Strong Laws for Randomly Weighted Sums of Random Variables and Applications in the Bootstrap and Random Design Regression

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Supplementary Material

In this supplement, we provide the proofs of Theorems 1-4, Corollaries 1 and 2, Lemmas 1 and 2, and Remark 5 in the main paper. Throughout this supplement, the symbol C denotes a positive constant which is not necessarily the same one in each appearance, $I(A)$ denotes the indicator function of the event A. It proves convenient in defining that $\log x = \max\{1, \ln x\}$ for $x > 0$, where $\ln x$ denotes the natural logarithm.

Proof of Lemma 1. If $z_1, \cdots, z_n$ are all nonnegative, we have that

$$\begin{aligned}
&P\left(X_1Y_1 \le z_1, \cdots, X_nY_n \le z_n\right) \\
&= \int \cdots \int I\left(x_1y_1 \le z_1, \cdots, x_ny_n \le z_n\right)\ dF_{X_1,\cdots,X_n,Y_1,\cdots,Y_n}(x_1, \cdots, x_n, y_1, \cdots, y_n) \\
&= \int \cdots \int I\left(x_1y_1 \le z_1, \cdots, x_ny_n \le z_n\right)\ dF_{X_1,\cdots,X_n}(x_1, \cdots, x_n) dF_{Y_1,\cdots,Y_n}(y_1, \cdots, y_n) \\
&\qquad \text{(by independence of } \{X_n\} \text{ and } \{Y_n\}\text{)} \\
&= \int \cdots \int P\left(x_1Y_1 \le z_1, \cdots, x_nY_n \le z_n\right)\ dF_{X_1,\cdots,X_n}(x_1, \cdots, x_n)
\end{aligned}$$

$$\le \int \cdots \int P\left(x_1 Y_1 \le z_1\right) \cdots P\left(x_n Y_n \le z_n\right) \, dF_{X_1, \cdots, X_n}(x_1, \cdots, x_n) \quad (\text{by NOD of } \{Y_n\})$$

$$= E\left[F_{Y_1}(z_1/X_1) \cdots F_{Y_n}(z_n/X_n)\right]$$

$$\le E\left[F_{Y_1}(z_1/X_1)\right] \cdots E\left[F_{Y_n}(z_n/X_n)\right] \quad (\text{by NOD of } \{X_n\})$$

$$= \iint I(x_1 y_1 \le z_1) \, dF_{Y_1}(y_1) dF_{X_1}(x_1) \cdots \iint I(x_n y_n \le z_n) \, dF_{Y_n}(y_n) dF_{X_n}(x_n)$$

$$= \iint I(x_1 y_1 \le z_1) \, dF_{X_1, Y_1}(x_1, y_1) \cdots \iint I(x_n y_n \le z_n) \, dF_{X_n, Y_n}(x_n, y_n)$$

(by independence of $\{X_n\}$ and $\{Y_n\}$)

$$= P(X_1 Y_1 \le z_1) \cdots P(X_n Y_n \le z_n).$$

Otherwise, we have that

$$P\left(X_1 Y_1 \le z_1, \cdots, X_n Y_n \le z_n\right) = P(X_1 Y_1 \le z_1) \cdots P(X_n Y_n \le z_n) = 0.$$

Similarly, we also have that

$$P\left(X_1 Y_1 > z_1, \cdots, X_n Y_n > z_n\right) \le P(X_1 Y_1 > z_1) \cdots P(X_n Y_n > z_n).$$

Therefore, $X_1 Y_1, \cdots, X_n Y_n$ are NOD. □

To prove Theorem 1, we need the following lemma which is the Rosenthal moment inequality for sums of NOD random variables.

Lemma A. (Asadian et al., 2006). *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^s < \infty$ for some $s \ge 2$ and all $n \ge 1$. Then there exists a positive constant C depending only on s such that for all $n \ge 1$,*

$$E\left|\sum_{k=1}^{n} X_k\right|^s \le C\left\{\sum_{k=1}^{n} E|X_k|^s + \left(\sum_{k=1}^{n} EX_k^2\right)^{s/2}\right\}.$$

Proof of Theorem 1. We can rewrite $\sum_{k=1}^{n}(w_{nk}X_k - Ew_{nk}EX_k)$ as

$$\begin{aligned}&\sum_{k=1}^{n}(w_{nk}X_k - Ew_{nk}EX_k)\\ &= \sum_{k=1}^{n}(w_{nk}^{+}X_k^{+} - Ew_{nk}^{+}EX_k^{+}) - \sum_{k=1}^{n}(w_{nk}^{+}X_k^{-} - Ew_{nk}^{+}EX_k^{-})\\ &\quad + \sum_{k=1}^{n}(w_{nk}^{-}X_k^{-} - Ew_{nk}^{-}EX_k^{-}) - \sum_{k=1}^{n}(w_{nk}^{-}X_k^{+} - Ew_{nk}^{-}EX_k^{+}).\end{aligned}$$

