\theta}_i$, the benchmarked Bayes estimator derived in Datta et al. (2011) is

$$\hat{\theta}_i^{BM1} = \hat{\theta}_i^B + (\bar{\hat{\theta}}_w - \bar{\hat{\theta}}_w^B), \quad i = 1, \ldots, m, \tag{2.2}$$

where $\bar{\hat{\theta}}_w^B = \sum_i w_i \hat{\theta}_i^B$. In more realistic settings, σ_u^2 is unknown. Let $P_X = X(X^TX)^{-1}X^T$, $h_{ij} = \boldsymbol{x}_i^T(X^TX)^{-1}\boldsymbol{x_j}$, $\hat{u}_i = \hat{\theta}_i - \boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\beta}} = (X^TX)^{-1}X^T\hat{\boldsymbol{\theta}}$. In this paper, we consider the simple moment estimator given by $\hat{\sigma}_u^2 = \max\{0, \tilde{\sigma}_u^2\}$ where $\tilde{\sigma}_u^2 = (m-p)^{-1}\left[\sum_{i=1}^m \hat{u}_i^2 - \sum_{i=1}^m D_i(1 - h_{ii})\right]$, which is given in Prasad and Rao (1990). Then the benchmarked EB estimator of θ_i is

$$\hat{\theta}_i^{EBM1} = \hat{\theta}_i^{EB} + (\bar{\hat{\theta}}_w - \bar{\hat{\theta}}_w^{EB}), \tag{2.3}$$

where $\hat{\theta}_i^{EB} = (1 - \hat{B}_i)\hat{\theta}_i + \hat{B}_i \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\hat{\sigma}_u^2)$, $\hat{B}_i = D_i(\hat{\sigma}_u^2 + D_i)^{-1}$, $i = 1, \ldots, m$. The objective of the next two sections will be to obtain the MSE of the benchmarked EB estimator correct up to $O(m^{-1})$ and also to find an estimator of the MSE correct to the same order.

3. Second-Order Approximation to MSE

Wang, Fuller, and Qu (2008) construct a simulation study to compare the MSE of the benchmarked EB estimator to the MSE of the EB estimator. In this section, we derive a second order expansion for the MSE of the benchmarked Bayes estimator under the same regularity conditions and assuming the standard benchmarking constraint. That is, for the model proposed in Section 2, we obtain a second-order approximation to the MSE of the empirical benchmarked Bayes estimator derived in Section 2. Take $h_{ij}^V = \boldsymbol{x}_i^T(X^TV^{-1}X)^{-1}\boldsymbol{x_j}$ and assume that $\sigma_u^2 > 0$. Establishing Theorem 1 requires the regularity conditions

(i) $0 < D_L \leq \inf_{1\leq i\leq m} D_i \leq \sup_{1\leq i\leq m} D_i \leq D_U < \infty$;

(ii) $\max_{1\le i\le m} h_{ii} = O(m^{-1})$; and

(iii) $\max_{1\le i\le m} w_i = O(m^{-1})$.

Condition (iii) requires a kind of homogeneity of the small areas, and in particular, it assumes there are not a few large areas that dominate the others in terms of the w_i. Conditions (i) and (ii) are similar to those of Prasad and Rao (1990) and are often assumed in the small area estimation literature.

Before stating Theorem 1, we first present some lemmas whose proofs are provided in the supplementary material and are used in the proof of Theorem 1. The proof of Theorem 1 can be found in Appendix B.

Lemma 1. *Let $r > 0$ be arbitrary. Then*

(i) $E\left[\left\{\dfrac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\right\}^{2r}\right] = O(1)$, *and*

(ii) $E\left[\sup_{\sigma_u^2\ge 0}\left|\dfrac{\partial^2\hat{\theta}_i^B}{\partial(\sigma_u^2)^2}\right|^{2r}\right] = O(1)$.

Recall that $\boldsymbol{u} = \hat{\boldsymbol{\theta}} - X\boldsymbol{\beta} \sim N(0, V)$. The results below then follow.

Lemma 2. *Let $r > 0$ and assume $\max_{1\le i\le m} \boldsymbol{x}_i^T\boldsymbol{\beta} = O(1)$. Then*

$$||\hat{\boldsymbol{\theta}} - X\tilde{\boldsymbol{\beta}}||^{2r} = O_p(m^r) \quad \textit{and} \quad E\left[||\hat{\boldsymbol{\theta}} - X\tilde{\boldsymbol{\beta}}||^{2r}\right] = O(m^r).$$

Lemma 3. *Let $\boldsymbol{z} \sim N_p(\boldsymbol{0}, \Sigma)$. For matrices $A_{p\times p}$ and $B_{p\times p}$, where B symmetric, we have*

(i) $\mathrm{Cov}(\boldsymbol{z}^T A\boldsymbol{z}, \boldsymbol{z}^T B\boldsymbol{z}) = 2\mathrm{tr}(A\Sigma B\Sigma)$.

(ii) $\mathrm{Cov}(\boldsymbol{z}^T A\boldsymbol{z}, (\boldsymbol{z}^T B\boldsymbol{z})^2) = 4\mathrm{tr}(A\Sigma B\Sigma)\mathrm{tr}(B\Sigma) + 8\mathrm{tr}(A\Sigma B\Sigma B\Sigma)$.

Lemma 4. $E[(\tilde{\sigma}_u^2 - \sigma_u^2)^2] = 2(m-p)^{-2}\sum_{i=1}^m(\sigma_u^2 + D_i)^2 + O(m^{-2})$.

Theorem 1. *If regularity conditions* (i)$-$(iii) *hold, then* $E[(\hat{\theta}_i^{EBM1} - \theta_i)^2] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + g_4(\sigma_u^2) + o(m^{-1})$, *where*

$$\begin{aligned} g_{1i}(\sigma_u^2) &= B_i\sigma_u^2 \\ g_{2i}(\sigma_u^2) &= B_i^2 h_{ii}^V \\ g_{3i}(\sigma_u^2) &= B_i^3 D_i^{-1}\, Var(\tilde{\sigma}_u^2) \\ g_4(\sigma_u^2) &= \sum_{i=1}^m w_i^2 B_i^2 V_i - \sum_{i=1}^m\sum_{j=1}^m w_i w_j B_i B_j h_{ij}^V, \end{aligned}$$

and where $Var(\tilde{\sigma}_u^2) = 2(m-p)^{-2}\sum_{k=1}^m(\sigma_u^2 + D_k)^2 + o(m^{-1})$.

Remark 1. We note that the the MSE of the benchmarked EB estimator in Theorem 1 is always non-negative. It is clear that $g_{1i}(\sigma_u^2)$, $g_{2i}(\sigma_u^2)$, and $g_{3i}(\sigma_u^2)$ are non-negative. To establish the non-negativity of $g_4(\sigma_u^2)$, let $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m)$, where $q_i = w_i B_i V_i^{1/2}$. We can write $g_4(\sigma_u^2) = \boldsymbol{q}^T(I - \tilde{P}_X^T)\boldsymbol{q}$, where $\tilde{P}_X^T = V^{-1/2}X(X^TV^{-1}X)^{-1}X^TV^{-1/2}$. Thus, $g_4(\sigma_u^2) \geq 0$, and hence, the MSE in Theorem 1 is always non-negative.

4. Estimator of MSE Approximation

We now obtain an estimator of the MSE approximation for the Fay-Herriot model (assuming normality). Theorem 2 shows that the expectation of the MSE estimator is correct up to $O(m^{-1})$.

