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MAX-CUSUM CHART FOR AUTOCORRELATED PROCESSES

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Abstract: A Cumulative Sum (CUSUM) control chart capable of detecting changes in both the mean and the standard deviation for autocorrelated data, referred to as the Max-CUSUM chart for Autocorrelated Process chart (MCAP chart), is proposed. This chart is based on fitting a time series model to the data, and then calculating the residuals. The observations are represented as a first-order autoregressive process plus a random error term. The Average Run Lengths (ARL's) for fixed decision intervals and reference values, (h, k) are calculated. The proposed chart is compared with the combined Shewhart-EWMA chart for autocorrelated data proposed by Lu and Reynolds (1999). Comparisons are based on the out-of-control ARL's. The MCAP chart detects small shifts in the mean and standard deviation at both low and high levels of autocorrelation more quickly than the combined Shewhart-EWMA chart. This makes the MCAP chart useful to modern production processes where high quality goods are produced with a low fraction of nonconforming products.

Key words and phrases: AR(1) model, autocorrelation, autoregressive, Markov chain, Max-CUSUM chart, random error and residual.

1. Introduction

Statistical process control (SPC) charts such as the Shewhart chart proposed by Shewhart (1924), the cumulative sum chart by Page (1956), and the exponentially weighted moving average chart by Roberts (1959), are used to monitor product quality and detect special events that may be indicators of out-of-control situations. These charts are based on the assumption that a process being monitored will produce measurements that are independent and identically distributed over time when only the inherent sources of variability are present in the system. However, in some applications the dynamics of the process will induce correlations in observations that are closely spaced in time. If the sampling interval used for process monitoring in these applications is short enough for the process dynamics to produce significant correlation, this can have a serious effect on the properties of standard control charts, (see VanBrackle and Reynolds (1997), Lu and Reynolds (1999, 2001) and Runger, Willemain and Prabhu (1995)).

Positive autocorrelation in observations can result in severe negative bias in traditional estimators of the standard deviation. This bias produces control limits for standard control charts that are much tighter than desired. Lu and Reynolds (2001) observed that tight control limits, combined with autocorrelation in the observations plotted, could result in an average false alarm rate much higher than expected. This results in effort being wasted searching for unavailable special causes of variation in the process. It can also result in loss of confidence in control charts and practitioners may abandon their use. Furthermore, when using control charts for residuals, if observations are positively autocorrelated when there is a shift in the process mean, only a fraction of the shift will be transferred to the residual means, resulting in the chart not quickly detecting the shift. This is an undesirable situation in process monitoring. It is therefore important to take autocorrelation among observations into consideration when designing a process-monitoring scheme, in particular control charts, in order to achieve full benefit.

Recently, new control charts have been proposed along two lines for dealing with autocorrelated data. The first approach uses standard control charts on original observations, but adjusts the control limits and methods of estimating parameters to account for the autocorrelation in the observations (see VanBackle and Reynolds (1997) and Lu and Reynolds (1999)). This approach is particularly applicable when the level of autocorrelation is not high.

A second approach for dealing with autocorrelation fits time series model to the process observations. The procedure forecasts observations from previous values and then computes the forecast errors or residuals. These residuals are then plotted on standard control charts because when the fitted time series model is the same as the true process model, and the parameters are estimated without error, the residuals are independent and identically distributed normal random variables when the process is in control. (See Alwan and Roberts (1988), Montgomery and Mastrangelo (1991), Wadell, Moskowitz, and Plante (1994), Lu and Reynolds (1999) and Runger, Willemain and Prabhu (1995)).

Yashchin (1993) recommends charting raw data directly when the level of autocorrelation is low and, at high level of autocorrelation, recommends some transformation procedures that create residuals. An allowance is made for autocorrelation in the residuals due to model misspecification. If a shift in the mean and/or standard deviation of the process occurs, this will cause a shift in the mean and/or standard deviation of the residuals. Control charts based on residuals seem to work well when the level of correlation is high. When the level of correlation is low, forecasting is more difficult and residual charts are not very effective in detecting process changes.

The studies mentioned above used several methods, such as simulation, asymptotic approximation and direct calculation to evaluate properties of the control charts. Their simplest message is that correlation between observations has a significant effect on the properties of the control charts. In particular, control charts run for a long time before detecting shifts in the process parameters from in-control values when the autocorrelations are large.

Here we propose a cumulative sum (CUSUM) control chart for autocorrelated data that can simultaneously monitor shifts in the mean and standard deviation using a single plotting variable. This investigation is done for the case of processes that can be modeled as a first order autoregressive AR(1) process plus an additional random error which can correspond to sampling or measurement error. This model has been used in several charts dealing with autocorrelated data. It allows relatively accurate numerical techniques to be used to evaluate properties of the charts. Lu and Reynolds (1999) proposed a simultaneous EWMA chart that uses two charts concurrently.

Markov chain methods are used to evaluate the ARL for different levels of correlation. Our proposed chart monitors the process by simultaneously monitoring the residual means and standard deviations. The results show that by adjusting the reference value of the standard CUSUM chart to take the autocorrelation structure into consideration, the CUSUM chart can effectively detect small shifts in the process mean and/or spread.

2. The AR(1) Process with an Additional Random Error

This model has been used previously in a number of contexts and can account for correlations between observations that are close together in time, for variability in the process mean over time, and for additional variability due to sampling or measurement error.

Suppose the observations are taken from a process at regularly spaced times, and let X_t represent the observation taken at sampling time t. The properties of control charts are usually investigated under the assumption that the observations are independent normal random variables with constant mean and variance. When the observations are independent identically distributed normal random variables, we write $X_t = \mu + \varepsilon_t$, $t = 1, 2, \ldots$, where μ is the process mean and the ε_t's are independent normal random variables with mean 0 and variance σ_ε^2. It is assumed that μ is at a target value when the process is in control, but can change to some value given cause.

