ASSESSMENT OF NONIGNORALBE LOG-LINEAR MODELS FOR AN INCOMPLETE CONTINGENCY TABLE

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Supplementary Material

In the Supplemental Material, we provide some details on the eight log-linear models given in Table 2 of the the paper, the details on the ignorable nonresponse models for an incomplete $I \times J$ table, detailed results of simulation studies and data analysis that were omitted in Section 5 of the the paper, the proofs of Theorem 1-3) and 4) in the paper, the details on the log-linear models for incomplete three-way contingency tables, and the details on a nonignorable selection model for an incomplete two-way contingency table.

This Supplement is organized as follows. Section S1 provides the expressions of α_{ij}, β_{ij} and g defined in Eq (2.1) of the the paper for each of the eight log-linear models given in Table 2 of the the paper. Section S2 gives the details on the ignorable nonresponse models (MCAR log-linear model and MAR selection model) for an incomplete $I \times J$ table. Section S3 presents the closed forms of $\hat{\pi}_{ij11}$, $\hat{\pi}_{i+12}$ and $\hat{\pi}_{+j21}$ in Eq (4.1) and (4.2) of the the paper. Section S4 specifies the values of parameters for simulation studies in Section 5.1 of the the paper and presents the results of simulation studies for N=10000. Section S5 provides the estimates of the expected cell counts for the eight nonignorable log-linear models and the ignorable nonresponse models fitted to two real data sets, the first data set used in Section 5.2 of the the paper and the second data set an additional real data example. The proofs of Theorem 1-3) and 4) in the paper are given in Section S6. Section S7 provides the details on the nonresponse log-linear models for incomplete three-way contingency tables and the model selection guideline. Section S8 provides the details on a nonignorable selection model for an incomplete two-way contingency table and illustrates its performance for two real data sets. Finally, the proofs of Theorems given in Section S7 and Section S8 are presented in Section S9.

S1 α_{ij} and β_{ij} for the eight nonignorable log-linear models in Table 2

Table S1 presents the expressions of α_{ij} and β_{ij} defined in Eq (2.1) of the the paper for the eight nonignorable log-linear models in Table 2 of the the paper. Note that the expression of g is the same for all eight log-linear models, $g = \exp[4\lambda^{11}_{R_1R_2}]$.

Table S1: α_{ij} and β_{ij} for the eight nonignorable log-linear models in Table 2

Model notation	
$(\alpha_{i\cdot}, \beta_{i\cdot})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_1}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_2}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_2}}]$
$(\alpha_{\cdot j}, \beta_{\cdot j})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_1}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_2}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_2}}]$
$(\alpha_{i\cdot}, \beta_{\cdot j})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_1}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_2}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_2}}]$
$(\alpha_{\cdot j}, \beta_{i\cdot})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_1}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_2}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_2}}]$
$(\alpha_{\cdot\cdot}, \beta_{i\cdot})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_2}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_2}}]$
$(\alpha_{\cdot\cdot}, \beta_{\cdot j})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_2}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_2}}]$
$(\alpha_{i\cdot}, \beta_{\cdot\cdot})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{i2}}_{\mathbf{Y_1R_1}} - \lambda^{\mathbf{i1}}_{\mathbf{Y_1R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2}]$
$(\alpha_{\cdot j}, \beta_{\cdot\cdot})$	$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2} + \lambda^{\mathbf{j2}}_{\mathbf{Y_2R_1}} - \lambda^{\mathbf{j1}}_{\mathbf{Y_2R_1}}]$ $\beta_{ij} = \exp[\lambda^2_{R_2} - \lambda^1_{R_2} + \lambda^{12}_{R_1R_2} - \lambda^{11}_{R_1R_2}]$

S2 Ignorable Nonresponse Models for an incomplete $I \times J$ Table in Table 1

As the important baseline models for an incomplete $I \times J$ of Table 1 in the paper, one can consider the ignorable nonresponse models, the MCAR model and the MAR model.

S2.1 MCAR log-linear model

The MCAR model for the cell probabilities $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ is represented as the log-linear model without any interaction term(s) between the row and column variables (Y_1 and Y_2) and the two missingness variables (R_1 and R_2),

$$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{R_1} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{k\ell}_{R_1R_2}. \tag{S2.1}$$

where $\sum_i \lambda^i_{Y_1} = \sum_j \lambda^j_{Y_2} = \sum_k \lambda^k_{R_1} = \sum_\ell \lambda^\ell_{R_2} = \sum_i \lambda^{ij}_{Y_1Y_2} = \sum_j \lambda^{ij}_{Y_1Y_2} = \sum_k \lambda^{k\ell}_{R_1R_2} = \sum_\ell \lambda^{k\ell}_{R_1R_2} = 0$.

The missing data mechanism assumed in Eq. (S2.1) is MCAR because the probability that an observation is missing does not depend on the unobserved realization of a random variable (Baker, Rosenberger and Dersimonian, 1992). That is, the parameterizations concerning the dependence of R_1 and R_2 on the two response variables, denoted as $\alpha_{ij} = \frac{\pi_{ij21}}{\pi_{ij11}}$ and $\beta_{ij} = \frac{\pi_{ij12}}{\pi_{ij11}}$, are both independent of both subscripts i and j:

$$\alpha_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2}],\ \beta_{ij} = \exp[\lambda^2_{R_1} - \lambda^1_{R_1} + \lambda^{21}_{R_1R_2} - \lambda^{11}_{R_1R_2}].$$

We denote this MCAR log-linear model as $(\alpha_{\cdot\cdot}, \beta_{\cdot\cdot})$.

We have also studied the property of the MCAR model $(\alpha_{\cdot\cdot}, \beta_{\cdot,\cdot})$ in Eq. (S2.1) in terms of the nonresponse odds, $\nu(j,j^{'})$ and $\omega(i,i^{'})$, and the response odds intervals, $OI^{\nu}(j,j^{'})$ and $OI^{\omega}(i,i^{'})$ (i.e., Eq (3.1), (3.2) and (3.3) in the paper). Since the model $(\alpha_{\cdot\cdot}, \beta_{\cdot,\cdot})$ has the same forms of $\nu_i(j,j')$, $\nu(j,j')$, $\omega_j(i,i')$ and $\omega(i,i')$ as the model $(\alpha_{i\cdot}, \beta_{\cdot j})$ in the absence of all $\lambda^{ik}_{Y_1R_1}$'s and $\lambda^{j\ell}_{Y_2R_2}$'s, both $H^{\nu}(j,j')$ in Table 8 of the paper and $H^{\omega}(i,i')$ in Table S12 of this Supplemental Material (Section S6) are 1. In addition, $M^{\nu}_m(j,j') > 1$ and $M^{\nu}_n(j,j') < 1$ for all pairs (j,j') of Y_2, and $M^{\omega}_m(i,i') > 1$ and $M^{\omega}_n(i,i') < 1$ for all pairs (i,i') of Y_1. These results mean that $\nu_m(j,j')/\nu(j,j') = H^{\nu}(j,j')M^{\nu}_m(j,j') > 1$ and $\nu_n(j,j')/\nu(j,j') = H^{\nu}(j,j')M^{\nu}_n(j,j') < 1$ for all pairs (j,j') of Y_2. Similarly, $\omega_m(i,i')/\omega(i,i') = H^{\omega}(i,i')M^{\omega}_m(i,i') > 1$ and $\omega_n(i,i')/\omega(i,i') = H^{\omega}(i,i')M^{\omega}_n(i,i') < 1$ for all pairs (i,i') of Y_1. Therefore, the MCAR log-linear model $(\alpha_{\cdot\cdot}, \beta_{\cdot,\cdot})$ has the following property:

Suppose that $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ for an $I \times I \times 2 \times 2$ table is modeled by the MCAR log-linear model, $(\alpha_{\cdot\cdot}, \beta_{\cdot\cdot})$. Then

$$\omega(i,i^{'}) \in OI^{\omega}(i,i^{'}) \text{ for any given pair } (i,i^{'}) \text{ of } Y_1,$$
$$\nu(j,j^{'}) \in OI^{\nu}(j,j^{'}) \text{ for any given pair } (j,j^{'}) \text{ of } Y_2.$$

Note that, for an $I \times J \times 2 \times 2$ table with $I \neq J$, the properties above still hold for the identifiable MCAR log-linear model.

The ML estimators of π_{ij11}, π_{i+12} and π_{+j21} under the MCAR log-linear model need to be obtained by numerically solving the likelihood equations because their closed forms are not available.

S2.2 MAR selection model

Any MAR model for an $I \times J \times 2 \times 2$ table must satisfy all the three conditions (Molenberghs *et al.*, 1999):

$$P(R_1 = 1, R_2 = 2 \mid Y_1 = i, Y_2 = j) = P(R_1 = 1, R_2 = 2 \mid Y_1 = i), \tag{S2.2}$$
$$P(R_1 = 2, R_2 = 1 \mid Y_1 = i, Y_2 = j) = P(R_1 = 2, R_2 = 1 \mid Y_2 = j), \tag{S2.3}$$
$$P(R_1 = 2, R_2 = 2 \mid Y_1 = i, Y_2 = j) = P(R_1 = 2, R_2 = 2). \tag{S2.4}$$

It can be shown that when any log-linear model for an $I \times J \times 2 \times 2$ table contains interaction term(s) between the two outcome variables (Y_1 and Y_2) and the two missingness variables (R_1 and R_2), such a model cannot satisfy at least two of the three conditions above and thus the assumed missing data mechanism cannot be MAR. That is, any MAR model for an $I \times J \times 2 \times 2$ table cannot be represented as a log-linear model.

Therefore, we have employed a selection model approach to the MAR mechanism for an $I \times J \times 2 \times 2$ table. That is, a MAR selection model decomposes the cell probabilities $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ as follows:

$$\pi_{ijk\ell} = \pi_{ij}\phi_{k,\ell|i,j} \tag{S2.5}$$

where $\pi_{ij} = Pr[Y_1 = i, Y_2 = j]$ and $\phi_{k,\ell|i,j} = Pr[R_1 = k, R_2 = \ell \mid Y_1 = i, Y_2 = j]$. The MAR mechanism for the missingness of Y_1 and Y_2 assumes that the probabilities of missingness depend on only observed outcomes (i.e., missingness of one variable is only affected by observed outcomes of the other variable). Then the nonresponse model in Eq. (S2.5) can be written as

$$\phi_{1,2|i,j} = \phi_{1,2|i},\ \phi_{2,1|i,j} = \phi_{2,1|j},\ \phi_{2,2|i,j} = \phi_{2,2},\ \phi_{1,1|i,j} = 1 - \phi_{1,2|i} - \phi_{2,1|j} - \phi_{2,2}.$$

Note that the MAR selection model in Eq. (S2.5) is saturated.

The log likelihood for $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ is represented as

$$\begin{aligned}
\ell &= \sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij11}\log\pi_{ij11} + \sum_{i=1}^{I} y_{i+12}\log\pi_{i+12} + \sum_{j=1}^{J} y_{+j21}\log\pi_{+j21} + y_{++22}\log\pi_{++22} \\
&= \sum_{i}^{I}\sum_{j}^{J} y_{ij11}\{\log\pi_{ij} + \log(1-\phi_{1,2|i}-\phi_{2,1|j}-\phi_{2,2})\} + \sum_{i}^{I} y_{i+12}\{\log\pi_{i+} + \log\phi_{1,2|i}\} \\
&+ \sum_{j}^{J} y_{+j21}\{\log\pi_{+j} + \log\phi_{2,1|j}\} + y_{++22}\log\phi_{2,2},
\end{aligned}$$

where

$$\begin{aligned}
\pi_{i+12} &= \sum_j \pi_{ij12} = \sum_j \pi_{ij}\phi_{1,2|i,j} = \sum_j \pi_{ij}\phi_{1,2|i} = \pi_{i+}\phi_{1,2|i}, \\
\pi_{+j21} &= \sum_i \pi_{ij21} = \sum_i \pi_{ij}\phi_{2,1|i,j} = \sum_i \pi_{ij}\phi_{2,1|j} = \pi_{+j}\phi_{2,1|j}, \\
\pi_{++22} &= \sum_i\sum_j \pi_{ij22} = \sum_i\sum_j \pi_{ij}\phi_{2,2|i,j} = \sum_i\sum_j \pi_{ij}\phi_{2,2} = \phi_{2,2}.
\end{aligned}$$

The closed form of the ML estimator for π_{ij} is unavailable, and Chen and Fienberg (1974) used an iterative scheme: the estimates of π_{ij} at the $(v+1)$th step, denoted by $\hat{\pi}_{ij}^{(v+1)}$, is

$$\hat{\pi}_{ij}^{(v+1)} = \frac{1}{N - y_{++22}}\left\{ y_{ij11} + y_{i+12}\frac{\hat{\pi}_{ij}^{(v)}}{\hat{\pi}_{i+}^{(v)}} + y_{+j12}\frac{\hat{\pi}_{ij}^{(v)}}{\hat{\pi}_{+j}^{(v)}} \right\}$$

where $N = \sum_{i,j,k,\ell} y_{ijk\ell}$. Using the ML estimates of π_{ij}, denoted by $\hat{\pi}_{ij}$, Molenberghs *et al.* (2008) provided the closed forms of the ML estimates of $\pi_{ijk\ell}$:

$$\hat{\pi}_{ij11} = \frac{y_{ij11}}{N},\quad \hat{\pi}_{ij12} = \frac{y_{i+12}}{N}\frac{\hat{\pi}_{ij}}{\hat{\pi}_{i+}},\quad \hat{\pi}_{ij21} = \frac{y_{+j21}}{N}\frac{\hat{\pi}_{ij}}{\hat{\pi}_{+j}},\quad \hat{\pi}_{ij22} = \frac{y_{++22}}{N}\hat{\pi}_{ij}.$$

Note that we could not study the property of the MAR selection model above with respect to the proposed nonresponse odds and the response odds intervals. This is because the conditional probabilities of missingness $\phi_{k,\ell|i,j}$ in the MAR selection model did not need any specific parameterization, unlike the nonignorable log-linear models in Table 2 of the paper and the nonignorable selection model presented in Section S8 of the Supplementary Material.

S3 ML estimators for π_{ij11}, π_{i+12} and π_{+j21} in Eq (4.1) and (4.2)

The ML estimators of π_{ij11}, π_{i+12} and π_{+j21} in Eq (4.1) and (4.2) of the the paper, denoted by $\hat{\pi}_{ij11}$, $\hat{\pi}_{i+12}$, and $\hat{\pi}_{+j21}$, can be obtained by the closed forms of the ML estimators for π_{ij11}, α_{ij} and β_{ij} in Eq (2.1) of the the paper, which are given in Baker, Rosenberger and Dersimonian (1992). Table S2 presents the closed forms of $\hat{\pi}_{ij11}$, $\hat{\pi}_{i+12}$ and $\hat{\pi}_{+j21}$ for the eight nonignorable log-linear models.

Table S2: Forms of $\hat{\pi}_{ij11}$, $\hat{\pi}_{i+12}$ and $\hat{\pi}_{+j21}$

Model	$\hat{\pi}_{ij11}$	$\hat{\pi}_{i+12}$	$\hat{\pi}_{+j21}$
$(\alpha_{i\cdot},\ \beta_{i\cdot})$, $(\alpha_{i\cdot},\ \beta_{\cdot j})$ $(\alpha_{\cdot j},\ \beta_{i\cdot})$, $(\alpha_{\cdot j},\ \beta_{\cdot j})$	y_{ij11}/N	y_{i+12}/N	y_{+j21}/N
$(\alpha_{\cdot\cdot},\ \beta_{i\cdot})$, $(\alpha_{\cdot\cdot},\ \beta_{\cdot j})$	$\frac{y_{ij11}}{N}\frac{y_{+j+1}y_{++11}}{y_{+j11}y_{+++1}}$	y_{i+12}/N	$\frac{\sum_i y_{ij11}}{N}\frac{y_{+j+1}y_{++21}}{y_{+j11}y_{+++1}}$
$(\alpha_{i\cdot},\ \beta_{\cdot\cdot})$, $(\alpha_{\cdot j},\ \beta_{\cdot\cdot})$	$\frac{y_{ij11}}{N}\frac{y_{i+1+}y_{++11}}{y_{i+11}y_{++1+}}$	$\frac{\sum_j y_{ij11}}{N}\frac{y_{i+1+}y_{++12}}{y_{i+11}y_{++1+}}$	y_{+j21}/N

S4 Simulation studies

Section S4.1 provides the parameter values used for the four saturated nonignorable log-linear models, $(\alpha_{\cdot j},\ \beta_{\cdot j})$, $(\alpha_{i\cdot},\ \beta_{i\cdot})$, $(\alpha_{\cdot j},\ \beta_{i\cdot})$ and $(\alpha_{i\cdot},\ \beta_{\cdot j})$, considered in the simulations studies of the paper. In Section S4.2, the simulation results for N=10000 are presented.

S4.1 Parameter values used in Section 5.1

Tables S3, S4, S5 and S6 show the parameter values used in the four simulation models $(\alpha_{\cdot j},\ \beta_{\cdot j})$, $(\alpha_{i\cdot},\ \beta_{i\cdot})$, $(\alpha_{\cdot j},\ \beta_{i\cdot})$ and $(\alpha_{i\cdot},\ \beta_{\cdot j})$ for the $2 \times 2 \times 2 \times 2$ table. Note that the parameter values are chosen so that the response rate and the nonresponse rate are about 77.5% and 22.5%, respectively (i.e., $\pi_{++11} \approx 0.775$, $\pi_{++12} \approx 0.1$, $\pi_{++21} \approx 0.1$ and $\pi_{++22} \approx 0.025$), and the odds ratio between Y_1 and Y_2 for the respondents ($R_1 = R_2 = 1$) is 2.23.

The magnitude of the nonignorable missingness assumed in each model is determined by the magnitudes of two interactions between (Y_1, Y_2) and (R_1, R_2), each with four levels, 0.05, 0.1, 0.2 and 0.4. The odds ratios of the corresponding values of the interactions are 1.22, 1.49, 2.23 and 4.95, respectively. For example, for the model $(\alpha_{\cdot j},\ \beta_{\cdot j})$ with both $\lambda^{11}_{Y_2R_1}$ and $\lambda^{11}_{Y_2R_2}$, there are 16 pairs of $(\lambda^{11}_{Y_2R_1},\ \lambda^{11}_{Y_2R_2})$ and the odds ratios of the interactions $\lambda^{11}_{Y_2R_1}$ and $\lambda^{11}_{Y_2R_2}$ are $\frac{\pi_{1111}/\pi_{1211}}{\pi_{1121}/\pi_{1221}}$ and $\frac{\pi_{1111}/\pi_{1211}}{\pi_{1112}/\pi_{1212}}$, respectively.

Table S3: Parameter values for the model $(\alpha_{\cdot j},\ \beta_{\cdot j})$

$\lambda^1_{Y_1}$	$\lambda^1_{Y_2}$	$\lambda^1_{R_1}$	$\lambda^1_{R_2}$	$\lambda^{11}_{Y_1Y_2}$	$\lambda^{11}_{R_1R_2}$	$\lambda^{11}_{Y_2R_1}$	$\lambda^{11}_{Y_2R_2}$
0.2	0	0.89	0.9	0.2	0.1	0.05	0.05
0.2	-0.05	0.89	0.9	0.2	0.1	0.1	0.05
0.2	-0.15	0.91	0.9	0.2	0.1	0.2	0.05
0.2	-0.35	0.99	0.9	0.2	0.1	0.4	0.05
0.2	-0.05	0.89	0.9	0.2	0.1	0.05	0.1
0.2	-0.1	0.89	0.9	0.2	0.1	0.1	0.1
0.2	-0.2	0.91	0.9	0.2	0.1	0.2	0.1
0.2	-0.4	1	0.9	0.2	0.1	0.4	0.1
0.2	-0.15	0.88	0.93	0.2	0.1	0.05	0.2
0.2	-0.2	0.88	0.93	0.2	0.1	0.1	0.2
0.2	-0.3	0.905	0.93	0.2	0.1	0.2	0.2
0.2	-0.5	1	0.93	0.2	0.1	0.4	0.2
0.2	-0.35	0.89	1	0.2	0.1	0.05	0.4
0.2	-0.4	0.9	1	0.2	0.1	0.1	0.4
0.2	-0.5	0.92	1	0.2	0.1	0.2	0.4
0.2	-0.7	1.03	1	0.2	0.1	0.4	0.4

Table S4: Parameter values for the model $(\alpha_{i\cdot},\ \beta_{i\cdot})$

$\lambda^1_{Y_1}$	$\lambda^1_{Y_2}$	$\lambda^1_{R_1}$	$\lambda^1_{R_2}$	$\lambda^{11}_{Y_1Y_2}$	$\lambda^{11}_{R_1R_2}$	$\lambda^{11}_{Y_1R_1}$	$\lambda^{11}_{Y_1R_2}$
0.1	0.1	0.89	0.89	0.2	0.1	0.05	0.05
0.05	0.1	0.89	0.89	0.2	0.1	0.1	0.05
-0.05	0.1	0.9	0.89	0.2	0.1	0.2	0.05
-0.25	0.1	0.97	0.89	0.2	0.1	0.4	0.05
0.05	0.1	0.88	0.89	0.2	0.1	0.05	0.1
0	0.1	0.88	0.89	0.2	0.1	0.1	0.1
-0.1	0.1	0.9	0.89	0.2	0.1	0.2	0.1
-0.3	0.1	0.97	0.89	0.2	0.1	0.4	0.1
-0.05	0.1	0.89	0.9	0.2	0.1	0.05	0.2
-0.1	0.1	0.89	0.9	0.2	0.1	0.1	0.2
-0.2	0.1	0.905	0.9	0.2	0.1	0.2	0.2
-0.4	0.1	0.985	0.9	0.2	0.1	0.4	0.2
-0.25	0.1	0.88	0.98	0.2	0.1	0.05	0.4
-0.3	0.1	0.88	0.98	0.2	0.1	0.1	0.4
-0.4	0.1	0.905	0.98	0.2	0.1	0.2	0.4
-0.6	0.1	0.99	0.98	0.2	0.1	0.4	0.4

Table S5: Parameter values for the model $(\alpha_{\cdot j},\ \beta_{i\cdot})$

$\lambda^1_{Y_1}$	$\lambda^1_{Y_2}$	$\lambda^1_{R_1}$	$\lambda^1_{R_2}$	$\lambda^{11}_{Y_1Y_2}$	$\lambda^{11}_{R_1R_2}$	$\lambda^{11}_{Y_2R_1}$	$\lambda^{11}_{Y_1R_2}$
0.15	0.05	0.89	0.89	0.2	0.1	0.05	0.05
0.15	0	0.89	0.89	0.2	0.1	0.1	0.05
0.15	-0.1	0.91	0.89	0.2	0.1	0.2	0.05
0.15	-0.3	0.99	0.89	0.2	0.1	0.4	0.05
0.1	0.05	0.89	0.89	0.2	0.1	0.05	0.1
0.1	0	0.89	0.89	0.2	0.1	0.1	0.1
0.1	-0.1	0.9	0.89	0.2	0.1	0.2	0.1
0.1	-0.3	0.99	0.89	0.2	0.1	0.4	0.1
0	0.05	0.89	0.9	0.2	0.1	0.05	0.2
0	0	0.89	0.9	0.2	0.1	0.1	0.2
0	-0.1	0.9	0.9	0.2	0.1	0.2	0.2
0	-0.3	0.99	0.9	0.2	0.1	0.4	0.2
-0.2	0.05	0.88	0.98	0.2	0.1	0.05	0.4
-0.2	0	0.88	0.98	0.2	0.1	0.1	0.4
-0.2	-0.1	0.9	0.98	0.2	0.1	0.2	0.4
-0.2	-0.3	0.99	0.98	0.2	0.1	0.4	0.4

Table S6: Parameter values for the model $(\alpha_{i\cdot},\ \beta_{\cdot j})$

$\lambda^1_{Y_1}$	$\lambda^1_{Y_2}$	$\lambda^1_{R_1}$	$\lambda^1_{R_2}$	$\lambda^{11}_{Y_1Y_2}$	$\lambda^{11}_{R_1R_2}$	$\lambda^{11}_{Y_1R_1}$	$\lambda^{11}_{Y_2R_2}$
0.15	0.05	0.9	0.88	0.2	0.1	0.05	0.05
0.1	0.05	0.9	0.88	0.2	0.1	0.1	0.05
0	0.05	0.905	0.88	0.2	0.1	0.2	0.05
-0.2	0.05	0.985	0.88	0.2	0.1	0.4	0.05
0.15	0	0.9	0.88	0.2	0.1	0.05	0.1
0.1	0	0.9	0.88	0.2	0.1	0.1	0.1
0	0	0.905	0.88	0.2	0.1	0.2	0.1
-0.2	0	0.985	0.88	0.2	0.1	0.4	0.1
0.15	-0.1	0.9	0.9	0.2	0.1	0.05	0.2
0.1	-0.1	0.9	0.9	0.2	0.1	0.1	0.2
0	-0.1	0.905	0.9	0.2	0.1	0.2	0.2
-0.2	-0.1	0.985	0.9	0.2	0.1	0.4	0.2
0.15	-0.3	0.89	0.99	0.2	0.1	0.05	0.4
0.1	-0.3	0.89	0.99	0.2	0.1	0.1	0.4
0	-0.3	0.905	0.99	0.2	0.1	0.2	0.4
-0.2	-0.3	0.98	0.99	0.2	0.1	0.4	0.4

S4.2 Simulation results for N=10000 in Section 5.1

Table S7 shows the number of the cases in which each of the four saturated models was chosen by AIC, BIC, G^2, and the proposed method in Corollary 1 when 1000 tables of size N=10000 are generated from each of the four saturated simulation models (i.e., four types of nonignorable missingness).