Since $\{X_n^{+}, n \geq 1\}$ and $\{X_n^{-}, n \geq 1\}$ are still sequences of NOD random variables, and $\{w_{nk}^{+}, n \geq 1, 1 \leq k \leq n\}$ and $\{w_{nk}^{-}, n \geq 1, 1 \leq k \leq n\}$ are still arrays of NOD random variables, we may assume that $\{X_n, n \geq 1\}$ and $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ are all nonnegative. Note that for all $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\left\{\max_{1\leq k\leq n} w_{nk} > \varepsilon n^{1/p}\right\} &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{w_{nk} > \varepsilon n^{1/p}\}\\ &\leq C\sum_{n=1}^{\infty} n^{-\alpha/p}\sum_{k=1}^{n} Ew_{nk}^{\alpha}\\ &\leq C\sum_{n=1}^{\infty} n^{1-\alpha/p} < \infty.\end{aligned}$$

By the Borel-Cantelli lemma,

$$n^{-1/p}\max_{1\leq k\leq n} w_{nk} \to 0 \quad \text{a.s.} \tag{1}$$

The moment condition $EX^{\beta} < \infty$ is equivalent to $\sum_{n=1}^{\infty} P\{X > n^{1/\beta}\} < \infty$, and hence

$$\sum_{n=1}^{\infty} P\{X_n > n^{1/\beta}\} < \infty.$$

By the Borel-Cantelli lemma, the series

$$\sum_{n=1}^{\infty} X_n I(X_n > n^{1/\beta})$$

converges almost surely. Then we have by (1) that

$$\begin{aligned} n^{-1/p}\sum_{k=1}^{n} w_{nk}X_kI(X_k > n^{1/\beta}) &\le \left(n^{-1/p}\max_{1\le k\le n} w_{nk}\right)\sum_{k=1}^{n} X_kI(X_k > n^{1/\beta}) \\ &\le \left(n^{-1/p}\max_{1\le k\le n} w_{nk}\right)\sum_{k=1}^{n} X_kI(X_k > k^{1/\beta}) \\ &\le \left(n^{-1/p}\max_{1\le k\le n} w_{nk}\right)\sum_{k=1}^{\infty} X_kI(X_k > k^{1/\beta}) \\ &\to 0 \quad \text{a.s.} \end{aligned}$$

Since $n^{1/\beta}I(X_k > n^{1/\beta}) \le X_kI(X_k > n^{1/\beta})$, we also have that

$$n^{-1/p}\sum_{k=1}^{n} w_{nk}n^{1/\beta}I(X_k > n^{1/\beta}) \to 0 \quad \text{a.s.}$$

On the other hand, by Remark 2,

$$\begin{aligned} n^{-1/p}\sum_{k=1}^{n} Ew_{nk}EX_kI(X_k > n^{1/\beta}) &= n^{-1/p}\left(\sum_{k=1}^{n} Ew_{nk}\right)EX^{\beta+1-\beta}I(X > n^{1/\beta}) \\ &\le Cn^{-1/\alpha}EX^{\beta}I(X > n^{1/\beta}) \\ &\to 0, \end{aligned}$$

and hence

$$n^{-1/p}\sum_{k=1}^{n} Ew_{nk}E\left(n^{1/\beta}I(X_k > n^{1/\beta})\right) \le n^{-1/p}\sum_{k=1}^{n} Ew_{nk}EX_kI(X_k > n^{1/\beta}) \to 0.$$

Hence, to prove the result, it suffices to prove that

$$n^{-1/p}\sum_{k=1}^{n}\left(w_{nk}X_k(n^{1/\beta}) - Ew_{nk}EX_k(n^{1/\beta})\right) \to 0 \quad \text{a.s.}, \tag{2}$$

where $X_k(n^{1/\beta}) := X_kI(X_k \le n^{1/\beta}) + n^{1/\beta}I(X_k > n^{1/\beta})$. Note that $\{X_k(n^{1/\beta}), n \ge 1, 1 \le k \le n\}$ is still an array of nonnegative rowwise NOD random variables. By the Borel-Cantelli lemma, to prove (2), it suffices to show that

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n}\left(w_{nk}X_k(n^{1/\beta}) - Ew_{nk}EX_k(n^{1/\beta})\right)\right| > \varepsilon n^{1/p}\right\} < \infty, \quad \forall\, \varepsilon > 0. \tag{3}$$

Set

$$X_{nk} = w_{nk}X_k(n^{1/\beta})I(w_{nk}X_k(n^{1/\beta}) \le n^{1/p}) + n^{1/p}I(w_{nk}X_k(n^{1/\beta}) > n^{1/p}).$$

Then, by Lemma 1, $\{w_{nk}X_k(n^{1/\beta}), n \ge 1, 1 \le k \le n\}$ is an array of rowwise NOD random variables. Since X_{nk} is an increasing transformation of $w_{nk}X_k(n^{1/\beta})$, $\{X_{nk}, n \ge 1, 1 \le k \le n\}$ is still an array of rowwise NOD random variables. Note that

$$\begin{aligned}
&\left\{\left|\sum_{k=1}^{n}\left(w_{nk}X_k(n^{1/\beta}) - Ew_{nk}EX_k(n^{1/\beta})\right)\right| > \varepsilon n^{1/p}\right\} \\
&\subset \cup_{k=1}^{n}\left\{w_{nk}X_k(n^{1/\beta}) > n^{1/p}\right\} \cup \left\{\left|\sum_{k=1}^{n}\left(X_{nk} - Ew_{nk}EX_k(n^{1/\beta})\right)\right| > \varepsilon n^{1/p}\right\}. \qquad (4)
\end{aligned}$$