To model observations from an autocorrelated process, we use a model that has been discussed previously in quality control by authors such as Lu and Reynolds (1999, 2001) and VanBackle and Reynolds (1997). Here

$$X_t = \mu_t + \varepsilon_t, \qquad t = 1, 2, \ldots, \tag{1}$$

where μ_t is the random process mean at sampling time t, and the ε_t's are independent normal random errors with mean 0 and variance σ_ε^2. It is assumed that μ_t is an AR(1) process:

$$\mu_t = (1-\phi)\xi + \phi\mu_{t-1} + \alpha_t, \qquad t = 1, 2, \ldots, \tag{2}$$

where ξ is the overall process mean, the α's are independent normal random variables with mean 0 and variance σ_α^2, and ϕ is the autoregressive parameter satisfying $|\phi| < 1$ so that the process is stationary. We assume that the starting value μ_0 follows a normal distribution with mean ξ and variance $\sigma_\mu^2 = \sigma_\alpha^2/(1-\phi^2)$. The distribution of X_t is therefore constant with mean ξ and variance given by

$$\sigma_X^2 = \sigma_\mu^2 + \sigma_\varepsilon^2 = \frac{\sigma_\alpha^2}{1-\phi^2} + \sigma_\varepsilon^2. \tag{3}$$

In this case σ_μ^2 represents long-term variability and σ_ε^2 represents a combination of short-term variability and the variability associated with measurement error. When assessing processes following models (1) and (2), it is often convenient to consider the portion of total process variability that is due to variation in μ_t and the portion due to error variability. The portion of the process variability due to variation in μ_t is $\psi = \sigma_\mu^2/(\sigma_\mu^2 + \sigma_\varepsilon^2)$, and the proportion of the variance due to ε_t is $1-\psi$. The covariance between two observations that are i units apart is $\phi^i\sigma_\mu^2$, and the correlation between two adjacent observations is $\rho = \phi\psi$.

Autocorrelation in the process may at times be assigned to causes that can be eliminated; this will reduce variability in the process. In other processes, the autocorrelations are inherent characteristics of the process and cannot be removed in the short-run. In these situations, the process is said to be in control when the process mean continuously wanders around the target value but within the acceptable region. Thus the process mean is not constant as in the case of independent observations.

The AR(1) process with an additional random error is equivalent to a first order autoregressive moving average, ARMA(1,1) process (Box, Jenkins and Reinsel (1994)):

$$(1-\phi B)X_t = (1-\phi)\xi + (1-\theta B)\gamma_t, \tag{4}$$

where γ's are independent normal random variables with mean 0 and variance σ_γ^2, θ is the moving average parameter, ϕ is the autoregressive parameter defined in (2), and B is a backshift operator such that $BX_t = X_{t-1}$. If $\theta > 0$, Koons and Foutz (1990) derived θ and σ_γ^2 as

$$\theta = \frac{\sigma_\alpha^2 + (1-\phi^2)\sigma_\varepsilon^2}{2\phi\sigma_\varepsilon^2} - \frac{1}{2}\sqrt{\Big(\frac{\sigma_\alpha^2 + (1-\phi^2)\sigma_\varepsilon^2}{\phi\sigma_\varepsilon^2}\Big)^2 - 4}, \tag{5}$$

$$\sigma_\gamma^2 = \frac{\phi\sigma_\varepsilon^2}{\theta}. \tag{6}$$

Standard time series estimation techniques can be used to estimate the parameters in the ARMA(1,1) model.

In some production processes, a large number of items is produced in a single lot. In this situation, more than one observation is sampled each time. Let X_{ti} be the ith observation at sampling time t. We assume that X_{ti} can be represented as

$$X_{ti} = \mu_t + \varepsilon'_{ti}, \tag{7}$$

where the ε'_{ti}'s are independent and identically distributed normal random variables with mean 0 and variance $\sigma^2_{\varepsilon'}$, and μ_t follows model (2). Lu and Reynolds (1999) show that the sample means from this process follow (1) and (2) with $\sigma^2_\varepsilon = \sigma^2_{\varepsilon'}/n$.

Here we monitor the process mean and standard deviation by monitoring the residuals from a forecast. To do this, we first determine the distribution of the residuals when the process is in control. When the process is in control, the residual at observation t from the minimum mean square error forecast made at observation $t-1$ is

$$e_t = X_t - \xi_0 - \phi(X_{t-1} - \xi_0) + \theta e_{t-1}, \tag{8}$$

where ϕ and θ are parameters in the ARMA(1,1) model given at (4). That is the residual at time t is the difference between X_t and the prediction of X_t based on the previous data.

If the fitted time series model is the same as the true process model and the parameters are estimated without error, then the residuals are independent and identically distributed normal random variables when the process is in control. We can then monitor the process by using standard control charts for independent observations using these residuals. If there is a step change in the process mean from the in-control value ξ_0 to ξ_1 between time $t = \tau - 1$ and $t = \tau$ the expectations of the residuals for various times are (see Lu and Reynolds (1999)) $E(e_t) = 0$ for $t < \tau$,

$$E(e_t) = \frac{1 - \phi + \phi^l(\phi - \theta)}{1 - \theta}(\xi_1 - \xi_0), \qquad t = \tau + 1,\ l = 0, 1, \ldots. \tag{9}$$

The asymptotic mean of these residuals is

$$\frac{1 - \phi}{1 - \theta}(\xi_1 - \xi_0). \tag{10}$$

These residuals are independent and normally distributed with variance σ^2_γ. The expectation of the residuals after the shift occurs is a decreasing function of time. As ϕ increases, a smaller fraction of shift in the process mean is transferred to the mean of the residuals. When the process mean shift, only a fraction of this shift is transferred to the means of the residuals. As a result the chart for the residuals' ability to detect the mean shift is reduced. On the other hand, the residuals chart is theoretically appealing because it takes serial correlation into account, and it reduces the problem to the well-known case of a shift in the process mean for independent observations.

