Statistica Sinica **29** (2019), 1739-1749
doi:https://doi.org/10.5705/ss.202017.0106

STRONG LAWS FOR RANDOMLY WEIGHTED SUMS OF RANDOM VARIABLES AND APPLICATIONS IN THE BOOTSTRAP AND RANDOM DESIGN REGRESSION

Pingyan Chen[1], Tao Zhang[1] and Soo Hak Sung[2]

[1]*Jinan University and* [2]*Pai Chai University*

Abstract: In this study, we establish the Marcinkiewicz–Zygmund strong law of large numbers for randomly weighted sums of negatively orthant dependent random variables. The law of the single law of logarithm for randomly weighted sums of negatively orthant dependent random variables is also established. Finally, the results are applied to bootstrap sample means and least-squares estimators in a simple linear regression with random design.

Key words and phrases: Bootstrap sample mean, law of the single logarithm, Marcinkiewicz–Zygmund strong law, randomly weighted sum, regression with random design.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and let $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ be an array of random variables independent of $\{X_n, n \geq 1\}$. Then a randomly weighted sum is defined by

$$\sum_{k=1}^{n} w_{nk} X_k. \tag{1.1}$$

Here, $w_{nk}, n \geq 1, 1 \leq k \leq n$, are called random weights. In the special case of constant random variables $w_{nk}, n \geq 1, 1 \leq k \leq n$, (1.1) is referred to as a non-randomly weighted sum or as a weighted sum. Many useful linear statistics, for example, least-squares estimators, nonparametric regression function estimators, and jackknife estimates, are weighted sums. Randomly weighted sums play an important role in various applied and theoretical problems. For example, in queueing theory, $\sum_{k=1}^{n} w_{nk} X_k$ represents the total output for a customer being served by n machines, where w_{nk} is the service time of the kth machine and X_k is the output from the kth machine. In statistics, bootstrap sample means and least-squares estimators in simple linear regressions with random design are randomly weighted sums (see Section 3).

We first consider strong laws for the weighted sums $\sum_{k=1}^n a_{nk}X_k$ of independent and identically distributed (i.i.d.) random variables $\{X, X_n, n \geq 1\}$, where $\{a_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of constants, that is, each random weight w_{nk} has a constant value a_{nk} with probability one. Chow (1966) proved the Kolmogorov strong law $n^{-1}\sum_{k=1}^n a_{nk}X_k \to 0$ almost surely (a.s.) when $EX = 0$, $E|X|^2 < \infty$, and $\{a_{nk}\}$ is an array of second Cesàro uniformly bounded constants, that is, $\sup_{n\geq 1} n^{-1}\sum_{k=1}^n |a_{nk}|^2 < \infty$. Choi and Sung (1987) showed the Kolmogorov strong law under the conditions that $EX = 0$ and the weights a_{nk} are uniformly bounded, that is, $\sup_{n\geq 1}\max_{1\leq k\leq n}|a_{nk}| < \infty$. Here, we compare these two results. If a_{nk} are uniformly bounded, then they are second Cesàro uniformly bounded. Hence, the condition on the weights of Chow (1966) is weaker than that of Choi and Sung (1987), but the moment conditions of Chow (1966) are stronger. Cuzick (1995) proved Chow's result (1966) when the weights a_{nk} are α-th Cesàro uniformly bounded for some $1 < \alpha < \infty$ (i.e., $\sup_{n\geq 1} n^{-1}\sum_{k=1}^n |a_{nk}|^\alpha < \infty$), $EX = 0$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1$. If $\alpha = \beta = 2$, the result of Cuzick (1995) reduces to that of Chow (1966). Bai and Cheng (2000) extended Cuzick's result (1995) and obtained the Marcinkiewicz–Zygmund strong law:

$$n^{-1/p}\sum_{k=1}^n a_{nk}X_k \to 0 \quad \text{a.s.}, \tag{1.2}$$

where $1 \leq p < 2$, $\{a_{nk}\}$ is an array of α-th Cesàro uniformly bounded constants for some $p < \alpha < \infty$, $EX = 0$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/p$.