By the Markov inequality and a standard computation,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{w_{nk}X_k(n^{1/\beta}) > n^{1/p}\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha/p} E\left(w_{nk}X_k(n^{1/\beta})\right)^{\alpha} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha/p} Ew_{nk}^{\alpha} EX^{\alpha}(n^{1/\beta}) \\
&\le C\sum_{n=1}^{\infty} n^{1-\alpha/p}\left\{EX^{\alpha}I(X \le n^{1/\beta}) + n^{\alpha/\beta}P\left\{X > n^{1/\beta}\right\}\right\} \\
&= C\sum_{n=1}^{\infty} n^{1-\alpha/p}\sum_{i=1}^{n} EX^{\alpha}I(i-1 < X^{\beta} \le i) + C\sum_{n=1}^{\infty} P\left\{X^{\beta} > n\right\} \\
&= C\sum_{i=1}^{\infty} EX^{\alpha}I(i-1 < X^{\beta} \le i)\sum_{n=i}^{\infty} n^{-\alpha/\beta} + C\sum_{n=1}^{\infty} P\left\{X^{\beta} > n\right\} \\
&\le CEX^{\beta} < \infty, \qquad (5)
\end{aligned}$$

and by Remark 2,

$$\begin{aligned}
& n^{-1/p}\left|\sum_{k=1}^{n} Ew_{nk}EX_k(n^{1/\beta}) - \sum_{k=1}^{n} EX_{nk}\right| \\
& \le n^{-1/p}\sum_{k=1}^{n} Ew_{nk}X_k(n^{1/\beta})I(w_{nk}X_k(n^{1/\beta}) > n^{1/p}) + \sum_{k=1}^{n} P\left\{w_{nk}X_k(n^{1/\beta}) > n^{1/p}\right\} \\
& \le 2n^{-1/p}\sum_{k=1}^{n} Ew_{nk}X_k(n^{1/\beta})I(w_{nk}X_k(n^{1/\beta}) > n^{1/p}) \\
& = 2n^{-1/p}\sum_{k=1}^{n} E\left(w_{nk}X_k(n^{1/\beta})\right)^{\beta+1-\beta} I(|w_{nk}X_k(n^{1/\beta})| > n^{1/p}) \\
& \le 2n^{-\beta/p}\left(\sum_{k=1}^{n} Ew_{nk}^{\beta}\right) EX^{\beta} \\
& \le Cn^{1-\beta/p} = Cn^{-\beta/\alpha} \to 0. \qquad (6)
\end{aligned}$$

Hence by (4)-(6), to prove (3), it suffices to show that

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n}(X_{nk} - EX_{nk})\right| > \varepsilon n^{1/p}\right\} < \infty, \quad \forall\ \varepsilon > 0. \qquad (7)$$

By the Markov inequality and Lemma A, we have that for any $q > 2$,

$$\begin{aligned}
& P\left\{\left|\sum_{k=1}^{n}(X_{nk} - EX_{nk})\right| > \varepsilon n^{1/p}\right\} \\
& \le C\left(n^{-2/p}\sum_{k=1}^{n} EX_{nk}^2\right)^{q/2} + Cn^{-q/p}\sum_{k=1}^{n} EX_{nk}^q. \qquad (8)
\end{aligned}$$

If $\beta < 2$, then

$$\begin{aligned}
& \sum_{k=1}^{n} EX_{nk}^2 \\
& = \sum_{k=1}^{n} E\left(w_{nk}X_k(n^{1/\beta})\right)^2 I(w_{nk}X_k(n^{1/\beta}) \le n^{1/p}) + n^{2/p}\sum_{k=1}^{n} P\left\{w_{nk}X_k(n^{1/\beta}) > n^{1/p}\right\} \\
& \le \sum_{k=1}^{n} E\left(w_{nk}X_k(n^{1/\beta})\right)^{\beta+2-\beta} I(w_{nk}X_k(n^{1/\beta}) \le n^{1/p}) + n^{2/p}n^{-\beta/p}\sum_{k=1}^{n} E\left(w_{nk}X_k(n^{1/\beta})\right)^{\beta} \\
& \le 2n^{(2-\beta)/p}\left(\sum_{k=1}^{n} Ew_{nk}^{\beta}\right) EX^{\beta} \\
& \le Cn^{1+(2-\beta)/p} = Cn^{2/p-\beta/\alpha}.
\end{aligned}$$

Choosing $q > 2\alpha/\beta$, we have

$$\sum_{n=1}^{\infty}\left(n^{-2/p}\sum_{k=1}^{n}EX_{nk}^{2}\right)^{q/2} \leq C\sum_{n=1}^{\infty}n^{-q\beta/(2\alpha)} < \infty. \tag{9}$$