A change in the process variance can be attributed to changes in the autoregressive parameter ϕ, the individual observation random shock variance σ_ε^2 and/or change in variability of the random shocks associated with the means σ_α^2. If σ_α^2 increases from its nominal value of $\sigma_{\alpha 0}^2$ to $\sigma_{\alpha 1}^2$ and σ_ε^2 increases from its nominal value $\sigma_{\varepsilon 0}^2$ to $\sigma_{\varepsilon 1}^2$ the residual variances increase to

$$\operatorname{Var}(e_t) = \alpha_{\gamma 0}^2 + \frac{\phi^2 - 2\phi\theta_0 + 1}{1-\theta_0^2}(\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + \frac{\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2}{1-\theta_0^2}. \tag{11}$$

The residuals after these shifts are correlated normal random variables with asymptotic mean at (10) and asymptotic variance at (11). From (11), we can see that changes in σ_α^2 and σ_ε^2 have different impact on the variability of the residuals. Given the parameters in the ARMA(1,1) model, for $\phi > 0$, σ_α^2 and σ_ε^2 are (see Reynolds, Arnold and Baik (1996))

$$\sigma_\alpha^2 = \frac{\sigma_\gamma^2(\phi-\theta)(1-\phi\theta)}{\phi}, \tag{12}$$

$$\sigma_\varepsilon^2 = \frac{\theta\sigma_\gamma^2}{\phi}. \tag{13}$$

We can then fit the AR(1) plus random error model in (1) and (2), which is the model considered in this paper. We consider the more prevalent case of positive autocorrelation. The objective of monitoring the process is to detect the situation in which one or more process parameters has changed from its target values.

3. The New Control Chart

Let $X_i = X_{i1}, \ldots, X_{in}$, $i = 1, 2, \ldots$, denote a sequence of samples of size n taken on a quality characteristic X. It is assumed that, for each i, $X_{i1}, \ldots, X_{in}$ are autocorrelated and satisfy (7). We monitor the process by first fitting the time series model the process observations and then compute residuals. Let ξ_0 and $\sigma_{\gamma 0}$ be the nominal process mean and standard deviation of the residuals for this fitted model. Assume that the process residual parameters ξ and σ_γ can be expressed as $\xi = \xi_0 + a\sigma_{\gamma 0}$ and $\sigma_\gamma = b\sigma_{\gamma 0}$, where $a = 0$ and $b = 1$ when the process is in-control; otherwise, the process has changed due to some assignable cause. Then a represents the shift in the process mean and b represents the shift in the process standard deviation.

Let $\overline{\xi}_i = (\xi_{i1} + \cdots + \xi_{in_i})/n_i$ and $\text{MSE}_i = \sum_{j=1}^{n_i}(\xi_{ij} - \overline{\xi}_i)^2/n_i$ be the mean and variance for the ith sample residuals respectively. These statistics are independently distributed, as are the sample residual values when the process is in-control. These two statistics follow different distributions. The CUSUM charts for the mean and standard deviation are based on $\overline{\xi}_i$ and MSE_i, respectively.

To develop a CUSUM chart for the process mean and process standard deviation using residuals, we carry out the following transformations:

$$Z_i = \sqrt{n}\frac{(\overline{\xi}_i - \xi_0)}{\sigma_{\gamma 0}}, \tag{14}$$

$$Y_i = \Phi^{-1}\Big\{H\Big[\frac{(n_i)\,\text{MSE}\ _i}{\sigma^2_{\gamma 0}}; n_i\Big]\Big\}, \tag{15}$$

where $\Phi(z) = P(Z \leq z)$, for $Z \sim N(0,1)$. $\Phi^{-1}(\cdot)$ is the inverse function of the cumulative distribution function of $N(0,1)$, and $H(w;p) = P(W \leq w|p)$ for $W \sim \chi^2_p$, the chi-square distribution with p degrees of freedom.

Z_i and Y_i are independent and, when $a = 0$ and $b = 1$, they follow the standard normal distribution. The CUSUM statistics based on Z_i and Y_i are

$$C_i^+ = \max[0, Z_i - k + C_{i-1}^+], \tag{16}$$

$$C_i^- = \max[0, -Z_i - k + C_{i-1}^-], \tag{17}$$

$$S_i^+ = \max[0, Y_i - k + S_{i-1}^+], \tag{18}$$

$$S_i^- = \max[0, -Y_i - k + S_{i-1}^-], \tag{19}$$

where C_0 and S_0 are starting points. Because Z_i and Y_i follow the same distribution, a new statistic for the single control chart can be defined as

$$M_i = \max[C_i^+, C_i^-, S_i^+, S_i^-]. \tag{20}$$

If the process has gone out of control, the M_i's will be plotted outside the control limits, otherwise the M_i values are within the limits. Since $M_i > 0$, we plot only the upper control limit for this chart, and consider the process to be out of control if an M_i value is plotted above the upper control limit.

In SPC, we use the ARL or the average time to signal (ATS) of the chart to assess the performance of the scheme. This is the expected number of samples (or observations if we take a single observation each time) required by the chart to signal an out-of-control situation. For a change in variability, we consider the effects of changes in σ^2_α and σ^2_ε separately to calculate the ARL. This is because the two parameters have different impact on the level of variability of the process as shown in equation (11). The shifts in these parameters are considered for different values of ϕ the correlation between μ_i and μ_{t-1}.

4. Design of a Max-CUSUM Chart for Autocorrelated Process (MCAP Chart)

Because M_i is the maximum of four statistics, we call this new chart the Maximum Cumulative Sum chart for Autocorrelated Process (MCAP chart). Lucas (1982) showed that a CUSUM chart for independent normal data is tuned to be most sensitive to a shift of magnitude δ by choosing $k = \delta/2$. Runger, Willemain and Prabhu (1995) proposed a modified procedure that takes into consideration the autocorrelation structure of the data; they proposed $k = \delta(1 - \phi)/2$ for the AR(1) process.