Lemma 5. *Suppose that*

$$\sup_{t \in T} |h'(t)| = O(m^{-1}) \tag{4.1}$$

for some interval $T \subseteq \mathbb{R}$. *If* $\hat{\sigma}_u^2, \sigma_u^2 \in T$ *w.p.* 1, *then* $E[h(\hat{\sigma}_u^2)] = h(\sigma_u^2) + o(m^{-1})$.

Proof. Consider the expansion $h(\hat{\sigma}_u^2) = h(\sigma_u^2) + h'(\sigma_u^{*2})(\hat{\sigma}_u^2 - \sigma_u^2)$ for some σ_u^{*2} between σ_u^2 and $\hat{\sigma}_u^2$. Then $\sigma_u^{*2} \in T$ a.s., and $h'(\sigma_u^{*2}) \leq \sup_{t \in T} |h'(t)|$ a.s. as well. This implies $E[h'(\sigma_u^{*2})(\hat{\sigma}_u^2 - \sigma_u^2)] \leq \sup_{t \in T} |h'(t)| E|\hat{\sigma}_u^2 - \sigma_u^2| = O(m^{-3/2})$ by equation (4.1) and since $E|\hat{\sigma}_u^2 - \sigma_u^2| \leq E^{1/2}[(\hat{\sigma}_u^2 - \sigma_u^2)^2]$. Hence, if (4.1) holds, then $E[h(\hat{\sigma}_u^2)] = h(\sigma_u^2) + o(m^{-1})$.

Theorem 2. $E[g_{1i}(\hat{\sigma}_u^2) + g_{2i}(\hat{\sigma}_u^2) + 2g_{3i}(\hat{\sigma}_u^2) + g_4(\hat{\sigma}_u^2)] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + g_4(\sigma_u^2) + o(m^{-1})$, *where* $g_{1i}(\sigma_u^2), g_{2i}(\sigma_u^2), g_{3i}(\sigma_u^2)$, *and* $g_4(\sigma_u^2)$ *are defined in Theorem* 1.

Proof. By Theorem A.3 in Prasad and Rao (1990), $E[g_{1i}(\hat{\sigma}_u^2) + g_{2i}(\hat{\sigma}_u^2) + 2g_{3i}(\hat{\sigma}_u^2)] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + o(m^{-1})$. In addition, we consider $E[g_4(\hat{\sigma}_u^2)]$, where $g_4(\sigma_u^2) = \sum_{i=1}^m w_i^2 B_i^2 V_i - \sum_{i=1}^m \sum_{j=1}^m w_i w_j B_i B_j h_{ij}^V =: g_{41}(\sigma_u^2) + g_{42}(\sigma_u^2)$. We first show that the derivatives of $g_{41}(\sigma_u^2)$ and $g_{42}(\sigma_u^2)$ satisfy (4.1). Let $T = [0, \infty)$. Consider

$$\sup_{\sigma_u^2 \geq 0} \left| \frac{\partial g_{41}(\sigma_u^2)}{\partial \sigma_u^2} \right| = \sup_{\sigma_u^2 \geq 0} \sum_{i=1}^m w_i^2 B_i^2 = O(m^{-1}).$$

It can be shown that $\dfrac{\partial B_i B_j}{\partial \sigma_u^2} = -B_i B_j^2 D_j^{-1} - B_i^2 B_j D_i^{-1}$ and $(X^TV^{-1}X)^{-1} \leq (X^TV^{-2}X)^{-1} D_L^{-1}$. Observe that

$$\left| \frac{\partial g_{42}(\sigma_u^2)}{\partial \sigma_u^2} \right| \leq \sum_{i=1}^m \sum_{j=1}^m w_i w_j \Big[|B_i D_L^{-1} h_{ij}^V| + |B_j D_L^{-1} h_{ij}^V|$$

$$
\begin{aligned}
&\quad +B_iB_j\boldsymbol{x}_i^T(X^TV^{-1}X)^{-1}X^TV^{-2}X(X^TV^{-1}X)^{-1}\boldsymbol{x}_i\Big]\\
&\le 3m^2(\max_{1\le i\le m} w_i)^2 D_L^{-1}B_i(\sigma_u^2+D_U)(\max_{1\le i\le m} h_i)\\
&\le 3m^2(\max_{1\le i\le m} w_i)^2 D_L^{-1}D_U(\sigma_u^2+D_L)^{-1}(\sigma_u^2+D_U)(\max_{1\le i\le m} h_i)\\
&= 3m^2(\max_{1\le i\le m} w_i)^2 D_L^{-1}D_U(1+D_UD_L^{-1})(\max_{1\le i\le m} h_i) = O(m^{-1}).
\end{aligned}
$$

This implies that $\sup_{\sigma_u^2\ge 0}\left|\frac{\partial g_{42}(\sigma_u^2)}{\partial \sigma_u^2}\right| = O(m^{-1})$. Since the derivatives of $g_{41}(\sigma_u^2)$ and $g_{42}(\sigma_u^2)$ satisfy (4.1), we know that $E[g_4(\hat\sigma_u^2)] = g_4(\sigma_u^2) + o(m^{-1})$.

5. Parametric Bootstrap Estimator of the MSE of the Benchmarked Empirical Bayes Estimator

In this section, we extend the methods of Butar and Lahiri (2003) to find a parametric bootstrap estimator of the MSE of the benchmarked EB estimator. Under the proposed model, the expectation of the proposed measure of uncertainty of the benchmarked EB estimator is correct up to order $O(m^{-1})$.

To introduce the parametric bootstrap method, consider the model

$$
\begin{aligned}
\hat\theta_i^{\,*}|u_i^* &\overset{ind.}{\sim} N(\boldsymbol{x}_i^T\tilde{\boldsymbol{\beta}} + u_i^*, D_i),\\
u_i^* &\overset{ind.}{\sim} N(0,\hat\sigma_u^2).
\end{aligned}
\tag{5.1}
$$

Following Butar and Lahiri (2003), we use the parametric bootstrap twice. We first use it to estimate $g_{1i}(\sigma_u^2), g_{2i}(\sigma_u^2)$, and $g_4(\sigma_u^2)$ by correcting the bias of $g_{1i}(\hat\sigma_u^2), g_{2i}(\hat\sigma_u^2)$, and $g_4(\hat\sigma_u^2)$. We then use it again to estimate $E[(\hat\theta_i^{EB} - \hat\theta_i^{B})^2] = g_{3i}(\sigma_u^2) + o(m^{-1})$.

Butar and Lahiri (2003) derived a parametric bootstrap estimator for the MSE of the EB estimator under the Fay and Herriot (1979) model. Using Theorem A.1 of their paper, they show that the bootstrap estimator $V_i^{\rm BOOT}$ is

$$
V_i^{\rm BOOT} = 2[g_{1i}(\hat\sigma_u^2)+g_{2i}(\hat\sigma_u^2)] - E_*\Big[g_{1i}(\hat\sigma_u^{*2})+g_{2i}(\hat\sigma_u^{*2})\Big] + E_*[(\hat\theta_i^{EB*}-\hat\theta_i^{EB})^2], \tag{5.2}
$$

where E_* denotes the expectation computed with respect to the model given in (5.1), and $\hat\theta_i^{EB*} = (1-B_i(\hat\sigma_u^{*2}))\hat\theta_i + B_i(\hat\sigma_u^{*2})\boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}$. Following their work, we propose a parametric bootstrap estimator of the MSE of the benchmarked EB estimator that is a simple extension of (5.2).