Table S7: Number of the cases where each of the four saturated models was chosen by AIC, BIC, G^2 and the proposed method in Corollary 1 under the four simulation models with N=10000.

Simulation $(\alpha_{i\cdot},\beta_{i\cdot})$		Fitted saturated models				Simulation $(\alpha_{\cdot j},\beta_{\cdot j})$		Fitted saturated models			
$\lambda^{11}_{Y_1R_1}$	$\lambda^{11}_{Y_1R_2}$	$(\alpha_{i\cdot},\beta_{i\cdot})$	$(\alpha_{i\cdot},\beta_{\cdot j})$	$(\alpha_{\cdot j},\beta_{i\cdot})$	$(\alpha_{\cdot j},\beta_{\cdot j})$	$\lambda^{11}_{Y_2R_1}$	$\lambda^{11}_{Y_2R_2}$	$(\alpha_{i\cdot},\beta_{i\cdot})$	$(\alpha_{i\cdot},\beta_{\cdot j})$	$(\alpha_{\cdot j},\beta_{i\cdot})$	$(\alpha_{\cdot j},\beta_{\cdot j})$
0.05	0.05	1000	1000	1000	1000	0.05	0.05	999	999	1000	1000
0.1	0.05	1000	1000	1000	1000	0.1	0.05	882	882	1000	1000
0.2	0.05	1000	999	1000	999	0.2	0.05	0	0	1000	1000
0.4	0.05	1000	1000	1000	1000	0.4	0.05	0	0	1000	1000
0.05	0.1	1000	718	1000	718	0.05	0.1	1000	1000	1000	1000
0.1	0.1	1000	732	1000	732	0.1	0.1	870	870	1000	1000
0.2	0.1	1000	708	1000	708	0.2	0.1	0	0	1000	1000
0.4	0.1	1000	716	1000	716	0.4	0.1	0	0	1000	1000
0.05	0.2	1000	0	1000	0	0.05	0.2	1000	1000	1000	1000
0.1	0.2	1000	0	1000	0	0.1	0.2	862	862	1000	1000
0.2	0.2	1000	0	1000	0	0.2	0.2	0	0	1000	1000
0.4	0.2	1000	0	1000	0	0.4	0.2	0	0	1000	1000
0.05	0.4	1000	0	1000	0	0.05	0.4	1000	1000	1000	1000
0.1	0.4	1000	0	1000	0	0.1	0.4	852	852	1000	1000
0.2	0.4	1000	0	1000	0	0.2	0.4	0	0	1000	1000
0.4	0.4	1000	0	1000	0	0.4	0.4	0	0	1000	1000

Simulation $(\alpha_{i\cdot},\beta_{\cdot j})$		Fitted saturated models				Simulation $(\alpha_{\cdot j},\beta_{i\cdot})$		Fitted saturated models			
$\lambda^{11}_{Y_1R_1}$	$\lambda^{11}_{Y_2R_2}$	$(\alpha_{i\cdot},\beta_{i\cdot})$	$(\alpha_{i\cdot},\beta_{\cdot j})$	$(\alpha_{\cdot j},\beta_{i\cdot})$	$(\alpha_{\cdot j},\beta_{\cdot j})$	$\lambda^{11}_{Y_2R_1}$	$\lambda^{11}_{Y_1R_2}$	$(\alpha_{i\cdot},\beta_{i\cdot})$	$(\alpha_{i\cdot},\beta_{\cdot j})$	$(\alpha_{\cdot j},\beta_{i\cdot})$	$(\alpha_{\cdot j},\beta_{\cdot j})$
0.05	0.05	1000	1000	1000	1000	0.05	0.05	1000	1000	1000	1000
0.1	0.05	1000	1000	1000	1000	0.1	0.05	846	845	1000	999
0.2	0.05	1000	1000	1000	1000	0.2	0.05	0	0	1000	1000
0.4	0.05	1000	1000	1000	1000	0.4	0.05	0	0	1000	1000
0.05	0.1	1000	1000	1000	1000	0.05	0.1	1000	708	1000	708
0.1	0.1	1000	1000	1000	1000	0.1	0.1	857	629	1000	719
0.2	0.1	1000	1000	1000	1000	0.2	0.1	0	0	1000	712
0.4	0.1	1000	1000	1000	1000	0.4	0.1	0	0	1000	712
0.05	0.2	1000	1000	1000	1000	0.05	0.2	1000	0	1000	0
0.1	0.2	1000	1000	1000	1000	0.1	0.2	861	0	1000	0
0.2	0.2	1000	1000	1000	1000	0.2	0.2	0	0	1000	0
0.4	0.2	1000	1000	1000	1000	0.4	0.2	0	0	1000	0
0.05	0.4	1000	1000	1000	1000	0.05	0.4	1000	0	1000	0
0.1	0.4	1000	1000	1000	1000	0.1	0.4	864	0	1000	0
0.2	0.4	1000	1000	1000	1000	0.2	0.4	0	0	1000	0
0.4	0.4	1000	1000	1000	1000	0.4	0.4	0	0	1000	0

S5 Data analysis

This section presents the results of real data analysis omitted in Section 5.2 of the paper and an additional real data example.

S5.1 Bone mineral density and family income

Table S8 presents the estimates for the expected counts $\hat{\mathbf{m}} = \{\hat{m}_{ijk\ell}\}$ under the eight nonignorable log-linear models in Table 2, and two ignorable nonresponse models (MAR selection model and MCAR log-linear model). Note that the log-linear models with either one of $\alpha_{i\cdot}$ and $\beta_{\cdot j}$, or both produced undesirable results such as poor model fit to the observed data (i.e., m_{ij11}) and/or occurrence of nonresponse boundary solutions in estimation. When the models with $\alpha_{i\cdot}$ are fitted, $\hat{m}_{i+2+} = \sum_{j,\ell} \hat{m}_{ij2\ell} = 0$ for at least one and at most two values of Y_1. Similarly, $\hat{m}_{+j+2} = \sum_{i,k} \hat{m}_{ijk2} = 0$ for at least one and at most two values of Y_2 when the models with $\beta_{\cdot j}$ are used. Note that the two models with perfect fit to observed data are $(\alpha_{\cdot j}, \beta_{i\cdot})$ and the MAR selection model although they produced different predictions of the unobserved outcomes (i.e., different values of $\hat{m}_{ij12}$, $\hat{m}_{ij21}$ and $\hat{m}_{ij22}$).

S5.2 Smoking and birth weight

An incomplete 2×2 table in Table S9 is the data from a study of pregnant women to investigate the association between perinatal factors and the subsequent development and course of abnormalities in the offspring (Baker, Rosenberger and Dersimonian, 1992). This table is classified by newborn's weight Y_1 and mother's smoking status Y_2 with supplemental margins.

To assess the types of nonignorable mechanism assumed in the saturated log-linear models for the data, we check the two conditions C_1 and C_2 of Corollary 1 by computing $\hat{\nu}_i(j, j')$ and $\hat{\nu}(j, j')$ for the models $(\alpha_{i\cdot},\ \beta_{\square})$ and $(\alpha_{\cdot j},\ \beta_{\square})$, and $\hat{\omega}_j(i, i')$ and $\hat{\omega}(i, i')$ for the models $(\alpha_{\square},\ \beta_{i\cdot})$ and $(\alpha_{\square},\ \beta_{\cdot j})$. We found that the condition C_1 does not hold, but C_2 holds: $\hat{\nu}(1,2)$=1049/1135=0.924 $\in \widehat{OI}^{\nu}(1,2)$=(21009/24132, 4512/3394)=(0.871, 1.329) for a pair $(1,2)$ of Y_2, but $\hat{\omega}(1,2)$=142/464=0.306 $\notin \widehat{OI}^{\omega}(1,2)$=(3394/24132, 4512/21009)=(0.141, 0.215) for a pair (1,2) of Y_1. By Corollary 1, thus, two saturated models are plausible, $(\alpha_{i\cdot},\ \beta_{i\cdot})$ and $(\alpha_{\cdot j},\ \beta_{i\cdot})$.

To assess the uncertainty of the accuracy of the proposed method, we perform the bootstrap resampling by generating 100,000 samples from each selected saturated model. We then compute the percentages of bootstrap samples satisfying the conditions C_1 and C_2 in Corollary 1 and the values are (C_1, C_2) = (8.68,99.97) for $(\alpha_{i\cdot},\ \beta_{i\cdot})$ and (8.83, 99.97) for $(\alpha_{\cdot j},\ \beta_{i\cdot})$. These results confirm the accuracy of the proposed method.

The result of the proposed method above also is consistent with the model selection results of G^2, AIC and BIC, as shown in Table S10. That is, the two saturated models $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$ have a zero value of G^2 and they have the smallest values of AIC and BIC among all saturated models.

Table S8: Estimates for the expected counts $\hat{\mathbf{m}} = \{\hat{m}_{ijk\ell}\}$ for BMD and FI data

Model			R_2=1			R_2=2		
			Y_2=1	Y_2=2	Y_2=3	Y_2=1	Y_2=2	Y_2=3
$(\alpha_{i\cdot}, \beta_{i\cdot})$	R_1=1	Y_1=1	620.8	257.1	317.1	70.1	29.0	35.8
		Y_1=2	260.0	131.0	117.0	35.3	17.8	15.9
		Y_1=3	93.0	30.0	18.0	17.8	5.7	3.4
	R_1=2	Y_1=1	456.1	188.9	232.9	23.4	9.7	11.9
		Y_1=2	0.0	0.0	0.0	0.0	0.0	0.0
		Y_1=3	0.0	0.0	0.0	0.0	0.0	0.0
$(\alpha_{\cdot j}, \beta_{\cdot j})$	R_1=1	Y_1=1	611.1	290.0	284.0	144.9	0.0	0.0
		Y_1=2	265.9	131.0	117.0	63.1	0.0	0.0
		Y_1=3	97.0	30.0	18.0	23.0	0.0	0.0
	R_1=2	Y_1=1	286.1	100.3	180.3	28.2	0.0	0.0
		Y_1=2	124.5	45.3	74.3	12.3	0.0	0.0
		Y_1=3	45.4	10.4	11.4	4.5	0.0	0.0
$(\alpha_{i\cdot}, \beta_{\cdot j})$	R_1=1	Y_1=1	613.3	255.8	315.9	134.8	10.2	0.0
		Y_1=2	265.1	131.5	117.0	58.3	5.2	0.0
		Y_1=3	97.2	30.2	18.0	21.4	1.2	0.0
	R_1=2	Y_1=1	454.4	189.5	234.1	41.8	3.2	0.0
		Y_1=2	0.0	0.0	0.0	0.0	0.0	0.0
		Y_1=3	0.0	0.0	0.0	0.0	0.0	0.0
$(\alpha_{\cdot j}, \beta_{i\cdot})$	R_1=1	Y_1=1	621.0	290.0	284.0	70.2	32.8	32.1
		Y_1=2	260.0	131.0	117.0	35.3	17.8	15.9
		Y_1=3	93.0	30.0	18.0	17.8	5.7	3.4
	R_1=2	Y_1=1	290.7	100.3	180.3	13.5	4.6	8.3
		Y_1=2	121.7	45.3	74.3	6.8	2.5	4.1
		Y_1=3	43.5	10.4	11.4	3.4	0.8	0.9
$(\alpha_{\cdot\cdot}, \beta_{i\cdot})$	R_1=1	Y_1=1	617.6	264.4	314.5	69.7	29.8	35.5
		Y_1=2	258.6	119.4	129.6	35.2	16.2	17.6
		Y_1=3	92.5	27.4	19.9	17.9	5.3	3.9
	R_1=2	Y_1=1	294.1	125.9	149.8	13.6	5.8	6.9
		Y_1=2	123.1	56.9	61.7	6.8	3.2	3.4
		Y_1=3	44.0	13.0	9.5	3.5	1.0	0.8
$(\alpha_{\cdot\cdot}, \beta_{\cdot j})$	R_1=1	Y_1=1	607.8	264.4	314.5	144.9	0.0	0.0
		Y_1=2	264.5	119.4	129.6	63.1	0.0	0.0
		Y_1=3	96.5	27.4	19.9	23.0	0.0	0.0
	R_1=2	Y_1=1	289.4	125.9	149.8	28.2	0.0	0.0
		Y_1=2	125.9	56.9	61.7	12.3	0.0	0.0
		Y_1=3	45.9	13.0	9.5	4.5	0.0	0.0
$(\alpha_{i\cdot}, \beta_{\cdot\cdot})$	R_1=1	Y_1=1	614.1	254.3	313.6	76.9	31.9	39.3
		Y_1=2	262.4	132.2	118.1	32.9	16.6	14.8
		Y_1=3	98.5	31.8	19.1	12.3	4.0	2.4
	R_1=2	Y_1=1	456.1	188.9	232.9	23.4	9.7	11.9
		Y_1=2	0.0	0.0	0.0	0.0	0.0	0.0
		Y_1=3	0.0	0.0	0.0	0.0	0.0	0.0
$(\alpha_{\cdot j}, \beta_{\cdot\cdot})$	R_1=1	Y_1=1	614.2	286.8	280.9	76.9	35.9	35.2
		Y_1=2	262.4	132.2	118.1	32.9	16.6	14.8
		Y_1=3	98.5	31.8	19.1	12.3	4.0	2.4
	R_1=2	Y_1=1	287.2	99.3	178.7	14.7	5.1	9.2
		Y_1=2	122.7	45.8	75.1	6.3	2.3	3.9
		Y_1=3	46.0	11.0	12.1	2.4	0.6	0.6
MAR	R_1=1	Y_1=1	621.0	290.0	284.0	69.7	29.9	35.5
		Y_1=2	260.0	131.0	117.0	35.1	16.2	17.6
		Y_1=3	93.0	30.0	18.0	17.9	5.3	3.9
	R_1=2	Y_1=1	287.2	99.3	178.6	14.9	6.4	7.6
		Y_1=2	122.7	45.7	75.2	6.4	2.9	3.2
		Y_1=3	46.1	11.0	12.2	2.4	0.7	0.5
$(\alpha_{\cdot\cdot}, \beta_{\cdot\cdot})$	R_1=1	Y_1=1	610.6	261.8	311.0	76.5	32.8	39.0
		Y_1=2	260.9	120.5	131.1	32.7	15.1	16.4
		Y_1=3	98.0	28.9	21.3	12.3	3.6	2.7
	R_1=2	Y_1=1	290.7	124.6	148.1	14.9	6.4	7.6
		Y_1=2	124.2	57.4	62.4	6.4	2.9	3.2
		Y_1=3	46.7	13.7	10.1	2.4	0.7	0.5

For the significance of the interaction parameters in the selected saturated models, we fit the two selected saturated models, $(\alpha_{i\cdot},\ \beta_{i\cdot})$ and $(\alpha_{\cdot j},\ \beta_{i\cdot})$, and the corresponding nested models, $(\alpha_{\cdot\cdot},\ \beta_{i\cdot})$, $(\alpha_{i\cdot},\ \beta_{\cdot\cdot})$ and $(\alpha_{\cdot j},\ \beta_{\cdot\cdot})$. From the results of LRT, G^2, AIC and BIC, the nested model $(\alpha_{\cdot\cdot},\ \beta_{i\cdot})$ is also plausible. Note that the other two nested models $(\alpha_{i\cdot},\ \beta_{\cdot\cdot})$ and $(\alpha_{\cdot j},\ \beta_{\cdot\cdot})$ have large values of G^2, AIC and BIC, compared to $(\alpha_{\cdot\cdot},\ \beta_{i\cdot})$. The results from these

Table S9: Birth weight (Y_1) and Smoking status (Y_2)

		$R_2 = 1$		$R_2 = 2$
		$Y_2 = 1$ (smoker)	$Y_2 = 2$ (non-smoker)	Y_2
$R_1 = 1$	$Y_1 = 1$ (< 2500 grams)	4512	3394	142
	$Y_1 = 2$ (≥ 2500 grams)	21009	24132	464
$R_1 = 2$	Y_1	1049	1135	1224

analyses indicate that the models ($\alpha_{\square}$, $\beta_{i\cdot}$) are suitable for the data. Note that the models ($\alpha_{\square}$, $\beta_{\cdot j}$) that are not considered by the proposed method have large values of G^2 (due to poor fits to observed data or occurrence of the boundary solutions for the estimates for the expected counts). The estimates for the expected counts under the eight log-linear models in Table 1 are provided in Table S11.

Table S10: Model selection for smoking data. Here, (C_1, C_2) represents the percentage of bootstrap samples satisfying the conditions C_1 and C_2 in Corollary 1

Saturated model	Nested model	Proposed method (C_1, C_2)	P-value for LRT	G^2	AIC	BIC
$(\alpha_{i\cdot}, \beta_{\cdot j})$				12.5	158,451.0	158,451.7
	$(\alpha_{i\cdot}, \beta_{\cdot\cdot})$		< 0.001	30.1	158,466.7	158,467.3
	$(\alpha_{\cdot\cdot}, \beta_{\cdot j})$		0.888	12.5	158,449.1	158,449.6
$(\alpha_{\cdot j}, \beta_{i\cdot})$		✓ (8.83, 99.97)		**0**	**158,438.6**	**158,439.2**
	$(\alpha_{\cdot j}, \beta_{\cdot\cdot})$	✓	< 0.001	30.1	158,466.7	158,467.3
	$(\alpha_{\cdot\cdot}, \beta_{i\cdot})$	✓	**0.888**	**0.02**	**158,436.6**	**158,437.2**
$(\alpha_{i\cdot}, \beta_{i\cdot})$		✓(8.68, 99.97)		**0**	**158,438.6**	**158,439.2**
	$(\alpha_{i\cdot}, \beta_{\cdot\cdot})$	✓	< 0.001	30.1	158,466.7	158,467.3
	$(\alpha_{\cdot\cdot}, \beta_{i\cdot})$	✓	**0.888**	**0.02**	**158,436.6**	**158,437.2**
$(\alpha_{\cdot j}, \beta_{\cdot j})$				12.5	158,451.0	158,451.7
	$(\alpha_{\cdot j}, \beta_{\cdot\cdot})$		< 0.001	30.1	158,466.7	158,467.3
	$(\alpha_{\cdot\cdot}, \beta_{\cdot j})$		0.888	12.5	158,449.1	158,449.6

We also fitted two ignorable nonresponse models to the data, the MCAR log-linear model ($\alpha_{\cdot\cdot}$, $\beta_{\cdot\cdot}$) and the MAR selection model. The values of the G^2, AIC and BIC are (30.12, 158,464.7, 158,465.2) for ($\alpha_{\cdot\cdot}$, $\beta_{\cdot\cdot}$) and (0, 158,438.6, 158,439.2) for the MAR model, which clearly shows that the MAR model performs better than the MCAR model. The estimates for the expected counts under the MCAR and MAR models are given in Table S11.

From all the analysis above, we can see that the plausible nonignorable log-linear models are ($\alpha_{\square}$, $\beta_{i\cdot}$). We also observe that the MAR selection model has the same fit to the data as the saturated nonignorable log-linear models selected by our proposed method, $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$, with respect to G^2, AIC, BIC and the estimates for the observed data, $\hat{m}_{ij11}$. Thus, the fitted MAR model is the MAR counterpart of the selected saturated log-linear models $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$.

Table S11: Estimates for the expected counts $\hat{\mathbf{m}} = \{\hat{m}_{ijk\ell}\}$ for smoking and birthweight data

Model			$R_2 = 1$		$R_2 = 2$	
			$Y_2 = 1$	$Y_2 = 2$	$Y_2 = 1$	$Y_2 = 2$
$(\alpha_{i\cdot}, \beta_{i\cdot})$	$R_1 = 1$	$Y_1 = 1$	4512.0	3394.0	81.0	61.0
		$Y_1 = 2$	21009.0	24132.0	215.9	248.1
	$R_1 = 2$	$Y_1 = 1$	176.3	132.6	156.2	117.5
		$Y_1 = 2$	872.7	1002.4	442.3	508.1
$(\alpha_{\cdot j}, \beta_{\cdot j})$	$R_1 = 1$	$Y_1 = 1$	4546.1	3394.0	107.9	0.0
		$Y_1 = 2$	20974.9	24132.0	498.0	0.0
	$R_1 = 2$	$Y_1 = 1$	186.9	139.9	218.0	0.0
		$Y_1 = 2$	862.1	995.1	1005.9	0.0
$(\alpha_{i\cdot}, \beta_{\cdot j})$	$R_1 = 1$	$Y_1 = 1$	4546.1	3394.0	107.9	0.0
		$Y_1 = 2$	20975.0	24132.0	498.0	0.0
	$R_1 = 2$	$Y_1 = 1$	177.9	132.8	207.5	0.0
		$Y_1 = 2$	871.1	1002.2	1016.4	0.0
$(\alpha_{\cdot j}, \beta_{i\cdot})$	$R_1 = 1$	$Y_1 = 1$	4512.0	3394.0	81.0	61.0
		$Y_1 = 2$	21009.0	24132.0	215.9	248.1
	$R_1 = 2$	$Y_1 = 1$	185.5	139.9	163.4	123.3
		$Y_1 = 2$	863.5	995.1	435.5	501.8
$(\alpha_{\cdot\cdot}, \beta_{i\cdot})$	$R_1 = 1$	$Y_1 = 1$	4511.7	3394.2	81.0	61.0
		$Y_1 = 2$	21007.6	24133.5	215.9	248.1
	$R_1 = 2$	$Y_1 = 1$	185.8	139.7	163.7	123.1
		$Y_1 = 2$	864.9	993.6	436.1	501.0
$(\alpha_{\cdot\cdot}, \beta_{\cdot j})$	$R_1 = 1$	$Y_1 = 1$	4545.8	3394.2	107.9	0.0
		$Y_1 = 2$	20973.6	24133.5	498.0	0.0
	$R_1 = 2$	$Y_1 = 1$	187.2	139.7	218.0	0.0
		$Y_1 = 2$	863.5	993.6	1005.9	0.0
$(\alpha_{i\cdot}, \beta_{\cdot\cdot})$	$R_1 = 1$	$Y_1 = 1$	4541.2	3415.9	51.9	39.0
		$Y_1 = 2$	20985.2	24104.7	239.7	275.4
	$R_1 = 2$	$Y_1 = 1$	176.5	132.8	98.9	74.4
		$Y_1 = 2$	872.5	1002.2	489.0	561.7
$(\alpha_{\cdot j}, \beta_{\cdot\cdot})$	$R_1 = 1$	$Y_1 = 1$	4541.2	3415.9	51.9	39.0
		$Y_1 = 2$	20985.2	24104.7	239.7	275.4
	$R_1 = 2$	$Y_1 = 1$	186.6	140.9	104.6	79.0
		$Y_1 = 2$	862.4	994.1	483.3	557.1
MAR	$R_1 = 1$	$Y_1 = 1$	4512.0	3394.0	81.0	61.0
		$Y_1 = 2$	21009.0	24132.0	215.9	248.1
	$R_1 = 2$	$Y_1 = 1$	186.6	140.9	104.8	78.8
		$Y_1 = 2$	862.4	994.1	484.2	556.2
$(\alpha_{\cdot\cdot}, \beta_{\cdot\cdot})$	$R_1 = 1$	$Y_1 = 1$	4540.8	3416.2	51.9	39.0
		$Y_1 = 2$	20983.7	24106.3	239.7	275.4
	$R_1 = 2$	$Y_1 = 1$	187.0	140.6	104.8	78.8
		$Y_1 = 2$	863.9	992.5	484.2	556.2

S6 Proof of Theorem 1 in Section 3

In this section, the proofs of Theorem 1-3) and 4) in Section 3 of the paper are provided. To prove them, we first establish the following lemma, and then accomplish the proof based on the lemma.

Lemma 1. *For a $I \times I \times 2 \times 2$ incomplete contingency table, the following inequalities hold under all nonresponse models in Table 2 of the the paper. For all pairs (i, i') of Y_1,*

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} \text{ and } \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2}$$

where m and n are subscripts corresponding to $\omega_m(i, i')$ and $\omega_n(i, i')$, respectively.