On the other hand, Bai, Cheng and Zhang (1997) obtained the law of the single logarithm:

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n a_{nk}X_k|}{\sqrt{2n\log n}} \leq \limsup_{n\to\infty}\left(n^{-1}\sum_{k=1}^n |a_{nk}|^2\right)^{1/2} \quad \text{a.s.}, \tag{1.3}$$

where $\{a_{nk}\}$ is an array of α-th Cesàro uniformly bounded constants for some $2 < \alpha < \infty$, $EX = 0$, $EX^2 = 1$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/2$. The result of Bai, Cheng and Zhang (1997) has since been improved and extended by several works. Chen and Gan (2007) weakened the moment condition to $E|X|^\beta/(\log|X|)^{\beta/2} < \infty$. Then, Sung (2009) and Chen and Chen (2010) extended the result of Chen and Gan (2007) from real-valued random variables to Banach-valued random elements. Chen, Ma and Sung (2014) and Chen, Kong and Sung (2017) subsequently obtained more generalized results that include the result of Chen and Gan (2007).

We next consider the strong laws for randomly weighted sums $\sum_{k=1}^n w_{nk}X_k$

of i.i.d. random variables $\{X, X_n, n \geq 1\}$. These strong laws are less studied than those for non-randomly weighted sums. Cuzick (1995) proved the Kolmogorov strong law for randomly weighted sums:

$$n^{-1}\sum_{k=1}^{n}(w_{nk}X_k - Ew_{nk}EX) \to 0 \quad \text{a.s.} \tag{1.4}$$

if $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of i.i.d. random variables independent of $\{X_n, n \geq 1\}$, $E|w_{11}|^\alpha < \infty$ for some $2 \leq \alpha < \infty$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1$. Cuzick (1995) also proved the law of the single logarithm for randomly weighted sums:

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} w_{nk}X_k}{\sqrt{2n\log n}} = 1 \quad \text{a.s.} \tag{1.5}$$

provided that $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of independent Rademacher random variables and $EX^2 = 1$. Li, Rao and Wang (1995) proved (1.5) under the general conditions that $\{w_{nk}\}$ is an array of bounded i.i.d. random variables independent of $\{X_n, n \geq 1\}$, $E(w_{11}X_1) = 0$, $E(w_{11} - Ew_{11})^2 = 1$, and $EX^2 = 1$.

The Marcinkiewicz–Zygmund strong law is between the Kolmogorov strong law (1.4) and the law of the single logarithm (1.5). However, the Marcinkiewicz–Zygmund strong law for randomly weighted sums of i.i.d. random variables is not proved.

Bootstrap samples were first investigated by Efron (1979) for a sequence of i.i.d. random variables. However, in general, bootstrap samples are defined for a sequence of random variables that are not necessarily i.i.d. Moreover, in the bootstrap sample mean, the weights w_{nk} are negatively associated, not independent (see Section 3). Therefore, it is more interesting to study the Marcinkiewicz–Zygmund strong law for randomly weighted sums of dependent random variables under dependent random weights.

For randomly weighted sums of dependent random variables, Rosalsky and Sreehari (1998), Thanh, Yin and Wang (2011), and Csörgő and Nasari (2013) provided sufficient conditions for some kinds of strong laws. However, their findings do not imply the Marcinkiewicz–Zygmund strong law. In this study, we prove the Marcinkiewicz–Zygmund strong law for randomly weighted sums of negatively orthant dependent (NOD) random variables. The weights are also NOD random variables.

First, we define the concept of NOD random variables, as originally introduced by Lehmann (1966).

Definition 1. *A finite family of random variables $\{X_1, \ldots, X_n\}$ is said to be*

NOD if the following two inequalities hold:

$$P(X_1 \le x_1, \ldots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i)$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i),$$

for all real numbers $x_1, \ldots, x_n$. *An infinite family of random variables is NOD if every finite subfamily is NOD.*

It is well known that if $\{X_n, n \ge 1\}$ is a sequence of NOD random variables and $\{f_n, n \ge 1\}$ is a sequence of Borel functions, all of which are monotone increasing (or all monotone decreasing), then $\{f_n(X_n), n \ge 1\}$ is still a sequence of NOD random variables. It is also well known that if $X_1, \ldots, X_n$ are nonnegative NOD random variables, then $E[X_1 \ldots X_n] \le EX_1 \ldots EX_n$.

The next dependence notion is that of negative association, as first introduced by Alam and Saxena (1981).