We propose to estimate $g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_4(\sigma_u^2)$ by

$$
2[g_{1i}(\hat\sigma_u^2) + g_{2i}(\hat\sigma_u^2) + g_4(\hat\sigma_u^2)] - E_*\Big[g_{1i}(\hat\sigma_u^{*2}) + g_{2i}(\hat\sigma_u^{*2}) + g_4(\hat\sigma_u^{*2})\Big]
$$

and then to estimate $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})^2]$ by $E_*[(\hat{\theta}_i^{EB*} - \hat{\theta}_i^{EB})^2]$. Thus, our proposed estimator of $\text{MSE}[\hat{\theta}_i^{\text{EBM1}}]$ is

$$V_i^{\text{B-BOOT}} = 2[g_{1i}(\hat{\sigma}_u^2) + g_{2i}(\hat{\sigma}_u^2) + g_4(\hat{\sigma}_u^2)] - E_*\Big[g_{1i}(\hat{\sigma}_u^{*2}) + g_{2i}(\hat{\sigma}_u^{*2}) + g_4(\hat{\sigma}_u^{*2})\Big]$$
$$+E_*[(\hat{\theta}_i^{EB*} - \hat{\theta}_i^{EB})^2].$$

Theorem 3. $E[V_i^{\textit{B-BOOT}}] = \textit{MSE}[\hat{\theta}_i^{\textit{EBM1}}] + o(m^{-1})$.

Proof. First, by Theorem A.1 in Butar and Lahiri (2003), we note that

$$E_*[g_{1i}(\hat{\sigma}_u^{*2})] = g_{1i}(\hat{\sigma}_u^2) - g_{3i}(\hat{\sigma}_u^2) + o_p(m^{-1}),$$
$$E_*[g_{2i}(\hat{\sigma}_u^{*2})] = g_{2i}(\hat{\sigma}_u^2) + o_p(m^{-1}), \text{ and}$$
$$E_*[(\hat{\theta}_i^{EB*} - \hat{\theta}_i^{EB})^2] = g_{5i}(\hat{\sigma}_u^2) + o_p(m^{-1}),$$

where $g_{5i}(\hat{\sigma}_u^2) = [B_i(\hat{\sigma}_u^2)]^4 D_i^{-2}\left(\hat{\theta}_i - \boldsymbol{x}_i^T\tilde{\boldsymbol{\beta}}(\hat{\sigma}_u^2)\right)^2$. Also, $E_*[g_4(\hat{\sigma}_u^{*2})] = g_4(\hat{\sigma}_u^2) + o_p(m^{-1})$, which follows along the lines of the proof of Theorem A.2(b) of Datta and Lahiri (2000). Applying these results and our Theorem 2, we find

$$V_i^{\text{B-BOOT}} = g_{1i}(\hat{\sigma}_u^2) + g_{2i}(\hat{\sigma}_u^2) + g_{3i}(\hat{\sigma}_u^2) + g_4(\hat{\sigma}_u^2) + g_{5i}(\hat{\sigma}_u^2) + o_p(m^{-1}).$$

This implies that

$$E[V_i^{\text{B-BOOT}}] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + g_4(\sigma_u^2) + o(m^{-1})$$

since $E[g_{5i}(\hat{\sigma}_u^2)] = g_{3i}(\sigma_u^2) + o(m^{-1})$ by Butar and Lahiri (2003), and by applying the results of Prasad and Rao (1990).

6. Two Applications

In this section, we consider a data set and report on a simulation study in order to compare the performance of the estimator of the MSE of the benchmarked EB estimator and the parametric bootstrap estimator of the MSE of the benchmarked EB estimator. Tables and figures that result from this can be found in Appendix A.

We consider data from the Small Area Income and Poverty Estimates (SAIPE) program at the U.S. Census Bureau, which produces model-based estimates of the number of poor school-aged children (5–17 years old) at the national, state, county, and district levels. The school district estimates are benchmarked to the state estimates by the Department of Education to allocate funds under the No Child Left Behind Act of 2001. Specifically, we consider year 1997. In the SAIPE program, the model-based state estimates are benchmarked to the national school-aged poverty rate using the benchmarked estimator in (2.3). The number of poor school-aged children has been collected from the Annual Social and Economic Supplement (ASEC) of the Current Population Survey (CPS) from 1995 to 2004, while the American Community Survey (ACS) estimates have been used since 2005. Additionally, the model-based county estimates are benchmarked to the model-based state estimates using the the benchmarked estimator in (2.3).

In the SAIPE program, the state model for poverty rates in school-aged children follows the basic Fay and Herriot (1979) framework where $\hat{\theta}_i = \theta_i + e_i$ and $\theta_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + u_i$. Here θ_i is the true state level poverty rate, $\hat{\theta}_i$ is the direct survey estimate (from CPS ASEC), e_i is the sampling error term with assumed known variance $D_i > 0$, $\boldsymbol{x}_i$ are the predictors, $\boldsymbol{\beta}$ is the unknown vector of regression coefficients, and u_i is the model error with unknown variance σ_u^2. The explanatory variables in the model are the IRS income tax–based pseudo-estimate of the child poverty rate, IRS non-filer rate, food stamp rate, and the residual term from the regression of the 1990 Census estimated child poverty rate. We estimate $\boldsymbol{\beta}$ using the weighted least squares type estimator $\tilde{\boldsymbol{\beta}}(\hat{\sigma}_u^2) = (X'V^{-1}X)^{-1}X'V^{-1}\hat{\boldsymbol{\theta}}$, and we estimate σ_u^2 using the modified moment estimator $\hat{\sigma}_u^2$ from Section 2.

As shown in Table A.1, the estimated MSE of the EB estimator, $\text{mse}(\hat{\theta}_i^{EB})$, compared to the estimated MSE of the benchmarked EB estimator, $\text{mse}(\hat{\theta}_i^{EBM1})$, differs by the constant $g_4(\sigma_u^2)$, 0.025. This constant is effectively the increase in MSE that we suffer from benchmarking, and we see that in this case it is small (compared to the values of the MSEs). Generally speaking, it is expected to be small since $g_4(\sigma_u^2) = O(m^{-1})$.

In Table A.1, we write mse^B and mse^{BB} as the bootstrap estimates of the MSE of the EB estimator and the benchmarked EB estimator, respectively. As mentioned, we consider year 1997 for illustrative purposes. When we performed the bootstrapping, we resampled $\tilde{\sigma}_u^{*2}$ 10,000 times in order to calculate mse^B and mse^{BB}. This is best understood through the concept behind our bootstrapping approach. Consider the behavior of $g_{1i}(\sigma_u^2)$, the only term that is $O(1)$. Ordinarily, $g_{1i}(\hat{\sigma}_u^2)$ underestimates $g_{1i}(\sigma_u^2)$, and $E_*[g_{1i}(\hat{\sigma}_u^2)]$ underestimates $g_{1i}(\hat{\sigma}_u^2)$. The basic idea is that we use the amount by which $E_*[g_{1i}(\hat{\sigma}_u^2)]$ underestimates $g_{1i}(\hat{\sigma}_u^2)$ as an approximation of the amount by which $g_{1i}(\hat{\sigma}_u^2)$ underestimates $g_{1i}(\sigma_u^2)$.

We run into a problem with the 1997 data, where $g_{1i}(\hat{\sigma}_u^2)$ is 0, since in this case $E_*[g_{1i}(\hat{\sigma}_u^2)]$ overestimates $g_{1i}(\hat{\sigma}_u^2)$. Recall that

$$V_i^{\text{B-BOOT}} = g_{1i}(\hat{\sigma}_u^2) + \{g_{1i}(\hat{\sigma}_u^2) - E_*[g_{1i}(\hat{\sigma}_u^{*2})]\} + O(m^{-1}).$$

Since $g_{1i}(\hat{\sigma}_u^2)$ is 0 and is the dominating term of $V_i^{\text{B-BOOT}}$, many of the estimated MSEs of the benchmarked bootstrapped estimator (mse^{BB}) are negative. Also, observe this same behavior holds true for the bootstrapped estimator proposed by Butar and Lahiri (2003), which we denote by mse^{B}. Hence, we do not recommend using bootstrapping when $\hat{\sigma}_u^2$ is too close to zero because of the form of $\hat{\sigma}_u^2$. We also note that the MSE of the benchmarked EB estimator is always non-negative as explained in Remark 1 of Section 3.