Proof. For sake of space, we only show the proof for the model $(\alpha_{i\cdot}, \beta_{\cdot j})$, as the proofs for the other models are similar. Suppose that $\pi_{ijk\ell}$ is parameterized by the model $(\alpha_{i\cdot}, \beta_{\cdot j})$. Then, $\omega_j(i, i')$ can be expressed by

$$\omega_j(i, i') = \frac{\pi_{ij11}}{\pi_{i'j11}} = \exp\left(\lambda^{i}_{Y_1} - \lambda^{i'}_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{i'j}_{Y_1Y_2} + \lambda^{i1}_{Y_1R_1} - \lambda^{i'1}_{Y_1R_1}\right) \tag{S6.1}$$

for each pair (i, i') of Y_1. Comparing $\omega_j(i, i')$, $\omega_m(i, i')$ and $\omega_n(i, i')$ in terms of Eq. (S6.1) gives $\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2}$ because $\omega_m(i, i') \geq \omega_j(i, i')$ for all j by the definition of $\omega_m(i, i')$. Similarly, we also have $\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2}$ because $\omega_n(i, i') \leq \omega_j(i, i')$ for all j by the definition of $\omega_n(i, i')$. □

We are now ready to prove Theorem 1-3) and 4).

Proof. For all models in Table 2 of the the paper, let $R^{\omega}_m(i, i') = \omega_m(i, i')/\omega(i, i')$, and $R^{\omega}_n(i, i') = \omega_n(i, i')/\omega(i, i')$. Then, $R^{\omega}_m(i, i')$ and $R^{\omega}_n(i, i')$ are decomposed into two parts according to each model in Table S12 : $R^{\omega}_m(i, i') = H^{\omega}(i, i')M^{\omega}_m(i, i')$ and $R^{\omega}_n(i, i') = H^{\omega}(i, i')M^{\omega}_n(i, i')$. By Lemma 1, for all eight models, $M^{\omega}_m(i, i') > 1$ and $M^{\omega}_n(i, i') < 1$ for all pairs (i, i') of Y_1. Note that $M^{\omega}_m(i, i')$ and $M^{\omega}_n(i, i')$ are equal to $R^{\omega}_m(i, i')$ and $R^{\omega}_n(i, i')$, respectively, under the assumption of $\lambda^{i\ell}_{Y_1R_2} = 0$ for all i's (which makes $H^{\omega}(i, i')$ equal to 1 for all i's).

Under the models $(\alpha_{\square}, \beta_{i\cdot})$ (i.e., $(\alpha_{\cdot j}, \beta_{i\cdot})$, $(\alpha_{i\cdot}, \beta_{i\cdot})$, and $(\alpha_{\cdot\cdot}, \beta_{i\cdot})$), we show below the necessary and sufficient condition for $\omega(i, i') \in OI^{\omega}(i, i')$. First, $\omega(i, i') \in OI^{\omega}(i, i')$ is the same as "$R^{\omega}_m(i, i') > 1$ and $R^{\omega}_n(i, i') < 1$". Since $R^{\omega}_m(i, i') = H^{\omega}(i, i')M^{\omega}_m(i, i')$ and $R^{\omega}_n(i, i') = H^{\omega}(i, i')M^{\omega}_n(i, i')$, "$R^{\omega}_m(i, i') > 1$ and $R^{\omega}_n(i, i') < 1$" is equivalent to that $H^{\omega}(i, i')$ is larger than $M^{\omega}_m(i, i')^{-1}$ and less than $M^{\omega}_n(i, i')^{-1}$, and thus $-\log M^{\omega}_m(i, i') < 2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) = \log H^{\omega}(i, i') < -\log M^{\omega}_n(i, i')$. Since $\omega(i, i') \notin OI^{\omega}(i, i')$ is the complement of $\omega(i, i') \in OI^{\omega}(i, i')$, it is straightforward to show that the necessary and sufficient condition for $\omega(i, i') \notin OI^{\omega}(i, i')$ is "$\log H^{\omega}(i, i') = 2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) < -\log M^{\omega}_m(i, i')$ or $\log H^{\omega}(i, i') = 2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) > -\log M^{\omega}_n(i, i')$".

For the models $(\alpha_{\square}, \beta_{\cdot j})$ (i.e., $(\alpha_{i\cdot}, \beta_{\cdot j})$, $(\alpha_{\cdot j}, \beta_{\cdot j})$, and $(\alpha_{\cdot\cdot}, \beta_{\cdot j})$), $R^{\omega}_m(i, i') = M^{\omega}_m(i, i')$ and $R^{\omega}_n(i, i') = M^{\omega}_n(i, i')$ because $H^{\omega}(i, i') = 1$ for all pairs (i, i') of Y_1 as shown in Table S12. Since $M^{\omega}_m(i, i') > 1$ and $M^{\omega}_m(i, i') < 1$, we have $\omega(i, i') \in OI^{\omega}(i, i')$ for all pairs (i, i') of Y_1.

Table S12: Decomposition of $R^{\omega}_{m}(i,i')$ and $R^{\omega}_{n}(i,i')$

Model	$H^{\omega}(i,i')$	$M^{\omega}_{m}(i,i')$	$M^{\omega}_{n}(i,i')$
$(\alpha_{i\cdot},\beta_{i\cdot})$ $(\alpha_{\cdot\cdot},\beta_{i\cdot})$	$\exp\left(2\lambda^{i'2}_{Y_1R_2}-2\lambda^{i2}_{Y_1R_2}\right)$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$
$(\alpha_{\cdot j},\beta_{i\cdot})$	$\exp\left(2\lambda^{i'2}_{Y_1R_2}-2\lambda^{i2}_{Y_1R_2}\right)$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$
$(\alpha_{i\cdot},\beta_{\cdot\cdot})$	1	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$
$(\alpha_{\cdot j},\beta_{\cdot\cdot})$	1	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$
$(\alpha_{i\cdot},\beta_{\cdot j})$ $(\alpha_{\cdot\cdot},\beta_{\cdot j})$	1	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$
$(\alpha_{\cdot j},\beta_{\cdot j})$	1	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{j2}_{Y_2R_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}$	$\frac{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{j2}_{Y_2R_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}$

For the models $(\alpha_{i\cdot},\beta_{\cdot\cdot})$ and $(\alpha_{\cdot j},\beta_{\cdot\cdot})$, $H^{\omega}(i,i')$=1 for all pairs (i,i') of Y_1. So, $R^{\omega}_{m}(i,i')$=$M^{\omega}_{m}(i,i')$ and $R^{\omega}_{n}(i,i')$=$M^{\omega}_{n}(i,i')$, which means $\omega(i,i') \in OI^{\omega}(i,i')$.

□

S7 Nonresponse Log-linear Models for Incomplete Three-Way Contingency Table and Model Assessment

To study the generalizability of the proposed method for a more than two-dimensional contingency table with supplemental margins, we considered hierarchical log-linear models for the two types of incomplete three-way contingency tables: the first type is a three-way contingency table with only one variable subject to missingness (S7.1) and the second type is a three-way contingency table with two supplemental margins summarizing missing data on two variables (S7.2).

S7.1 Model assessment for an $I \times J \times K \times 2$ table

S7.1.1 Nonresponse log-linear models

Let Y_2 be an incompletely observed variable with J categories, and Y_1 and Y_3 be completely observed variables with I and K categories, respectively. Let R_2 be an index variable of missingness for Y_2 such that $R_2 = 1$ if Y_2 is observed and $R_2 = 2$ otherwise. The full array of Y_1, Y_2, Y_3 and R_2 produces an $I \times J \times K \times 2$ table with the cell probabilities $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ and the cell counts $\mathbf{y} = \{y_{ijk\ell}\}$ where $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ and $\ell = 1, 2$. But, we observe only $\mathbf{y}_{obs} = \{y_{ijk1}, y_{i+k2}\}$ where a '+' in the subscript denotes summation over j. Table S13 shows the $2 \times 2 \times 2 \times 2$ table.

Table S13: $2 \times 2 \times 2 \times 2$ table

		$R_2 = 1$		$R_2 = 2$
		$Y_2 = 1$	$Y_2 = 2$	
$Y_3 = 1$	$Y_1 = 1$	y_{1111}	y_{1211}	y_{1+12}
	$Y_1 = 2$	y_{2111}	y_{2211}	y_{2+12}
$Y_3 = 2$	$Y_1 = 1$	y_{1121}	y_{1221}	y_{1+22}
	$Y_1 = 2$	y_{2121}	y_{2221}	y_{2+22}

Assuming that the observed cell counts are generated from a multinomial distribution with the cell probabilities $\boldsymbol{\pi}$ and the fixed total count $N = \sum_{i,j,k,\ell} y_{ijk\ell}$, the log likelihood of $\boldsymbol{\pi}$ is

$$\ell = \sum_i \sum_j \sum_k y_{ijk1} \log \pi_{ijk1} + \sum_i \sum_k y_{i+k2} \log \pi_{i+k2}$$

where $\pi_{ijk\ell} = m_{ijk\ell} / \sum_{i,j,k,\ell} m_{ijk\ell}$ and $\mathbf{m} = \{m_{ijk\ell}\}$ are the expected cell counts. To take account of plausible patterns of missing data mechanism, as shown in Table S14, we consider six nonresponse log-linear models for the expected cell counts $\mathbf{m}$ in the $I \times J \times K \times 2$ table. The sum of each λ-term over each of the superscripts in each model is constrained to be zero. Note that the first three models containing three-way interactions are saturated for an $I \times I \times I \times 2$ table (i.e., $I = J = K$).

Table S14: Nonresponse log-linear models

Notation	Model
$(Y_1Y_2Y_3, Y_1Y_3R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{i\ell}_{Y_1R_2} + \lambda^{k\ell}_{Y_3R_2} + \lambda^{ik\ell}_{Y_1Y_3R_2}$
$(Y_1Y_2Y_3, Y_2Y_3R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{j\ell}_{Y_2R_2} + \lambda^{k\ell}_{Y_3R_2} + \lambda^{jk\ell}_{Y_2Y_3R_2}$
$(Y_1Y_2Y_3, Y_1Y_2R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{i\ell}_{Y_1R_2} + \lambda^{j\ell}_{Y_2R_2} + \lambda^{ij\ell}_{Y_1Y_2R_2}$
$(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{i\ell}_{Y_1R_2} + \lambda^{k\ell}_{Y_3R_2}$
$(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{j\ell}_{Y_2R_2} + \lambda^{k\ell}_{Y_3R_2}$
$(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$	$\log m_{ijk\ell} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} + \lambda^{i\ell}_{Y_1R_2} + \lambda^{j\ell}_{Y_2R_2}$

S7.1.2 Properties of nonresponse log-linear models in Table S14

For a detailed analysis of the features of the log-linear models in Table S14, we first define the response odds and nonresponse odds based on the cell probabilities $\boldsymbol{\pi}$ in the $I \times I \times I \times 2$ table.

For a given pair (i, i') of Y_1 and k of Y_3, we define the response odds and nonresponse odds to be

$$\omega_j(i, i', k) = \frac{\pi_{ijk1}}{\pi_{i'jk1}}, \ \omega(i, i', k) = \frac{\pi_{i+k2}}{\pi_{i'+k2}}. \tag{S7.1}$$

Similarly, for given a pair (k,k') of Y_3 and i of Y_1, the response odds and nonresponse odds are defined to be

$$\gamma_j(k,k',i)=\frac{\pi_{ijk1}}{\pi_{ijk'1}},\ \gamma(k,k',i)=\frac{\pi_{i+k2}}{\pi_{i+k'2}}. \tag{S7.2}$$

We also denote the response odds intervals containing $\omega_j(i,i',k)$ in Eq. (S7.1) and $\gamma_j(k,k',i)$ in Eq. (S7.2) by, respectively,

$$OI^{\omega}(i,i',k)=\big(\omega_n(i,i',k),\omega_m(i,i',k)\big)\ ,\ OI^{\gamma}(k,k',i)=\big(\gamma_n(k,k',i),\gamma_m(k,k',i)\big) \tag{S7.3}$$

where $\omega_n(i,i',k)$=$\min_j \omega_j(i,i',k)$, $\omega_m(i,i',k)$=$\max_j \omega_j(i,i',k)$, $\gamma_n(k,k',i)$=$\min_j \gamma_j(k,k',i)$ and $\gamma_m(k,k',i)$=$\max_j \gamma_j(k,k',i)$.

Theorem S1 below presents the theoretical properties with respect to the inequalities relating the nonresponse odds (Eq. (S7.1) and Eq. (S7.2)) to the response odds intervals (Eq. (S7.3)) to enable identification of the informative missingness assumed in the nonresponse models of Table S14.

Theorem S1. Suppose that $\boldsymbol{\pi}$=$\{\pi_{ijk\ell}\}$ in an $I\times I\times I\times 2$ table is modeled by

1) the model $(Y_1Y_2Y_3, Y_1Y_3R_2)$. Then

(1) for each pair pair (i,i') of Y_1 and k of Y_3, one and only one of the following must hold:

(i) $\omega(i,i',k)\in OI^{\omega}(i,i',k)$ if and only if
$$-0.5\log M_m^{\omega}(i,i',k)<(\lambda_{Y_1R_2}^{i'2}-\lambda_{Y_1R_2}^{i2})+(\lambda_{Y_1Y_3R_2}^{i'k2}-\lambda_{Y_1Y_3R_2}^{ik2})<-0.5\log M_n^{\omega}(i,i',k)$$

(ii) $\omega(i,i',k)\notin OI^{\omega}(i,i',k)$ if and only if
$$(\lambda_{Y_1R_2}^{i'2}-\lambda_{Y_1R_2}^{i2})+(\lambda_{Y_1Y_3R_2}^{i'k2}-\lambda_{Y_1Y_3R_2}^{ik2})<-0.5\log M_m^{\omega}(i,i',k) \text{ or}$$
$$(\lambda_{Y_1R_2}^{i'2}-\lambda_{Y_1R_2}^{i2})+(\lambda_{Y_1Y_3R_2}^{i'k2}-\lambda_{Y_1Y_3R_2}^{ik2})>-0.5\log M_n^{\omega}(i,i',k)$$

where $M_m^{\omega}(i,i',k)=\omega_m(i,i',k)/\omega(i,i',k)>1$ and $M_n^{\omega}(i,i',k)=\omega_n(i,i',k)/\omega(i,i',k)<1$ in the absence of all $\lambda_{Y_1R_2}^{i\ell}$'s and $\lambda_{Y_1Y_3R_2}^{ik\ell}$'s.

(2) for each pair pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i)\in OI^{\gamma}(k,k',i)$ if and only if
$$-0.5\log M_m^{\gamma}(k,k',i)<(\lambda_{Y_3R_2}^{k'2}-\lambda_{Y_3R_2}^{k2})+(\lambda_{Y_1Y_3R_2}^{ik'2}-\lambda_{Y_1Y_3R_2}^{ik2})<-0.5\log M_n^{\gamma}(k,k',i)$$

(ii) $\gamma(k,k',i)\notin OI^{\gamma}(k,k',i)$ if and only if
$$(\lambda_{Y_3R_2}^{k'2}-\lambda_{Y_3R_2}^{k2})+(\lambda_{Y_1Y_3R_2}^{ik'2}-\lambda_{Y_1Y_3R_2}^{ik2})<-0.5\log M_m^{\gamma}(k,k',i) \text{ or}$$
$$(\lambda_{Y_3R_2}^{k'2}-\lambda_{Y_3R_2}^{k2})+(\lambda_{Y_1Y_3R_2}^{ik'2}-\lambda_{Y_1Y_3R_2}^{ik2})>-0.5\log M_n^{\gamma}(k,k',i)$$

where $M_m^{\gamma}(k,k',i)=\gamma_m(k,k',i)/\gamma(k,k',i)>1$ and $M_n^{\gamma}(k,k',i)=\gamma_n(k,k',i)/\gamma(k,k',i)<1$ in the absence of all $\lambda_{Y_3R_2}^{k\ell}$'s and $\lambda_{Y_1Y_3R_2}^{ik\ell}$'s.

2) the model $(Y_1Y_2Y_3, Y_2Y_3R_2)$. Then

(1) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ for any given pair (i,i') of Y_1 and k of Y_3

(2) for each pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if
$$-0.5\log M_m^{\gamma}(k,k',i) < \lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} < -0.5\log M_n^{\gamma}(k,k',i)$$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if
$$\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} < -0.5\log M_m^{\gamma}(k,k',i) \text{ or } \lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} > -0.5\log M_n^{\gamma}(k,k',i)$$

where $M_m^{\gamma}(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i)$ and $M_n^{\gamma}(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i)$ do not depend on any $\lambda_{Y_3R_2}^{k\ell}$ but do depend on $\lambda_{Y_2Y_3R_2}^{jk'\ell}$ and $\lambda_{Y_2Y_3R_2}^{jk\ell}$ for all j's, and $M_n^{\gamma}(k,k',i) > 1$ ($M_m^{\gamma}(k,k',i) < 1$) guarantees existence of positively (negatively) large value of $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2}$ for at least one j.

3) the model $(Y_1Y_2Y_3, Y_1Y_2R_2)$. Then

(1) for each pair (i,i') of Y_1 and k of Y_3, one and only one of the following must hold :

(i) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ if and only if
$$-0.5\log M_m^{\omega}(i,i',k) < \lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} < -0.5\log M_n^{\omega}(i,i',k)$$

(ii) $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ if and only if
$$\lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} < -0.5\log M_m^{\omega}(i,i',k) \text{ or } \lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} > -0.5\log M_n^{\omega}(i,i',k)$$

where $M_m^{\omega}(i,i',k) = \omega_m(i,i',k)/\omega(i,i',k)$ and $M_n^{\omega}(i,i',k) = \omega_n(i,i',k)/\omega(i,i',k)$ do not depend on any $\lambda_{Y_1R_2}^{i\ell}$, but do depend on $\lambda_{Y_1Y_2R_2}^{i'j\ell}$ and $\lambda_{Y_1Y_2R_2}^{ij\ell}$ for all j's, and $M_n^{\omega}(i,i',k) > 1$ ($M_m^{\omega}(i,i',k) < 1$) guarantees existence of positively (negatively) large value of $\lambda_{Y_1Y_2R_2}^{i'j2} - \lambda_{Y_1Y_2R_2}^{ij2}$ for at least one j.

(2) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ for any given pair (k,k') of Y_3 and i of Y_1.

4) the model $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$. Then

(1) for each pair pair (i,i') of Y_1 and k of Y_3, one and only one of the following must hold:

(i) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ if and only if
$$-0.5\log M_m^{\omega}(i,i',k) < \lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} < -0.5\log M_n^{\omega}(i,i',k)$$

(ii) $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ if and only if
$$\lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} < -0.5\log M_m^{\omega}(i,i',k) \text{ or } \lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2} > -0.5\log M_n^{\omega}(i,i',k)$$

where $M_m^{\omega}(i,i',k) = \omega_m(i,i',k)/\omega(i,i',k) > 1$ and $M_n^{\omega}(i,i',k) = \omega_n(i,i',k)/\omega(i,i',k) < 1$ in the absence of all $\lambda_{Y_1R_2}^{i\ell}$'s.

(2) for each pair pair (k,k') of Y_3 and i of Y_1, one and only one of the following must

hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if

$-0.5 \log M^{\gamma}_m(k,k',i) < \lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5 \log M^{\gamma}_n(k,k',i)$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if

$\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5 \log M^{\gamma}_m(k,k',i)$ or $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} > -0.5 \log M^{\gamma}_n(k,k',i)$

where $M^{\gamma}_m(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i) > 1$ and $M^{\gamma}_n(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i) < 1$ in the absence of all $\lambda^{k\ell}_{Y_3R_2}$'s.

5) the model $(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$. Then
(1) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ for any given pair (i,i') of Y_1 and k of Y_3

(2) for each pair pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if

$-0.5 \log M^{\gamma}_m(k,k',i) < \lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5 \log M^{\gamma}_n(k,k',i)$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if

$\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5 \log M^{\gamma}_m(k,k',i)$ or $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} > -0.5 \log M^{\gamma}_n(k,k',i)$

where $M^{\gamma}_m(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i) > 1$ and $M^{\gamma}_n(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i) < 1$ in the absence of all $\lambda^{k\ell}_{Y_3R_2}$'s.

6) the model $(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$. Then
(1) for each pair pair (i,i') of Y_1 and k of Y_3, one and only one of the following must hold:

(i) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ if and only if

$-0.5 \log M^{\omega}_m(i,i',k) < \lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} < -0.5 \log M^{\omega}_n(i,i',k)$

(ii) $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ if and only if

$\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} < -0.5 \log M^{\omega}_m(i,i',k)$ or $\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} > -0.5 \log M^{\omega}_n(i,i',k)$

where $M^{\omega}_m(i,i',k) = \omega_m(i,i',k)/\omega(i,i',k) > 1$ and $M^{\omega}_n(i,i',k) = \omega_n(i,i',k)/\omega(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s.

(2) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ for any given pair (k,k') of Y_3 and i of Y_1.

Proof. See **S9.1**. □

From Theorem S1 1)-3), we have two interesting observations: First, the nonresponse odds $\omega(i,i',k)$ and the response odds interval $OI^{\omega}(i,i',k)$ are designed to identify the models including (two-way/three-way) interactions associated with Y_1 and R_2. Second, the nonresponse odds $\gamma(k,k',i)$ and the response odds interval $OI^{\gamma}(k,k',i)$ is able to identify the models with (two-way/three-way) interactions concerning Y_3 and R_2. These observations indicate that the first three saturated nonresponse models in Table S14 can be differentiated by the inequalities relating the nonresponse odds in Eq. (S7.1) and Eq. (S7.2) to the response odds intervals in Eq. (S7.3).

For the model $(Y_1Y_2Y_3, Y_1Y_3R_2)$, if there exist at least one pair (i,i') of Y_1 and k of Y_3 satisfying $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ and at least one pair (k,k') of Y_3 and i of Y_1 satisfying $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$, the model has strong (two-way and/or three way) interactions associated with both Y_1R_2 and Y_3R_2, compared to the models with zero values of $\lambda^{i\ell}_{Y_1R_2}$'s, $\lambda^{k\ell}_{Y_3R_2}$'s and $\lambda^{ik\ell}_{Y_1Y_3R_2}$'s. In terms of the model $(Y_1Y_2Y_3, Y_2Y_3R_2)$, the existence of at least one pair (k,k') of Y_3 and i of Y_1 satisfying $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ is equivalent to the existence of a subset of parameter space such that the value of $\mid \lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} \mid$ (and possibly $\mid \lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2} \mid$) is far from zero, and thus the presence of strong interaction effects concerning Y_3R_2. In the same way, if there exist at least one pair (i,i') of Y_1 and k of Y_3 satisfying $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ under the model $(Y_1Y_2Y_3, Y_1Y_2R_2)$, then the model has strong (two-way and possibly three-way) interactions related to Y_1R_2.

Theorem S1 4)-6) also show that the last four unsaturated nonresponse models in Table S14 are distinguishable with respect to the inequalities relating the nonresponse odds in Eq. (S7.1) and Eq. (S7.2) to the response odds intervals in Eq. (S7.3). For the models $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$, if there exist at least one pair (i,i') of Y_1 and k of Y_3 satisfying $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ and at least one pair (k,k') of Y_3 and i of Y_1 satisfying $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$, the model has strong two two-way interactions associated with Y_1R_2 and Y_3R_2. Under the model $(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$, the existence of at least one pair (k,k') of Y_3 and i of Y_1 satisfying $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ is equivalent to the existence of a subset of parameter space such that the value of $\mid \lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} \mid$ is far from zero and thus the presence of strong two-way interaction effects concerning Y_3R_2. Similarly, if there exist at least one pair (i,i') of Y_1 and k of Y_3 satisfying $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ under the model $(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$, then the model has strong two-way interactions related to Y_1R_2.

Under the first three saturated models of Table S14 for an $I \times I \times I \times 2$ table, we can easily obtain the ML estimators of the response odds and nonresponse odds in Eq. (S7.1) and Eq. (S7.2) :

$$\hat{\omega}_j(i,i',k) = \frac{y_{ijk1}}{y_{i'jk1}},\ \hat{\omega}(i,i',k) = \frac{y_{i+k2}}{y_{i'+k2}}, \tag{S7.4}$$

$$\hat{\gamma}_j(k,k',i) = \frac{y_{ijk1}}{y_{ijk'1}},\ \hat{\gamma}(k,k',i) = \frac{y_{i+k2}}{y_{i+k'2}}. \tag{S7.5}$$

We let the estimators for the response odds intervals in Eq. (S7.3) be denoted by, respectively,

$$\widehat{OI}^{\omega}(i,i',k) = \left(\hat{\omega}_n(i,i',k), \hat{\omega}_m(i,i',k)\right), \widehat{OI}^{\gamma}(k,k',i) = \left(\hat{\gamma}_n(k,k',i), \hat{\gamma}_m(k,k',i)\right) \tag{S7.6}$$

where $\hat{\omega}_n(i,i',k)$=$\min_j \hat{\omega}_j(i,i',k)$, $\hat{\omega}_m(i,i',k)$=$\max_j \hat{\omega}_j(i,i',k)$, $\hat{\gamma}_n(k,k',i)$=$\min_j \hat{\gamma}_j(k,k',i)$ and $\hat{\gamma}_m(k,k',i)$=$\max_j \hat{\gamma}_j(k,k',i)$.

Using Theorem S1 above and the closed-forms of the ML estimators of the nonresponse odds and response odd intervals in Eq. (S7.4), (S7.5) and (S7.6), we can identify the informative missingness assumed in the nonresponse models of Table S14, as shown in the following Corollary S1.