Definition 2. *A finite family of random variables* $\{X_i, 1 \le i \le n\}$ *is said to be negatively associated if, for every pair of disjoint subsets* A *and* B *of* $\{1, 2, \ldots, n\}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0$$

whenever f_1 *and* f_2 *are coordinatewise increasing (or coordinatewise decreasing) and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.*

As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariate distributions possess the negative association property, including the multinomial distribution, convolution of unlike multinomial distribution, multivariate hypergeometric distribution, Dirichlet distribution, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks.

Because a negative association implies NOD, the above multivariate distributions also possess NOD property.

The following example shows that the random variables X_1, X_2, X_3 are NOD, but not negatively associated.

Example 1. Let (X_1, X_2, X_3) have the joint distribution described in Table 1. For $x_1, x_2, x_3 = 0$ or 1, it is easy to show that

$$P\{X_1 \le x_1, X_2 \le x_2, X_3 \le x_3\} \le P\{X_1 \le x_1\}P\{X_2 \le x_2\}P\{X_3 \le x_3\},$$

Table 1. The joint distribution of (X_1, X_2, X_3).

(x_1, x_2, x_3)	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
probability	1/16	3/16	3/16	1/16	1/8	1/8	1/8	1/8

$$P\{X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3\} \leq P\{X_1 \geq x_1\}P\{X_2 \geq x_2\}P\{X_3 \geq x_3\}.$$

For example, if $(x_1, x_2, x_3) = (1, 0, 0)$, then

$$P\{X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3\} = \frac{3}{16},\ P\{X_1 \leq x_1\}P\{X_2 \leq x_2\}P\{X_3 \leq x_3\} = \frac{1}{4},$$
$$P\{X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3\} = \frac{1}{2},\ P\{X_1 \geq x_1\}P\{X_2 \geq x_2\}P\{X_3 \geq x_3\} = \frac{1}{2}.$$

Therefore, X_1, X_2, X_3 are NOD. However, $P\{X_1 \leq 0, X_2 + X_3 \leq 1\} > P\{X_1 \leq 0\}P\{X_2 + X_3 \leq 1\}$, because $P\{X_1 \leq 0, X_2 + X_3 \leq 1\} = 7/16$, $P\{X_1 \leq 0\}P\{X_2 + X_3 \leq 1\} = 13/32$. Hence, X_1, X_2, X_3 are not negatively associated.

The rest of this paper is organized as follows. In Section 2, we present the main results and remarks. The applications of our main results to bootstrap sample means and least-squares estimators are given in Section 3. The technical details are provided in the online supplementary material.

Note that throughout this paper, $I(A)$ denotes the indicator function of the event A. It proves convenient in defining that $\log x = \max\{1, \ln x\}$ for $x > 0$, where $\ln x$ denotes the natural logarithm.

2. Main Results and Remarks

In this section, we present our main results and remarks. We first provide a lemma that plays an important role in the proofs of our main results.

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative NOD random variables, and let $\{Y_n, n \geq 1\}$ be a sequence of nonnegative NOD random variables. Assume that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent. Then $\{X_nY_n, n \geq 1\}$ is a sequence of NOD random variables.*

The following theorem is the Marcinkiewicz–Zygmund strong law for randomly weighted sums of NOD random variables.

Theorem 1. *Let $1 \leq p < 2$. Let $\{X, X_n, n \geq 1\}$ be a sequence of NOD and identically distributed random variables, and let $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ be an array of rowwise NOD random variables, with*

$$\sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha} < \infty, \tag{2.1}$$

for some $\alpha > 2p$. Assume that $\{X_n\}$ and $\{w_{nk}\}$ are independent. If $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/p$, then

$$n^{-1/p} \sum_{k=1}^{n} (w_{nk} X_k - E w_{nk} EX) \to 0 \quad \text{a.s..} \tag{2.2}$$

If we further assume that all weights w_{nk} have the same distribution, then the case $\alpha = 2p$ is possible (see the following corollary).