In the second example, we ran a simulation study, using the same covariates from the SAIPE dataset from 1997. We generated our data from the model

$$\begin{aligned} \hat{\theta}_i|\theta_i &\overset{ind.}{\sim} N(\theta_i, D_i), \\ \theta_i &\overset{ind.}{\sim} N(X^T\boldsymbol{\beta}, \sigma_u^2), \end{aligned} \tag{6.1}$$

where D_i comes from the SAIPE dataset. We first simulated 10,000 sets of values for θ_i and $\hat{\theta}_i$ using (6.1). We then used each set of $\hat{\theta}_i$ values as the data and computed the EB and benchmarked EB estimators according to (2.3) and the EB formula given below it. In order to use EB, we took $\boldsymbol{\beta} = (-3, 0.5, 1, 1, 0.5)^T$ and $\sigma_u^2 = 5$.

In Figure 1, we compare the estimator of the theoretical MSE of the benchmarked EB estimator and the bootstrap estimator of the MSE of the benchmarked EB estimator with the true value, i.e., the average of the squared difference between the estimator values and the true θ_i, generated according to model (6.1). In the upper plot, we see that the estimator of the theoretical MSE of the benchmarked EB estimator overshoots the truth very slightly, which shows that our estimator is slightly conservative. We find the opposite behavior to be true of the bootstrap estimator of the MSE of the benchmarked Bayes estimator, meaning that it undershoots the truth slightly.

In practice, it seems safer to use a MSE estimator that overestimates than one that underestimates, and hence, we recommend our proposed MSE estimator over the bootstrapped MSE estimator. Using the lower plot, we compared the theoretical Prasad Rao (PR) MSE estimator with the associated true value. We find the same behavior in the PR estimator as we did in our proposed theoretical MSE of the benchmarked EB estimator. The overshoot occurs in the terms that the estimators have in common, i.e., $g_{1i}(\sigma_u^2); g_{2i}(\sigma_u^2)$; and $g_{3i}(\sigma_u^2)$. We see that for this particular simulation study where m is particularly large at 10,000, the difference between the two MSEs is indistinguishable.

7. Summary and Conclusion

We have shown that the increase in MSE due to benchmarking under our modeling assumptions is quite small for the Fay-Herriot model, specifically $O(m^{-1})$. We have derived an asymptotically unbiased estimate of the MSE of the benchmarked EB estimator (EBLUP) under the same assumptions which is correct to order $O(m^{-1})$. We have derived a parametric bootstrap estimator of the benchmarked EB estimator based on work done by Butar and Lahiri

(2003). Furthermore, we have illustrated our methodology for a data set for fixed m using U.S. Census data. Since our theoretical estimator of the MSE under benchmarking is guaranteed to be positive, we recommend it over the one derived by bootstrapping. We also performed a simulation study that suggests use of the theoretical estimator of the MSE under benchmarking. In closing, it is important to pursue further work for more complex models, and, in particular, when it is necessary to achieve multi-stage benchmarking.

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Appendix A

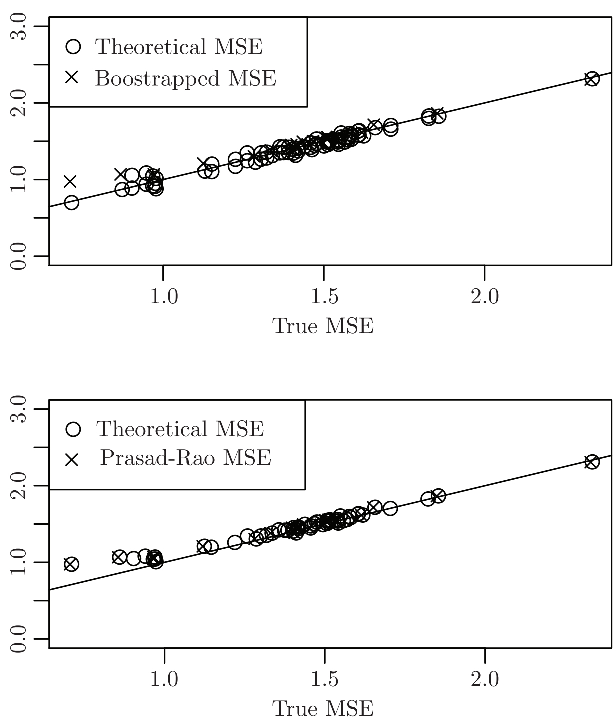

Figure 1. Comparing Simulated MSEs with True MSEs

Table 1. Table of estimates for 1997.

i	$\hat{\theta}_i$	$\hat{\theta}_i^{EB}$	$\hat{\theta}_i^{EBM1}$	mse($\hat{\theta}_i$)	mse($\hat{\theta}_i^{EB}$)	mse($\hat{\theta}_i^{EBM1}$)	mse^B	mse^{BB}
1	25.16	21.38	21.56	15.72	1.38	1.41	0.02	0.04
2	10.99	14.94	15.11	10.44	2.12	2.14	0.66	0.68
3	23.35	20.89	21.06	11.84	1.68	1.70	0.00	0.01
4	23.32	22.18	22.35	13.85	1.90	1.92	0.37	0.38
5	23.55	22.71	22.88	2.39	5.92	5.94	1.12	1.13
6	9.14	13.12	13.29	6.38	2.19	2.22	0.36	0.38
7	10.34	13.39	13.56	9.85	2.08	2.10	0.39	0.41
8	15.54	13.06	13.23	17.56	0.91	0.94	-0.47	-0.45
9	35.85	32.43	32.60	32.35	4.92	4.95	3.49	3.50
10	18.34	19.59	19.76	3.70	3.71	3.74	0.40	0.41
11	23.52	20.53	20.70	12.93	1.16	1.19	-0.38	-0.37
12	18.98	13.72	13.89	20.87	2.45	2.48	1.24	1.26
13	17.56	13.64	13.82	12.38	1.70	1.73	0.23	0.25
14	14.57	15.72	15.89	3.56	3.45	3.47	-0.06	-0.05
15	11.07	12.53	12.70	7.58	1.84	1.86	-0.23	-0.22
16	11.09	11.21	11.38	8.49	1.74	1.76	-0.24	-0.22
17	11.01	13.48	13.65	9.34	1.61	1.63	-0.15	-0.14
18	23.12	20.78	20.95	13.98	1.37	1.40	-0.12	-0.11
19	21.08	24.15	24.32	15.19	1.80	1.82	0.40	0.42
20	13.18	12.44	12.61	13.63	2.09	2.11	0.56	0.57
21	9.90	13.16	13.33	9.28	1.65	1.67	-0.03	-0.01
22	19.66	14.38	14.56	7.66	2.46	2.48	1.02	1.04
23	13.78	16.86	17.03	4.04	3.11	3.13	0.38	0.39
24	14.34	10.11	10.28	9.91	1.64	1.67	0.16	0.17
25	20.58	22.30	22.47	15.07	2.42	2.45	0.97	0.99
26	18.90	15.11	15.28	15.24	1.00	1.03	-0.37	-0.35
27	17.00	18.60	18.77	12.95	1.37	1.40	-0.21	-0.19
28	9.72	9.62	9.79	7.18	2.24	2.26	0.09	0.10
29	14.06	12.94	13.12	10.23	1.71	1.74	-0.06	-0.04
30	10.94	6.72	6.89	11.35	1.88	1.91	0.50	0.52
31	14.66	13.28	13.45	5.52	2.48	2.51	-0.03	-0.01
32	29.69	24.44	24.61	13.18	2.62	2.65	1.38	1.40
33	23.76	22.85	23.02	3.10	4.76	4.79	0.94	0.95
34	13.90	16.58	16.75	5.70	2.29	2.31	-0.01	0.01
35	18.19	13.64	13.81	11.92	1.81	1.84	0.48	0.50
36	13.91	13.64	13.81	3.95	3.07	3.10	-0.25	-0.23
37	16.09	21.50	21.68	11.14	1.52	1.54	0.24	0.26
38	12.60	13.43	13.60	10.35	2.53	2.56	0.83	0.84
39	14.61	13.92	14.09	3.73	3.40	3.42	-0.01	0.00
40	20.37	14.60	14.77	18.53	1.04	1.07	-0.15	-0.14
41	18.74	21.21	21.38	14.57	1.49	1.52	0.02	0.04
42	12.87	15.77	15.94	12.94	1.98	2.01	0.46	0.47
43	16.09	16.10	16.27	11.94	1.92	1.95	0.28	0.30
44	21.95	21.38	21.55	3.38	4.05	4.07	0.38	0.40
45	11.27	9.76	9.93	9.45	2.28	2.31	0.50	0.51
46	11.15	10.10	10.27	11.95	2.45	2.48	0.86	0.88
47	16.40	14.96	15.13	11.51	1.20	1.22	-0.49	-0.47
48	12.26	13.17	13.34	9.33	1.85	1.87	0.01	0.02
49	18.76	22.25	22.42	13.73	3.81	3.83	2.46	2.48
50	7.60	11.87	12.04	6.41	2.74	2.76	0.97	0.98
51	11.74	11.70	11.87	8.86	2.08	2.10	0.17	0.19