Corollary S1. Given an $I \times I \times I \times 2$ table, suppose that the estimates in Eq. (S7.4), (S7.5) and (S7.6) are computed. Define the two events E_ω and E_γ as

$$E_\omega = \{\hat{\omega}(i,i',k) \in \widehat{OI}^{\omega}(i,i',k) \text{ for all pairs } (i,i^{'}) \text{ of } Y_1 \text{ and all values of } k \text{ of } Y_3\},$$
$$E_\gamma = \{\hat{\gamma}(k,k',i) \in \widehat{OI}^{\gamma}(k,k',i) \text{ for all pairs } (k,k^{'}) \text{ of } Y_3 \text{ and all values of } i \text{ of } Y_1\}.$$

Then,

1) if neither E_ω nor E_γ occur (i.e., "$\hat{\omega}(i,i',k) \notin \widehat{OI}^{\omega}(i,i',k)$ for at least one pair $(i,i^{'})$ of Y_1 and one value of k of Y_3", and "$\hat{\gamma}(k,k',i) \notin \widehat{OI}^{\gamma}(k,k',i)$ for at least one pair $(k,k^{'})$ of Y_3 and one value of i of Y_1"), the plausible models would be $(Y_1Y_2Y_3, Y_1Y_3R_2)$ and $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$.

2) if E_ω does not occur and E_γ occurs, the plausible models would be $(Y_1Y_2Y_3, Y_1Y_3R_2)$, $(Y_1Y_2Y_3, Y_1Y_2R_2)$, $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$ and $(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$.

3) if E_ω occurs and E_γ does not occur, the plausible models would be $(Y_1Y_2Y_3, Y_1Y_3R_2)$, $(Y_1Y_2Y_3, Y_2Y_3R_2)$, $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$ and $(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$.

4) if both E_ω and E_γ occur, the plausible models would be $(Y_1Y_2Y_3, Y_1Y_3R_2)$, $(Y_1Y_2Y_3, Y_1Y_2R_2)$, $(Y_1Y_2Y_3, Y_2Y_3R_2)$, $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$, $(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$, and $(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$.

S7.2 Model assessment for an $I \times J \times K \times 2 \times 2$ table

S7.2.1 Nonresponse log-linear models

Let Y_1 and Y_2 be incompletely observed variables with I and J categories, respectively, and Y_3 be a completely observed variable with K categories. Let R_1 be an index variable of missingness for Y_1 such that $R_1 = 1$ if Y_1 is observed and $R_1 = 2$ otherwise. Let R_2 be an index variable of missingness for Y_2 such that $R_2 = 1$ if Y_2 is observed and $R_2 = 2$ otherwise. The full array of Y_1, Y_2, Y_3, R_1 and R_2 produces the $I \times J \times K \times 2 \times 2$ table with the cell probabilities $\boldsymbol{\pi} = \{\pi_{ijk\ell m}\}$ and the cell counts $\mathbf{y} = \{y_{ijk\ell m}\}$ where $i = 1,\dots,I$, $j = 1,\dots,J$, $k = 1,\dots,K$, $\ell = 1,2$ and $m = 1,2$. But, we observe only $\mathbf{y}_{obs} = \{y_{ijk11}, y_{i+k12}, y_{+jk21}, y_{++k22}\}$. Table S15 presents a 2 $\times$ 2 $\times$ 2 $\times$ 2 $\times$ 2 table.

For observed cell counts, we assume a multinomial distribution with the cell probabilities $\boldsymbol{\pi}$ and the fixed total count $N = \sum_{i,j,k,\ell,m} y_{ijk\ell m}$. Then, the log likelihood of $\boldsymbol{\pi}$ is

$$\begin{aligned}\ell &= \sum_i \sum_j \sum_k y_{ijk11} \log \pi_{ijk11} + \sum_i \sum_k y_{i+k12} \log \pi_{i+k12} \\ &+ \sum_j \sum_k y_{+jk21} \log \pi_{+jk21} + \sum_k y_{++k22} \log \pi_{++k22}\end{aligned}$$

Table S15: $2 \times 2 \times 2 \times 2 \times 2$ table

			$R_2 = 1$		$R_2 = 2$
			$Y_2 = 1$	$Y_2 = 2$	
$Y_3 = 1$	$R_1 = 1$	$Y_1 = 1$	y_{11111}	y_{12111}	y_{1+112}
		$Y_1 = 2$	y_{21111}	y_{22111}	y_{2+112}
	$R_1 = 2$		y_{+1121}	y_{+2121}	y_{++122}
$Y_3 = 2$	$R_1 = 1$	$Y_1 = 1$	y_{11211}	y_{12211}	y_{1+212}
		$Y_1 = 2$	y_{21211}	y_{22211}	y_{2+212}
	$R_1 = 2$		y_{+1221}	y_{+2221}	y_{++221}

where $\pi_{ijk\ell m} = m_{ijk\ell m} / \sum_{i,j,k,\ell,m} m_{ijk\ell m}$ and $\mathbf{m} = \{m_{ijk\ell m}\}$ are the expected cell counts.

For illustrative purpose, we consider the four hierarchical log-linear models for the expected cell counts $\mathbf{m}$ as given in Table S16. The sum of each λ-term over each of the superscripts in each model is constrained to be zero. Note that these four models are saturated for an $I \times I \times I \times 2 \times 2$ table (i.e., $I = J = K$). The four models in Table S16 are the extensions of the four saturated models for an incomplete two-way contingency table considered in Table 1 of the paper because Y_3 is additionally considered in each of the four saturated models in Table 1.

Table S16: Nonresponse log-linear models

Notation	Model
$(Y_1Y_2Y_3, Y_1Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$	$\log m_{ijk\ell m} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_1} + +\lambda^m_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3}$ $+\lambda^{i\ell}_{Y_1R_1} + \lambda^{k\ell}_{Y_3R_1} + \lambda^{ik\ell}_{Y_1Y_3R_1} + \lambda^{im}_{Y_1R_2} + \lambda^{km}_{Y_3R_2} + \lambda^{ikm}_{Y_1Y_3R_2} + \lambda^{\ell m}_{R_1R_2} + \lambda^{k\ell m}_{Y_3R_1R_2}$
$(Y_1Y_2Y_3, Y_2Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$	$\log m_{ijk\ell m} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_1} + +\lambda^m_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3}$ $+\lambda^{j\ell}_{Y_2R_1} + \lambda^{k\ell}_{Y_3R_1} + \lambda^{ik\ell}_{Y_1Y_3R_1} + \lambda^{jm}_{Y_2R_2} + \lambda^{km}_{Y_3R_2} + \lambda^{jkm}_{Y_2Y_3R_2} + \lambda^{\ell m}_{R_1R_2} + \lambda^{k\ell m}_{Y_3R_1R_2}$
$(Y_1Y_2Y_3, Y_1Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$	$\log m_{ijk\ell m} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_1} + +\lambda^m_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3}$ $+\lambda^{i\ell}_{Y_1R_1} + \lambda^{k\ell}_{Y_3R_1} + \lambda^{ik\ell}_{Y_1Y_3R_1} + \lambda^{jm}_{Y_2R_2} + \lambda^{km}_{Y_3R_2} + \lambda^{jkm}_{Y_2Y_3R_2} + \lambda^{\ell m}_{R_1R_2} + \lambda^{k\ell m}_{Y_3R_1R_2}$
$(Y_1Y_2Y_3, Y_2Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$	$\log m_{ijk\ell m} = \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^k_{Y_3} + \lambda^\ell_{R_1} + +\lambda^m_{R_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3}$ $+\lambda^{j\ell}_{Y_2R_1} + \lambda^{k\ell}_{Y_3R_1} + \lambda^{jk\ell}_{Y_2Y_3R_1} + \lambda^{im}_{Y_1R_2} + \lambda^{km}_{Y_3R_2} + \lambda^{ikm}_{Y_1Y_3R_2} + \lambda^{\ell m}_{R_1R_2} + \lambda^{k\ell m}_{Y_3R_1R_2}$

S7.2.2 Properties of nonresponse log-linear models in Table S16

For a detailed analysis of the features of the nonresponse log-linear models in Table S16, we first define the response odds and nonresponse odds based on the cell probabilities $\boldsymbol{\pi}$ in the $I \times I \times I \times 2 \times 2$ table :

For a given pair (i, i') of Y_1 and k of Y_3,

$$\omega_j(i, i', k) = \frac{\pi_{ijk11}}{\pi_{i'jk11}}, \ \omega(i, i', k) = \frac{\pi_{i+k12}}{\pi_{i'+k12}}. \tag{S7.7}$$

For a given pair (j, j') of Y_2 and k of Y_3,

$$\nu_i(j, j', k) = \frac{\pi_{ijk11}}{\pi_{ij'k11}}, \ \nu(j, j', k) = \frac{\pi_{+jk21}}{\pi_{+j'k21}}. \tag{S7.8}$$

For given a pair (k, k') of Y_3 and i of Y_1,

$$\gamma_j(k,k',i) = \frac{\pi_{ijk11}}{\pi_{ijk'11}},\ \gamma(k,k',i) = \frac{\pi_{i+k12}}{\pi_{i+k'12}}. \tag{S7.9}$$

For given a pair (k, k') of Y_3 and j of Y_2,

$$\eta_i(k,k',j) = \frac{\pi_{ijk11}}{\pi_{ijk'11}},\ \eta(k,k',j) = \frac{\pi_{+jk21}}{\pi_{+jk'21}}. \tag{S7.10}$$

We denote the response odds intervals containing $\omega_j(i,i',k)$, $\nu_i(j,j',k)$, $\gamma_j(k,k',i)$ and $\eta_i(k,k',j)$ by, respectively,

$$\begin{aligned} OI^{\omega}(i,i',k) &= \big(\omega_n(i,i',k), \omega_m(i,i',k)\big) \\ OI^{\nu}(j,j',k) &= \big(\nu_n(j,j',k), \nu_m(j,j',k)\big) \\ OI^{\gamma}(k,k',i) &= \big(\gamma_n(k,k',i), \gamma_m(k,k',i)\big) \\ OI^{\eta}(k,k',j) &= \big(\eta_n(k,k',j), \eta_m(k,k',j)\big) \end{aligned} \tag{S7.11}$$

where $\omega_n(i,i',k) = \min_j \omega_j(i,i',k)$, $\omega_m(i,i',k) = \max_j \omega_j(i,i',k)$, $\nu_n(j,j',k) = \min_i \nu_i(j,j',k)$, $\nu_m(j,j',k) = \max_i \nu_i(j,j',k)$, $\gamma_n(k,k',i) = \min_j \gamma_j(k,k',i)$, $\gamma_m(k,k',i) = \max_j \gamma_j(k,k',i)$, $\eta_n(k,k',j) = \min_i \eta_i(k,k',j)$ and $\eta_m(k,k',j) = \max_i \eta_i(k,k',j)$.

Theorem S2 below presents the theoretical properties with respect to the inequalities relating the nonresponse odds (Eq. (S7.7), (S7.8), (S7.9), (S7.10)) to the response odds intervals (Eq. (S7.11)) for identification of the informative missingness assumed in the nonresponse models of Table S16.

Theorem S2. Suppose that $\boldsymbol{\pi}$=$\{\pi_{ijk\ell}\}$ in an $I \times I \times I \times 2 \times 2$ table is modeled by

1) the model $(Y_1Y_2Y_3, Y_1Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$. Then

   (1) for each pair (i, i') of Y_1 and k of Y_3, one and only one of the following must hold :

   (i) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ if and only if
   $-0.5\log M_m^{\omega}(i,i',k) < (\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) < -0.5\log M_n^{\omega}(i,i',k)$

   (ii) $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ if and only if
   $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) < -0.5\log M_m^{\omega}(i,i',k)$ or
   $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) > -0.5\log M_n^{\omega}(i,i',k)$

   where $M_m^{\omega}(i,i',k) = \omega_m(i,i',k)/\omega(i,i',k) > 1$ and $M_n^{\omega}(i,i',k) = \omega_n(i,i',k)/\omega(i,i',k) < 1$ in the absence of all $\lambda^{im}_{Y_1R_2}$'s and $\lambda^{ikm}_{Y_1Y_3R_2}$'s.

   (2) $\nu(j,j',k) \in OI^{\nu}(j,j',k)$ for any given pair (j,j') of Y_2 and k of Y_3

(3) for each pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if

$$-0.5\log M_m^{\gamma}(k,k',i) < (\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + (\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + (\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2}) < -0.5\log M_n^{\gamma}(k,k',i)$$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if

$$(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + (\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + (\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2}) < -0.5\log M_m^{\gamma}(k,k',i) \text{ or}$$
$$(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + (\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + (\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2}) > -0.5\log M_n^{\gamma}(k,k',i)$$

where $M_m^{\gamma}(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i) > 1$ and $M_n^{\gamma}(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i) <$ 1 in the absence of all $\lambda^{km}_{Y_3R_2}$'s, $\lambda^{ikm}_{Y_1Y_3R_2}$'s and $\lambda^{k\ell m}_{Y_3R_1R_2}$'s.

(4) for each pair (k,k') of Y_3 and j of Y_2, one and only one of the following must hold :

(i) $\eta(k,k',j) \in OI^{\eta}(k,k',j)$ if and only if

$$-0.5\log M_m^{\eta}(k,k',j) < (\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + (\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2}) < -0.5\log M_n^{\eta}(k,k',j)$$

(ii) $\eta(k,k',j) \notin OI^{\eta}(k,k',j)$ if and only if

$$(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + (\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2}) < -0.5\log M_m^{\eta}(k,k',j) \text{ or}$$
$$(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + (\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2}) > -0.5\log M_n^{\eta}(k,k',j)$$

where $M_m^{\eta}(k,k',j) = \eta_m(k,k',j)/\eta(k,k',j)$ and $M_n^{\eta}(k,k',j) = \eta_n(k,k',j)/\eta(k,k',j)$ do not depend on any $\lambda^{k\ell}_{Y_3R_1}$ and $\lambda^{k\ell m}_{Y_3R_1R_2}$ but do depend on $\lambda^{ik'\ell}_{Y_1Y_3R_1}$ and $\lambda^{ik\ell}_{Y_1Y_3R_1}$ for all i's, and $M_n^{\eta}(k,k',j) > 1$ ($M_m^{\eta}(k,k',j) < 1$) guarantees existence of positively (negatively) large value of $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1}$ for at least one i.

2) the model $(Y_1Y_2Y_3, Y_2Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$. Then

(1) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ for any given pair (i,i') of Y_1 and k of Y_3

(2) for each pair (j,j') of Y_2 and k of Y_3, one and only one of the following must hold :

(i) $\nu(j,j',k) \in OI^{\nu}(j,j',k)$ if and only if

$$-0.5\log M_m^{\nu}(j,j',k) < (\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) < -0.5\log M_n^{\nu}(j,j',k)$$

(ii) $\nu(j,j',k) \notin OI^{\nu}(j,j',k)$ if and only if

$$(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) < -0.5\log M_m^{\nu}(j,j',k) \text{ or}$$
$$(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) > -0.5\log M_n^{\nu}(j,j',k)$$

where $M_m^{\nu}(j,j',k) = \nu_m(j,j',k)/\nu(j,j',k) > 1$ and $M_n^{\nu}(j,j',k) = \nu_n(j,j',k)/\nu(j,j',k) <$ 1 in the absence of all $\lambda^{i\ell}_{Y_2R_1}$'s and $\lambda^{jk\ell}_{Y_2Y_3R_1}$'s.

(3) for each pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if
$-0.5\log M_m^{\gamma}(k,k',i) < (\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_n^{\gamma}(k,k',i)$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if
$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_m^{\gamma}(k,k',i)$ or
$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) > -0.5\log M_n^{\gamma}(k,k',i)$

where $M_m^{\gamma}(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i)$ and $M_n^{\gamma}(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i)$ do not depend on any $\lambda_{Y_3R_2}^{km}$ and $\lambda_{Y_3R_1R_2}^{k\ell m}$ but do depend on $\lambda_{Y_2Y_3R_2}^{jk'm}$ and $\lambda_{Y_2Y_3R_2}^{jkm}$ for all j's, and $M_n^{\gamma}(k,k',i) > 1$ ($M_m^{\gamma}(k,k',i) < 1$) guarantees existence of positively (negatively) large value of $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2}$ for at least one j.

(4) for each pair (k,k') of Y_3 and j of Y_2, one and only one of the following must hold :

(i) $\eta(k,k',j) \in OI^{\eta}(k,k',j)$ if and only if
$-0.5\log M_m^{\eta}(k,k',j) < (\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21})$
$< -0.5\log M_n^{\eta}(k,k',j)$

(ii) $\eta(k,k',j) \notin OI^{\eta}(k,k',j)$ if and only if
$(\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) < -0.5\log M_m^{\eta}(k,k',j)$ or
$(\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) > -0.5\log M_n^{\eta}(k,k',j)$

where $M_m^{\eta}(k,k',j) = \eta_m(k,k',j)/\eta(k,k',j) > 1$ and $M_n^{\eta}(k,k',j) = \eta_n(k,k',j)/\eta(k,k',j) < 1$ in the absence of all $\lambda_{Y_3R_1}^{k\ell}$'s, $\lambda_{Y_2Y_3R_1}^{jkm}$'s and $\lambda_{Y_3R_1R_2}^{k\ell m}$'s.

3) the model $(Y_1Y_2Y_3, Y_1Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$. Then

(1) $\omega(i,i',k) \in OI^{\omega}(i,i',k)$ for any given pair (i,i') of Y_1 and k of Y_3

(2) $\nu(j,j',k) \in OI^{\nu}(j,j',k)$ for any given pair (j,j') of Y_2 and k of Y_3

(3) for each pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if
$-0.5\log M_m^{\gamma}(k,k',i) < (\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_n^{\gamma}(k,k',i)$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if
$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_m^{\gamma}(k,k',i)$ or
$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) > -0.5\log M_n^{\gamma}(k,k',i)$

where $M_m^{\gamma}(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i)$ and $M_n^{\gamma}(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i)$ do not depend on any $\lambda_{Y_3R_2}^{km}$ and $\lambda_{Y_3R_1R_2}^{k\ell m}$ but do depend on $\lambda_{Y_2Y_3R_2}^{jk'm}$ and $\lambda_{Y_2Y_3R_2}^{jkm}$ for all j's, and $M_n^{\gamma}(k,k',i) > 1$ ($M_m^{\gamma}(k,k',i) < 1$) guarantees existence of positively (negatively) large value of $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2}$ for at least one j.

(4) for each pair (k,k') of Y_3 and j of Y_2, one and only one of the following must hold :

$$
\begin{aligned}
(i)\quad & \eta(k,k',j) \in OI^{\eta}(k,k',j) \ \text{ if and only if} \\
& -0.5\log M_m^{\eta}(k,k',j) < (\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) < -0.5\log M_n^{\eta}(k,k',j) \\
(ii)\quad & \eta(k,k',j) \notin OI^{\eta}(k,k',j) \ \text{ if and only if} \\
& (\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) < -0.5\log M_m^{\eta}(k,k',j) \text{ or} \\
& (\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) > -0.5\log M_n^{\eta}(k,k',j)
\end{aligned}
$$

where $M_m^{\eta}(k,k',j) = \eta_m(k,k',j)/\eta(k,k',j)$ and $M_n^{\eta}(k,k',j) = \eta_n(k,k',j)/\eta(k,k',j)$ do not depend on any $\lambda_{Y_3R_1}^{k\ell}$ and $\lambda_{Y_3R_1R_2}^{k\ell m}$ but do depend on $\lambda_{Y_1Y_3R_1}^{ik'\ell}$ and $\lambda_{Y_1Y_3R_1}^{ik\ell}$ for all i's, and $M_n^{\eta}(k,k',j) > 1$ ($M_m^{\eta}(k,k',j) < 1$) guarantees existence of positively (negatively) large value of $\lambda_{Y_1Y_3R_1}^{ik'2} - \lambda_{Y_1Y_3R_1}^{ik2}$ for at least one i.

4) the model $(Y_1Y_2Y_3, Y_2Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$. Then

(1) for each pair (i,i') of Y_1 and k of Y_3, one and only one of the following must hold :

$$
\begin{aligned}
(i)\quad & \omega(i,i',k) \in OI^{\omega}(i,i',k) \ \text{ if and only if} \\
& -0.5\log M_m^{\omega}(i,i',k) < (\lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2}) + (\lambda_{Y_1Y_3R_2}^{i'k2} - \lambda_{Y_1Y_3R_2}^{ik2}) < -0.5\log M_n^{\omega}(i,i',k) \\
(ii)\quad & \omega(i,i',k) \notin OI^{\omega}(i,i',k) \ \text{ if and only if} \\
& (\lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2}) + (\lambda_{Y_1Y_3R_2}^{i'k2} - \lambda_{Y_1Y_3R_2}^{ik2}) < -0.5\log M_m^{\omega}(i,i',k) \text{ or} \\
& (\lambda_{Y_1R_2}^{i'2} - \lambda_{Y_1R_2}^{i2}) + (\lambda_{Y_1Y_3R_2}^{i'k2} - \lambda_{Y_1Y_3R_2}^{ik2}) > -0.5\log M_n^{\omega}(i,i',k)
\end{aligned}
$$

where $M_m^{\omega}(i,i',k) = \omega_m(i,i',k)/\omega(i,i',k) > 1$ and $M_n^{\omega}(i,i',k) = \omega_n(i,i',k)/\omega(i,i',k) < 1$ in the absence of all $\lambda_{Y_1R_2}^{im}$'s and $\lambda_{Y_1Y_3R_2}^{ikm}$'s.

(2) for each pair (j,j') of Y_2 and k of Y_3, one and only one of the following must hold :

$$
\begin{aligned}
(i)\quad & \nu(j,j',k) \in OI^{\nu}(j,j',k) \ \text{ if and only if} \\
& -0.5\log M_m^{\nu}(j,j',k) < (\lambda_{Y_2R_1}^{j'2} - \lambda_{Y_2R_1}^{j2}) + (\lambda_{Y_2Y_3R_1}^{j'k2} - \lambda_{Y_2Y_3R_1}^{jk2}) < -0.5\log M_n^{\nu}(j,j',k) \\
(ii)\quad & \nu(j,j',k) \notin OI^{\nu}(j,j',k) \ \text{ if and only if} \\
& (\lambda_{Y_2R_1}^{j'2} - \lambda_{Y_2R_1}^{j2}) + (\lambda_{Y_2Y_3R_1}^{j'k2} - \lambda_{Y_2Y_3R_1}^{jk2}) < -0.5\log M_m^{\nu}(j,j',k) \text{ or} \\
& (\lambda_{Y_2R_1}^{j'2} - \lambda_{Y_2R_1}^{j2}) + (\lambda_{Y_2Y_3R_1}^{j'k2} - \lambda_{Y_2Y_3R_1}^{jk2}) > -0.5\log M_n^{\nu}(j,j',k)
\end{aligned}
$$

where $M_m^{\nu}(j,j',k) = \nu_m(j,j',k)/\nu(j,j',k) > 1$ and $M_n^{\nu}(j,j',k) = \nu_n(j,j',k)/\nu(j,j',k) < 1$ in the absence of all $\lambda_{Y_2R_1}^{i\ell}$'s and $\lambda_{Y_2Y_3R_1}^{jk\ell}$'s.

(3) for each pair (k,k') of Y_3 and i of Y_1, one and only one of the following must hold :

(i) $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$ if and only if
$$-0.5\log M_m^{\gamma}(k,k',i) < (\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_1Y_3R_2}^{ik'2} - \lambda_{Y_1Y_3R_2}^{ik2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_n^{\gamma}(k,k',i)$$

(ii) $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ if and only if
$$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_1Y_3R_2}^{ik'2} - \lambda_{Y_1Y_3R_2}^{ik2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) < -0.5\log M_m^{\gamma}(k,k',i) \text{ or}$$
$$(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_1Y_3R_2}^{ik'2} - \lambda_{Y_1Y_3R_2}^{ik2}) + (\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12}) > -0.5\log M_n^{\gamma}(k,k',i)$$

where $M_m^{\gamma}(k,k',i) = \gamma_m(k,k',i)/\gamma(k,k',i) > 1$ and $M_n^{\gamma}(k,k',i) = \gamma_n(k,k',i)/\gamma(k,k',i) < 1$ in the absence of all $\lambda_{Y_3R_2}^{km}$'s, $\lambda_{Y_1Y_3R_2}^{ikm}$'s and $\lambda_{Y_3R_1R_2}^{k\ell m}$'s.

(4) for each pair (k,k') of Y_3 and j of Y_2, one and only one of the following must hold :

(i) $\eta(k,k',j) \in OI^{\eta}(k,k',j)$ if and only if
$$-0.5\log M_m^{\eta}(k,k',j) < (\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) < -0.5\log M_n^{\eta}(k,k',j)$$

(ii) $\eta(k,k',j) \notin OI^{\eta}(k,k',j)$ if and only if
$$(\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) < -0.5\log M_m^{\eta}(k,k',j) \text{ or}$$
$$(\lambda_{Y_3R_1}^{k'2} - \lambda_{Y_3R_1}^{k2}) + (\lambda_{Y_2Y_3R_1}^{jk'2} - \lambda_{Y_2Y_3R_1}^{jk2}) + (\lambda_{Y_3R_1R_2}^{k'21} - \lambda_{Y_3R_1R_2}^{k21}) > -0.5\log M_n^{\eta}(k,k',j)$$

where $M_m^{\eta}(k,k',j) = \eta_m(k,k',j)/\eta(k,k',j) > 1$ and $M_n^{\eta}(k,k',j) = \eta_n(k,k',j)/\eta(k,k',j) < 1$ in the absence of all $\lambda_{Y_3R_1}^{k\ell}$'s, $\lambda_{Y_2Y_3R_1}^{jkm}$'s and $\lambda_{Y_3R_1R_2}^{k\ell m}$'s.