If $\beta \geq 2$, then

$$\sum_{k=1}^{n}EX_{nk}^{2} \leq \left(\sum_{k=1}^{n}Ew_{nk}^{2}\right)EX^{2} \leq Cn.$$

Choosing $q > 2/(2/p-1)$, we have

$$\sum_{n=1}^{\infty}\left(n^{-2/p}\sum_{k=1}^{n}EX_{nk}^{2}\right)^{q/2} \leq C\sum_{n=1}^{\infty}n^{-q(2/p-1)/2} < \infty. \tag{10}$$

For any $q \geq \alpha$,

$$\begin{aligned}\sum_{k=1}^{n}EX_{nk}^{q} &= \sum_{k=1}^{n}E\left(w_{nk}X_{k}(n^{1/\beta})\right)^{q}I(w_{nk}X_{k}(n^{1/\beta}) \leq n^{1/p}) \\ &\quad + n^{q/p}\sum_{k=1}^{n}P\left\{w_{nk}X_{k}(n^{1/\beta}) > n^{1/p}\right\} \\ &\leq \sum_{k=1}^{n}E\left(w_{nk}X_{k}(n^{1/\beta})\right)^{\alpha+q-\alpha}I(w_{nk}X_{k}(n^{1/\beta}) \leq n^{1/p}) \\ &\quad + n^{q/p}n^{-\alpha/p}\sum_{k=1}^{n}E\left(w_{nk}X_{k}(n^{1/\beta})\right)^{\alpha} \\ &\leq 2n^{(q-\alpha)/p}\left(\sum_{k=1}^{n}Ew_{nk}^{\alpha}\right)EX^{\alpha}(n^{1/\beta}) \\ &\leq Cn^{1+(q-\alpha)/p}EX^{\alpha}(n^{1/\beta}) \\ &= Cn^{q/p-\alpha/\beta}EX^{\alpha}(n^{1/\beta}),\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-q/p} \sum_{k=1}^{n} EX_{nk}^{q} &\le C \sum_{n=1}^{\infty} n^{-\alpha/\beta} EX^{\alpha}(n^{1/\beta}) \\ &= C \sum_{n=1}^{\infty} n^{-\alpha/\beta} \left\{ EX^{\alpha} I(X \le n^{1/\beta}) + n^{\alpha/\beta} P\left\{X > n^{1/\beta}\right\}\right\} \\ &= C \sum_{n=1}^{\infty} n^{-\alpha/\beta} \sum_{i=1}^{n} EX^{\alpha} I(i-1 < X^{\beta} \le i) + C \sum_{n=1}^{\infty} P\left\{X^{\beta} > n\right\} \\ &= C \sum_{i=1}^{\infty} EX^{\alpha} I(i-1 < X^{\beta} \le i) \sum_{n=i}^{\infty} n^{-\alpha/\beta} + C \sum_{n=1}^{\infty} P\left\{X^{\beta} > n\right\} \\ &\le CEX^{\beta} < \infty. \qquad (11)\end{aligned}$$

Thus (7) follows from (8)-(11). The proof is completed. □

Proof of Corollary 1. If $\alpha > 2p$, then the result holds at once by Theorem 1. Now we consider the case $\alpha = 2p$. Then $\beta = 2p$ and $E|wX|^{2p} = E|w|^{2p}E|X|^{2p} < \infty$. As in the proof of Theorem 1, we may assume that $X_n, n \ge 1$, and w_{nk}, $n \ge 1$ and $1 \le k \le n$, are nonnegative. By Lemma 1, $\{w_{nk}X_k, n \ge 1, 1 \le k \le n\}$ is an array of rowwise NOD random variables. By a result of Taylor et al. (2002),

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n}(w_{nk}X_k - EwEX)\right| > \varepsilon n^{1/p}\right\} < \infty, \quad \forall\, \varepsilon > 0,$$

which ensures the result by the Borel-Cantelli lemma. □

To prove Theorem 2, we need the following Fuk-Nagaev inequality for NOD random variables. One can refer to Chen and Sung (2017).

Lemma B. *Let $\{\xi_k, 1 \le k \le n\}$ be a sequence of NOD random variables such that for some $q \ge 2$, $E|\xi_k|^q < \infty$ for $1 \le k \le n$. Then, for any $\varepsilon > 0$ and $\delta > 0$,*

$$P\left\{\left|\sum_{k=1}^{n}(\xi_k - E\xi_k)\right| > \varepsilon\right\} \le 2\exp\left\{-\frac{\varepsilon^2}{(2+\delta)\sum_{k=1}^{n} E\xi_k^2}\right\} + C\sum_{k=1}^{n} E|\xi_k|^q/\varepsilon^q,$$

where C is a positive constant depending only on δ and q.