Appendix B

Proof of Theorem 1. Observe that

$$\begin{aligned}
&E[(\hat{\theta}_i^{EBM1} - \theta_i)^2] \\
&\quad = E[(\hat{\theta}_i^B - \theta_i)^2] + E[(\hat{\theta}_i^{EBM1} - \hat{\theta}_i^B)^2] \\
&\quad = E[(\hat{\theta}_i^B - \theta_i)^2] + E[(\hat{\theta}_i^B - \hat{\theta}_i^{EB} - t + \bar{\hat{\theta}}_w^{EB})^2] \\
&\quad = E[(\hat{\theta}_i^B - \theta_i)^2] + E[(\hat{\theta}_i^B - \hat{\theta}_i^{EB} + \bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B + \bar{\hat{\theta}}_w^B - t)^2] \\
&\quad = E[(\hat{\theta}_i^B - \theta_i)^2] + E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2] + E[(\bar{\hat{\theta}}_w^B - t)^2] + E[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B)^2] \\
&\qquad -2E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B)] - 2E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\bar{\hat{\theta}}_w^B - t)] \\
&\qquad +2E[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B)(\bar{\hat{\theta}}_w^B - t)]. \qquad\qquad (B.1)
\end{aligned}$$

Next, observe that $E[(\hat{\theta}_i^B - \theta_i)^2] + E[\hat{\theta}_i^{EB} - \hat{\theta}_i^B]^2 = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + o(m^{-1})$, by Prasad and Rao (1990), where

$$\begin{aligned}
g_{1i}(\sigma_u^2) &= B_i \sigma_u^2, \\
g_{2i}(\sigma_u^2) &= B_i^2 h_{ii}^V, \\
g_{3i}(\sigma_u^2) &= B_i^3 D_i^{-1} \mathrm{Var}(\tilde{\sigma}_u^2).
\end{aligned}$$

It may be noted that while $g_{1i}(\sigma_u^2) = O(1)$, both $g_{2i}(\sigma_u^2)$ and $g_{3i}(\sigma_u^2)$ are of order $O(m^{-1})$, as shown in Prasad and Rao (1990). We show that $E[(\bar{\hat{\theta}}_w^B - t)^2] = g_4(\sigma_u^2) = O(m^{-1})$, whereas the remaining four terms of expression (B.1) are of order $o(m^{-1})$.

First, we show that $E[(\bar{\hat{\theta}}_w^B - t)^2] = g_4(\sigma_u^2)$. We write $\bar{\hat{\theta}}_w^B - t = -\sum_{i=1}^m w_i B_i (\hat{\theta}_i - \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}})$ and consider

$$\begin{aligned}
E[(\bar{\hat{\theta}}_w^B - t)^2] &= E\left[\left\{\sum_{i=1}^m w_i B_i (\hat{\theta}_i - \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}})\right\}^2\right] \\
&= \sum_{i=1}^m w_i^2 B_i^2 E[(\hat{\theta}_i - \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}})^2] + \sum_{i \neq j} w_i w_j B_i B_j E[(\hat{\theta}_i - \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}})(\hat{\theta}_j - \boldsymbol{x}_j^T \tilde{\boldsymbol{\beta}})] \\
&= \sum_{i=1}^m w_i^2 B_i^2 (V_i - h_{ii}^V) + \sum_{i \neq j} w_i w_j B_i B_j (-h_{ij}^V) \\
&= \sum_{i=1}^m w_i^2 B_i^2 V_i - \sum_{i=1}^m \sum_{j=1}^m w_i w_j B_i B_j h_{ij}^V. \qquad\qquad (B.2)
\end{aligned}$$

Note that the expression on the right hand side of (B.2) is $O(m^{-1})$ since $\max_{1 \le i \le m} h_{ii} = O(m^{-1})$, which implies that $\max_{1 \le i \le j \le m} h_{ij}^V = O(m^{-1})$.

Next, we return to (B.1) and show that $E[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^{B})^2] = o(m^{-1})$. Consider that

$$\begin{aligned} E[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^{B})^2] &= \sum_i w_i^2 E\left[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2\right] \\ &\quad +2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} w_i w_j E\left[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\hat{\theta}_j^{EB} - \hat{\theta}_j^B)\right] \\ &= 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} w_i w_j E\left[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\hat{\theta}_j^{EB} - \hat{\theta}_j^B)\right] + o(m^{-1}), \quad \text{(B.3)} \end{aligned}$$

since $\sum_i w_i^2 E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2] = o(m^{-1})$. The latter holds because $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2] = g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) = O(m^{-1})$, $\max_{1\le i\le m} w_i = O(m^{-1})$, and $\sum_i w_i = 1$. Thus, it suffices to show $E\left[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\hat{\theta}_j^{EB} - \hat{\theta}_j^B)\right] = o(m^{-1})$ for all $i \ne j$, and we do so by expanding $\hat{\theta}_i^{EB}$ about $\hat{\theta}_i^B$. For simplicity of notation, denote

$$\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2} = \frac{\partial\hat{\theta}_i^B(\sigma_u^2)}{\partial\sigma_u^2} \quad \text{and} \quad \frac{\partial^2\hat{\theta}_{i*}^B}{\partial(\sigma_u^2)^2} = \frac{\partial^2\hat{\theta}_i^B(\sigma_u^{*2})}{\partial(\sigma_u^2)^2}.$$

Then

$$\hat{\theta}_i^{EB} - \hat{\theta}_i^B = \frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2 - \sigma_u^2) + \frac{1}{2}\frac{\partial^2\hat{\theta}_{i*}^B}{\partial(\sigma_u^2)^2}(\hat{\sigma}_u^2 - \sigma_u^2)^2$$

for some σ_u^{*2} between σ_u^2 and $\hat{\sigma}_u^2$. The expansion of $\hat{\theta}_j^{EB}$ about $\hat{\theta}_j^B$ is similar.