Proof. See **S9.2** □

From Theorem S2, we see that each nonresponse odds and its associated response odds interval are able to identify the models with (two-way/three-way) interactions concerning the missingness of an incompletely observed outcome variable and other outcome variable : (i) $(\omega(i,i',k),\, OI^{\omega}(i,i',k))$ for interactions associated with R_2 and Y_1, (ii) $(\nu(j,j',k),\, OI^{\nu}(j,j',k))$ for interactions associated with R_1 and Y_2, (iii) $(\gamma(k,k',i),\, OI^{\gamma}(k,k',i))$ for interactions associated with R_2 and Y_3, (iv) $(\eta(k,k',j),\, OI^{\eta}(k,k',j))$ for interactions associated with R_1 and Y_3. Therefore, as summarized in Table S17, the nonresponse models in Table S16 can be differentiated by the inequalities relating the nonresponse odds in Eq. (S7.7), (S7.8), (S7.9), (S7.10) to the response odds intervals in Eq. (S7.11).

Table S17: Properties of the nonresponse log-linear models in Table S16

Notation	$(\omega(i,i',k),\, OI^{\omega}(i,i',k))$	$(\nu(j,j',k),\, OI^{\nu}(j,j',k))$	$(\gamma(k,k',i),\, OI^{\gamma}(k,k',i))$	$\eta(k,k',j),\, OI^{\eta}(k,k',j))$
$(Y_1Y_2Y_3, Y_1Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$	△	○	△	△
$(Y_1Y_2Y_3, Y_2Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$	○	△	△	△
$(Y_1Y_2Y_3, Y_1Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$	○	○	△	△
$(Y_1Y_2Y_3, Y_2Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$	△	△	△	△

In Table S17 ◯ means the nonresponse odds always lies in the response odds interval. The symbol △ represents that whether or not the nonresponse odds lies outside the response odds interval depends on the existence and magnitudes of the (two-way/three-way) interactions between the missingness of an incompletely observed outcome variable and other outcome variable.

As the nonresponse models in Table S16 are saturated for an $I \times I \times I \times 2 \times 2$ table, we can easily obtain the ML estimators of the response odds and nonresponse odds in Eq. (S7.7), (S7.8), (S7.9), (S7.10) :

$$\hat{\omega}_j(i,i',k) = \frac{y_{ijk11}}{y_{i'jk11}},\ \hat{\omega}(i,i',k) = \frac{y_{i+k12}}{y_{i'+k12}}, \tag{S7.12}$$

$$\hat{\nu}_i(j,j',k) = \frac{y_{ijk11}}{y_{ij'k11}},\ \hat{\nu}(j,j',k) = \frac{y_{+jk21}}{y_{+j'k21}}, \tag{S7.13}$$

$$\hat{\gamma}_j(k,k',i) = \frac{y_{ijk11}}{y_{ijk'11}},\ \hat{\gamma}(k,k',i) = \frac{y_{i+k12}}{y_{i+k'12}}, \tag{S7.14}$$

$$\hat{\eta}_i(k,k',j) = \frac{y_{ijk11}}{y_{ijk'11}},\ \hat{\eta}(k,k',j) = \frac{y_{+jk21}}{y_{+jk'21}}. \tag{S7.15}$$

We let the estimators for the response odds intervals in Eq. (S7.11) be denoted by, respectively,

$$\begin{aligned} \widehat{OI}^{\omega}(i,i',k) &= \big(\hat{\omega}_n(i,i',k), \hat{\omega}_m(i,i',k)\big) \\ \widehat{OI}^{\nu}(j,j',k) &= \big(\hat{\nu}_n(j,j',k), \hat{\nu}_m(j,j',k)\big) \\ \widehat{OI}^{\gamma}(k,k',i) &= \big(\hat{\gamma}_n(k,k',i), \hat{\gamma}_m(k,k',i)\big) \\ \widehat{OI}^{\eta}(k,k',j) &= \big(\hat{\eta}_n(k,k',j), \hat{\eta}_m(k,k',j)\big) \end{aligned} \tag{S7.16}$$

where $\hat{\omega}_n(i,i',k) = \min_j \hat{\omega}_j(i,i',k)$, $\hat{\omega}_m(i,i',k) = \max_j \hat{\omega}_j(i,i',k)$, $\hat{\nu}_n(j,j',k) = \min_i \hat{\nu}_i(j,j',k)$, $\hat{\nu}_m(j,j',k) = \max_i \hat{\nu}_i(j,j',k)$, $\hat{\gamma}_n(k,k',i) = \min_j \hat{\gamma}_j(k,k',i)$, $\hat{\gamma}_m(k,k',i) = \max_j \hat{\gamma}_j(k,k',i)$, $\hat{\eta}_n(k,k',j) = \min_i \hat{\eta}_i(k,k',j)$ and $\hat{\eta}_m(k,k',j) = \max_i \hat{\eta}_i(k,k',j)$.

Using Table S17 (based on Theorem S2) and the closed-forms of the ML estimators of the nonresponse odds and response odds intervals in Eq. (S7.12), (S7.13), (S7.14), and (S7.15), we can identify the informative missingness assumed in the nonresponse models of Table S16.

To further develop a complete data analytic guideline to distinguish between plausible nonresponse log-linear models for an $I \times J \times K \times 2 \times 2$ table and a multi-way incomplete contingency table, it is required to find a complete list of estimable hierarchical nonresponse log-linear models. We think this task is beyond the scope of the present paper and calls for future studies.

S8 Nonignorable Selection Model for An Incomplete Two-way Contingency Table

S8.1 Nonignorable selection model

In order to investigate the applicability of the proposed method to nonignorable nonresponse models outside the framework of the log-linear model, we consider a nonignorable selection model proposed in Fay (1986) and Molenberghs, Goetghebeur, Lipsitz and Kenward (1999). We denote this model as NNS-FM.

A selection model decomposes the cell probabilities $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ of an incomplete two-way contingency table (such as Table 1 of the paper) as follows:

$$\pi_{ijk\ell} = \pi_{ij}\phi_{k,\ell|i,j} \tag{S8.1}$$

where $\pi_{ijk\ell} = Pr(Y_1 = i, Y_2 = j, R_1 = k, R_2 = \ell)$, $\pi_{ij} = Pr(Y_1 = i, Y_2 = j)$ and $\phi_{k,\ell|i,j} = Pr(R_1 = k, R_2 = \ell \mid Y_1 = i, Y_2 = j)$. In the NNS-FM model, $\phi_{k,\ell|i,j}$ in Eq. (S8.1) is defined using the conditional decomposition components, $p_1(i,j)$, $p_{21}(i,j)$ and $p_{20}(i,j)$ (Fay, 1986):

$$\begin{aligned} \phi_{1,1|i,j} &= p_1(i,j)p_{21}(i,j), & \phi_{1,2|i,j} &= p_1(i,j)[1-p_{21}(i,j)], \\ \phi_{2,1|i,j} &= [1-p_1(i,j)]p_{20}(i,j), & \phi_{2,2|i,j} &= [1-p_1(i,j)][1-p_{20}(i,j)], \end{aligned}$$

where $p_1(i,j) = Pr(R_1 = 1 \mid Y_1 = i, Y_2 = j)$, $p_{21}(i,j) = Pr(R_2 = 1 \mid Y_1 = i, Y_2 = j, R_1 = 1)$, and $p_{20}(i,j) = Pr(R_2 = 1 \mid Y_1 = i, Y_2 = j, R_1 = 2)$. For the three decomposition components $p_1(i,j)$, $p_{21}(i,j)$ and $p_{20}(i,j)$, a logistic parameterization is used (Molenberghs *et al.*, 1999):

$$\begin{aligned} \log m_{ij} &= \lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2}, & \log\frac{p_1(i,j)}{1-p_1(i,j)} &= \alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}, \\ \log\tfrac{p_{21}(i,j)}{1-p_{21}(i,j)} &= \alpha_2 + \alpha^i_{Y_1} + \alpha^j_{Y_2}, & \log\frac{p_{20}(i,j)}{1-p_{20}(i,j)} &= \alpha_3, \end{aligned}$$

where $\pi_{ij} = m_{ij}/\sum_{i,j} m_{ij}$, $\mathbf{m} = \{m_{ij}\}$ is the expected cell count for the $I \times J$ table obtained by collapsing the hypothetical complete data over the four missingness patterns, and $\sum_i \alpha^i_{Y_1} = \sum_j \alpha^j_{Y_2} = \sum_i \lambda^i_{Y_1} = \sum_j \lambda^j_{Y_2} = \sum_i \lambda^{ij}_{Y_1Y_2} = \sum_j \lambda^{ij}_{Y_1Y_2} = 0$. Note that the NNS-FM model above is saturated and each missingness is assumed to depend on both Y_1 and Y_2.

We will examine the NNS-FM model with respect to the odds of cell probabilities in Eq. (2.1) of the paper and the inequalities relating the nonresponse odds to the response odds intervals in Eq. (3.1) - Eq. (3.4) of the paper.

S8.2 Odds of the cell probabilities

Under the NNS-FM model, the three odds of the cell probabilities can be expressed as

$$\begin{aligned}
\alpha_{ij} &= \frac{\pi_{ij21}}{\pi_{ij11}} = \frac{\pi_{ij}[1-p_1(i,j)]p_{20}(i,j)}{\pi_{ij}p_1(i,j)p_{21}(i,j)} = \frac{\exp(\alpha_3)(1+\exp(\alpha_2+\alpha^i_{Y_1}+\alpha^j_{Y_2}))}{\exp(\alpha_1+\alpha_2+2\alpha^i_{Y_1}+2\alpha^j_{Y_2})(1+\exp(\alpha_3))}, \\
\beta_{ij} &= \frac{\pi_{ij12}}{\pi_{ij11}} = \frac{\pi_{ij}p_1(i,j)[1-p_{21}(i,j)]}{\pi_{ij}p_1(i,j)p_{21}(i,j)} = \frac{1}{\exp(\alpha_2+\alpha^i_{Y_1}+\alpha^j_{Y_2})}, \\
g_{ij} &= \frac{\pi_{ij11}\pi_{ij22}}{\pi_{ij12}\pi_{ij21}} = \frac{p_{21}(i,j)}{1-p_{21}(i,j)}\frac{1-p_{20}(i,j)}{p_{20}(i,j)} = \exp(\alpha_2-\alpha_3+\alpha^i_{Y_1}+\alpha^j_{Y_2}).
\end{aligned}$$

We can see that α_{ij}, β_{ij} and g_{ij} depend on both i and j, unlike the eight nonignorable log linear models shown in Table S1.

S8.3 Inequalities relating the nonresponse odds to the response odds intervals

In the following Theorem S3 we present the theoretical behaviors of the NNS-FM model with respect to the nonresponse odds and the response odds intervals in Eq.(3.1), (3.2), (3.3) and (3.4) of the paper.

Theorem S3. Suppose that $\boldsymbol{\pi} = \{\pi_{ijk\ell}\}$ for an $I \times J \times 2 \times 2$ table is modeled by the NNS-FM model in **S8.1**. Then,

1) for each pair (j, j') of Y_2, one and only one of the following must hold

$$\begin{aligned}
&(1) \quad \nu(j,j') \in OI^{\nu}(j,j') \ \text{ if and only if } \ |\, \alpha^j_{Y_2} - \alpha^{j'}_{Y_2} \,| \le A_2 \\
&(2) \quad \nu(j,j') \notin OI^{\nu}(j,j') \ \text{ if and only if } \ |\, \alpha^j_{Y_2} - \alpha^{j'}_{Y_2} \,| > A_2
\end{aligned}$$

where A_2 is a constant over $\alpha^j_{Y_2}$.

2) for each pair (i, i') of Y_1 , one and only one of the following must hold

$$\begin{aligned}
&(1) \quad \omega(i,i') \in OI^{\omega}(i,i') \ \text{ if and only if } \ |\, \alpha^i_{Y_1} - \alpha^{i'}_{Y_1} \,| \le A_1 \\
&(2) \quad \omega(i,i') \notin OI^{\omega}(i,i') \ \text{ if and only if } \ |\, \alpha^i_{Y_1} - \alpha^{i'}_{Y_1} \,| > A_1
\end{aligned}$$

where A_1 is a constant over $\alpha^i_{Y_1}$.

Proof. See **S9.3.1** and **S9.3.2** below. □

From Theorem S3, we can see that, for the NNS-FM model, the existence of at least one pair (j, j') of Y_2 satisfying $\nu(j,j') \notin OI^{\nu}(j,j')$ is equivalent to the existence of a subset of parameter space such that the value of $|\, \alpha^j_{Y_2} - \alpha^{j'}_{Y_2} \,|$ is far from zero and thus the presence of a strong effect of $\alpha^j_{Y_2}$ (i.e., positively/negatively large value of $\alpha^j_{Y_2}$). In the same way, if there exist at least one pair (i, i') of Y_1 satisfying $\omega(i,i') \notin OI^{\omega}(i,i')$, then the selection model has a strong effect of $\alpha^i_{Y_1}$.

As the NNS-FM model is saturated, the ML estimators of $\nu_i(j,j')$, $\nu(j,j')$, $OI^{\nu}(j,j')$, $\omega j(i,i')$, $\omega(i,i')$, and $OI^{\omega}(i,i')$ are

$$\hat{\nu}_i(j,j') = \frac{y_{ij11}}{y_{ij'11}},\ \hat{\nu}(j,j') = \frac{y_{+j21}}{y_{+j'21}},\ \widehat{OI}^{\nu}(j,j') = \left(\hat{\nu}_n(j,j'), \hat{\nu}_m(j,j')\right) \tag{S8.2}$$

$$\hat{\omega}_j(i,i') = \frac{y_{ij11}}{y_{i'j11}},\ \hat{\omega}(i,i') = \frac{y_{i+12}}{y_{i'+12}},\ \widehat{OI}^{\omega}(i,i') = \left(\hat{\omega}_n(i,i'), \hat{\omega}_m(i,i')\right), \tag{S8.3}$$

where $\hat{\nu}_n(j,j')$=$\min_i \hat{\nu}_i(j,j')$, $\hat{\nu}_m(j,j')$=$\max_i \hat{\nu}_i(j,j')$, $\hat{\omega}_n(i,i')$=$\min_j \hat{\omega}_j(i,i')$ and $\hat{\omega}_m(i,i')$=$\max_j \hat{\omega}_j(i,i')$.

Using Theorem S3 and the closed forms of the ML estimators in Eq. (S8.2) and (S8.3), therefore, we can identify the informative missingness assumed in the NNS-FM model, as shown in the following Corollary S3.

Corollary S3. Given an $I \times I \times 2 \times 2$ table, suppose that the estimates in Eq. (S8.2) and (S8.3) are computed.

1) If $\hat{\nu}(j,j') \notin \widehat{OI}^{\nu}(j,j')$ for at least one pair (j,j') of Y_2, the NNS-FM model (with large $|\alpha^{j}_{Y_2}|$ values) would be plausible,

2) If $\hat{\omega}(i,i') \notin \widehat{OI}^{\omega}(i,i')$ for at least one pair (i,i') of Y_1, the NNS-FM model (with large $|\alpha^{i}_{Y_1}|$ values) would be plausible.

Note that if one is interested in nonignorable nonresponse models different from the NNS-FM model above (e.g., different logistic parameterizations for $p_1(i,j)$, $p_{21}(i,j)$ and $p_{20}(i,j)$ in **S8.1**, as given in Molenberghs, Goetghebeur, Lipsitz and Kenward (1999)), one needs to analytically check the applicability of the proposed method for each model one by one.

S8.4 Data analysis using the NNS-FM model : BMD data and Smoking data

We here illustrate the performance of the proposed method under the model NNS-FM in **S8.1** with two public health data sets, the first data set used in Section 5.2 of the the paper and the second data set analyzed in S5.2 of the Supplemental material.

First, we evaluate the suitability of the NNS-FM model for the BMD data by computing the nonresponse odds and the response odds intervals in Eq. (S8.2) and (S8.3). We observe the same results as those in Table 6 of the paper (Page 15) because the NNS-FM model is saturated : $\hat{\nu}(j,j') \notin \widehat{OI}^{\nu}(j,j')$ for two pairs (j,j') of Y_2 and $\hat{\omega}(i,i') \notin \widehat{OI}^{\omega}(i,i')$ for three pairs (i,i') of Y_1. By Corollary S3, thus, the NNS-FM model with strong effects of $\alpha^{i}_{Y_1}$ and $\alpha^{j}_{Y_2}$ would be a plausible nonignorable model, along with the selected saturated log-linear models $(\alpha_{\cdot j}, \beta_{i\cdot})$.

To verify our finding, we also applied the BMD data to the NNS-FM model, and computed the ML estimates for the parameters : $(\hat{\lambda}^{1}_{Y_1}, \hat{\lambda}^{2}_{Y_1}, \hat{\lambda}^{1}_{Y_2}, \hat{\lambda}^{2}_{Y_2}, \hat{\lambda}^{11}_{Y_1Y_2}, \hat{\lambda}^{12}_{Y_1Y_2},\ \hat{\lambda}^{21}_{Y_1Y_2}, \hat{\lambda}^{22}_{Y_1Y_2}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}^{1}_{Y_1}, \hat{\alpha}^{2}_{Y_1}, \hat{\alpha}^{1}_{Y_2}, \hat{\alpha}^{2}_{Y_2})$=(0.938, 0.166, 0.629, -0.339, -0.128, -0.010, -0.149, 0.030, 0.685, 1.944, 2.971, 0.250, 0.060, 0.015, 0.274). Note that the corresponding estimated standard errors are (0.055, 0.061, 0.042, 0.052, 0.051, 0.063, 0.058, 0.070, 0.068, 0.091, 0.153, 0.103, 0.115, 0.051, 0.064). We can see that the estimates for $\alpha^1_{Y_1}$ and $\alpha^2_{Y_2}$ are positively large relative to their standard errors, i.e., 2.427(=0.250/0.103) and 4.281(=0.274/0.064).

Table S18: Estimates for the expected count for BMD data

Model			R_2=1			R_2=2		
			Y_2=1	Y_2=2	Y_2=3	Y_2=1	Y_2=2	Y_2=3
Selection	R_1=1	Y_1=1	621.0	290.0	284.0	68.2	24.6	42.3
		Y_1=2	260.0	131.0	117.0	34.5	13.4	21.1
		Y_1=3	93.0	30.0	18.0	17.9	4.4	4.7
	R_1=2	Y_1=1	253.6	89.3	162.8	13.0	4.6	8.3
		Y_1=2	131.1	49.6	83.3	6.7	2.5	4.3
		Y_1=3	71.4	17.1	19.8	3.7	0.9	1.0

The estimates for the expected counts for the BMD data under the NNS-FM model are given in Table S18. As the NNS-FM model is saturated, the corresponding maximized log likelihood, G^2, AIC and BIC are exactly the same as those of the selected saturated log-linear model $(\alpha_{\cdot j}, \beta_{i\cdot})$. However, the estimates of the expected counts for (R_1, R_2)=(1,2), (2,1), (2,2) are different for both nonignorable models (see Table S8 of Section S5 in the Supplemental Material).

We now assess the suitability of the NNS-FM model for the smoking data by computing the nonresponse odds and the response odds intervals in Eq. (S8.2) and (S8.3). We found that $\hat{\nu}(1,2)$=0.924 $\in \widehat{OI}^{\nu}(1,2)$=(0.871, 1.329) for a pair (1,2) of Y_2, but $\hat{\omega}(1,2)$=0.306 $\notin \widehat{OI}^{\omega}(1,2)$=(0.141, 0.215) for a pair (1,2) of Y_1. This indicates that, by Corollary S3, the NNS-FM model with strong effect of $\alpha^i_{Y_1}$ would be a plausible nonignorable model, along with the selected saturated log-linear models $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$ in S5.2 of the Supplemental material.

To confirm this result, we fitted the NNS-FM model to the smoking data and obtained the ML estimates for the parameters: $(\hat{\lambda}^1_{Y_1}, \hat{\lambda}^1_{Y_2}, \hat{\lambda}^{11}_{Y_1Y_2}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}^1_{Y_1},\ \hat{\alpha}^1_{Y_2})$=(-0.851, 0.035, 0.105, 2.583, 4.298, 0.579, -0.281, 0.020). Note that the corresponding estimated standard errors are (0.007, 0.006, 0.006, 0.031, 0.048, 0.036, 0.049, 0.022). We can see that the estimate for $\alpha^1_{Y_1}$ is negatively large relative to its standard error, i.e., -5.73(=-0.281/0.049).

Table S19: Estimates for the expected count for smoking and birthweight data

Model			$R_2 = 1$		$R_2 = 2$	
			$Y_2 = 1$	$Y_2 = 2$	$Y_2 = 1$	$Y_2 = 2$
Selection	$R_1 = 1$	$Y_1 = 1$	4512.0	3394.0	79.7	62.3
		$Y_1 = 2$	21009.0	24132.0	211.4	252.6
	$R_1 = 2$	$Y_1 = 1$	288.7	226.1	161.8	126.7
		$Y_1 = 2$	760.3	908.9	426.1	509.4

Table S19 shows the estimates of the expected counts for the smoking data under the NNS-FM model. As the NNS-FM model is saturated, the corresponding maximized log likelihood, G^2, AIC and BIC are exactly the same as those of the selected saturated log-linear models, $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$. However, the predictions for the unobserved outcomes, the estimates of the expected counts for (R_1, R_2)=(1,2), (2,1), (2,2), are different. Please see Table S11 of

Section S5.2 in the Supplemental Material for the estimates of the expected counts under the models $(\alpha_{\cdot j}, \beta_{i\cdot})$ and $(\alpha_{i\cdot}, \beta_{i\cdot})$.

S9 Proofs of Theorems in Section S7 and Section S8

S9.1 Proof of Theorem S1 in S7.1.2

1) Model $(Y_1Y_2Y_3, Y_1Y_3R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.1) and (S9.2):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.1}$$

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.2}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\begin{aligned}\frac{\omega_m(i,i',k)}{\omega(i,i',k)} &= \exp\Big(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\Big) \times M^{\omega}_m(i,i',k),\\ \frac{\omega_n(i,i',k)}{\omega(i,i',k)} &= \exp\Big(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\Big) \times M^{\omega}_n(i,i',k).\end{aligned}$$

where

$$\begin{aligned}M^{\omega}_m(i,i',k) &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3})},\\ M^{\omega}_n(i,i',k) &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3})}.\end{aligned}$$

By Eq. (S9.1) and (S9.2), $M^{\omega}_m(i,i',k) > 1$ and $M^{\omega}_n(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s and $\lambda^{ik\ell}_{Y_1Y_3R_2}$'s. Thus, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) < -0.5\log M^{\omega}_m(i,i',k)$ or $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) > -0.5\log M^{\omega}_n(i,i',k)$.

(2) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.3) and (S9.4):

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} > \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.3}$$

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} < \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.4}$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\begin{aligned}\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} &= \exp\Big(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\Big) \times M^{\gamma}_m(k,k',i),\\ \frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} &= \exp\Big(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\Big) \times M^{\gamma}_n(k,k',i).\end{aligned}$$

where

$$M_m^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk})},$$

$$M_n^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink})}.$$

By Eq. (S9.3) and (S9.4), $M_m^\gamma(k,k',i) > 1$ and $M_n^\gamma(k,k',i) < 1$ in the absence of all $\lambda_{Y_3R_2}^{k\ell}$'s and $\lambda_{Y_1Y_3R_2}^{ik\ell}$'s. Thus, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^\gamma(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_1Y_3R_2}^{ik'2} - \lambda_{Y_1Y_3R_2}^{ik2}) < -0.5 \log M_m^\gamma(k,k',i)$ or $(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + (\lambda_{Y_1Y_3R_2}^{ik'2} - \lambda_{Y_1Y_3R_2}^{ik2}) > -0.5 \log M_n^\gamma(k,k',i)$.

2) Model $(Y_1Y_2Y_3, Y_2Y_3R_2)$

(1) Since $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$, we have two inequalities in Eq. (S9.5) and (S9.6):

$$\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'm} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'mk} > \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{im} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk}, \tag{S9.5}$$

$$\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'n} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'nk} < \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{in} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink}. \tag{S9.6}$$

Now we can represent $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3R_2}^{jk2} + \lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'm} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'mk})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3R_2}^{jk2} + \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{im} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk})},$$

$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3R_2}^{jk2} + \lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'n} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'nk})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3R_2}^{jk2} + \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{in} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink})}.$$

By Eq. (S9.5) and (S9.6), we can see $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} > 1$ and $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} < 1$, and thus $\omega(i,i',k) \in OI^\omega(i,i',k)$.