Corollary 1. *Let $1 \le p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of NOD and identically distributed random variables, and let $\{w, w_{nk}, n \ge 1, 1 \le k \le n\}$ be an array of rowwise NOD random variables identically distributed as w. Assume that $\{X_n\}$ and $\{w_{nk}\}$ are independent. If $E|w|^\alpha < \infty$, for some $\alpha \ge 2p$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/p$, then*

$$n^{-1/p} \sum_{k=1}^{n} (w_{nk} X_k - EwEX) \to 0 \quad \text{a.s..} \tag{2.3}$$

Remark 1. The assumption $\alpha \ge 2p$ is needed. Let $P(X = 1) = 1$. Then, the sufficient moment condition for (2.3) is $E|w|^{2p} < \infty$ (see Taylor, Patterson and Bozorgnia (2002)). Furthermore, assume that $\{w, w_{nk}, n \ge 1, 1 \le k \le n\}$ is an array of i.i.d. random variables. Then, by the Borel–Cantelli lemma, (2.3) is equivalent to $\sum_{n=1}^{\infty} P(|\sum_{k=1}^{n}(w_{nk} - Ew)| > n^{1/p}\varepsilon) < \infty, \forall \varepsilon > 0$. The latter is also equivalent to $E|w|^{2p} < \infty$ (see Katz (1963)). Hence, the necessary and sufficient moment condition for (2.3) is $E|w|^{2p} < \infty$.

Remark 2. For any $\alpha' \in (0, \alpha)$, by Hölder's inequality and Jensen's inequality,

$$\sup_{n \ge 1} n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha'} \le \sup_{n \ge 1} \left(n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha} \right)^{\alpha'/\alpha} < \infty.$$

Hence, condition (2.1) becomes increasingly stronger as α increases.

Remark 3. Let $1/\alpha + 1/\beta = 1/p$. If $\alpha > 2p$, then $\beta < 2p$. Hence, $\alpha > \beta$. Conversely, if $\alpha > \beta$, then $\alpha > 2p$ (if $\alpha \le 2p$, then $\beta \ge 2p$, and so $\alpha \le \beta$). Hence, under the condition $1/\alpha + 1/\beta = 1/p$, $\alpha > 2p$ and $\alpha > \beta$ are equivalent.

Remark 4. Thanh, Yin and Wang (2011) obtained a randomly weighted version of Theorem 3.1 in Li et al. (1995). However, the result of Thanh, Yin and Wang (2011) does not include Theorem 1.

If $p = 2$, we have the law of the single logarithm for randomly weighted sums of NOD random variables.

Theorem 2. *Let $\{X, X_n, n \ge 1\}$ be a sequence of NOD and identically distributed* *random variables, and let* $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ *be an array of rowwise NOD random variables satisfying* (2.1)*, for some* $\alpha > 4$*. Assume that* $\{X_n\}$ *and* $\{w_{nk}\}$ *are independent,* $Ew_{nk}X_k = 0$ *for all* $n \geq 1$ *and* $1 \leq k \leq n$*,* $EX^2 = 1$*, and* $E|X|^\beta/(\log|X|)^{\beta/2} < \infty$*, where* $1/\alpha + 1/\beta = 1/2$*. If* $X \geq 0$ *a.s. and* $w_{nk} \geq 0$ *a.s. for all* $n \geq 1$ *and* $1 \leq k \leq n$*, then*

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n w_{nk}X_k|}{\sqrt{2n\log n}} \leq \rho \quad \text{a.s.}, \tag{2.4}$$

where $\rho = \inf\{u > 0 : \sum_{n=1}^\infty \exp(-u^2 n \log n / \sum_{k=1}^n Ew_{nk}^2) < \infty\}$*.*

Remark 5. The following inequalities hold (the proof is provided in the online supplementary material):

$$\liminf_{n\to\infty} \left(n^{-1}\sum_{k=1}^n Ew_{nk}^2\right)^{1/2} \leq \rho \leq \limsup_{n\to\infty} \left(n^{-1}\sum_{k=1}^n Ew_{nk}^2\right)^{1/2}.$$

Hence $\rho = \lim_{n\to\infty} \left(n^{-1}\sum_{k=1}^n Ew_{nk}^2\right)^{1/2}$ whenever the limit exists.