To calculate the ARL of the new chart, we make use of the modified Markov chain procedure proposed by Runger, Willemain and Prabhu (1995). For the AR(1) plus random error model investigated here for shifts in mean and/or standard deviation, we use the asymptotic mean given at (10). The expected residual mean after the shift is $\delta(1 - \phi)/(1 - \theta)$. For a given in-control ARL and a shift of the mean and/or standard deviation intended to be detected by the chart, the reference value (k) is $(\delta/2)(1 - \phi)/(1 - \theta)$. This guideline takes into consideration the autocorrelation structure between the variables. For values (ARL, k), the value of the decision interval (h) is chosen to achieve the specified ARL. Then we use the procedure for a CUSUM chart with standard (h, k) values for a normal distribution with mean (10) and variance (11) to calculate the ARL's.

Table 1 and Table 2 give the optimal combinations of h and k for an in-control ARL fixed at 370 and the autoregressive parameter $\phi = 0.25$, with 80% of process variability due to variation in μ_t and the correlation between adjacent observations $\rho = 0.2$. Without loss of generality, we take $\xi_0 = 0$ and $\sigma_{\gamma 0} = 1$. We calculate the out-of-control ARL for the effect of changes in the standard deviation that is due to changes in σ_ε and σ_α. The smallest value of an out-of-control ARL is calculated with respect to a pair of specified shifts in both mean and standard deviation using the optimal in-control ARL CUSUM chart parameters. We assume that the process starts in an in-control state and thus the initial value of the CUSUM statistic is set at zero. For example, if one wants to have an in-control ARL of 370, and to guard against a $3\sigma_{\gamma 0}$ increase in the process mean and a $2\sigma_{\gamma 0}$ increase in the process standard deviation due to an increase in σ_ε, i.e., $a = 3$ and $b = 2$, the optimal in-control chart parameter values are $h = 1.359$ and $k = 1.187$. The ARL is approximately two, which means that these shifts can be detected, on average, on the second sample inspection.

Table 3 and Table 4 give the optimal combinations of h and k for an in-control ARL fixed at 370 and an autoregressive parameter $\phi = 0.75$, with 80% of process variability due to variation in μ_t and the correlation between adjacent observations equal to 0.6. We use the same procedure to calculate the ARL for these tables as for Tables 1 and 2. Tables 1 and 3 show the chart's performance for different shifts in the process mean and/or standard deviation, with shifts in the standard deviation due to shifts in σ_α; Tables 2 and 4 corresponds to changes in these parameters with shifts in standard deviation due to shifts in σ_ε.

Table 1. (k, h) combinations and the corresponding ARL's for the MCAP chart, with $\phi = 0.25$ and $\psi = 0.8$ for shifts in the process standard deviation due to shifts in σ_α.

$\mathrm{ARL}_0 = 370$									
		a							
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	370.27	34.31	13.31	4.59	2.75	2.26	1.56	1.38
1.25	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	26.44	30.93	12.49	4.39	2.59	2.14	1.48	1.41
1.50	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	18.50	27.54	11.59	4.09	2.45	2.03	1.42	1.33
2.00	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	11.46	21.74	9.87	3.64	2.22	1.84	1.33	1.21
2.50	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	8.39	17.45	8.42	3.27	2.05	1.70	1.27	1.18
3.00	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	6.74	14.38	7.26	2.99	1.92	1.60	1.23	1.02
4.00	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	5.04	10.45	5.65	2.58	1.75	1.47	1.18	1.02

Comparing these tables, it can be seen that at a low level of autocorrelation, the chart more quickly detects small shifts in the parameters than at a high level. The scheme is slightly more sensitive to small shifts in the standard deviation due to shifts in σ_ε than it is to shifts in the process standard deviation resulting from shifts in σ_α. At a high level of autocorrelation, the chart is also more sensitive to small shifts in σ_ε than in σ_α, and more sensitive to large shifts in the process mean and σ_α than shifts in the process mean and σ_ε. When only the process variability shifts, the scheme is more sensitive to shift in σ_ε than it is to shifts in the process variability due to changes in σ_α. This is due to the fact that an increase in σ_α increases the level of correlation between observations since, as the variance of μ_t increases, the proportion of total process variability due to the variation of the autocorrelated mean μ_t increases.

Table 2. (k, h) combinations and the corresponding ARL's for the MCAP chart, with $\phi = 0.25$ and $\psi = 0.8$ for shifts in the process standard deviation due to shifts in σ_ε.

$\mathrm{ARL}_0 = 370$									
		a							
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
1.00	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	370.27	34.31	13.31	4.59	2.75	2.26	1.56	1.38
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
1.25	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	27.43	31.27	12.63	4.75	2.73	2.19	1.49	1.41
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
1.50	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	19.46	28.15	11.87	4.61	2.71	2.11	1.42	1.33
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
2.00	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	12.13	22.60	10.33	4.31	2.66	1.98	1.33	1.21
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
2.50	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	8.87	18.37	8.98	4.02	2.35	1.75	1.30	1.18
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
3.00	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	7.09	15.62	7.88	3.77	2.10	1.60	1.23	1.02
	h	4.097	9.755	6.712	4.097	2.894	2.176	1.710	1.359
4.00	k	0.396	0.099	0.198	0.396	0.593	0.791	0.989	1.187
	ARL	5.24	11.22	6.30	3.38	1.85	1.34	1.18	1.02

An increase in σ_ε decreases the level of correlation between observations. This is particularly evident at higher levels of autocorrelations. This improves the performance of the MCAP chart. Overall, the scheme is more sensitive at low levels of autocorrelation than at higher levels for shifts in both mean and standard deviation. In the next section, we compare this scheme with simultaneous control charts for autocorrelated processes discussed in the literature.