Proof of Theorem 2. Set $a_n = \sqrt{2n\log n}, b_n = n^{1/\beta}(\log n)^{1/2}$, $n \geq 1$. Note that for all $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\left\{\max_{1\leq k\leq n} w_{nk} > \varepsilon a_n\right\} &\leq \sum_{n=1}^{\infty} P\left\{\max_{1\leq k\leq n} w_{nk} > \varepsilon n^{1/2}\right\} \\ &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{w_{nk} > \varepsilon n^{1/2}\} \\ &\leq C\sum_{n=1}^{\infty} n^{-\alpha/2}\sum_{k=1}^{n} Ew_{nk}^{\alpha} \\ &\leq C\sum_{n=1}^{\infty} n^{1-\alpha/2} < \infty.\end{aligned}$$

By the Borel-Cantelli lemma,

$$a_n^{-1}\max_{1\leq k\leq n} w_{nk} \to 0 \quad \text{a.s.} \tag{12}$$

The moment condition $EX^{\beta}/(\log X)^{\beta/2} < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} P\{X > b_n\} < \infty.$$

Then by the Borel-Cantelli lemma, the series

$$\sum_{n=1}^{\infty} X_n I(X_n > b_n)$$

converges almost surely. Then by (12),

$$\begin{aligned}a_n^{-1}\sum_{k=1}^{n} w_{nk}X_k I(X_k > b_n) &\leq \left(a_n^{-1}\max_{1\leq k\leq n} w_{nk}\right)\sum_{k=1}^{n} X_k I(X_k > b_n) \\ &\leq \left(a_n^{-1}\max_{1\leq k\leq n} w_{nk}\right)\sum_{k=1}^{n} X_k I(X_k > b_k) \\ &\leq \left(a_n^{-1}\max_{1\leq k\leq n} w_{nk}\right)\sum_{k=1}^{\infty} X_k I(X_k > b_k) \\ &\to 0 \quad \text{a.s.}\end{aligned}$$

Since $b_n I(X_k > b_n) \leq X_k I(X_k > b_n)$, we also have that

$$a_n^{-1}\sum_{k=1}^{n} w_{nk}b_n I(X_k > b_n) \to 0 \quad \text{a.s.}$$

On the other hand, by Remark 2,

$$\begin{aligned}
&a_n^{-1}\sum_{k=1}^{n} Ew_{nk}EX_kI(X_k>b_n)\\
&\le a_n^{-1}\left(\sum_{k=1}^{n} Ew_{nk}\right)E\left[\frac{X^\beta}{(\log X)^{\beta/2}}\cdot X^{1-\beta}(\log X)^{\beta/2}I(X>b_n)\right]\\
&\le Cn^{-1/\alpha}E\left[\frac{X^\beta}{(\log X)^{\beta/2}}I(X>b_n)\right]\\
&\to 0,
\end{aligned}$$

and hence

$$a_n^{-1}\sum_{k=1}^{n} Ew_{nk}Eb_nI(X_k>b_n)\le a_n^{-1}\sum_{k=1}^{n} Ew_{nk}EX_kI(X_k>b_n)\to 0.$$

Let $X_k(b_n):=X_kI(X_k\le b_n)+b_nI(X_k>b_n)$. Then, to prove the result, it suffices to show that

$$\limsup_{n\to\infty}\frac{|\sum_{k=1}^{n}(w_{nk}X_k(b_n)-Ew_{nk}X_k(b_n))|}{a_n}\le\rho\ \text{ a.s.}\tag{13}$$

By the Borel-Cantelli lemma, it suffices to show that

$$\sum_{n=1}^{\infty}P\left\{\left|\sum_{k=1}^{n}(w_{nk}X_k(b_n)-Ew_{nk}X_k(b_n))\right|>\varepsilon a_n\right\}<\infty,\ \forall\ \varepsilon>\rho.\tag{14}$$

Note that $\{X_k(b_n),1\le k\le n\}$ is still a sequence of nonnegative NOD random variables. By Lemmas 1 and B, we have that for any $\delta>0$,

$$\begin{aligned}
&P\left\{\left|\sum_{k=1}^{n}(w_{nk}X_k(b_n)-Ew_{nk}X_k(b_n))\right|>\varepsilon a_n\right\}\\
&\le 2\exp\left\{-\frac{\varepsilon^2a_n^2}{(2+\delta)\sum_{k=1}^{n}E\left(w_{nk}X_k(b_n)\right)^2}\right\}+Ca_n^{-\alpha}\sum_{k=1}^{n}E\left(w_{nk}X_k(b_n)\right)^\alpha\\
&\le 2\exp\left\{-\frac{2\varepsilon^2n\log n}{(2+\delta)\sum_{k=1}^{n}Ew_{nk}^2}\right\}+Ca_n^{-\alpha}\sum_{k=1}^{n}E\left(w_{nk}X_k(b_n)\right)^\alpha,
\end{aligned}\tag{15}$$

the last inequality follows from the fact that $EX_k^2(b_n)\le EX^2=1$ for all $n\ge1$ and $1\le k\le n$. Since $\varepsilon>\rho$, we can choose δ closed to zero enough such that $\sqrt{2\varepsilon^2/(2+\delta)}>\rho$, which ensures that