Remark 6. The nonnegative conditions on X and w_{nk} ensure that $\{w_{nk}X_k, 1 \leq k \leq n\}$ is also a sequence of NOD random variables, by Lemma 1. If one of the nonnegative conditions is excluded, we can apply Theorem 2 to $\{X^+, X_n^+, n \geq 1\}$ and $\{X^-, X_n^-, n \geq 1\}$ (where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$), or to $\{w_{nk}^+, n \geq 1, 1 \leq k \leq n\}$ and $\{w_{nk}^-, n \geq 1, 1 \leq k \leq n\}$, respectively. Hence, (2.4) can be replaced by

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n w_{nk}X_k|}{\sqrt{2n\log n}} \leq 2\rho \quad \text{a.s.}.$$

Then, the upper bound has increased by a factor of two. Similarly, if all of the nonnegative conditions are excluded, (2.4) can be replaced by

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n w_{nk}X_k|}{\sqrt{2n\log n}} \leq 4\rho \quad \text{a.s.}.$$

Then, the upper bound has increased by a factor of four.

Remark 7. If $\{X, X_n, n \geq 1\}$ and $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ are all independent, then (2.4) also holds without the nonnegative conditions on X and w_{nk}, because $\{w_{nk}X_k, 1 \leq k \leq n\}$ is a sequence of independent, and hence NOD.

If we further assume that all weights w_{nk} are i.i.d. and $\{X_n, n \geq 1\}$ are independent, then the reverse inequality of (2.4) holds (see the following corollary). Thus, the upper bound of (2.4) is optimal.

Corollary 2. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables, and let* $\{w, w_{nk}, n \geq 1, 1 \leq k \leq n\}$ *be an array of i.i.d. random variables independent* *of* $\{X, X_n, n \geq 1\}$. *If* $E(wX) = 0$, $E|w|^\alpha < \infty$ *for some* $\alpha > 4$, $EX^2 = 1$, *and* $E|X|^\beta/(\log|X|)^{\beta/2} < \infty$, *where* $1/\alpha + 1/\beta = 1/2$, *then*

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n w_{nk}X_k|}{\sqrt{2n\log n}} = \sqrt{E(w - Ew)^2} \quad \text{a.s..} \tag{2.5}$$

3. Applications

In this section, we apply our main results to bootstrap sample means and least-squares estimators.

We first consider bootstrap samples, which were introduced by Efron (1979). Let $Y_{n1}, Y_{n2}, \ldots, Y_{nm(n)}$ be a sample selected randomly, with replacement, from the set of random variables $\{X_i, 1 \leq i \leq n\}$. Here, $\{Y_{n1}, Y_{n2}, \ldots, Y_{nm(n)}\}$ is called a bootstrap sample from $\{X_i, 1 \leq i \leq n\}$, with bootstrap sample size $m(n)$. We can write $Y_{ni} = X_{Z_{ni}}$, where $Z_{n1}, Z_{n2}, \ldots, Z_{nm(n)}$ are independent and uniformly distributed on $\{1, 2, \ldots, n\}$ and independent of $\{X_i, 1 \leq i \leq n\}$. Setting the weights w_{nk} as

$$w_{nk} = \frac{1}{m(n)} \sum_{i=1}^{m(n)} I(Z_{ni} = k), \quad 1 \leq k \leq n,$$

the bootstrap sample mean is

$$\bar{X}_n^* = \frac{1}{m(n)} \sum_{i=1}^{m(n)} Y_{ni} = \sum_{k=1}^{n} w_{nk} X_k.$$

Note that $m(n)(w_{n1}, \ldots, w_{nn})$ follows a multinomial distribution with parameters $(m(n), 1/n, \ldots, 1/n)$.

By the application of Theorems 1 and 2, we obtain the convergence rate of the bootstrap strong law of large numbers for NOD random variables. To prove it, the following lemma is needed. The proofs of the results in this section are also provided in the online supplementary material.

Lemma 2. *Let* X_n *follow a binomial distribution with parameters* n *and* p_n. *Then the following statements hold.*
(i) If $np_n \leq c$ *for some positive constant* c, *then* $EX_n^k \leq C_k np_n$ *for all* $k \geq 1$, *where* C_k *is a positive constant depending only on* k.
(ii) If $np_n \geq d$ *for some positive constant* d, *then* $EX_n^k \leq D_k(np_n)^k$ *for all* $k \geq 1$, *where* D_k *is a positive constant depending only on* k.