5. Comparison with Other Charts

The performance of control charts for monitoring a process is usually assessed using the ARL. The chart that has low ARL when the process has shifted and high

ARL when the process is running at the target value is considered better than the one that has high out-of-control ARL and low in-control ARL. Most of the charts discussed in the literature are for monitoring shifts in the process mean and variability for autocorrelated observations using separate charts. We compare the MCAP chart with the combined Shewhart-EWMA chart for autocorrelated data proposed by Lu and Reynolds (1999). The combined Shewhart-EWMA chart's ARL were obtained from Table 3 and Table 4 of Lu and Reynolds (1999). The combined Shewhart-EWMA charts were run by simultaneously running the two charts: one chart designed primarily to detect shifts in the mean, and the other designed primarily to detect shifts in the process variability. The decision rule is that a signal is given if one of the two charts signals. The value of ϕ, the correlation between μ_t and μ_{t-1}, were taken to be 0.4 and 0.8 and the proportions of variation in the process attributed to variation in μ_t, ψ, are 0.1 and 0.9.

Table 3. (k, h) combinations and the corresponding ARL's for the MCAP chart, with $\phi = 0.75$ and $\psi = 0.8$ for shifts in the process standard deviation due to shift in σ_α.

$ARL_0 = 370$									
		a							
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
1.00	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	370.33	34.37	13.41	4.99	2.90	2.39	2.03	1.80
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
1.25	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	33.42	32.81	13.06	4.93	2.89	2.38	2.02	1.64
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
1.50	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	26.27	31.05	12.66	4.86	2.87	2.38	2.02	1.60
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
2.00	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	17.64	27.28	11.71	4.68	2.83	2.37	2.01	1.52
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
2.50	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	13.00	23.69	10.71	4.48	2.79	2.36	1.98	1.47
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
3.00	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	10.24	20.58	9.75	4.27	2.74	2.34	1.96	1.34
	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
4.00	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	7.27	15.84	8.13	3.89	2.64	2.31	1.82	1.33

Table 4. (k, h) combinations and the corresponding ARL's for the MCAP chart, with $\phi = 0.75$ and $\psi = 0.8$ for shifts in the process standard deviation due to shift in σ_ε.

$ARL_0 = 370$									
		a							
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	370.33	34.37	13.41	4.99	2.90	2.39	2.03	1.80
1.25	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	30.32	31.23	12.78	4.69	2.66	2.36	2.02	1.64
1.50	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	23.88	30.03	12.33	4.31	2.55	2.36	2.02	1.60
2.00	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	15.98	25.76	10.83	4.01	2.41	2.34	2.01	1.52
2.50	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	11.54	20.00	10.00	3.79	2.27	2.34	1.98	1.47
3.00	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	8.70	18.96	9.24	3.33	2.06	2.34	1.96	1.34
4.00	h	4.098	9.754	6.712	4.098	2.886	2.181	1.710	1.359
	k	0.164	0.041	0.082	0.164	0.246	0.329	0.411	0.493
	ARL	6.95	13.88	7.92	3.19	2.00	2.30	1.82	1.33

Comparison of ARL's of these charts are listed in Tables 5−8. The two charts are compatible, with the MCAP chart performing better than the combined Shewhart-EWMA chart for small shifts in the process mean and/or variability. This is particularly evident at high levels of correlations where the combined Shewhart-EWMA chart is negatively affected by the level of autocorrelation while the effect of autocorrelation is not very strong for the MCAP chart. For very larger shifts in the process mean and standard deviation, the combined Shewhart-EWMA chart perform slightly better than the MCAP chart, particularly at high level of correlations.

We recommend the use of our MCAP chart for detecting small shifts in the process mean and/or variability. For detecting large shifts, we can use a combined Shewhart-MCAP chart, as the Shewhart chart is very effective in detecting large shifts in the process parameters even in the presence of autocorrelation.

Table 5. Comparison of the MCAP chart, with a combined Shewhart-EWMA chart, with $\phi = 0.4$ and $\psi = 0.1$.

		a							
		0		1		2		3	
b	Parameter	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW
1	σ_ε^2	367.0	367.9	5.0	12.0	2.8	3.7	2.0	2.0
	σ_α^2	367.0	365.8	5.0	12.1	2.8	3.8	2.2	2.0
2	σ_ε^2	12.0	30.1	4.3	8.7	2.9	3.4	2.1	2.0
	σ_α^2	12.0	25.8	4.4	9.3	2.9	3.8	2.1	2.1
3	σ_ε^2	7.0	12.6	3.8	6.6	2.8	3.3	2.2	2.0
	σ_α^2	7.1	12.0	3.8	7.4	2.8	3.7	2.8	2.2
10	σ_ε^2	2.9	3.1	2.5	2.8	2.3	2.4	2.1	2.0
	σ_α^2	2.9	3.3	2.5	3.1	2.3	2.6	2.1	2.2

MCAP: Max-CUSUM chart for autocorrelated process. S-EW: Combined Shewhart-EWMA charts of residual.

Table 6. Comparison of the MCAP chart, with a combined Shewhart-EWMA chart, with $\phi = 0.8$ and $\psi = 0.1$.

		a							
		0		1		2		3	
b	Parameter	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW
1	σ_ε^2	367.6	364.4	5.6	16.8	3.1	4.4	2.3	2.2
	σ_α^2	367.6	367.0	5.6	16.9	3.1	4.1	2.3	2.2
2	σ_ε^2	12.9	30.7	4.7	10.2	3.1	3.7	2.4	2.0
	σ_α^2	15.3	24.9	4.9	13.3	3.1	4.9	2.3	2.2
3	σ_ε^2	7.2	12.3	4.0	7.2	3.0	3.4	2.4	2.0
	σ_α^2	8.3	13.9	4.2	10.4	3.0	5.0	2.4	2.5
10	σ_ε^2	2.9	3.0	2.6	2.8	2.4	2.4	2.2	1.9
	σ_α^2	3.1	5.0	2.7	4.6	2.4	3.7	2.2	2.7

MCAP: Max-CUSUM chart for autocorrelated process. S-EW: Combined Shewhart-EWMA charts of residual.