Consider $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\hat{\theta}_j^{EB} - \hat{\theta}_j^B)]$ for $i \ne j$. Notice that

$$\begin{aligned} &E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\hat{\theta}_j^{EB} - \hat{\theta}_j^B)] \\ &= E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial\hat{\theta}_j^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2 - \sigma_u^2)^2\right] + \frac{1}{2}E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\hat{\sigma}_u^2 - \sigma_u^2)^3\right] \\ &\quad +\frac{1}{2}E\left[\frac{\partial^2\hat{\theta}_{i*}^B}{\partial(\sigma_u^{*2})^2}\frac{\partial\hat{\theta}_j^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2 - \sigma_u^2)^3\right] + \frac{1}{4}E\left[\frac{\partial^2\hat{\theta}_{i*}^B}{(\partial\sigma_u^2)^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial^2(\sigma_u^2)^2}(\hat{\sigma}_u^2 - \sigma_u^2)^4\right] \\ &:= R_0 + R_1 + R_2 + R_3. \end{aligned}$$

In R_1,

$$\begin{aligned} E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\hat{\sigma}_u^2 - \sigma_u^2)^3\right] &= E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\tilde{\sigma}_u^2 - \sigma_u^2)^3 I(\tilde{\sigma}_u^2 > 0)\right] \\ &\quad - E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\sigma_u^2)^3 I(\tilde{\sigma}_u^2 \le 0)\right]. \quad \text{(B.4)} \end{aligned}$$

Observe that

$$
\begin{aligned}
&E\Big[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\sigma_u^2)^3 I(\tilde{\sigma}_u^2 \le 0)\Big] \\
&\quad\le \sigma_u^6 E^{1/4}\bigg[\Big\{\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\Big\}^4\bigg] E^{1/4}\bigg[\Big\{\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}\Big\}^4\bigg] P^{1/2}(\tilde{\sigma}_u^2 \le 0) \\
&\quad\le \sigma_u^6 E^{1/4}\bigg[\Big\{\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\Big\}^4\bigg] E^{1/4}\bigg[\sup_{\sigma_u^2\ge 0}\Big\{\frac{\partial^2\hat{\theta}_j^B}{\partial(\sigma_u^2)^2}\Big\}^4\bigg] P^{1/2}(\tilde{\sigma}_u^2 \le 0) \\
&\quad= o(m^{-r})
\end{aligned}
$$

for all $r > 0$ by Lemmas 1 (ii) and 2, which we have proved in Appendix A. Also, $P(\tilde{\sigma}_u^2 \le 0) = O(m^{-r})\ \forall\ r > 0$, as proved in Lemma A.6 of Prasad and Rao (1990). Now

$$
\begin{aligned}
&E\Big[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\tilde{\sigma}_u^2-\sigma_u^2)^3 I(\tilde{\sigma}_u^2 > 0)\Big] \\
&\quad= E\Big[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\tilde{\sigma}_u^2-\sigma_u^2)^3\Big] - E\Big[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\tilde{\sigma}_u^2-\sigma_u^2)^3 I(\tilde{\sigma}_u^2 \le 0)\Big], \qquad \text{(B.5)}
\end{aligned}
$$

where the second term expression in (B.5) is $O(m^{-r})$ since $P(\tilde{\sigma}_u^2 \le 0) = O(m^{-r})$ $\forall\ r > 0$. We next observe that

$$
\begin{aligned}
&E\Big[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}(\tilde{\sigma}_u^2-\sigma_u^2)^3\Big] \\
&\quad\le E^{1/4}\bigg[\Big\{\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\Big\}^4\bigg] E^{1/4}\bigg[\Big\{\frac{\partial^2\hat{\theta}_{j*}^B}{\partial(\sigma_u^2)^2}\Big\}^4\bigg] E^{1/2}[(\tilde{\sigma}_u^2-\sigma_u^2)^6] \\
&\quad\le E^{1/4}\bigg[\Big\{\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}\Big\}^4\bigg] E^{1/4}\bigg[\sup_{\sigma_u^2\ge 0}\Big\{\frac{\partial^2\hat{\theta}_j^B}{\partial(\sigma_u^2)^2}\Big\}^4\bigg] E^{1/2}[(\tilde{\sigma}_u^2-\sigma_u^2)^6] \\
&\quad= O(m^{-3/2})
\end{aligned}
$$

since $E[(\tilde{\sigma}_u^2-\sigma_u^2)^{2r}] = O(m^{-r})$ for any $r \ge 1$ by Lemma A.5 in Prasad and Rao (1990). This proves that $R_1 = o(m^{-1})$ since $\max_{1\le i\le m} w_i = O(m^{-1})$. By symmetry, R_2 is also $o(m^{-1})$. Finally, we show that R_3 is $o(m^{-1})$. Using a similar calculation involving R_1, we can show that

$$
\begin{aligned}
&E\Big[\frac{\partial^2\hat{\theta}_{i*}^B}{(\partial\sigma_u^2)^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial^2(\sigma_u^2)^2}(\hat{\sigma}_u^2-\sigma_u^2)^4\Big] \\
&\quad= E\Big[\frac{\partial^2\hat{\theta}_{i*}^B}{(\partial\sigma_u^2)^2}\frac{\partial^2\hat{\theta}_{j*}^B}{\partial^2(\sigma_u^2)^2}(\tilde{\sigma}_u^2-\sigma_u^2)^4\Big] + o(m^{-r}). \qquad \text{(B.6)}
\end{aligned}
$$

Observe now that

$$\begin{aligned}
&E\left[\frac{\partial^2\hat{\theta}^B_{i*}}{(\partial\sigma^2_u)^2}\frac{\partial^2\hat{\theta}^B_{j*}}{\partial^2(\sigma^2_u)^2}(\tilde{\sigma}^2_u-\sigma^2_u)^4\right]\\
&\le E^{1/4}\left[\left\{\frac{\partial^2\hat{\theta}^B_{i*}}{(\partial\sigma^2_u)^2}\right\}^4\right]E^{1/4}\left[\left\{\frac{\partial^2\hat{\theta}^B_{j*}}{\partial^2(\sigma^2_u)^2}\right\}^4\right]E^{1/2}\left[(\tilde{\sigma}^2_u-\sigma^2_u)^8\right]\\
&\le E^{1/4}\left[\sup_{\sigma^2_u\ge 0}\left\{\frac{\partial^2\hat{\theta}^B_{i}}{(\partial\sigma^2_u)^2}\right\}^4\right]E^{1/4}\left[\sup_{\sigma^2_u\ge 0}\left\{\frac{\partial^2\hat{\theta}^B_{j}}{\partial^2(\sigma^2_u)^2}\right\}^4\right]E^{1/2}\left[(\tilde{\sigma}^2_u-\sigma^2_u)^8\right]\\
&=O(m^{-2}).
\end{aligned}$$