Theorem 3. *Let* $1 \leq p \leq 2$, $\{X, X_n, n \geq 1\}$ *be a sequence of NOD and identically distributed random variables with* $E|X|^\beta < \infty$, *for some* $\beta > p$. *If* $n = O(m(n))$, *then we have the following:*

(i) If $1 \le p < 2$*, then*

$$n^{1-1/p}\left(\bar{X}_n^* - EX\right) \to 0 \quad \text{a.s..} \tag{3.1}$$

(ii) If $p = 2$*, then*

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log n}}\left|\bar{X}_n^* - EX\right| \le \sqrt{(r+1)EX^2} \quad \text{a.s.,} \tag{3.2}$$

where $r = \limsup_{n\to\infty} n/m(n)$*.*

Remark 8. To the best of our knowledge, no studies have investigated the convergence rate of the unconditional strong law of large numbers for the bootstrap mean. Thus, Theorem 3 is a new result.

Remark 9. Arenal-Gutiérrez, Matrán and Cuesta-Albertos (1996) proved the bootstrap strong law of large numbers (i.e., $\bar{X}_n^* \to EX$ a.s.) under the conditions that $n \log n = o(m(n))$ and $\{X, X_n, n \ge 1\}$ is a sequence of pairwise i.i.d. random variables, with $E|X| < \infty$. The moment condition is weaker than that of Theorem 3, but the condition on the bootstrap sample size is stronger.

We next consider a simple linear regression model with random design:

$$Y_{nk} = a + bX_{nk} + \epsilon_k, \quad 1 \le k \le n. \tag{3.3}$$

Here, a and b are unknown parameters, the random design points $\{X_{nk}, n \ge 1, 1 \le k \le n\}$ form an array of rowwise NOD random variables identically distributed as a random variable X, $\{Y_{nk}, n \ge 1, 1 \le k \le n\}$ is an array of observable variables, and the errors $\{\epsilon_n, n \ge 1\}$ form a sequence of NOD and identically distributed random variables independent of $\{X_{nk}, n \ge 1, 1 \le k \le n\}$ and with the same distribution as a random variable ϵ. Then the least-square estimators of b and a are

$$\hat{b}_n = \frac{\sum_{k=1}^n (Y_{nk} - \bar{Y}_n)(X_{nk} - \bar{X}_n)}{S_n^2}, \quad \hat{a}_n = \bar{Y}_n - \hat{b}_n \bar{X}_n, \tag{3.4}$$

respectively, where $\bar{X}_n = n^{-1}\sum_{k=1}^n X_{nk}$, $\bar{Y}_n = n^{-1}\sum_{k=1}^n Y_{nk}$, and $S_n^2 = \sum_{k=1}^n (X_{nk} - \bar{X}_n)^2$, $n \ge 1$.

In the following, as applications of Corollaries 1 and 2, we obtain the convergence rates for the strong consistency of the least-square estimators of the unknown parameters.

Theorem 4. *Let* $1 \le p \le 2$*. Under model* (3.3)*, we have the following:*
(i) When $1 \le p < 2$*, we assume that* $E|X|^\alpha < \infty$ *for some* $\alpha \ge 2p$*,* $E\epsilon = 0$*, and* $E|\epsilon|^\beta < \infty$*, where* $1/\alpha + 1/\beta = 1/p$*. Then,*

$$n^{1-1/p}|\hat{b}_n - b| \to 0 \quad \text{a.s.} \tag{3.5}$$

and

$$n^{1-1/p}|\hat{a}_n - a| \to 0 \quad \text{a.s..} \tag{3.6}$$

(ii) When $p = 2$, *we assume that* $E|X|^\alpha < \infty$ *for some* $\alpha > 4$, $E\epsilon = 0$, *and* $E|\epsilon|^\beta/(\log|\epsilon|)^{\beta/2} < \infty$, *where* $1/\alpha + 1/\beta = 1/2$, *and further assume that* $\{X_{nk}, n \geq 1, 1 \leq k \leq n\}$ *is an array of i.i.d. random variables and* $\{\epsilon_n, n \geq 1\}$ *is a sequence of i.i.d. random variables. Then,*

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log n}}\ |\hat{b}_n - b| = \sqrt{\frac{E\epsilon^2}{E(X-EX)^2}} \quad \text{a.s.} \tag{3.7}$$

and

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log n}}\ |\hat{a}_n - a| = |EX| \cdot \sqrt{\frac{E\epsilon^2}{E(X-EX)^2}} \quad \text{a.s..} \tag{3.8}$$

Supplementary Materials

The online supplementary material contains technical details.