Table 7. Comparison of the MCAP chart, with a combined Shewhart-EWMA chart, with $\phi = 0.4$ and $\psi = 0.9$.

		a							
		0		1		2		3	
b	Parameter	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW
1	σ_ε^2	370.8	368.9	6.0	23.3	3.2	5.4	2.4	2.3
	σ_α^2	370.8	370.9	6.0	22.8	3.2	5.4	2.4	2.3
2	σ_ε^2	12.6	26.0	4.9	11.2	3.2	4.2	2.5	2.1
	σ_α^2	15.2	31.2	5.1	12.5	3.3	4.8	2.5	2.3
3	σ_ε^2	6.9	10.6	4.1	7.0	3.1	3.6	2.5	2.1
	σ_α^2	8.1	12.9	4.3	8.3	3.1	4.2	2.5	2.3
10	σ_ε^2	2.9	2.9	2.6	2.6	2.4	2.3	2.2	1.9
	σ_α^2	3.0	3.2	2.7	3.0	2.7	2.6	2.3	2.1

MCAP: Max-CUSUM chart for autocorrelated process. S-EW: Combined Shewhart-EWMA charts of residual.

Table 8. Comparison of the MCAP chart, with a combined Shewhart-EWMA chart, with $\phi = 0.8$ and $\psi = 0.9$.

		a							
		0		1		2		3	
b	Parameter	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW	MCAP	S-EW
1	σ_ε^2	374.5	374.4	7.7	82.1	3.8	12.1	2.8	1.8
	σ_α^2	374.5	375.5	7.7	83.4	3.8	12.1	2.8	1.8
2	σ_ε^2	12.5	10.1	5.6	7.6	3.7	3.7	2.8	1.7
	σ_α^2	37.2	38.5	6.9	22.4	3.8	7.6	2.9	2.1
3	σ_ε^2	6.5	5.0	4.3	4.2	3.3	2.7	2.8	1.7
	σ_α^2	16.5	16.1	6.0	12.1	3.7	5.8	2.9	2.3
10	σ_ε^2	2.8	2.1	2.6	2.0	2.4	1.8	2.3	1.5
	σ_α^2	3.8	3.7	3.2	3.4	2.8	2.7	2.6	2.1

MCAP: Max-CUSUM chart for autocorrelated process. S-EW: Combined Shewhart-EWMA charts of residual.

6. Charting Procedures

Since the residuals are independent normal random variables when the process is in control, the charting procedure for the MCAP chart is similar to that of the chart for uncorrelated data. The successive CUSUM values, Mi , are plotted against the sample numbers. If a point plots below the decision interval, the process is said to be in control and the point is plotted as a dot point. An out-of-control signal is given if any point plots above the decision interval and is plotted as one of the characters defined below. The MCAP chart is a combination of two two-sided standard CUSUM charts. Use the following procedure to construct this chart.

1. Fit the time series model to the data.
2. Specify the in-control or target value of the mean ξ_0, and the in-control or target value of the standard deviation $\sigma_{\gamma 0}$.
3. If ξ_0 is not known, use the sample grand average $\overline{\overline{\xi}} = (\overline{\xi}_1 + \cdots + \overline{\xi}_m)/m$ to estimate it. If $\sigma_{\gamma 0}$ is unknown, use $\overline{R}/d_2$, where $\overline{R} = (R_1 + \cdots + R_m)/m$ is the average of the sample ranges. We can also use $\overline{S}/c_4$ to estimate $\sigma_{\gamma 0}$, where $\overline{S} = (S_1 + \cdots + S_m)/m$ is the average of the sample standard errors, $S_i = \sqrt{\text{MSE}_{\ i}}$ and d_2 and c_4 are statistically determined constants.
4. For each sample, compute Z_i and Y_i.
5. To detect specified changes in the process mean and standard deviation, choose an optimal (h, k) combination and calculate C_i^+, C_i^-, S_i^+ and S_i^-.
6. Compute the M_i's and compare them with h, the decision interval.
7. Denote the sample points with a dot and plot them against the sample number if $M_i \leq h$.
8. If any of the Mi*'s* is greater than the decision interval h, the following plotting characters may be used to show the direction as well as the statistic that is

plotting above the interval. (i) If $C_i^+ > h$, plot C_+. (ii) If $C_i^- > h$, plot C_-. (iii) If $S_i^+ > h$, plot S_+. (iv) If $S_i^- > h$, plot S_-. (v) If both $C_i^+ > h$ and $S_i^+ > h$, plot B_{++}. (vi) If $C_i^+ > h$ and $S_i^- > h$, plot B_{+-}. (vii) If $C_i^- > h$ and $S_i^+ > h$, plot B_{-+}. (viii) If $C_i^- > h$ and $S_i^- > h$, plot B_{--}.

9. Investigate the cause(s) of shift for each out-of-control point in the chart and carry out the remedial measure(s) needed to bring the process back to an in-control state.

7. An example

To provide a visual picture of how the MCAP chart responds to various kinds of process changes, a set of simulated data is used. Specific process changes are introduced into the data, and the chart is plotted to monitor these changes in the parameters. The data set was generated using the first order autoregressive models in equations (1) and (2). The data were simulated by simulating sequences of α_t's and ε_t's, using Matlab.

For a fixed sequence of α_t's and ε_t's, a shift in σ_α can be introduced by multiplying α_t in (2) by a constant. A change in σ_ε can be introduced by multiplying ε_t in (1) by a constant, and a change in the mean is introduced by adding a constant to the generated observations. This approach is discussed by Lu and Reynolds (1999). And allows different types of process changes to be investigated on the same basic sequence of α_t's and ε_t's. In this example, we assume the autoregressive parameter ϕ remains constant.