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{2\varepsilon^2 n\log n}{(2+\delta)\sum_{k=1}^{n} Ew_{nk}^2}\right\} < \infty. \tag{16}$$

By a standard computation,

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n^{-\alpha} \sum_{k=1}^{n} E\left(w_{nk}X_k(b_n)\right)^{\alpha} \\
&= \sum_{n=1}^{\infty} a_n^{-\alpha} \left(\sum_{k=1}^{n} Ew_{nk}^{\alpha}\right)\left(EX^{\alpha}I(X\le b_n) + b_n^{\alpha}P\{X>b_n\}\right) \\
&\le C\sum_{n=1}^{\infty} n^{1-\alpha/2}(\log n)^{-\alpha/2}\left(EX^{\alpha}I(X\le b_n) + b_n^{\alpha}P\{X>b_n\}\right) \\
&= C\sum_{n=1}^{\infty} n^{-\alpha/\beta}(\log n)^{-\alpha/2}EX^{\alpha}I(X\le b_n) + C\sum_{n=1}^{\infty} P\{X>b_n\} \\
&\le CEX^{\beta}/(\log X)^{\beta/2} < \infty.
\end{aligned} \tag{17}$$

Thus (14) follows from (15)-(17). The proof is completed. ☐

Proof of Remark 5. To prove the first inequality, let $a = \liminf_{n\to\infty}\left(n^{-1}\sum_{k=1}^{n} Ew_{nk}^2\right)^{1/2} = \lim_{m\to\infty}\inf_{n\ge m}\left(n^{-1}\sum_{k=1}^{n} Ew_{nk}^2\right)^{1/2}$. Then, for any $\varepsilon > 0$, there exists a positive integer N such that

$$\left|\inf_{n\ge m}\left(n^{-1}\sum_{k=1}^{n} Ew_{nk}^2\right)^{1/2} - a\right| < \varepsilon \quad \text{if } m \ge N,$$

which implies that

$$\left(n^{-1}\sum_{k=1}^{n} Ew_{nk}^2\right)^{1/2} > a-\varepsilon \quad \text{if } n\ge N.$$

It follows that

$$\sum_{n=N}^{\infty} \exp\left(-\frac{u^2 n\log n}{\sum_{k=1}^{n} Ew_{nk}^2}\right) > \sum_{n=N}^{\infty} \exp\left(-\frac{u^2\log n}{(a-\varepsilon)^2}\right).$$

If $u \le a-\varepsilon$, the second series diverges, and hence the first series also diverges. By the definition of ρ, we have that $\rho \ge a-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain that $\rho \ge a$. Hence the first inequality holds. Similarly, the second inequality also holds. ☐

Proof of Corollary 2. By $E(wX)=0$,

$$\sum_{k=1}^{n} w_{nk}X_k = \sum_{k=1}^{n}(w_{nk}-Ew_{nk})X_k + (Ew)\sum_{k=1}^{n}(X_k - EX_k),$$

and by the classical Hartman-Wintner law of iterated logarithm (see Hartman and Wintner, 1941),

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n}(X_k - EX_k)|}{\sqrt{2n\log\log n}} = \sqrt{E(X-EX)^2} \quad \text{a.s.},$$

which ensures that

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n}(X_k - EX_k)|}{\sqrt{2n\log n}} = 0 \quad \text{a.s.}$$

Thus, to prove the result, it suffices to show that

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n}(w_{nk}-Ew_{nk})X_k|}{\sqrt{2n\log n}} = \sqrt{E(w-Ew)^2} \quad \text{a.s.} \tag{18}$$

So we can assume that $Ew=0$. For any $M>0$, set

$$w'_{nk} = w_{nk}I(|w_{nk}|\le M) - Ew_{nk}I(|w_{nk}|\le M),$$

$$w''_{nk} = w_{nk}I(|w_{nk}|> M) - Ew_{nk}I(|w_{nk}|> M).$$

Then $w_{nk} = w'_{nk} + w''_{nk}$ and

$$\frac{|\sum_{k=1}^{n} w'_{nk}X_k|}{\sqrt{2n\log n}} - \frac{|\sum_{k=1}^{n} w''_{nk}X_k|}{\sqrt{2n\log n}} \le \frac{|\sum_{k=1}^{n} w_{nk}X_k|}{\sqrt{2n\log n}} \le \frac{|\sum_{k=1}^{n} w'_{nk}X_k|}{\sqrt{2n\log n}} + \frac{|\sum_{k=1}^{n} w''_{nk}X_k|}{\sqrt{2n\log n}}. \tag{19}$$

By Theorem 2.3 of Li et al. (1995),

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} w'_{nk}X_k|}{\sqrt{2n\log n}} = \sqrt{E(wI(|w|\le M) - EwI(|w|\le M))^2} \quad \text{a.s.}, \tag{20}$$

and by Remark 7,

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} w''_{nk}X_k|}{\sqrt{2n\log n}} \le \sqrt{E(wI(|w|> M) - EwI(|w|> M))^2} \quad \text{a.s.}. \tag{21}$$

Since

$$E(wI(|w|\le M) - EwI(|w|\le M))^2 \to Ew^2, \quad E(wI(|w|> M) - EwI(|w|> M))^2 \to 0$$

as $M \to \infty$, (18) follows from (19)-(21). The proof is completed. $\square$

Proof of Lemma 2. We prove the result by induction on k.