(2) By the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.7) and (S9.8):

$$\begin{aligned} &\lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'} + \lambda_{Y_2Y_3R_2}^{jk'1} - \lambda_{Y_2Y_3R_2}^{mk'1} \\ &> \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{mk1}, \end{aligned} \tag{S9.7}$$

$$\begin{aligned} &\lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'} + \lambda_{Y_2Y_3R_2}^{jk'1} - \lambda_{Y_2Y_3R_2}^{nk'1} \\ &< \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{nk1}. \end{aligned} \tag{S9.8}$$

We express $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} = \exp(2\lambda_{Y_3R_2}^{k'2} - 2\lambda_{Y_3R_2}^{k2}) \times M_m^\gamma(k,k',i),$$

$$\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} = \exp(2\lambda_{Y_3R_2}^{k'2} - 2\lambda_{Y_3R_2}^{k2}) \times M_n^\gamma(k,k',i)$$

where

$$M_m^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda^m(i,j,k') + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk'2}})}{\sum_j \exp(\lambda^m(i,j,k) + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk2}})},\quad M_n^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda^n(i,j,k') + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk'2}})}{\sum_j \exp(\lambda^n(i,j,k) + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk2}})},$$

$$\lambda^m(i,j,k) = \lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk} + \lambda_{Y_2Y_3R2}^{jk1} - \lambda_{Y_2Y_3R_2}^{mk1},$$

$$\lambda^n(i,j,k) = \lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink} + \lambda_{Y_2Y_3R2}^{jk1} - \lambda_{Y_2Y_3R_2}^{nk1}.$$

Note that, by Eq. (S9.7) and (S9.8), $\lambda^m(i,j,k') > \lambda^m(i,j,k)$ and $\lambda^n(i,j,k') < \lambda^n(i,j,k)$ for all j's. However, when $M_m^\gamma(k,k',i) < 1$, there exists at least one j such that $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2}$ is negative (i.e., $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2} < 0.5(\lambda^m(i,j,k) - \lambda^m(i,j,k')) < 0$). Similarly, $M_n^\gamma(k,k',i) > 1$ guarantees the positivity of $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2}$ for at least one j (i.e., $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2} > 0.5(\lambda^n(i,j,k) - \lambda^n(i,j,k')) > 0$).

Therefore, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^\gamma(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5\log M_m^\gamma(k,k',i)$ or $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} > -0.5\log M_n^\gamma(k,k',i)$ where $M_m^\gamma(k,k',i)$ and $M_n^\gamma(k,k',i)$ depend on $\lambda^{jk'2}_{Y_2Y_3R_2}$ and $\lambda^{jk2}_{Y_2Y_3R_2}$ for all j's.

3) Model $(Y_1Y_2Y_3, Y_1Y_2R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.10) and (S9.11):

$$\begin{aligned}&\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} + \lambda^{i'j1}_{Y_1Y_2R_2} - \lambda^{i'm1}_{Y_1Y_2R_2} \\ &> \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3} + \lambda^{ij1}_{Y_1Y_2R_2} - \lambda^{im1}_{Y_1Y_2R_2},\end{aligned} \tag{S9.9}$$

$$\begin{aligned}&\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} + \lambda^{i'j1}_{Y_1Y_2R_2} - \lambda^{i'n1}_{Y_1Y_2R_2} \\ &< \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3} + \lambda^{ij1}_{Y_1Y_2R_2} - \lambda^{in1}_{Y_1Y_2R_2}.\end{aligned} \tag{S9.10}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\begin{aligned}\frac{\omega_m(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2} - 2\lambda^{i2}_{Y_1R_2}) \times M_m^\omega(i,i',k), \\ \frac{\omega_n(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2} - 2\lambda^{i2}_{Y_1R_2}) \times M_n^\omega(i,i',k),\end{aligned}$$

where

$$\begin{aligned}M_m^\omega(i,i',k) &= \frac{\sum_j \exp(\lambda^m(i',j,k) + \mathbf{2\lambda^{i'j2}_{Y_1Y_2R_2}})}{\sum_j \exp(\lambda^m(i,j,k) + \mathbf{2\lambda^{ij2}_{Y_1Y_2R_2}})},\ M_n^\omega(i,i',k) = \frac{\sum_j \exp(\lambda^n(i',j,k) + \mathbf{2\lambda^{i'j2}_{Y_1Y_2R_2}})}{\sum_j \exp(\lambda^n(i,j,k) + \mathbf{2\lambda^{ij2}_{Y_1Y_2R_2}})}, \\ \lambda^m(i,j,k) &= \lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3} + \lambda^{ij1}_{Y_1Y_2R_2} - \lambda^{im1}_{Y_1Y_2R_2}, \\ \lambda^n(i,j,k) &= \lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3} + \lambda^{ij1}_{Y_1Y_2R_2} - \lambda^{in1}_{Y_1Y_2R_2}.\end{aligned}$$

Note that, by Eq. (S9.10) and (S9.11), $\lambda^m(i',j,k) > \lambda^m(i,j,k)$ and $\lambda^n(i',j,k) < \lambda^n(i,j,k)$ for all j's. However, when $M_m^\omega(i,i',k) < 1$, there exists at least one j such that $\lambda^{i'j2}_{Y_1Y_2R_2} - \lambda^{ij2}_{Y_1Y_2R_2}$ is negative (i.e., $\lambda^{i'j2}_{Y_1Y_2R_2} - \lambda^{ij2}_{Y_1Y_2R_2} < 0.5(\lambda^m(i,j,k) - \lambda^m(i',j,k)) < 0$). Similarly, $M_n^\omega(i,i',k) > 1$ guarantees the positivity of $\lambda^{i'j2}_{Y_1Y_2R_2} - \lambda^{ij2}_{Y_1Y_2R_2}$ for at least one j (i.e., $\lambda^{i'j2}_{Y_1Y_2R_2} - \lambda^{ij2}_{Y_1Y_2R_2} > 0.5(\lambda^n(i,j,k) - \lambda^n(i',j,k)) > 0$).

Therefore, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^\omega(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} < -0.5\log M_m^\omega(i,i',k)$ or $\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} > -0.5\log M_n^\omega(i,i',k)$ where $M_m^\omega(i,i',k)$ and $M_n^\omega(i,i',k)$ depend on $\lambda^{i'j2}_{Y_1Y_2R_2}$ and $\lambda^{ij2}_{Y_1Y_2R_2}$ for all j's.

(2) Since $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(i,i',k) < \gamma_j(k,k',i)$ for all $j(\neq n)$, we have two inequalities in Eq. (S9.11) and (S9.12):

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} > \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.11}$$

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} < \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.12}$$

Now we can represent $\gamma_m(i,i',k)/\gamma(i,i',k)$ and $\gamma_n(i,i',k)/\gamma(i,i',k)$ as follows :

$$\begin{aligned}\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} &= \frac{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij2}_{Y_1Y_2R_2}+\lambda^{jk'}_{Y_2Y_3}-\lambda^{mk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{imk'}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij2}_{Y_1Y_2R_2}+\lambda^{jk}_{Y_2Y_3}-\lambda^{mk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{imk}_{Y_1Y_2Y_3})},\\ \frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} &= \frac{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij2}_{Y_1Y_2R_2}+\lambda^{jk'}_{Y_2Y_3}-\lambda^{nk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{ink'}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j2}_{Y_2R_2}+\lambda^{ij2}_{Y_1Y_2R_2}+\lambda^{jk}_{Y_2Y_3}-\lambda^{nk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{ink}_{Y_1Y_2Y_3})}.\end{aligned}$$

By Eq. (S9.11) and (S9.12), we can see $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} > 1$ and $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} < 1$, and thus $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$.

4) Model $(Y_1Y_2Y_3, Y_1R_2, Y_3R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.13) and (S9.14):

$$\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}+\lambda^{i'jk}_{Y_1Y_2Y_3}-\lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{imk}_{Y_1Y_2Y_3}, \quad \text{(S9.13)}$$
$$\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}+\lambda^{i'jk}_{Y_1Y_2Y_3}-\lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{ink}_{Y_1Y_2Y_3}. \quad \text{(S9.14)}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\begin{aligned}\frac{\omega_m(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2}-2\lambda^{i2}_{Y_1R_2})\times M^{\omega}_m(i,i',k),\\ \frac{\omega_n(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2}-2\lambda^{i2}_{Y_1R_2})\times M^{\omega}_n(i,i',k).\end{aligned}$$

where

$$\begin{aligned}M^{\omega}_m(i,i',k) &= \frac{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{jk}_{Y_2Y_3}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'm}_{Y_1Y_2}+\lambda^{i'jk}_{Y_1Y_2Y_3}-\lambda^{i'mk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{jk}_{Y_2Y_3}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{imk}_{Y_1Y_2Y_3})},\\ M^{\omega}_n(i,i',k) &= \frac{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{jk}_{Y_2Y_3}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}+\lambda^{i'jk}_{Y_1Y_2Y_3}-\lambda^{i'nk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^{j}_{Y_2}+\lambda^{jk}_{Y_2Y_3}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{ink}_{Y_1Y_2Y_3})}.\end{aligned}$$

By Eq. (S9.13) and (S9.14), $M^{\omega}_m(i,i',k) > 1$ and $M^{\omega}_n(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s. Thus, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $\lambda^{i'2}_{Y_1R_2}-\lambda^{i2}_{Y_1R_2} < -0.5\log M^{\omega}_m(i,i',k)$ or $\lambda^{i'2}_{Y_1R_2}-\lambda^{i2}_{Y_1R_2} > -0.5\log M^{\omega}_n(i,i',k)$.

(2) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.15) and (S9.16):

$$\lambda^{jk'}_{Y_2Y_3}-\lambda^{mk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{imk'}_{Y_1Y_2Y_3} > \lambda^{jk}_{Y_2Y_3}-\lambda^{mk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{imk}_{Y_1Y_2Y_3}, \quad \text{(S9.15)}$$
$$\lambda^{jk'}_{Y_2Y_3}-\lambda^{nk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{ink'}_{Y_1Y_2Y_3} < \lambda^{jk}_{Y_2Y_3}-\lambda^{nk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{ink}_{Y_1Y_2Y_3}. \quad \text{(S9.16)}$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\begin{aligned}\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} &= \exp(2\lambda^{k'2}_{Y_3R_2}-2\lambda^{k2}_{Y_3R_2})\times M^{\gamma}_m(k,k',i),\\ \frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} &= \exp(2\lambda^{k'2}_{Y_3R_2}-2\lambda^{k2}_{Y_3R_2})\times M^{\gamma}_n(k,k',i).\end{aligned}$$

where

$$M_m^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk})},$$

$$M_n^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink})}.$$

By Eq. (S9.15) and (S9.16), $M_m^\gamma(k,k',i) > 1$ and $M_n^\gamma(k,k',i) < 1$ in the absence of all $\lambda_{Y_3R_2}^{k\ell}$'s. Thus, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^\gamma(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} < -0.5 \log M_m^\gamma(k,k',i)$ or $\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} > -0.5 \log M_n^\gamma(k,k',i)$.

5) Model $(Y_1Y_2Y_3, Y_2R_2, Y_3R_2)$

(1) Since $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$, we have two inequalities in Eq. (S9.17) and (S9.18):

$$\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'm} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'mk} > \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{im} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk}, \tag{S9.17}$$

$$\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'n} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'nk} < \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{in} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink}. \tag{S9.18}$$

Now we can represent $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'm} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'mk})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{im} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk})},$$

$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'n} + \lambda_{Y_1Y_2Y_3}^{i'jk} - \lambda_{Y_1Y_2Y_3}^{i'nk})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_2Y_3}^{jk} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{in} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink})}.$$

By Eq. (S9.17) and (S9.18), we can see $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} > 1$ and $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} < 1$, and thus $\omega(i,i',k) \in OI^\omega(i,i',k)$.

(2) By the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.19) and (S9.20):

$$\lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'} > \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk}, \tag{S9.19}$$

$$\lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'} < \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink}. \tag{S9.20}$$

We express $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} = \exp(2\lambda_{Y_3R_2}^{k'2} - 2\lambda_{Y_3R_2}^{k2}) \times M_m^\gamma(k,k',i),$$

$$\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} = \exp(2\lambda_{Y_3R_2}^{k'2} - 2\lambda_{Y_3R_2}^{k2}) \times M_n^\gamma(k,k',i)$$

where

$$M_m^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk})},$$

$$M_n^\gamma(k,k',i) = \frac{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'})}{\sum_j \exp(\lambda_{Y_2}^j + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink})}.$$

By Eq. (S9.19) and (S9.20), the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} < -0.5\log M^{\gamma}_m(k,k',i)$ or $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} > -0.5\log M^{\gamma}_n(k,k',i)$.

6) Model $(Y_1Y_2Y_3, Y_1R_2, Y_2R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.21) and (S9.22):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.21}$$
$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.22}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\begin{aligned}\frac{\omega_m(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2} - 2\lambda^{i2}_{Y_1R_2}) \times M^{\omega}_m(i,i',k),\\ \frac{\omega_n(i,i',k)}{\omega(i,i',k)} &= \exp(2\lambda^{i'2}_{Y_1R_2} - 2\lambda^{i2}_{Y_1R_2}) \times M^{\omega}_n(i,i',k).\end{aligned}$$

where

$$\begin{aligned}M^{\omega}_m(i,i',k) &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3})},\\ M^{\omega}_n(i,i',k) &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3})}.\end{aligned}$$

By Eq. (S9.21) and (S9.22), $M^{\omega}_m(i,i',k) > 1$ and $M^{\omega}_n(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s. Thus, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} < -0.5\log M^{\omega}_m(i,i',k)$ or $\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2} > -0.5\log M^{\omega}_n(i,i',k)$.

(2) Since $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(i,i',k) < \gamma_j(k,k',i)$ for all $j(\neq n)$, we have two inequalities in Eq. (S9.23) and (S9.24):

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} > \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.23}$$
$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} < \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.24}$$

Now we can represent $\gamma_m(i,i',k)/\gamma(i,i',k)$ and $\gamma_n(i,i',k)/\gamma(i,i',k)$ as follows :

$$\begin{aligned}\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3})},\\ \frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} &= \frac{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3})}{\sum_j \exp(\lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3})}.\end{aligned}$$

By Eq. (S9.23) and (S9.24), we can see $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} > 1$ and $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} < 1$, and thus $\gamma(k,k',i) \in OI^{\gamma}(k,k',i)$.

S9.2 Proof of Theorem S2 in S7.2.2

1) Model $(Y_1Y_2Y_3, Y_1Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.25) and (S9.26):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} \quad > \quad \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \qquad \text{(S9.25)}$$

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} \quad < \quad \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \qquad \text{(S9.26)}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \exp\left(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\right) \times M^{\omega}_m(i,i',k),$$

$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \exp\left(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\right) \times M^{\omega}_n(i,i',k).$$

where

$$M^{\omega}_m(i,i',k) = \frac{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}\right)},$$

$$M^{\omega}_n(i,i',k) = \frac{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.25) and (S9.26), $M^{\omega}_m(i,i',k) > 1$ and $M^{\omega}_n(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s and $\lambda^{ik\ell}_{Y_1Y_3R_2}$'s. Thus, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) < -0.5 \log M^{\omega}_m(i,i',k)$ or $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) > -0.5 \log M^{\omega}_n(i,i',k)$.

(2) By the definition of $\nu_m(j,j',k)$ and $\nu_n(j,j',k)$, $\nu_m(j,j',k) > \nu_i(j,j',k)$ for all $i(\neq m)$ and $\nu_n(j,j',k) < \nu_i(j,j',k)$ for all $i(\neq n)$ lead to two inequalities in Eq. (S9.27) and (S9.28):

$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3} \quad > \quad \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}, \qquad \text{(S9.27)}$$

$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3} \quad < \quad \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}. \qquad \text{(S9.28)}$$

We now express $\nu_m(j,j',k)/\nu(j,j',k)$ and $\nu_n(j,j',k)/\nu(j,j',k)$ as follows :

$$\frac{\nu_m(j,j',k)}{\nu(j,j',k)} = \frac{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}\right)},$$

$$\frac{\nu_n(j,j',k)}{\nu(j,j',k)} = \frac{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.27) and (S9.28), we can see $\frac{\nu_m(j,j',k)}{\nu(j,j',k)} > 1$ and $\frac{\nu_n(j,j',k)}{\nu(j,j',k)} < 1$, and thus $\nu(j,j',k) \in OI^{\nu}(j,j',k)$.

(3) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in

Eq. (S9.29) and (S9.30):

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} > \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \quad \text{(S9.29)}$$

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} < \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \quad \text{(S9.30)}$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + 2(\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2})\Big) \times M^{\gamma}_m(k,k',i),$$

$$\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{ik'2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + 2(\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2})\Big) \times M^{\gamma}_n(k,k',i).$$

where

$$M^{\gamma}_m(j,j',k) = \frac{\sum_j \exp\Big(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3}\Big)}{\sum_j \exp\Big(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}\Big)},$$

$$M^{\gamma}_n(j,j',k) = \frac{\sum_j \exp\Big(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3}\Big)}{\sum_j \exp\Big(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}\Big)}.$$

By Eq. (S9.29) and (S9.30), $M^{\gamma}_m(k,k',i) > 1$ and $M^{\gamma}_n(k,k',i) < 1$ in the absence of all $\lambda^{km}_{Y_3R_2}$'s, $\lambda^{ikm}_{Y_1Y_3R_2}$'s and $\lambda^{k\ell m}_{Y_3R_1R_2}$'s. Thus, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',j)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + (\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2}) < -0.5\log M^{\gamma}_m(k,k',i)$ or $(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) + (\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2}) > -0.5\log M^{\gamma}_n(k,k',i)$.

(4) By the definition of $\eta_m(k,k',j)$ and $\eta_n(k,k',j)$, $\eta_m(k,k',j) > \eta_i(k,k',j)$ for all $i(\neq m)$ and $\eta_n(k,k',j) < \eta_i(k,k',j)$ for all $i(\neq n)$ result in two inequalities in Eq. (S9.31) and (S9.32):

$$\begin{aligned} & \lambda^{ik'}_{Y_1Y_3} - \lambda^{mk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{mjk'}_{Y_1Y_2Y_3} + \lambda^{ik'1}_{Y_1Y_3R_1} - \lambda^{mk'1}_{Y_1Y_3R_1} + \lambda^{ik'1}_{Y_1Y_3R_2} - \lambda^{mk'1}_{Y_1Y_3R_2} \\ > \; & \lambda^{ik}_{Y_1Y_3} - \lambda^{mk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{mk1}_{Y_1Y_3R_1} + \lambda^{ik1}_{Y_1Y_3R_2} - \lambda^{mk1}_{Y_1Y_3R_2}, \quad \text{(S9.31)} \\ & \lambda^{ik'}_{Y_1Y_3} - \lambda^{nk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{njk'}_{Y_1Y_2Y_3} + \lambda^{ik'1}_{Y_1Y_3R_1} - \lambda^{nk'1}_{Y_1Y_3R_1} + \lambda^{ik'1}_{Y_1Y_3R_2} - \lambda^{nk'1}_{Y_1Y_3R_2} \\ < \; & \lambda^{ik}_{Y_1Y_3} - \lambda^{nk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{nk1}_{Y_1Y_3R_1} + \lambda^{ik1}_{Y_1Y_3R_2} - \lambda^{nk1}_{Y_1Y_3R_2}. \quad \text{(S9.32)} \end{aligned}$$

We represent $\eta_m(k,k',j)/\eta(k,k',j)$ and $\eta_n(k,k',j)/\eta(k,k',j)$ as follows :

$$\frac{\eta_m(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\Big) \times M^{\eta}_m(k,k',j),$$

$$\frac{\eta_n(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\Big) \times M^{\eta}_n(k,k',j).$$

where

$$M^{\gamma}_m(k,k',j) = \frac{\sum_i \exp(\lambda^m(i,j,k') + \mathbf{2\lambda^{ik'2}_{Y_1Y_3R_1}})}{\sum_i \exp(\lambda^m(i,j,k) + \mathbf{2\lambda^{ik2}_{Y_1Y_3R_1}})}, \; M^{\gamma}_n(k,k',j) = \frac{\sum_i \exp(\lambda^n(i,j,k') + \mathbf{2\lambda^{ik'2}_{Y_1Y_3R_1}})}{\sum_i \exp(\lambda^n(i,j,k) + \mathbf{2\lambda^{ik2}_{Y_1Y_3R_1}})},$$

$$\lambda^m(i,j,k) = \lambda^{i}_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik}_{Y_1Y_3} - \lambda^{mk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{mk1}_{Y_1Y_3R_1} + \lambda^{ik1}_{Y_1Y_3R_2} - \lambda^{mk2}_{Y_1Y_3R_1},$$

$$\lambda^n(i,j,k) = \lambda^{i}_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik}_{Y_1Y_3} - \lambda^{nk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{nk1}_{Y_1Y_3R_1} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{nk1}_{Y_1Y_3R_1}.$$

Note that, by Eq. (S9.31) and (S9.32), $\lambda^m(i,j,k') > \lambda^m(i,j,k)$ and $\lambda^n(i,j,k') < \lambda^n(i,j,k)$ for all i's. However, when $M^{\eta}_m(k,k',j) < 1$, there exists at least one i such that $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1}$ is negative (i.e., $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1} < 0.5(\lambda^m(i,j,k) - \lambda^m(i,j,k')) < 0$). Similarly, $M^\eta_n(k,k',j) > 1$ guarantees the positivity of $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1}$ for at least one i (i.e., $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1} > 0.5(\lambda^n(i,j,k) - \lambda^n(i,j,k')) > 0$).

Therefore, the necessary and sufficient condition for $\eta(k,k',j) \notin OI^\eta(k,k',j)$ (i.e., $\frac{\eta_m(k,k',j)}{\eta(k,k',j)} < 1$ or $\frac{\eta_n(k,k',j)}{\eta(k,k',j)} > 1$) is $\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1} + \lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2} < -0.5\log M^\eta_m(k,k',j)$ or $\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1} + \lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2} > -0.5\log M^\eta_n(k,k',j)$ where $M^\eta_m(k,k',j)$ and $M^\eta_n(k,k',j)$ depend on $\lambda^{ik'2}_{Y_1Y_3R_1}$ and $\lambda^{ik2}_{Y_1Y_3R_1}$ for all i's.

2) Model $(Y_1Y_2Y_3, Y_2Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.33) and (S9.34):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \quad \text{(S9.33)}$$
$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \quad \text{(S9.34)}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp\left(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}\right)},$$
$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp\left(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.33) and (S9.34), we can see $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} > 1$ and $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} < 1$, and thus $\omega(i,i',k) \in OI^\omega(i,i',k)$.

(2) By the definition of $\nu_m(j,j',k)$ and $\nu_n(j,j',k)$, $\nu_m(j,j',k) > \nu_i(j,j',k)$ for all $i(\neq m)$ and $\nu_n(j,j',k) < \nu_i(j,j',k)$ for all $i(\neq n)$ lead to two inequalities in Eq. (S9.35) and (S9.36):

$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}, \quad \text{(S9.35)}$$
$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}. \quad \text{(S9.36)}$$

We now express $\nu_m(j,j',k)/\nu(j,j',k)$ and $\nu_n(j,j',k)/\nu(j,j',k)$ as follows :

$$\frac{\nu_m(j,j',k)}{\nu(j,j',k)} = \exp\left(2(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + 2(\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1})\right) \times M^\nu_m(j,j',k),$$
$$\frac{\nu_n(j,j',k)}{\nu(j,j',k)} = \exp\left(2(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + 2(\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1})\right) \times M^\nu_n(j,j',k).$$

where

$$M^\nu_m(j,j',k) = \frac{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}\right)},$$
$$M^\nu_n(j,j',k) = \frac{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.35) and (S9.36), $M_m^{\nu}(j,j',k) > 1$ and $M_n^{\nu}(j,j',k) < 1$ in the absence of all $\lambda_{Y_2R_1}^{j\ell}$'s and $\lambda_{Y_2Y_3R_2}^{jk\ell}$'s. Thus, the necessary and sufficient condition for $\nu(j,j',k) \notin OI^{\nu}(j,j',k)$ (i.e., $\frac{\nu_m(j,j',k)}{\nu(j,j',k)} < 1$ or $\frac{\nu_n(j,j',k)}{\nu(j,j',k)} > 1$) is $(\lambda_{Y_2R_1}^{j'2} - \lambda_{Y_2R_1}^{j2}) + (\lambda_{Y_2Y_3R_1}^{j'k2} - \lambda_{Y_2Y_3R_1}^{jk2}) < -0.5\log M_m^{\nu}(j,j',k)$ or $(\lambda_{Y_2R_1}^{j'2} - \lambda_{Y_2R_1}^{j2}) + (\lambda_{Y_2Y_3R_1}^{j'k2} - \lambda_{Y_2Y_3R_1}^{jk2}) > -0.5\log M_n^{\nu}(j,j',k)$.