We simulated 100 observations with the following parameters: $\xi = 0$, $\phi = 0.75$, $\sigma_\alpha = 0.59$ and $\sigma_\varepsilon = 0.5$. This give $\sigma_X = 1.02$ and $\psi = 0.76$, and implies that 76% of variability in the process is due to variation in μ_t, and that the correlation between the adjacent observations is $\rho = \phi\psi = 0.57$. Using (5) and (6), the corresponding parameters in the ARMA(1,1) model in (4) are $\theta = 0.27$ and $\sigma_\gamma = 0.83$.

The MCAP chart for these simulated observations is drawn in Figure 1. All points fall within the acceptable region, thus the simulated process is in control. This chart is designed to detect a $1\sigma_\gamma$ shift in the mean with an in-control ARL= 370.

Figure 2 shows the performance of this chart for a shift in the process standard deviation that is due to an increase in σ_α. Suppose that, due to a special cause immediately after observation 60, σ_α increases from 0.59 to 0.97 and stays at this value for the next 40 observations. We assume that other parameters in the model remain at their in-control values. This increase in σ_α results in an increase in the process standard deviation, σ_X, from 1.02 to 1.56. This leads to an increase in ψ from 0.76 to 0.90 and an increase in the correlation between adjacent observations from 0.57 to 0.68. Therefore 90% of variation in the process is due to variation in μ_t.

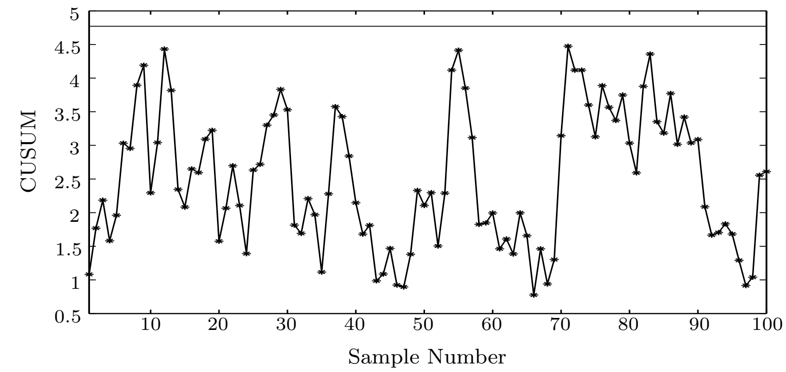

Figure 1. The MCAP chart for in-control simulated values.

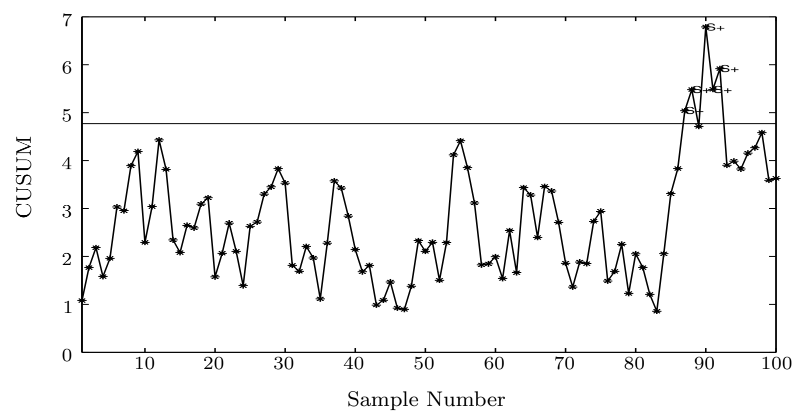

Figure 2. The MCAP chart for shift in the variability due to shift σ_α.

When applied to the simulated data, this shift in the standard deviation was signaled for the first time on the 86th observation. The delay in detecting this increase is caused by an increase in the correlation between observations, caused by an increase in σ_α as discussed above.

Figure 3 shows the performance of the MCAP chart for an increase in the process variability due to an increase in σ_ε. If σ_ε increases from its in-control value of 0.5 to 1.00 immediately after observation 60 and remains there for the rest of the process, and the rest of the process parameters remain at their in-control values, one has an increase in the process standard deviation from 1.02 to 1.34. Unlike the increase in σ_α, the increase in σ_ε results in a decrease in the correlation between observations from 0.57 to 0.33, the value of the proportion of total process variability that is due to the variability in μ_t also decreases from 56% to 44%. This increase in the process standard deviation is detected for the first time on the 64th observation. Though this corresponds to only a 30%

increase in the process standard deviation, it is more quickly detected than the increase in σ_α, which had the standard deviation increased by 52%.

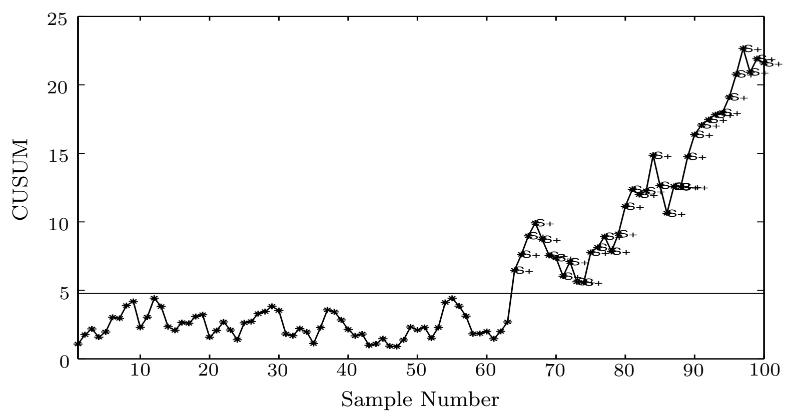

Figure 3. The MCAP chart for shift the variability due to shift in σ_ε.