Acknowledgment

The authors would like to thank the Co-Editor Zhiliang Ying, Associate Editor, and two anonymous reviewers for their careful reading of the manuscript and valuable suggestions that helped to improve an earlier version of this paper. The research of Pingyan Chen is supported by the National Natural Science Foundation of China (No. 11271161). The research of Soo Hak Sung is supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A1B03029898).

References

Alam, K. and Saxena, K. M. L. (1981). Positive dependence in multivariate distributions. *Comm. Statist. Theory Methods* **10**, 1183–1196.

Arenal-Gutiérrez, E., Matrán, C. and Cuesta-Albertos, J. A. (1996). On the unconditional strong law of large numbers for the bootstrap mean. *Statist. Probab. Lett.* **27**, 49–60.

Bai, Z. D. and Cheng, P. E. (2000). Marcinkiewicz strong laws for linear statistics. *Statist. Probab. Lett.* **46**, 105–112.

Bai, Z. D., Cheng, P. E. and Zhang, C. H. (1997). An extension of the Hardy-Littlewood strong law. *Statist. Sinica* **7**, 923–928.

Chen, P. and Chen, R. (2010). A remark of LSL for weighted sums of i.i.d. random elements. *Statist. Probab. Lett.* **80**, 1329–1334.

Chen, P. and Gan, S. (2007). Limiting behavior of weighted sums of i.i.d. random variables. *Statist. Probab. Lett.* **77**, 1589–1599.

Chen, P., Kong, N. and Sung, S. H. (2017). Complete convergence for weighted sums of i.i.d. random variables with applications in regression estimation and EV model. *Comm. Statist. Theory Methods* **46**, 3599–3613.

Chen, P., Ma, X. and Sung, S. H. (2014). On complete convergence and strong law for weighted sums of i.i.d. random variables. *Abstr. Appl. Anal.* **2014**, Article ID 251435, 7 pages.

Choi, B. D. and Sung, S. H. (1987). Almost sure convergence theorems of weighted sums of random variables. *Stoch. Anal. Appl.* **5**, 365–377.

Chow, Y. S. (1966). Some convergence theorems for independent random variables. *Ann. Math. Statist.* **37**, 1482–1493.

Csörgő, M. and Nasari, M. M. (2013). Asymptotics of randomly weighted u- and v-statistics: Application to bootstrap. *J. Multivariate Anal.* **121**, 176–192.

Cuzick, J. (1995). A strong law for weighted sums of i.i.d. random variables. *J. Theoret. Probab.* **8**, 625–641.

Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *Ann. Statist.* **7**, 1–26.

Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables with applications. *Ann. Statist.* **11**, 286–295.

Katz, M. (1963). The probability in the tail of a distribution. *Ann. Math. Statist.* **34**, 312–318.

Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37**, 1137–1153.

Li, D., Rao, M. B., Jiang, T. and Wang, X. (1995). Complete convergence and almost sure convergence of weighted sums of random variables. *J. Theoret. Probab.* **8**, 49–76.

Li, D., Rao, M. B. and Wang, X. C. (1995). On the strong law of large numbers and the law of the logarithm for weighted sums of independent random variables with multidimensional indices. *J. Multivariate Anal.* **52**, 181–198.

Rosalsky, A. and Sreehari, M. (1998). On the limiting behavior of randomly weighted partial sums. *Statist. Probab. Lett.* **40**, 403–410.

Sung, S. H. (2009). A law of the single logarithm for weighted sums of i.i.d. random elements. *Statist. Probab. Lett.* **79**, 1351–1357.

Taylor, R. L., Patterson, R. F. and Bozorgnia, A. (2002). A strong law of large numbers for arrays of rowwise negatively dependent random variables. *Stoch. Anal. Appl.* **20**, 643–656.

Thanh, L. V., Yin, G. G. and Wang L. Y. (2011). State observers with random sampling times and convergence analysis of double-indexed and randomly weighted sums of mixing processes. *SIAM J. Control Optim.* **49**, 106–124.

Department of Mathematics, Jinan University, Guangzhou, 510630, P.R. China.

E-mail: tchenpy@jnu.edu.cn

Department of Statistics, Jinan University, Guangzhou, 510630, P.R. China.

E-mail: zhangt.jnu@foxmail.com

Department of Applied Mathematics, Pai Chai University, Daejeon, 35345, South Korea.

E-mail: sungsh@pcu.ac.kr

(Received September 2016; accepted January 2018)