Plugging this back into (B.6), we find that $E\left[\frac{\partial^2\hat{\theta}^B_{i*}}{(\partial\sigma^2_u)^2}\frac{\partial^2\hat{\theta}^B_{j*}}{\partial^2(\sigma^2_u)^2}(\hat{\sigma}^2_u-\sigma^2_u)^4\right]=o(m^{-1})$. Hence, R_3 is $o(m^{-1})$. Finally, by calculations similar to those used for (B.4), we find that

$$R_0=E\left[\frac{\partial\hat{\theta}^B_i}{\partial\sigma^2_u}\frac{\partial\hat{\theta}^B_j}{\partial\sigma^2_u}(\hat{\sigma}^2_u-\sigma^2_u)^2\right]=E\left[\frac{\partial\hat{\theta}^B_i}{\partial\sigma^2_u}\frac{\partial\hat{\theta}^B_j}{\partial\sigma^2_u}(\tilde{\sigma}^2_u-\sigma^2_u)^2\right]+o(m^{-r}).$$

Take $\Sigma=V-X(X^TV^{-1}X)^{-1}X^T=(I-P^V_X)V$, where $P_X=X(X^TV^{-1}X)^{-1}X^T$, write $P^V_X=X(X^TV^{-1}X)^{-1}X^TV^{-1}$, and let $\boldsymbol{e_i}$ be the ith unit vector. We can show $\frac{\partial\hat{\theta}^B_i}{\partial\sigma^2_u}=B_i\boldsymbol{e_i}^T\Sigma V^{-2}\tilde{\boldsymbol{u}}$, where $\tilde{\boldsymbol{u}}=\hat{\boldsymbol{\theta}}-X\tilde{\boldsymbol{\beta}}$. Define $A_{ij}=B_iB_jV^{-2}\Sigma\boldsymbol{e_i}\boldsymbol{e_j}^T\Sigma V^{-2}$ and consider

$$\begin{aligned}
E\left[\frac{\partial\hat{\theta}^B_i}{\partial\sigma^2_u}\frac{\partial\hat{\theta}^B_j}{\partial\sigma^2_u}(\tilde{\sigma}^2_u-\sigma^2_u)^2\right]&=E[\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}}(\tilde{\sigma}^2_u-\sigma^2_u)^2]\\
&=\mathrm{Cov}\,(\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}},(\tilde{\sigma}^2_u-\sigma^2_u)^2)+E[\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}}]E[(\tilde{\sigma}^2_u-\sigma^2_u)^2].
\end{aligned}$$

Using Lemma 3 and the relation $(I-P_X)\Sigma=(I-P_X)V$,

$$\begin{aligned}
&\mathrm{Cov}\,(\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}},(\tilde{\sigma}^2_u-\sigma^2_u)^2)\\
&=(m-p)^{-2}\,\mathrm{Cov}\,(\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}},[\tilde{\boldsymbol{u}}^T(I-P_X)\tilde{\boldsymbol{u}}-\mathrm{tr}\{(I-P_X)V\}]^2)\\
&=(m-p)^{-2}\,\mathrm{Cov}\,(\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}},[\tilde{\boldsymbol{u}}^T(I-P_X)\tilde{\boldsymbol{u}}]^2)\\
&\quad-2(m-p)^{-2}\,\mathrm{Cov}\,(\tilde{\boldsymbol{u}}^TA_{ij}\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}}^T(I-P_X)\tilde{\boldsymbol{u}})\,\mathrm{tr}\{(I-P_X)V\}\\
&=(m-p)^{-2}\Big\{4\,\mathrm{tr}\{A_{ij}V(I-P_X)V\}\,\mathrm{tr}\{(I-P_X)V\}\\
&\quad+8\,\mathrm{tr}\{A_{ij}V(I-P_X)V(I-P_X)V\}\\
&\quad-4\,\mathrm{tr}\{A_{ij}V(I-P_X)V\}\,\mathrm{tr}\{(I-P_X)V\}\Big\}\\
&=8(m-p)^{-2}\,\mathrm{tr}\{A_{ij}V(I-P_X)V(I-P_X)V\}.
\end{aligned}$$

$$= 8(m-p)^{-2} B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T \Sigma V^{-1}(I-P_X)V(I-P_X)V^{-1}\Sigma \boldsymbol{e}_{\boldsymbol{i}}, \quad \text{(B.7)}$$

where tr denotes the trace. Observe that $(I-P_X)V^{-1}\Sigma = I-(P_X^V)^T$ and $(I-P_X^V)V(I-(P_X^V)^T) = \Sigma$. Then

$$\begin{aligned}\text{Cov}\,(\tilde{\boldsymbol{u}}^T A_{ij}\tilde{\boldsymbol{u}}, (\tilde{\sigma}_u^2-\sigma_u^2)^2) &= 8(m-p)^{-2} B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T \Sigma V^{-1}(I-P_X)V(I-P_X)V^{-1}\Sigma \boldsymbol{e}_{\boldsymbol{i}} \\ &= 8(m-p)^{-2} B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T (I-P_X^V)V(I-(P_X^V)^T)\boldsymbol{e}_{\boldsymbol{i}} \\ &= 8(m-p)^{-2} B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T \Sigma \boldsymbol{e}_{\boldsymbol{i}} \\ &= 8(m-p)^{-2} B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T V \boldsymbol{e}_{\boldsymbol{i}} + O(m^{-3}) = O(m^{-3}),\end{aligned}$$

since the first term is zero because $i \neq j$ and V is diagonal. We now calculate

$$E[\tilde{\boldsymbol{u}}^T A_{ij}\tilde{\boldsymbol{u}}] = \text{tr}\{B_i B_j V^{-2}\Sigma \boldsymbol{e}_{\boldsymbol{i}} \boldsymbol{e}_{\boldsymbol{j}}^T \Sigma V^{-2}\Sigma\} = B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T \Sigma V^{-2}\Sigma V^{-2}\Sigma \boldsymbol{e}_{\boldsymbol{i}}.$$

Observe that $\Sigma V^{-2}\Sigma = I-(P_X^V)^T - P_X^V + P_X^V(P_X^V)^T$. Then, after some computations, we find that $E[\tilde{\boldsymbol{u}}^T A_{ij}\tilde{\boldsymbol{u}}] = B_i B_j \boldsymbol{e}_{\boldsymbol{j}}^T V^{-1}\boldsymbol{e}_{\boldsymbol{i}} + O(m^{-1}) = O(m^{-1})$ since $i \neq j$. By Lemma 4, $E[(\tilde{\sigma}_u^2-\sigma_u^2)^2] = 2(m-p)^{-2}\sum_{k=1}^m(\sigma_u^2+D_k)^2 + O(m^{-2})$. Then

$$E[\tilde{\boldsymbol{u}}^T A_{ij}\tilde{\boldsymbol{u}}]E[(\tilde{\sigma}_u^2-\sigma_u^2)^2] = o(m^{-1}),$$

since $i \neq j$. This implies that $R_0 = o(m^{-1})$, which in turn implies that

$$E[(\hat{\theta}_i^{EB}-\hat{\theta}_i^B)(\hat{\theta}_j^{EB}-\hat{\theta}_j^B)] = o(m^{-1}) \;\text{ for }\; i \neq j, \quad \text{(B.8)}$$

since R_0, R_1, R_2, and R_3 are all $o(m^{-1})$. Finally, this and (B.3) establishes that $E[(\bar{\hat{\theta}}_w^{EB}-\bar{\hat{\theta}}_w^B)^2] = o(m^{-1})$.

We return to (B.1) to show that $E[(\hat{\theta}_i^{EB}-\hat{\theta}_i^B)(\bar{\hat{\theta}}_w^{EB}-\bar{\hat{\theta}}_w^B)] = o(m^{-1})$. By the Cauchy-Schwarz inequality, we find that

$$E[(\hat{\theta}_i^{EB}-\hat{\theta}_i^B)(\bar{\hat{\theta}}_w^{EB}-\bar{\hat{\theta}}_w^B)] \le E^{1/2}\left[(\hat{\theta}_i^{EB}-\hat{\theta}_i^B)^2\right]E^{1/2}\left[(\bar{\hat{\theta}}_w^{EB}-\bar{\hat{\theta}}_w^B)^2\right] = o(m^{-1}),$$

since the first term is $O(m^{-1/2})$ and the second term is $o(m^{-1/2})$.