(i) If $k = 1$, then $EX_n = np_n$ and we take $C_1 = 1$. Assume that $EX_n^i \le C_i np_n$ for $i \le k$. We can write the expansion of $X_n(X_n - 1)\cdots(X_n - k)$ as

$$X_n(X_n - 1)\cdots(X_n - k) = X_n^{k+1} + \sum_{i=1}^{k} a_i X_n^i.$$

Since $E\left[X_n(X_n - 1)\cdots(X_n - k)\right] = n(n-1)\cdots(n-k)p_n^{k+1}$, we have that

$$\begin{aligned}
EX_n^{k+1} &= n(n-1)\cdots(n-k)p_n^{k+1} - \sum_{i=1}^{k} a_i EX_n^i \\
&\le n(n-1)\cdots(n-k)p_n^{k+1} + \sum_{i=1}^{k} |a_i| EX_n^i \\
&\le (np_n)^{k+1} + \sum_{i=1}^{k} |a_i| C_i np_n \\
&= np_n\{(np_n)^k + \sum_{i=1}^{k} |a_i| C_i\} \\
&\le np_n\{c^k + \sum_{i=1}^{k} |a_i| C_i\}.
\end{aligned}$$

Hence, we can take $C_{k+1} = c^k + \sum_{i=1}^{k} |a_i| C_i$.

(ii) If $k = 1$, then $EX_n = np_n$ and we take $D_1 = 1$. Assume that $EX_n^i \le D_i(np_n)^i$ for $i \le k$. Then

$$\begin{aligned}
EX_n^{k+1} &\le n(n-1)\cdots(n-k)p_n^{k+1} + \sum_{i=1}^{k} |a_i| EX_n^i \\
&\le (np_n)^{k+1} + \sum_{i=1}^{k} |a_i| D_i (np_n)^i \\
&= (np_n)^{k+1}\{1 + \sum_{i=1}^{k} |a_i| D_i (np_n)^{-k-1+i}\} \\
&\le (np_n)^{k+1}\{1 + \sum_{i=1}^{k} |a_i| D_i d^{-k-1+i}\}.
\end{aligned}$$

Hence, we can take $D_{k+1} = 1 + \sum_{i=1}^{k} |a_i| D_i d^{-k-1+i}$. $\square$

Proof of Theorem 3. Since $m(n)(w_{n1}, w_{n2}, \cdots, w_{nn})$ has the multinomial distribution with parameters $(m(n), 1/n, 1/n, \cdots, 1/n)$, $m(n)w_{n1}, m(n)w_{n2}, \cdots, m(n)w_{nn}$ are negatively associated (see Joag-Dev and Proschan, 1983) and hence NOD. In particular, $m(n)w_{nk}$ has the binomial distribution with parameters $m(n)$ and $1/n$. Then by Lemma 2, for any $\alpha > 1$,

$$\begin{aligned}\sum_{k=1}^{n} E|nw_{nk}|^{\alpha} &= \frac{n^{\alpha+1}}{m(n)^{\alpha}} E|m(n)w_{n1}|^{\alpha} \\ &= \begin{cases} \frac{n^{\alpha+1}}{m(n)^{\alpha}} \cdot O(m(n)/n), & \text{if } m(n)/n \leq 1, \\ \frac{n^{\alpha+1}}{m(n)^{\alpha}} \cdot O(m^{\alpha}(n)/n^{\alpha}), & \text{if } m(n)/n > 1 \end{cases} \\ &= O(n). \end{aligned} \tag{22}$$

(i) We can rewrite $n^{1-1/p}\left(\bar{X}_n^* - EX\right)$ as

$$\begin{aligned} n^{1-1/p}\left(\bar{X}_n^* - EX\right) &= n^{1-1/p}\left(\sum_{k=1}^{n}(w_{nk}X_k - Ew_{nk}EX_k) + \sum_{k=1}^{n} Ew_{nk}EX_k - EX\right) \\ &= n^{1-1/p}\sum_{k=1}^{n}(w_{nk}X_k - Ew_{nk}EX_k) \\ &= n^{-1/p}\sum_{k=1}^{n}(nw_{nk}X_k - nEw_{nk}EX_k). \end{aligned}$$

Without loss of the generality, we can choose β closed to p enough such that $\alpha > 2p$ (if $E|X|^{\beta} < \infty$, then $E|X|^{\beta'} < \infty$ for $0 < \beta' < \beta$), where $1/\alpha + 1/\beta = 1/p$. Thus (3.1) holds by (22) and Theorem 1.