(3) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.37) and (S9.38):

$$\begin{aligned} & \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{imk'} + \lambda_{Y_2Y_3R_1}^{jk'1} - \lambda_{Y_2Y_3R_1}^{mk'1} + \lambda_{Y_2Y_3R_2}^{jk'1} - \lambda_{Y_2Y_3R_2}^{mk'1} \\ > \; & \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk} + \lambda_{Y_2Y_3R_1}^{jk1} - \lambda_{Y_2Y_3R_1}^{mk1} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{mk1}, \quad \text{(S9.37)} \\ & \lambda_{Y_2Y_3}^{jk'} - \lambda_{Y_2Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{ink'} + \lambda_{Y_2Y_3R_1}^{jk'1} - \lambda_{Y_2Y_3R_1}^{nk'1} + \lambda_{Y_2Y_3R_2}^{jk'1} - \lambda_{Y_2Y_3R_2}^{nk'1} \\ < \; & \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink} + \lambda_{Y_2Y_3R_1}^{jk1} - \lambda_{Y_2Y_3R_1}^{nk1} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{nk'1}. \quad \text{(S9.38)} \end{aligned}$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\begin{aligned} \frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} &= \exp\Big(2(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + 2(\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12})\Big) \times M_m^{\gamma}(k,k',i), \\ \frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} &= \exp\Big(2(\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2}) + 2(\lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12})\Big) \times M_n^{\gamma}(k,k',i). \end{aligned}$$

where

$$\begin{aligned} M_m^{\gamma}(k,k',i) &= \frac{\sum_j \exp(\lambda^m(i,j,k') + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk'2}})}{\sum_j \exp(\lambda^m(i,j,k) + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk2}})},\ M_n^{\gamma}(k,k',i) = \frac{\sum_j \exp(\lambda^n(i,j,k') + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk'2}})}{\sum_j \exp(\lambda^n(i,j,k) + \mathbf{2\lambda_{Y_2Y_3R_2}^{jk2}})}, \\ \lambda^m(i,j,k) &= \lambda_{Y_2}^{j} + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{imk} + \lambda_{Y_2Y_3R_1}^{jk1} - \lambda_{Y_2Y_3R_1}^{mk1} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{mk1}, \\ \lambda^n(i,j,k) &= \lambda_{Y_2}^{j} + \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_2R_2}^{j2} + \lambda_{Y_2Y_3}^{jk} - \lambda_{Y_2Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{ink} + \lambda_{Y_2Y_3R_1}^{jk1} - \lambda_{Y_2Y_3R_1}^{nk1} + \lambda_{Y_2Y_3R_2}^{jk1} - \lambda_{Y_2Y_3R_2}^{nk1}. \end{aligned}$$

Note that, by Eq. (S9.37) and (S9.38), $\lambda^m(i,j,k') > \lambda^m(i,j,k)$ and $\lambda^n(i,j,k') < \lambda^n(i,j,k)$ for all j's. However, when $M_m^{\gamma}(k,k',i) < 1$, there exists at least one j such that $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2}$ is negative (i.e., $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2} < 0.5(\lambda^m(i,j,k) - \lambda^m(i,j,k')) < 0$). Similarly, $M_n^{\gamma}(k,k',i) > 1$ guarantees the positivity of $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2}$ for at least one j (i.e., $\lambda_{Y_2Y_3R_2}^{jk'2} - \lambda_{Y_2Y_3R_2}^{jk2} > 0.5(\lambda^n(i,j,k) - \lambda^n(i,j,k')) > 0$).

Therefore, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^{\gamma}(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} + \lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12} < -0.5\log M_m^{\gamma}(k,k',i)$ or $\lambda_{Y_3R_2}^{k'2} - \lambda_{Y_3R_2}^{k2} + \lambda_{Y_3R_1R_2}^{k'12} - \lambda_{Y_3R_1R_2}^{k12} > -0.5\log M_n^{\gamma}(k,k',i)$ where $M_m^{\gamma}(k,k',i)$ and $M_n^{\gamma}(k,k',i)$ depend on $\lambda_{Y_2Y_3R_2}^{jk'2}$ and $\lambda_{Y_2Y_3R_2}^{jk2}$ for all j's.

(4) By the definition of $\eta_m(k,k',j)$ and $\eta_n(k,k',j)$, $\eta_m(k,k',j) > \eta_i(k,k',j)$ for all $i(\neq m)$ and $\eta_n(k,k',j) < \eta_i(k,k',j)$ for all $i(\neq n)$ result in two inequalities in Eq. (S9.39) and (S9.40):

$$\begin{aligned} \lambda_{Y_1Y_3}^{ik'} - \lambda_{Y_1Y_3}^{mk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{mjk'} &> \lambda_{Y_1Y_3}^{ik} - \lambda_{Y_1Y_3}^{mk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{mjk}, \quad \text{(S9.39)} \\ \lambda_{Y_1Y_3}^{ik'} - \lambda_{Y_1Y_3}^{nk'} + \lambda_{Y_1Y_2Y_3}^{ijk'} - \lambda_{Y_1Y_2Y_3}^{njk'} &> \lambda_{Y_1Y_3}^{ik} - \lambda_{Y_1Y_3}^{nk} + \lambda_{Y_1Y_2Y_3}^{ijk} - \lambda_{Y_1Y_2Y_3}^{njk}. \quad \text{(S9.40)} \end{aligned}$$

We represent $\eta_m(k,k',j)/\eta(k,k',j)$ and $\eta_n(k,k',j)/\eta(k,k',j)$ as follows :

$$\frac{\eta_m(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{jk'2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\Big) \times M^{\eta}_m(k,k',j),$$
$$\frac{\eta_n(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{jk'2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\Big) \times M^{\eta}_n(k,k',j).$$

where

$$M^{\eta}_m(k,k',j) = \frac{\sum_i \exp\Big(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik'}_{Y_1Y_3} - \lambda^{mk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{mjk'}_{Y_1Y_2Y_3}\Big)}{\sum_i \exp\Big(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} - \lambda^{mk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}\Big)},$$
$$M^{\eta}_n(k,k',j) = \frac{\sum_i \exp\Big(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik'}_{Y_1Y_3} - \lambda^{nk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{njk'}_{Y_1Y_2Y_3}\Big)}{\sum_i \exp\Big(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{ik}_{Y_1Y_3} - \lambda^{nk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}\Big)}.$$

By Eq. (S9.39) and (S9.40), $M^{\eta}_m(k,k',j) > 1$ and $M^{\eta}_n(k,k',j) < 1$ in the absence of all $\lambda^{k\ell}_{Y_3R_1}$'s, $\lambda^{jk\ell}_{Y_2Y_3R_1}$'s and $\lambda^{k\ell m}_{Y_3R_1R_2}$'s. Thus, the necessary and sufficient condition for $\eta(k,k',j) \notin OI^{\eta}(k,k',j)$ (i.e., $\frac{\eta_m(k,k',j)}{\eta(k,k',j)} < 1$ or $\frac{\eta_n(k,k',j)}{\eta(k,k',j)} > 1$) is $(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) + (\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2}) < -0.5\log M^{\eta}_m(k,k',j)$ or $(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) + (\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2}) > -0.5\log M^{\eta}_n(k,k',j)$.

3) Model $(Y_1Y_2Y_3, Y_1Y_3R_1, Y_2Y_3R_2, Y_3R_1R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.41) and (S9.42):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \quad \text{(S9.41)}$$
$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \quad \text{(S9.42)}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp\Big(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3}\Big)}{\sum_j \exp\Big(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}\Big)},$$
$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \frac{\sum_j \exp\Big(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3}\Big)}{\sum_j \exp\Big(\lambda^j_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk2}_{Y_2Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}\Big)}.$$

By Eq. (S9.41) and (S9.42), we can see $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} > 1$ and $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} < 1$, and thus $\omega(i,i',k) \in OI^{\omega}(i,i',k)$.

(2) By the definition of $\nu_m(j,j',k)$ and $\nu_n(j,j',k)$, $\nu_m(j,j',k) > \nu_i(j,j',k)$ for all $i(\neq m)$ and $\nu_n(j,j',k) < \nu_i(j,j',k)$ for all $i(\neq n)$ lead to two inequalities in Eq. (S9.43) and (S9.44):

$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}, \quad \text{(S9.43)}$$
$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}. \quad \text{(S9.44)}$$

We now express $\nu_m(j,j',k)/\nu(j,j',k)$ and $\nu_n(j,j',k)/\nu(j,j',k)$ as follows :

$$\frac{\nu_m(j,j',k)}{\nu(j,j',k)} = \frac{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}\right)},$$
$$\frac{\nu_n(j,j',k)}{\nu(j,j',k)} = \frac{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^i_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik2}_{Y_1Y_3R_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.43) and (S9.44), we can see $\frac{\nu_m(j,j',k)}{\nu(j,j',k)} > 1$ and $\frac{\nu_n(j,j',k)}{\nu(j,j',k)} < 1$, and thus $\nu(j,j',k) \in OI^\nu(j,j',k)$.

(3) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.46) and (S9.47):

$$\begin{aligned}
&\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_2} - \lambda^{mk'1}_{Y_2Y_3R_2} \qquad \text{(S9.45)}\\
&> \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3} + \lambda^{jk1}_{Y_2Y_3R_2} - \lambda^{mk1}_{Y_2Y_3R_2},\\
&\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_2} - \lambda^{nk'1}_{Y_2Y_3R_2} \qquad \text{(S9.46)}\\
&< \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_2} - \lambda^{nk'1}_{Y_2Y_3R_2}.
\end{aligned}$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} = \exp\left(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2})\right) \times M^\gamma_m(k,k',i),$$
$$\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} = \exp\left(2(\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2}) + 2(\lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2})\right) \times M^\gamma_n(k,k',i).$$

where

$$M^\gamma_m(k,k',i) = \frac{\sum_j \exp(\lambda^m(i,j,k') + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{jk'2}}_{\mathbf{Y_2Y_3R_2}})}{\sum_j \exp(\lambda^m(i,j,k) + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{jk2}}_{\mathbf{Y_2Y_3R_2}})},\ M^\gamma_n(k,k',i) = \frac{\sum_j \exp(\lambda^n(i,j,k') + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{jk'2}}_{\mathbf{Y_2Y_3R_2}})}{\sum_j \exp(\lambda^n(i,j,k) + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{jk2}}_{\mathbf{Y_2Y_3R_2}})},$$
$$\lambda^m(i,j,k) = \lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3} + \lambda^{jk1}_{Y_2Y_3R_2} - \lambda^{mk1}_{Y_2Y_3R_2},$$
$$\lambda^n(i,j,k) = \lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2} + \lambda^{j2}_{Y_2R_2} + \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3} + \lambda^{jk1}_{Y_2Y_3R_2} - \lambda^{nk1}_{Y_2Y_3R_2}.$$

Note that, by Eq. (S9.46) and (S9.47), $\lambda^m(i,j,k') > \lambda^m(i,j,k)$ and $\lambda^n(i,j,k') < \lambda^n(i,j,k)$ for all j's. However, when $M^\gamma_m(k,k',i) < 1$, there exists at least one j such that $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2}$ is negative (i.e., $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2} < 0.5(\lambda^m(i,j,k) - \lambda^m(i,j,k')) < 0$). Similarly, $M^\gamma_n(k,k',i) > 1$ guarantees the positivity of $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2}$ for at least one j (i.e., $\lambda^{jk'2}_{Y_2Y_3R_2} - \lambda^{jk2}_{Y_2Y_3R_2} > 0.5(\lambda^n(i,j,k) - \lambda^n(i,j,k')) > 0$).

Therefore, the necessary and sufficient condition for $\gamma(k,k',i) \notin OI^\gamma(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} < 1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} > 1$) is $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} + \lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2} < -0.5 \log M^\gamma_m(k,k',i)$ or $\lambda^{k'2}_{Y_3R_2} - \lambda^{k2}_{Y_3R_2} + \lambda^{k'12}_{Y_3R_1R_2} - \lambda^{k12}_{Y_3R_1R_2} > -0.5 \log M^\gamma_n(k,k',i)$ where $M^\gamma_m(k,k',i)$ and $M^\gamma_n(k,k',i)$ depend on $\lambda^{jk'2}_{Y_2Y_3R_2}$ and $\lambda^{jk2}_{Y_2Y_3R_2}$ for all j's.

(4) By the definition of $\eta_m(k,k',j)$ and $\eta_n(k,k',j)$, $\eta_m(k,k',j) > \eta_i(k,k',j)$ for all $i(\neq m)$

and $\eta_n(k,k',j) < \eta_i(k,k',j)$ for all $i(\neq n)$ result in two inequalities in Eq. (S9.48) and (S9.49):

$$\lambda^{ik'}_{Y_1Y_3} - \lambda^{mk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{mjk'}_{Y_1Y_2Y_3} + \lambda^{ik'1}_{Y_1Y_3R_1} - \lambda^{mk'1}_{Y_1Y_3R_1} > \lambda^{ik}_{Y_1Y_3} - \lambda^{mk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{mk1}_{Y_1Y_3R_1}, \tag{S9.47}$$

$$\lambda^{ik'}_{Y_1Y_3} - \lambda^{nk'}_{Y_1Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{njk'}_{Y_1Y_2Y_3} + \lambda^{ik'1}_{Y_1Y_3R_1} - \lambda^{nk'1}_{Y_1Y_3R_1} < \lambda^{ik}_{Y_1Y_3} - \lambda^{nk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{nk1}_{Y_1Y_3R_1}. \tag{S9.48}$$

We represent $\eta_m(k,k',j)/\eta(k,k',j)$ and $\eta_n(k,k',j)/\eta(k,k',j)$ as follows :

$$\frac{\eta_m(k,k',j)}{\eta(k,k',j)} = \exp\left(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\right) \times M^{\eta}_m(k,k',j),$$
$$\frac{\eta_n(k,k',j)}{\eta(k,k',j)} = \exp\left(2(\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1}) + 2(\lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2})\right) \times M^{\eta}_n(k,k',j).$$

where

$$M^{\gamma}_m(k,k',j) = \frac{\sum_i \exp(\lambda^m(i,j,k') + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{ik'2}}_{\mathbf{Y_1Y_3R_1}})}{\sum_i \exp(\lambda^m(i,j,k) + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{ik2}}_{\mathbf{Y_1Y_3R_1}})},\ M^{\gamma}_n(k,k',j) = \frac{\sum_i \exp(\lambda^n(i,j,k') + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{ik'2}}_{\mathbf{Y_1Y_3R_1}})}{\sum_i \exp(\lambda^n(i,j,k) + \mathbf{2}\boldsymbol{\lambda}^{\mathbf{ik2}}_{\mathbf{Y_1Y_3R_1}})},$$
$$\lambda^m(i,j,k) = \lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik}_{Y_1Y_3} - \lambda^{mk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{mk1}_{Y_1Y_3R_1},$$
$$\lambda^n(i,j,k) = \lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} + \lambda^{i2}_{Y_1R_1} + \lambda^{ik}_{Y_1Y_3} - \lambda^{nk}_{Y_1Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3} + \lambda^{ik1}_{Y_1Y_3R_1} - \lambda^{nk1}_{Y_1Y_3R_1}.$$

Note that, by Eq. (S9.48) and (S9.49), $\lambda^m(i,j,k') > \lambda^m(i,j,k)$ and $\lambda^n(i,j,k') < \lambda^n(i,j,k)$ for all i's. However, when $M^{\eta}_m(k,k',j) < 1$, there exists at least one i such that $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1}$ is negative (i.e., $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1} < 0.5(\lambda^m(i,j,k) - \lambda^m(i,j,k')) < 0$). Similarly, $M^{\eta}_n(k,k',j) > 1$ guarantees the positivity of $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1}$ for at least one i (i.e., $\lambda^{ik'2}_{Y_1Y_3R_1} - \lambda^{ik2}_{Y_1Y_3R_1} > 0.5(\lambda^n(i,j,k) - \lambda^n(i,j,k')) > 0$).

Therefore, the necessary and sufficient condition for $\eta(k,k',j) \notin OI^{\eta}(k,k',j)$ (i.e., $\frac{\eta_m(k,k',j)}{\eta(k,k',j)} < 1$ or $\frac{\eta_n(k,k',j)}{\eta(k,k',j)} > 1$) is $\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1} + \lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2} < -0.5\log M^{\eta}_m(k,k',j)$ or $\lambda^{k'2}_{Y_3R_1} - \lambda^{k2}_{Y_3R_1} + \lambda^{k'21}_{Y_3R_1R_2} - \lambda^{k21}_{Y_3R_1R_2} > -0.5\log M^{\eta}_n(k,k',j)$ where $M^{\eta}_m(k,k',j)$ and $M^{\eta}_n(k,k',j)$ depend on $\lambda^{ik'2}_{Y_1Y_3R_1}$ and $\lambda^{ik2}_{Y_1Y_3R_1}$ for all i's.

4) Model $(Y_1Y_2Y_3, Y_2Y_3R_1, Y_1Y_3R_2, Y_3R_1R_2)$

(1) By the definition of $\omega_m(i,i',k)$ and $\omega_n(i,i',k)$, $\omega_m(i,i',k) > \omega_j(i,i',k)$ for all $j(\neq m)$ and $\omega_n(i,i',k) < \omega_j(i,i',k)$ for all $j(\neq n)$ lead to two inequalities in Eq. (S9.49) and (S9.50):

$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}, \tag{S9.49}$$
$$\lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}. \tag{S9.50}$$

We now express $\omega_m(i,i',k)/\omega(i,i',k)$ and $\omega_n(i,i',k)/\omega(i,i',k)$ as follows :

$$\frac{\omega_m(i,i',k)}{\omega(i,i',k)} = \exp\left(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\right) \times M^{\omega}_m(i,i',k),$$
$$\frac{\omega_n(i,i',k)}{\omega(i,i',k)} = \exp\left(2(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + 2(\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2})\right) \times M^{\omega}_n(i,i',k).$$

where

$$M_m^{\omega}(i,i',k) = \frac{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'mk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3}\right)},$$
$$M_n^{\omega}(i,i',k) = \frac{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'n}_{Y_1Y_2} + \lambda^{i'jk}_{Y_1Y_2Y_3} - \lambda^{i'nk}_{Y_1Y_2Y_3}\right)}{\sum_j \exp\left(\lambda^{j}_{Y_2} + \lambda^{jk}_{Y_2Y_3} + \lambda^{j1}_{Y_2R_1} + \lambda^{jk1}_{Y_2Y_3R_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{in}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.49) and (S9.50), $M_m^{\omega}(i,i',k) > 1$ and $M_n^{\omega}(i,i',k) < 1$ in the absence of all $\lambda^{i\ell}_{Y_1R_2}$'s and $\lambda^{ik\ell}_{Y_1Y_3R_2}$'s. Thus, the necessary and sufficient condition for $\omega(i,i',k) \notin OI^{\omega}(i,i',k)$ (i.e., $\frac{\omega_m(i,i',k)}{\omega(i,i',k)} < 1$ or $\frac{\omega_n(i,i',k)}{\omega(i,i',k)} > 1$) is $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) < -0.5\log M_m^{\omega}(i,i',k)$ or $(\lambda^{i'2}_{Y_1R_2} - \lambda^{i2}_{Y_1R_2}) + (\lambda^{i'k2}_{Y_1Y_3R_2} - \lambda^{ik2}_{Y_1Y_3R_2}) > -0.5\log M_n^{\omega}(i,i',k)$.

(2) By the definition of $\nu_m(j,j',k)$ and $\nu_n(j,j',k)$, $\nu_m(j,j',k) > \nu_i(j,j',k)$ for all $i(\neq m)$ and $\nu_n(j,j',k) < \nu_i(j,j',k)$ for all $i(\neq n)$ lead to two inequalities in Eq. (S9.51) and (S9.52):

$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3} > \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}, \quad \text{(S9.51)}$$
$$\lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3} < \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}. \quad \text{(S9.52)}$$

We now express $\nu_m(j,j',k)/\nu(j,j',k)$ and $\nu_n(j,j',k)/\nu(j,j',k)$ as follows :

$$\frac{\nu_m(j,j',k)}{\nu(j,j',k)} = \exp\left(2(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + 2(\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1})\right) \times M_m^{\nu}(j,j',k),$$
$$\frac{\nu_n(j,j',k)}{\nu(j,j',k)} = \exp\left(2(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + 2(\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1})\right) \times M_n^{\nu}(j,j',k).$$

where

$$M_m^{\nu}(j,j',k) = \frac{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{mj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{mjk}_{Y_1Y_2Y_3}\right)},$$
$$M_n^{\nu}(j,j',k) = \frac{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} + \lambda^{ij'k}_{Y_1Y_2Y_3} - \lambda^{nj'k}_{Y_1Y_2Y_3}\right)}{\sum_i \exp\left(\lambda^{i}_{Y_1} + \lambda^{ik}_{Y_1Y_3} + \lambda^{i1}_{Y_1R_2} + \lambda^{ik1}_{Y_1Y_3R_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{njk}_{Y_1Y_2Y_3}\right)}.$$

By Eq. (S9.51) and (S9.52), $M_m^{\nu}(j,j',k) > 1$ and $M_n^{\nu}(j,j',k) < 1$ in the absence of all $\lambda^{j\ell}_{Y_2R_1}$'s and $\lambda^{jk\ell}_{Y_2Y_3R_2}$'s. Thus, the necessary and sufficient condition for $\nu(j,j',k) \notin OI^{\nu}(j,j',k)$ (i.e., $\frac{\nu_m(j,j',k)}{\nu(j,j',k)} < 1$ or $\frac{\nu_n(j,j',k)}{\nu(j,j',k)} > 1$) is $(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) < -0.5\log M_m^{\nu}(j,j',k)$ or $(\lambda^{j'2}_{Y_2R_1} - \lambda^{j2}_{Y_2R_1}) + (\lambda^{j'k2}_{Y_2Y_3R_1} - \lambda^{jk2}_{Y_2Y_3R_1}) > -0.5\log M_n^{\nu}(j,j',k)$.

(3) In a similar fashion, by the definition of $\gamma_m(k,k',i)$ and $\gamma_n(k,k',i)$, $\gamma_m(k,k',i) > \gamma_j(k,k',i)$ for all $j(\neq m)$ and $\gamma_n(k,k',i) < \gamma_j(k,k',i)$ for all $j(\neq n)$ result in two inequalities in Eq. (S9.54) and (S9.55):

$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{mk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{imk'}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_1} - \lambda^{mk'1}_{Y_2Y_3R_1} \quad \text{(S9.53)}$$
$$> \lambda^{jk}_{Y_2Y_3} - \lambda^{mk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{imk}_{Y_1Y_2Y_3} + \lambda^{jk1}_{Y_2Y_3R_1} - \lambda^{mk1}_{Y_2Y_3R_1},$$
$$\lambda^{jk'}_{Y_2Y_3} - \lambda^{nk'}_{Y_2Y_3} + \lambda^{ijk'}_{Y_1Y_2Y_3} - \lambda^{ink'}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_1} - \lambda^{nk'1}_{Y_2Y_3R_1} \quad \text{(S9.54)}$$
$$< \lambda^{jk}_{Y_2Y_3} - \lambda^{nk}_{Y_2Y_3} + \lambda^{ijk}_{Y_1Y_2Y_3} - \lambda^{ink}_{Y_1Y_2Y_3} + \lambda^{jk'1}_{Y_2Y_3R_1} - \lambda^{nk'1}_{Y_2Y_3R_1}.$$

We represent $\gamma_m(k,k',i)/\gamma(k,k',i)$ and $\gamma_n(k,k',i)/\gamma(k,k',i)$ as follows :

$$\frac{\gamma_m(k,k',i)}{\gamma(k,k',i)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_2}-\lambda^{k2}_{Y_3R_2})+2(\lambda^{ik'2}_{Y_1Y_3R_2}-\lambda^{ik2}_{Y_1Y_3R_2})+2(\lambda^{k'12}_{Y_3R_1R_2}-\lambda^{k12}_{Y_3R_1R_2})\Big)\times M^{\gamma}_m(k,k',i),$$

$$\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_2}-\lambda^{k2}_{Y_3R_2})+2(\lambda^{ik'2}_{Y_1Y_3R_2}-\lambda^{ik2}_{Y_1Y_3R_2})+2(\lambda^{k'12}_{Y_3R_1R_2}-\lambda^{k12}_{Y_3R_1R_2})\Big)\times M^{\gamma}_n(k,k',i).$$

where

$$M^{\gamma}_m(j,j',k)=\frac{\sum_j \exp\Big(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{jk'}_{Y_2Y_3}-\lambda^{mk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{imk'}_{Y_1Y_2Y_3}+\lambda^{jk'1}_{Y_2Y_3R_1}-\lambda^{mk'1}_{Y_2Y_3R_1}\Big)}{\sum_j \exp\Big(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{jk}_{Y_2Y_3}-\lambda^{mk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{imk}_{Y_1Y_2Y_3}+\lambda^{jk1}_{Y_2Y_3R_1}-\lambda^{mk1}_{Y_2Y_3R_1}\Big)},$$

$$M^{\gamma}_n(j,j',k)=\frac{\sum_j \exp\Big(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{jk'}_{Y_2Y_3}-\lambda^{nk'}_{Y_2Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{ink'}_{Y_1Y_2Y_3}+\lambda^{jk'1}_{Y_2Y_3R_1}-\lambda^{nk'1}_{Y_2Y_3R_1}\Big)}{\sum_j \exp\Big(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}+\lambda^{j1}_{Y_2R_1}+\lambda^{jk}_{Y_2Y_3}-\lambda^{nk}_{Y_2Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{ink}_{Y_1Y_2Y_3}+\lambda^{jk1}_{Y_2Y_3R_1}-\lambda^{nk1}_{Y_2Y_3R_1}\Big)}.$$

By Eq. (S9.54) and (S9.55), $M^{\gamma}_m(k,k',i)>1$ and $M^{\gamma}_n(k,k',i)<1$ in the absence of all $\lambda^{km}_{Y_3R_2}$'s, $\lambda^{ikm}_{Y_1Y_3R_2}$'s and $\lambda^{k\ell m}_{Y_3R_1R_2}$'s. Thus, the necessary and sufficient condition for $\gamma(k,k',i)\notin OI^{\gamma}(k,k',i)$ (i.e., $\frac{\gamma_m(k,k',i)}{\gamma(k,k',j)}<1$ or $\frac{\gamma_n(k,k',i)}{\gamma(k,k',i)}>1$) is $(\lambda^{k'2}_{Y_3R_2}-\lambda^{k2}_{Y_3R_2})+(\lambda^{i'k2}_{Y_1Y_3R_2}-\lambda^{ik2}_{Y_1Y_3R_2})+(\lambda^{k'12}_{Y_3R_1R_2}-\lambda^{k12}_{Y_3R_1R_2})<-0.5\log M^{\gamma}_m(k,k',i)$ or $(\lambda^{k'2}_{Y_3R_2}-\lambda^{k2}_{Y_3R_2})+(\lambda^{i'k2}_{Y_1Y_3R_2}-\lambda^{ik2}_{Y_1Y_3R_2})+(\lambda^{k'12}_{Y_3R_1R_2}-\lambda^{k12}_{Y_3R_1R_2})>-0.5\log M^{\gamma}_n(k,k',i)$.