For the next term of (B.1), we are interested in showing that $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\bar{\hat{\theta}}_w^B - t)] = o(m^{-1})$. First, by Taylor expansion, we find that

$$\hat{\theta}_i^{EB}-\hat{\theta}_i^B = \frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2-\sigma_u^2) + \frac{1}{2}\frac{\partial^2\hat{\theta}_{i*}^B}{\partial(\sigma_u^2)^2}(\hat{\sigma}_u^2-\sigma_u^2)^2$$

for some σ_u^{*2} between σ_u^2 and $\hat{\sigma}_u^2$. Consider that $\bar{\hat{\theta}}_w^B - t = -\sum_i w_i B_i(\hat{\theta}_i - \boldsymbol{x}_i^T\tilde{\boldsymbol{\beta}})$. Then

$$E[(\hat{\theta}_i^{EB}-\hat{\theta}_i^B)(\bar{\hat{\theta}}_w^B - t)] = -\sum_j w_j B_j E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j - \boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_j w_j B_j E\left[\frac{\partial^2 \hat{\theta}_{i*}^B}{\partial(\sigma_u^2)^2}(\hat{\sigma}_u^2-\sigma_u^2)^2(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]\\
&:= R_4+R_5.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\hat{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]\\
&\quad=-\sigma_u^2 E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2\le 0)\right]+E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2>0)\right]\\
&\quad=E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2>0)\right]+o(m^{-r})\\
&\quad=E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]\\
&\qquad-E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2\le 0)\right]+o(m^{-r})\\
&\quad=E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]+o(m^{-r}) \qquad \text{(B.9)}
\end{aligned}
$$

since we may observe that $E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2\le 0)\right]=o(m^{-r})$ and $E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})I(\tilde{\sigma}_u^2\le 0)\right]=o(m^{-r})$. Now, note that $\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}=B_i\boldsymbol{e}_{\boldsymbol{i}}^T\Sigma V^{-2}\tilde{\boldsymbol{u}}$, and write $D_{ij}=B_iV^{-2}\Sigma\boldsymbol{e}_{\boldsymbol{i}}\boldsymbol{e}_{\boldsymbol{j}}^T$. Then by calculations similar to those in (B.7), we find

$$
\begin{aligned}
&E\left[\frac{\partial\hat{\theta}_i^B}{\partial\sigma_u^2}(\tilde{\sigma}_u^2-\sigma_u^2)(\hat{\theta}_j-\boldsymbol{x}_{\boldsymbol{j}}^T\tilde{\boldsymbol{\beta}})\right]\\
&\quad=\operatorname{Cov}(\tilde{\boldsymbol{u}}^TD_{ij}\tilde{\boldsymbol{u}},\tilde{\sigma}_u^2-\sigma_u^2)\\
&\quad=(m-p)^{-1}\operatorname{Cov}(\tilde{\boldsymbol{u}}^TD_{ij}\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}}^T(I-P_X)\tilde{\boldsymbol{u}}-\operatorname{tr}\{(I-P_X)V\})\\
&\quad=2(m-p)^{-1}\operatorname{tr}\{D_{ij}V(I-P_X)V\}\\
&\quad=2(m-p)^{-1}\operatorname{tr}\{B_iV^{-2}\Sigma\boldsymbol{e}_{\boldsymbol{i}}\boldsymbol{e}_{\boldsymbol{j}}^TV(I-P_X)V\}\\
&\quad=2(m-p)^{-1}B_i\boldsymbol{e}_{\boldsymbol{j}}^TV(I-P_X)V^{-1}\Sigma\boldsymbol{e}_{\boldsymbol{i}}\\
&\quad=2(m-p)^{-1}B_i\boldsymbol{e}_{\boldsymbol{j}}^TV(I-(P_X^V)^T)\boldsymbol{e}_{\boldsymbol{i}}\\
&\quad=2(m-p)^{-1}B_i[\boldsymbol{e}_{\boldsymbol{j}}^TV\boldsymbol{e}_{\boldsymbol{i}}-h_{ij}^V]\\
&\quad=2(m-p)^{-1}B_i\boldsymbol{e}_{\boldsymbol{j}}^TV\boldsymbol{e}_{\boldsymbol{i}}+o(m^{-1}).
\end{aligned}
$$

With this, we find that

$$\sum_j w_j B_j E\left[\frac{\partial \hat{\theta}_i^B}{\partial \sigma_u^2}(\tilde{\sigma}_u^2 - \sigma_u^2)(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right]$$
$$= 2(m-p)^{-1} B_i^2 w_i(\sigma_u^2 + D_i) + o(m^{-1}) = o(m^{-1}).$$

Hence, R_4 is $o(m^{-1})$. We now show that $R_5 = o(m^{-1})$. By calculations similar to those in (B.9),

$$\sum_j w_j B_j E\left[\frac{\partial^2 \hat{\theta}_{i*}^B}{\partial (\sigma_u^2)^2}(\hat{\sigma}_u^2 - \sigma_u^2)^2(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right]$$
$$= \sum_j w_j B_j E\left[\frac{\partial^2 \hat{\theta}_{i*}^B}{\partial (\sigma_u^2)^2}(\tilde{\sigma}_u^2 - \sigma_u^2)^2(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right] + o(m^{-r}).$$

Recall that $E[\{\sum_j w_j B_j(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\}^2] = O(m^{-1})$ by (B.2). Now note that

$$\sum_j w_j B_j E\left[\frac{\partial^2 \hat{\theta}_{i*}^B}{\partial (\sigma_u^2)^2}(\tilde{\sigma}_u^2 - \sigma_u^2)^2(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right]$$
$$\leq E^{1/4}\left[\left\{\frac{\partial^2 \hat{\theta}_{i*}^B}{\partial (\sigma_u^2)^2}\right\}^4\right] E^{1/4}\left[(\tilde{\sigma}_u^2 - \sigma_u^2)^8\right] E^{1/2}\left[\left\{\sum_j w_j B_j(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right\}^2\right]$$
$$\leq E^{1/4}\left[\left\{\sup_{\sigma_u^2 \geq 0} \frac{\partial^2 \hat{\theta}_i^B}{\partial (\sigma_u^2)^2}\right\}^4\right] E^{1/4}\left[(\tilde{\sigma}_u^2 - \sigma_u^2)^8\right] E^{1/2}\left[\left\{\sum_j w_j B_j(\hat{\theta}_j - \boldsymbol{x_j^T}\tilde{\boldsymbol{\beta}})\right\}^2\right]$$
$$= O(m^{-3/2})$$

by Lemma 1(ii), by Theorem A.5 of Prasad and Rao (1990), and by expression (B.2). Thus, R_5 is $o(m^{-1})$, and $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)(\bar{\hat{\theta}}_w^B - t)] = o(m^{-1})$.

For the last term in (B.1), we use the the Cauchy-Schwartz inequality to show

$$E[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B)(\bar{\hat{\theta}}_w^B - t)] \leq E^{1/2}[(\bar{\hat{\theta}}_w^{EB} - \bar{\hat{\theta}}_w^B)^2] E^{1/2}[(\bar{\hat{\theta}}_w^B - t)^2] = o(m^{-1}).$$

This concludes the proof of the theorem.

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