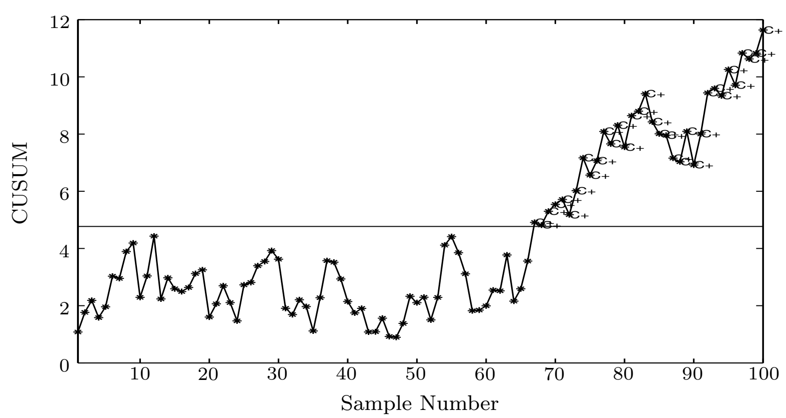

Figure 4. The MCAP chart for shift in the process mean.

A shift in the mean of the last 40 observations is shown in Figure 4. We assume the process standard deviation remained at its in-control value. This increase in the mean is signaled for the first time on the 68th observation.

Next we consider a simultaneous increase in the process mean and standard deviation. In Figure 5, we apply the MCAP chart to investigate an increase in the mean from 0 to 1 and an increase in σ_α from 0.59 to 0.97. This is for an increase in the mean and σ_α of the last 40 observations. These shifts are signaled for the first time on the 73rd observation, an increase in the mean only, and the 76th observation, an increase in both the mean and the standard deviation.

An increase in both the mean and σ_ε is shown in Figure 6. We consider an increase in mean from 0 to 1 and an increase in σ_ε from 0.5 to 1. Assume that these shifts occur immediately after the 60th observation and remain in effect for the rest of the process. The chart signals a shift for the first time on the 63th observation for an increase in the mean, and on the 64th observation for an increase in the standard deviation. It signals an increase in both parameters for the first time on the 67th observation. Therefore a combination of shifts in the mean and σ_ε is more quickly detected than a combination of shifts in the mean and σ_α, due to the variance components' effect on the level of autocorrelation.

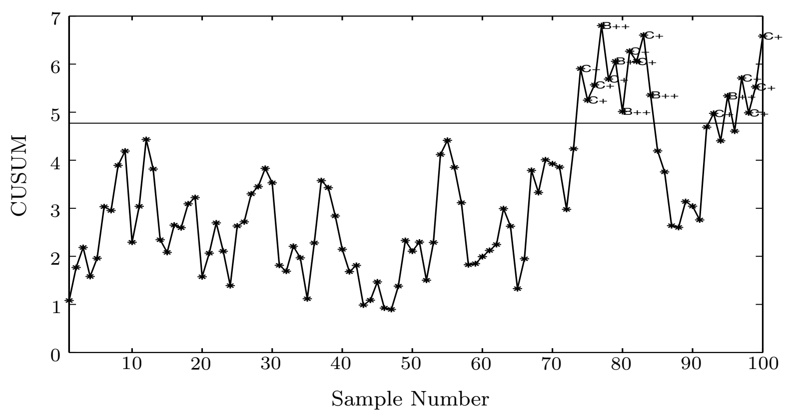

Figure 5. The MCAP chart for shift the mean and variability due to shift in σ_α.

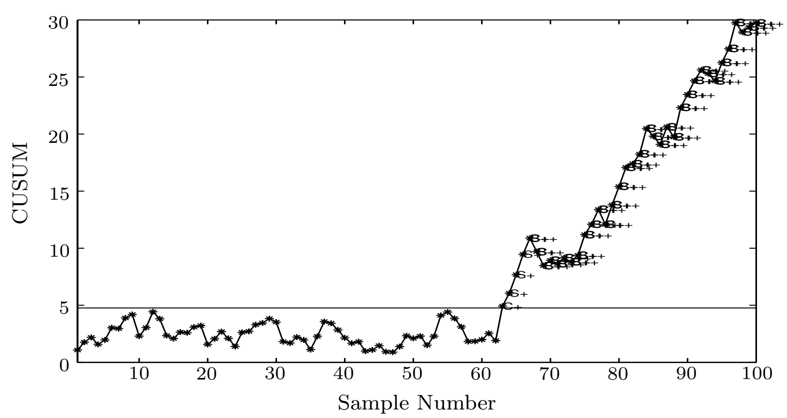

Figure 6. The MCAP chart for shift the mean and variability due to shift in σ_ε.

8. Conclusions

Although it is difficult to draw general conclusions based on one set of data corresponding to one set of process parameters, the ARL results, together with the charts plotted in Figures 1 to 6, allow some conclusions to be drawn.

Correlation among observations from a process can indeed have significant effect on the performance of the cumulative sum control charts. Computer simulation of individual data from an AR(1) plus random error model can be used to show a pictorial display of the MCAP chart. The monitoring problem for this model is complicated, as it requires more parameters than for the case when the observations are independent. We have shown how a change in any one of the two components of residual variance, coupled with a change in the process mean, has an impact on the overall process performance.

In many applications, a change in the process may be due to a combination of changes in these parameters. Then it becomes difficult to diagnose the variance component that has caused the process variability to change. It might be necessary to estimate the residual variance at the point of the shift to see which component has shifted.

The MCAP chart that simultaneously monitors both process mean and standard deviation performs better than its competitors at low to moderate shifts in the process parameters. The chart for residuals uses the standard CUSUM chart parameters, as residuals are independent when the process is in-control, we recommend this chart for autocorrelated data. The only adjustment to be made is to modify the reference value when calculating the out-of-control ARL in order to take autocorrelation into consideration. Standard time series procedure discussed in Box, Jenkins and Reinsel (1994) can be used to fit the model and calculate the residuals.

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