(ii) Note that

$$\begin{aligned} \sqrt{\frac{n}{2\log n}}\left|\bar{X}_n^* - EX\right| &= \sqrt{\frac{n}{2\log n}}\left|\sum_{k=1}^{n} w_{nk}(X_k - EX_k)\right| \\ &= \frac{1}{\sqrt{2n\log n}}\left|\sum_{k=1}^{n} nw_{nk}(X_k - EX_k)\right|, \end{aligned}$$

and it is easy to show that

$$\begin{aligned}
&\inf\left\{u>0:\sum_{n=1}^{\infty}\exp\left(-\frac{u^2 n\log n}{\sum_{k=1}^{n}E(nw_{nk})^2}\right)<\infty\right\}\\
&=\inf\left\{u>0:\sum_{n=1}^{\infty}\exp\left(-\frac{u^2\log n}{(n/m(n))(1-1/n)+1}\right)<\infty\right\}\\
&\le\sqrt{r+1}.
\end{aligned}$$

Without loss of the generality, we can choose β close 2 enough such that $\alpha>4$, where $1/\alpha+1/\beta=1/2$. Thus, (3.2) holds by (22) and Theorem 2. The proof is completed. □

Proof of Theorem 4. By (3.3) and (3.4), we have that for all $n\ge 1$,

$$\hat{b}_n-b=\frac{\sum_{k=1}^{n}X_{nk}\epsilon_k-\bar{X}_n\sum_{k=1}^{n}\epsilon_k}{S_n^2},\quad \hat{a}_n-a=-\bar{X}_n(\hat{b}_n-b)+\bar{\epsilon}_n, \tag{23}$$

where $\bar{\epsilon}_n=n^{-1}\sum_{k=1}^{n}\epsilon_k$.

(i) By (23), to prove (3.5), it suffices to prove that

$$n^{-1/p}\sum_{k=1}^{n}X_{nk}\epsilon_k\to 0 \quad \text{a.s.}, \tag{24}$$

$$n^{-1/p}\bar{X}_n\sum_{k=1}^{n}\epsilon_k\to 0 \quad \text{a.s.}, \tag{25}$$

$$\liminf_{n\to\infty}n^{-1}S_n^2>0 \quad \text{a.s.} \tag{26}$$

By Corollary 1, (24) holds. By the Kolmogorov strong law of large number for an array of rowwise NOD random variables (see Taylor et al., 2002), and the Marcinkiewicz-Zygmund strong law of large number for a sequence of NOD random variables (see Wu, 2010),

$$\bar{X}_n\to EX \quad \text{a.s.},\qquad n^{-1/p}\sum_{k=1}^{n}\epsilon_k\to 0 \quad \text{a.s.}, \tag{27}$$

which ensure (25). By the moment condition, EX^2 exists, and $EX^2>(EX)^2$ whenever X is non-degenerated. Thus there exists $M>0$ such that $EX^2(M)>(EX)^2$, where $X(M)=$

$XI(|X| \le M) + MI(X > M) - MI(X < -M)$. Then by the Kolmogorov strong law of large number for an array of rowwise NOD random variables (see Taylor et al., 2002) again,

$$\begin{aligned}\liminf_{n\to\infty} n^{-1}S_n^2 &= \liminf_{n\to\infty}\left(n^{-1}\sum_{k=1}^n X_{nk}^2 - \bar{X}_n^2\right)\\ &= \liminf_{n\to\infty} n^{-1}\sum_{k=1}^n X_{nk}^2 - (EX)^2\\ &= \liminf_{n\to\infty} n^{-1}\sum_{k=1}^n \left\{(X_{nk}^+)^2 + (X_{nk}^-)^2\right\} - (EX)^2\\ &\ge \liminf_{n\to\infty} n^{-1}\sum_{k=1}^n \left\{(X_{nk}^+(M))^2 + (X_{nk}^-(M))^2\right\} - (EX)^2\\ &= E(X^+(M))^2 + E(X^-(M))^2 - (EX)^2 \quad \text{a.s.}\\ &= EX^2(M) - (EX)^2 > 0 \quad \text{a.s.},\end{aligned}$$

which implies (26). Hence (3.5) holds. The equation (3.6) follows from (3.5), (23) and (27). The proof of (i) is completed.

(ii) By Corollary 2,

$$\limsup_{n\to\infty}\frac{|\sum_{k=1}^n X_{nk}\epsilon_k|}{\sqrt{2n\log n}} = \sqrt{E(X-EX)^2E\epsilon^2} \quad \text{a.s.} \tag{28}$$

By the Kolmogorov strong law of large numbers for an array of rowwise independent random variables (see Hu et al. 1989),

$$\bar{X}_n \to EX \quad \text{a.s.}, \quad n^{-1}S_n^2 = n^{-1}\sum_{k=1}^n X_{nk}^2 - \bar{X}_n^2 \to E(X^2) - (EX)^2 \quad \text{a.s.}, \tag{29}$$

and by the classical Hartman-Wintner law of iterated logarithm (see Hartman and Wintner, 1941),

$$\frac{\sum_{k=1}^n \epsilon_k}{\sqrt{2n\log n}} \to 0 \quad \text{a.s.} \tag{30}$$

Then (3.7) follows from (23) and (28)-(30). The equation (3.8) follows from (3.7), (23) and (30). The proof is completed. □

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