(4) By the definition of $\eta_m(k,k',j)$ and $\eta_n(k,k',j)$, $\eta_m(k,k',j)>\eta_i(k,k',j)$ for all $i(\neq m)$ and $\eta_n(k,k',j)<\eta_i(k,k',j)$ for all $i(\neq n)$ result in two inequalities in Eq. (S9.56) and (S9.57):

$$\begin{aligned}&\lambda^{ik'}_{Y_1Y_3}-\lambda^{mk'}_{Y_1Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{mjk'}_{Y_1Y_2Y_3}+\lambda^{ik'1}_{Y_1Y_3R_2}-\lambda^{mk'1}_{Y_1Y_3R_2}\\&>\lambda^{ik}_{Y_1Y_3}-\lambda^{mk}_{Y_1Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{mjk}_{Y_1Y_2Y_3}+\lambda^{ik1}_{Y_1Y_3R_2}-\lambda^{mk1}_{Y_1Y_3R_2},\end{aligned}\tag{S9.55}$$

$$\begin{aligned}&\lambda^{ik'}_{Y_1Y_3}-\lambda^{nk'}_{Y_1Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{njk'}_{Y_1Y_2Y_3}+\lambda^{ik'1}_{Y_1Y_3R_2}-\lambda^{nk'1}_{Y_1Y_3R_2}\\&>\lambda^{ik}_{Y_1Y_3}-\lambda^{nk}_{Y_1Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{njk}_{Y_1Y_2Y_3}+\lambda^{ik1}_{Y_1Y_3R_2}-\lambda^{nk1}_{Y_1Y_3R_2}.\end{aligned}\tag{S9.56}$$

We represent $\eta_m(k,k',j)/\eta(k,k',j)$ and $\eta_n(k,k',j)/\eta(k,k',j)$ as follows :

$$\frac{\eta_m(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1}-\lambda^{k2}_{Y_3R_1})+2(\lambda^{jk'2}_{Y_2Y_3R_1}-\lambda^{jk2}_{Y_2Y_3R_1})+2(\lambda^{k'21}_{Y_3R_1R_2}-\lambda^{k21}_{Y_3R_1R_2})\Big)\times M^{\eta}_m(k,k',j),$$

$$\frac{\eta_n(k,k',j)}{\eta(k,k',j)} = \exp\Big(2(\lambda^{k'2}_{Y_3R_1}-\lambda^{k2}_{Y_3R_1})+2(\lambda^{jk'2}_{Y_2Y_3R_1}-\lambda^{jk2}_{Y_2Y_3R_1})+2(\lambda^{k'21}_{Y_3R_1R_2}-\lambda^{k21}_{Y_3R_1R_2})\Big)\times M^{\eta}_n(k,k',j).$$

where

$$M^{\eta}_m(k,k',j)=\frac{\sum_i \exp\Big(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}+\lambda^{i1}_{Y_1R_2}+\lambda^{ik'}_{Y_1Y_3}-\lambda^{mk'}_{Y_1Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{mjk'}_{Y_1Y_2Y_3}+\lambda^{ik'1}_{Y_1Y_3R_2}-\lambda^{mk'1}_{Y_1Y_3R_2}\Big)}{\sum_i \exp\Big(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}+\lambda^{i1}_{Y_1R_2}+\lambda^{ik}_{Y_1Y_3}-\lambda^{mk}_{Y_1Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{mjk}_{Y_1Y_2Y_3}+\lambda^{ik1}_{Y_1Y_3R_2}-\lambda^{mk1}_{Y_1Y_3R_2}\Big)},$$

$$M^{\eta}_n(k,k',j)=\frac{\sum_i \exp\Big(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}+\lambda^{i1}_{Y_1R_2}+\lambda^{ik'}_{Y_1Y_3}-\lambda^{nk'}_{Y_1Y_3}+\lambda^{ijk'}_{Y_1Y_2Y_3}-\lambda^{njk'}_{Y_1Y_2Y_3}+\lambda^{ik'1}_{Y_1Y_3R_2}-\lambda^{nk'1}_{Y_1Y_3R_2}\Big)}{\sum_i \exp\Big(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}+\lambda^{i1}_{Y_1R_2}+\lambda^{ik}_{Y_1Y_3}-\lambda^{nk}_{Y_1Y_3}+\lambda^{ijk}_{Y_1Y_2Y_3}-\lambda^{njk}_{Y_1Y_2Y_3}+\lambda^{ik1}_{Y_1Y_3R_2}-\lambda^{nk1}_{Y_1Y_3R_2}\Big)}.$$

By Eq. (S9.56) and (S9.57), $M^{\eta}_m(k,k',j)>1$ and $M^{\eta}_n(k,k',j)<1$ in the absence of all $\lambda^{k\ell}_{Y_3R_1}$'s, $\lambda^{jk\ell}_{Y_2Y_3R_1}$'s and $\lambda^{k\ell m}_{Y_3R_1R_2}$'s. Thus, the necessary and sufficient condition for $\eta(k,k',j)\notin OI^{\eta}(k,k',j)$ (i.e., $\frac{\eta_m(k,k',j)}{\eta(k,k',j)}<1$ or $\frac{\eta_n(k,k',j)}{\eta(k,k',j)}>1$) is $(\lambda^{k'2}_{Y_3R_1}-\lambda^{k2}_{Y_3R_1})+(\lambda^{j'k2}_{Y_2Y_3R_1}-\lambda^{jk2}_{Y_2Y_3R_1})+(\lambda^{k'21}_{Y_3R_1R_2}-\lambda^{k21}_{Y_3R_1R_2})<-0.5\log M^{\eta}_m(k,k',j)$ or $(\lambda^{k'2}_{Y_3R_1}-\lambda^{k2}_{Y_3R_1})+(\lambda^{j'k2}_{Y_2Y_3R_1}-\lambda^{jk2}_{Y_2Y_3R_1})+(\lambda^{k'21}_{Y_3R_1R_2}-\lambda^{k21}_{Y_3R_1R_2})>-0.5\log M^{\eta}_n(k,k',j)$.

S9.3 Proof of Theorem S3 in S8.3

S9.3.1 Proof of Theorem S3-1)

Since $\pi_{ij11} = \pi_{ij} p_1(i,j) p_{21}(i,j)$,

$$\begin{aligned}\nu_i(j,j') = \frac{\pi_{ij11}}{\pi_{ij'11}} &= \exp\left(\lambda^{j}_{Y_2} - \lambda^{j'}_{Y_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{ij'}_{Y_1Y_2}\right) \times \exp(2\alpha^{j}_{Y_2} - 2\alpha^{j'}_{Y_2}) \\ &\times \frac{1+\exp(\alpha_1 + \alpha^{i}_{Y_1} + \alpha^{j'}_{Y_2})}{1+\exp(\alpha_1 + \alpha^{i}_{Y_1} + \alpha^{j}_{Y_2})} \times \frac{1+\exp(\alpha_2 + \alpha^{i}_{Y_1} + \alpha^{j'}_{Y_2})}{1+\exp(\alpha_2 + \alpha^{i}_{Y_1} + \alpha^{j}_{Y_2})}.\end{aligned}$$

By the definition of $\nu_m(j,j')$ and $\nu_n(j,j')$, $\nu_n(j,j') < \nu_i(j,j') < \nu_m(j,j')$ for all $i(\neq m,n)$. Thus, the followings hold:

$$\begin{aligned}\frac{\nu_m(j,j')}{\nu_i(j,j')} &= \exp\left(\lambda^{mj}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2} - \lambda^{ij}_{Y_1Y_2} + \lambda^{ij'}_{Y_1Y_2}\right) \\ &\times \frac{\frac{1+\exp\left(\alpha_1+\alpha^{m}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{m}_{Y_1}+\alpha^{j}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{m}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{m}_{Y_1}+\alpha^{j}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}} > 1 \\ \Leftrightarrow\ & \exp\left(\lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{m}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)} \\ >\ & \exp\left(\lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{m}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{m}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{m}_{Y_1}+\alpha^{j'}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}} \qquad \text{(S9.57)}\end{aligned}$$

and

$$\begin{aligned}\frac{\nu_n(j,j')}{\nu_i(j,j')} &= \exp\left(\lambda^{nj}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2} - \lambda^{ij}_{Y_1Y_2} + \lambda^{ij'}_{Y_1Y_2}\right) \\ &\times \frac{\frac{1+\exp\left(\alpha_1+\alpha^{n}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{n}_{Y_1}+\alpha^{j}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{n}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{n}_{Y_1}+\alpha^{j}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}} < 1 \\ \Leftrightarrow\ & \exp\left(\lambda^{ij'}_{Y_1Y_2} - \lambda^{nj'}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{n}_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)} \\ >\ & \exp\left(\lambda^{ij}_{Y_1Y_2} - \lambda^{nj}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{n}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{n}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{n}_{Y_1}+\alpha^{j'}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j'}_{Y_2}\right)}} \qquad \text{(S9.58)}\end{aligned}$$

As $\pi_{ij21} = \pi_{ij}(1-p_1(i,j))p_{20}(i,j)$, the nonresponse odds $\nu(j,j')$ is

$$\nu(j,j') = \frac{\pi_{+j21}}{\pi_{+j'21}} = \frac{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^j_{Y_2} + \lambda^{ij}_{Y_1Y_2}\right) \times \frac{1}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \times \frac{\exp(\alpha_3)}{1+\exp(\alpha_3)} \right\}}{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^{j'}_{Y_2} + \lambda^{ij'}_{Y_1Y_2}\right) \times \frac{1}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^{j'}_{Y_2}\right)} \times \frac{\exp(\alpha_3)}{1+\exp(\alpha_3)} \right\}}.$$

Now, by Eq. (S9.57) and (S9.58), the forms of $\nu_m(j,j')/\nu(j,j')$ and $\nu_n(j,j')/\nu(j,j')$ are expressed as follows:

$$\begin{aligned}
\frac{\nu_m(j,j')}{\nu(j,j')} &= \exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right) \times \frac{1+\exp(\alpha_2 + \alpha^m_{Y_1} + \alpha^{j'}_{Y_2})}{1+\exp(\alpha_2 + \alpha^m_{Y_1} + \alpha^j_{Y_2})} \\
&\times \frac{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^{ij'}_{Y_1Y_2} - \lambda^{mj'}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1 + \alpha^m_{Y_1} + \alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^{j'}_{Y_2}\right)} \right\}}{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1 + \alpha^m_{Y_1} + \alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \right\}}, \\
&\underset{(by\ Eq.\ (\text{S9.57}))}{>} \exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right) \times \frac{1+\exp(\alpha_2 + \alpha^m_{Y_1} + \alpha^{j'}_{Y_2})}{1+\exp(\alpha_2 + \alpha^m_{Y_1} + \alpha^j_{Y_2})} \\
&\times \frac{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1 + \alpha^m_{Y_1} + \alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \times \frac{\frac{1+\exp\left(\alpha_2 + \alpha^m_{Y_1} + \alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_2 + \alpha^m_{Y_1} + \alpha^{j'}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_2 + \alpha^i_{Y_1} + \alpha^{j'}_{Y_2}\right)}} \right\}}{\sum_i \left\{ \exp\left(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1 + \alpha^m_{Y_1} + \alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \right\}}, \\
&= \frac{\sum_i \left\{ \frac{\exp\left(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \times \exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right) \times \frac{1+\exp\left(\alpha_2 + \alpha^i_{Y_1} + \alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \right\}}{\sum_i \left\{ \frac{\exp\left(\lambda^i_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{mj}_{Y_1Y_2}\right)}{1+\exp\left(\alpha_1 + \alpha^i_{Y_1} + \alpha^j_{Y_2}\right)} \right\}} \qquad \text{(S9.59)}
\end{aligned}$$

$$\begin{aligned}
\frac{\nu_n(j,j')}{\nu(j,j')} &= \exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right) \times \frac{1+\exp(\alpha_2+\alpha^n_{Y_1}+\alpha^{j'}_{Y_2})}{1+\exp(\alpha_2+\alpha^n_{Y_1}+\alpha^j_{Y_2})} \\
&\times \frac{\sum_i \left\{\exp\left(\lambda^i_{Y_1}+\lambda^{ij'}_{Y_1Y_2}-\lambda^{nj'}_{Y_1Y_2}\right)\times\frac{1+\exp\left(\alpha_1+\alpha^n_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^{j'}_{Y_2}\right)}\right\}}{\sum_i \left\{\exp\left(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{nj}_{Y_1Y_2}\right)\times\frac{1+\exp\left(\alpha_1+\alpha^n_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}\right\}}, \\
\underset{(by\ Eq.\ (S9.58))}{<} &\exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right) \times \frac{1+\exp(\alpha_2+\alpha^n_{Y_1}+\alpha^{j'}_{Y_2})}{1+\exp(\alpha_2+\alpha^n_{Y_1}+\alpha^j_{Y_2})} \\
&\times \frac{\sum_i \left\{\exp\left(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{nj}_{Y_1Y_2}\right)\times\frac{1+\exp\left(\alpha_1+\alpha^n_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}\times\frac{\frac{1+\exp\left(\alpha_2+\alpha^n_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^n_{Y_1}+\alpha^{j'}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j'}_{Y_2}\right)}}\right\}}{\sum_i \left\{\exp\left(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{nj}_{Y_1Y_2}\right)\times\frac{1+\exp\left(\alpha_1+\alpha^n_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}\right\}}, \\
&= \frac{\sum_i \left\{\frac{\exp\left(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{nj}_{Y_1Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)}\times\exp\left(2\alpha^j_{Y_2}-2\alpha^{j'}_{Y_2}\right)\times\frac{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}\right\}}{\sum_i \left\{\frac{\exp\left(\lambda^i_{Y_1}+\lambda^{ij}_{Y_1Y_2}-\lambda^{nj}_{Y_1Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)}\right\}} \quad \text{(S9.60)}
\end{aligned}$$

We can see that the numerator and the denominator in Eq. (S9.59) and (S9.60) are the same except for an additional term in the numerator, $\exp\left(2\alpha^j_{Y_2} - 2\alpha^{j'}_{Y_2}\right)\frac{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j'}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^{j}_{Y_2}\right)}$, and the magnitudes of Eq. (S9.59) and (S9.60) depend on how far is the additional term from 1. Therefore, the necessary and sufficient condition for $\nu(j,j') \in OI^{\nu}(j,j') = (\nu_n(j,j'), \nu_m(j,j'))$ is $\mid \alpha^j_{Y_2} - \alpha^{j'}_{Y_2} \mid \leq A_2$ for a pair $(j, j^{'})$ of Y_2, where A_2 is a constant over $\alpha^j_{Y_2}$.

S9.3.2 Proof of Theorem S3-2)

Since $\pi_{ij11} = \pi_{ij} p_1(i,j) p_{21}(i,j)$,

$$\begin{aligned}
\omega_j(i,i') = \frac{\pi_{ij11}}{\pi_{i'j11}} &= \exp\left(\lambda^i_{Y_1} - \lambda^{i'}_{Y_1} + \lambda^{ij}_{Y_1Y_2} - \lambda^{i'j}_{Y_1Y_2}\right) \times \exp(2\alpha^i_{Y_1} - 2\alpha^{i'}_{Y_1}) \\
&\times \frac{1+\exp(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})} \times \frac{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})}.
\end{aligned}$$

By the definition of $\omega_m(i,i')$ and $\omega_n(i,i')$, $\omega_n(i,i') < \omega_j(i,i') < \omega_m(i,i')$ for all $j(\neq m,n)$.

Thus, the followings hold:

$$\begin{aligned}
\frac{\omega_m(i,i')}{\omega_j(i,i')} &= \exp\left(\lambda_{Y_1Y_2}^{im} - \lambda_{Y_1Y_2}^{i'm} - \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_1Y_2}^{i'j}\right) \\
&\times \frac{\frac{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{m}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{m}\right)}}{\frac{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{m}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{m}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}} > 1 \\
\Leftrightarrow &\quad \exp(\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'm}) \times \frac{1+\exp(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{m})}{1+\exp(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j})} \\
> &\quad \exp(\lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{im}) \times \frac{1+\exp(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{m})}{1+\exp(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j})} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{m}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{m}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}} \quad \text{(S9.61)} \\
\frac{\omega_n(i,i')}{\omega_j(i,i')} &= \exp\left(\lambda_{Y_1Y_2}^{in} - \lambda_{Y_1Y_2}^{i'n} - \lambda_{Y_1Y_2}^{ij} + \lambda_{Y_1Y_2}^{i'j}\right) \\
&\times \frac{\frac{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{n}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{n}\right)}}{\frac{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{n}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{n}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}} < 1 \\
\Leftrightarrow &\quad \exp(\lambda_{Y_1Y_2}^{i'j} - \lambda_{Y_1Y_2}^{i'n}) \times \frac{1+\exp(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{n})}{1+\exp(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j})} \\
< &\quad \exp(\lambda_{Y_1Y_2}^{ij} - \lambda_{Y_1Y_2}^{in}) \times \frac{1+\exp(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{n})}{1+\exp(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j})} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{n}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{n}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}} \quad \text{(S9.62)}
\end{aligned}$$

Since $\pi_{ij12} = \pi_{ij}p_1(i,j)(1-p_{21}(i,j))$, the nonresponse odds $\omega(i,i')$ is presented by

$$\omega(i,i') = \frac{\pi_{i+12}}{\pi_{i'+12}} = \frac{\sum_j \left\{\exp\left(\lambda_{Y_1}^{i} + \lambda_{Y_2}^{j} + \lambda_{Y_1Y_2}^{ij}\right) \times \frac{\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j}\right)} \times \frac{1}{1+\exp(\alpha_2+\alpha_{Y_1}^{i}+\alpha_{Y_2}^{j})}\right\}}{\sum_j \left\{\exp\left(\lambda_{Y_1}^{i'} + \lambda_{Y_2}^{j} + \lambda_{Y_1Y_2}^{i'j}\right) \times \frac{\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)}{1+\exp\left(\alpha_1+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j}\right)} \times \frac{1}{1+\exp(\alpha_2+\alpha_{Y_1}^{i'}+\alpha_{Y_2}^{j})}\right\}},$$

Now, by Eq. (S9.61) and (S9.62), the forms of $\omega_m(i,i')/\omega(i,i')$ and $\omega_n(i,i')/\omega(i,i')$ are expressed as follows:

$$
\begin{aligned}
\frac{\omega_m(i,i')}{\omega(i,i')} &= \exp\left(2\alpha^{i}_{Y_1} - 2\alpha^{i'}_{Y_1}\right) \times \frac{1+\exp(\alpha_2 + \alpha^{i'}_{Y_1} + \alpha^{m}_{Y_2})}{1+\exp(\alpha_2 + \alpha^{i}_{Y_1} + \alpha^{m}_{Y_2})} \\
&\times \frac{\sum_j \left\{ \exp\left(\lambda^{j}_{Y_2} + \lambda^{i'j}_{Y_1Y_2} - \lambda^{i'm}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^{m}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2})}{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2})} \right\}}{\sum_j \left\{ \exp\left(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{m}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2})} \right\}} \\
&\underset{(by\ Eq.\ \text{(S9.61)})}{>} \exp\left(2\alpha^{i}_{Y_1} - 2\alpha^{i'}_{Y_1}\right) \times \frac{1+\exp(\alpha_2 + \alpha^{i'}_{Y_1} + \alpha^{m}_{Y_2})}{1+\exp(\alpha_2 + \alpha^{i}_{Y_1} + \alpha^{m}_{Y_2})} \\
&\times \frac{\sum_j \left\{ \exp\left(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{m}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{m}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^{m}_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2}\right)}} \times \frac{\exp(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2})}{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^{j}_{Y_2})} \right\}}{\sum_j \left\{ \exp\left(\lambda^{j}_{Y_2} + \lambda^{ij}_{Y_1Y_2} - \lambda^{im}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{m}_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2})} \right\}}, \\
&= \exp(\alpha^{i}_{Y_1} - \alpha^{i'}_{Y_1}) \frac{\sum_j \left\{ \frac{\exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}{\left(1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)\right)\left(1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)\right)} \times \exp(\alpha_1+\alpha^{j}_{Y_2}) \right\}}{\sum_j \left\{ \frac{\exp\left(\lambda^{j}_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{im}_{Y_1Y_2}\right)}{\left(1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)\right)\left(1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^{j}_{Y_2}\right)\right)} \times \exp(\alpha_1+\alpha^{j}_{Y_2}) \right\}}, \\
&= \exp(\alpha^{i}_{Y_1} - \alpha^{i'}_{Y_1}). \qquad \text{(S9.63)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\omega_n(i,i')}{\omega(i,i')} &= \exp\left(2\alpha^i_{Y_1} - 2\alpha^{i'}_{Y_1}\right) \times \frac{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^n_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^n_{Y_2})} \\
&\times \frac{\sum_j \left\{\exp\left(\lambda^j_{Y_2}+\lambda^{i'j}_{Y_1Y_2}-\lambda^{i'n}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^n_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}\right\}}{\sum_j \left\{\exp\left(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^n_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})}\right\}} \\
\underset{(by\ Eq.\ \text{(S9.62)})}{<} & \exp\left(2\alpha^i_{Y_1} - 2\alpha^{i'}_{Y_1}\right) \times \frac{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^n_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^n_{Y_2})} \\
&\times \frac{\sum_j \left\{\exp\left(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^n_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2}\right)} \times \frac{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^n_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^n_{Y_2}\right)}}{\frac{1+\exp\left(\alpha_2+\alpha^{i}_{Y_1}+\alpha^j_{Y_2}\right)}{1+\exp\left(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2}\right)}} \times \frac{\exp(\alpha_1+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_2+\alpha^{i'}_{Y_1}+\alpha^j_{Y_2})}\right\}}{\sum_j \left\{\exp\left(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right) \times \frac{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^n_{Y_2}\right)}{1+\exp\left(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2}\right)} \times \frac{\exp(\alpha_1+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})}{1+\exp(\alpha_2+\alpha^{i}_{Y_1}+\alpha^j_{Y_2})}\right\}}, \\
&= \exp(\alpha^i_{Y_1}-\alpha^{i'}_{Y_1}) \frac{\sum_j \left\{\frac{\exp\left(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}{\left(1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)\right)\left(1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)\right)} \times \exp(\alpha_1+\alpha^j_{Y_2})\right\}}{\sum_j \left\{\frac{\exp\left(\lambda^j_{Y_2}+\lambda^{ij}_{Y_1Y_2}-\lambda^{in}_{Y_1Y_2}\right)}{\left(1+\exp\left(\alpha_1+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)\right)\left(1+\exp\left(\alpha_2+\alpha^i_{Y_1}+\alpha^j_{Y_2}\right)\right)} \times \exp(\alpha_1+\alpha^j_{Y_2})\right\}}, \\
&= \exp(\alpha^i_{Y_1}-\alpha^{i'}_{Y_1}). \qquad \text{(S9.64)}
\end{aligned}
$$

We can see from Eq. (S9.63) and (S9.64) that the necessary and sufficient condition for $\omega_n(i,i') < \omega(i,i') < \omega_m(i,i')$ (i.e., $\omega(i,i') \in OI^{\omega}(i,i')$) depends on the magnitude of the difference between $\alpha^i_{Y_1}$ and $\alpha^{i'}_{Y_1}$, i.e., $\mid \alpha^i_{Y_1} - \alpha^{i'}_{Y_1} \mid \leq A_1$ for a pair $(i, i^{'})$ of Y_1, where A_1 is a constant over $\alpha^i_{Y_1}$.