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CHANGE-POINT TESTS FOR THE TAIL PARAMETER OF LONG MEMORY STOCHASTIC VOLATILITY TIME SERIES

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Abstract: We consider a change-point test based on the Hill estimator to test for structural changes in the tail index of long memory stochastic volatility time series. In order to determine the asymptotic distribution of the corresponding test statistic, we prove a uniform reduction principle for the tail empirical process in a two-parameter Skorohod space. It is shown that such a process displays a dichotomous behavior according to an interplay between the Hurst parameter, that is, a parameter characterizing the dependence in the data, and the tail index. Our theoretical results are accompanied by simulation studies and an analysis of financial time series with regard to structural changes in the tail index.

Key words and phrases: Chaining, change-point tests, heavy tails, long-range dependence, stochastic volatility, tail empirical process.

1. Introduction

The tail behavior of the marginal distribution of time series is of major relevance for statistics in applied sciences such as econometrics and hydrology, where heavy-tailed data occur frequently. More precisely, time series from finance, such as the log-returns of exchange rates and stock market indices, display heavy tails; see Mandelbrot (1963). Furthermore, drastic events such as the financial crisis in 2008 substantiate the importance of studying time series models that underlie financial data. Against this background, identifying changes in the tail behavior of data-generating stochastic processes that result in an increase or decrease in the probability of extreme events is of utmost interest. In particular, analyzing of the tail behavior of financial data may pave the way for a corresponding adjustment of risk management for capital investments, thus preventing huge capital losses. Indeed, there is empirical evidence that the tail behavior of financial time series may change over time. Quintos, Fan and Phillips (2001) identify changes in the

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tail of Asian stock market indices, Galbraith and Zernov (2004) find evidence of changes in the tail behavior of returns on U.S. equities, and Werner and Upper (2004) detect structural breaks in high-frequency data of Bund future returns.

1.1. Tail index estimation and change-point problem

Let X_j, for $j \in \mathbb{N}$, be a stationary time series with a marginal tail distribution function $\bar{F}$ that is regularly varying with the index $-\alpha$, $\alpha > 0$; that is, $\mathbb{P}(X > x) = x^{-\alpha} L(x)$, where L is slowly varying at infinity. We recall that a measurable real-valued function is slowly varying at infinity if for all $t > 0$ $\lim_{x\to\infty} L(tx)/L(x) = 1$. Typical examples for L include constant functions or (iterated) logarithms. Because the tail behavior of $X_j, j \in \mathbb{N}$, is determined primarily by the value of the tail index α, identifying a change in the tail of data-generating processes corresponds to testing for a change-point in this parameter.

In particular, this means that, given a set of observations $X_1, \ldots, X_n$, with $\mathbb{P}(X_j > x) = x^{-\alpha_j} L(x)$, for $j = 1, \ldots, n$, we decide on the testing problem (H, A) with

$$\begin{aligned}
&H : \alpha_1 = \cdots = \alpha_n \\
&\text{and} \\
&A : \alpha_1 = \cdots = \alpha_k \neq \alpha_{k+1} = \cdots = \alpha_n, \text{ for some } \ k \in \{1, \ldots, n-1\}.
\end{aligned}$$

Test statistics that are designed to identify structural changes in the tail index are derived naturally from an estimation of the tail index α. For some general results on tail index estimation, see Drees (1998a) and Drees (1998b). In this study, we focus on two estimators that are motivated by the fact that, for a random variable X with tail index α,

$$\lim_{u\to\infty} \mathbb{E}\left[\log\left(\frac{X}{u}\right) \mid X > u\right] = \lim_{u\to\infty} \frac{\mathbb{E}\left[\log\left(X/u\right)\mathbf{1}\left\{X > u\right\}\right]}{\mathbb{P}\left(X > u\right)} = \frac{1}{\alpha} =: \gamma.$$

When we are given a set of observations $X_1, \ldots, X_n$, an approximation of the unknown distribution of X by its empirical analogue gives the following estimator for the tail index:

$$\widehat{\gamma} := \frac{1}{\sum_{j=1}^{n} \mathbf{1}\left\{X_j > u_n\right\}} \sum_{j=1}^{n} \log\left(\frac{X_j}{u_n}\right) \mathbf{1}\left\{X_j > u_n\right\}, \tag{1.1}$$

where u_n, for $n \in \mathbb{N}$, is a sequence with $u_n \to \infty$ and $n\bar{F}(u_n) \to \infty$. Replacing the deterministic levels u_n in the formula for $\widehat{\gamma}$ by $X_{n:n-k_n}$ for some k_n, $1 \leqslant k_n \leqslant n-1$

such that $k_n \to \infty$ and $k_n/n \to 0$, where $X_{n:n} \geqslant X_{n:n-1} \geqslant \cdots \geqslant X_{n:1}$ are the order statistics of the sample $X_1, \ldots, X_n$, yields the Hill estimator

$$\widehat{\gamma}_{\text{Hill}} = \frac{1}{k_n} \sum_{i=1}^{k_n} \log \left(\frac{X_{n:n-i+1}}{X_{n:n-k_n}} \right).$$

As the most popular estimator for the tail index, established in Hill (1975), the Hill estimator has been studied widely in the literature. Its limiting distribution has been obtained under various model assumptions, including linear processes (Resnick and Stărică (1997)), β-mixing processes (Drees (2000)), and long memory stochastic volatility (LMSV) models (Kulik and Soulier (2011)). The first study to establish a theory for change-point tests based on the Hill estimator seems to be that of Quintos, Fan and Phillips (2001). Whereas Quintos, Fan and Phillips (2001) consider independent and identically distributed (i.i.d.) observations, ARCH- and GARCH-type processes, Kim and Lee (2011) and Kim and Lee (2012) extend their results to β-mixing processes and residual-based change-point tests for AR(p) processes with heavy-tailed innovations. In contrast, we study change-point tests for the tail index of LMSV time series based on the two estimators $\widehat{\gamma}$ and $\widehat{\gamma}_{\text{Hill}}$. In fact, our results are the first to consider the change-point problem for stochastic volatility models and time series with long-range dependence.

To motivate the design of the test statistics for deciding on the change-point problem (H, A), we temporarily assume that the change-point location is known; that is, for a given $k \in \{1, \ldots, n-1\}$, we consider the testing problem (H, A_k), with

$$A_k : \alpha_1 = \cdots = \alpha_k \neq \alpha_{k+1} = \cdots = \alpha_n.$$

For this testing problem, change-point tests have been considered in Phillips and Loretan (1990) and Koedijk, Schafgans and De Vries (1990). In order to decide on (H, A_k), we compare an estimator $\widehat{\gamma}_k$ of the tail index based on the observations $X_1, \ldots, X_k$ with an estimator $\widehat{\gamma}_n$ of the tail index based on the whole sample $X_1, \ldots, X_n$. This idea leads to studying the following test statistic:

$$\Gamma_{k,n} = \frac{k}{n} \left| \frac{\widehat{\gamma}_k}{\widehat{\gamma}_n} - 1 \right|.$$

Under the assumption that the change-point location is unknown under the alternative, it seems natural to consider the statistic $\Gamma_{k,n}$ for every potential change-point location k, and to decide in favor of the alternative hypothesis A if the maximum of its values exceeds a predefined threshold. As a result, a change-point test for the testing problem (H, A) that rests on the estimator $\widehat{\gamma}$ defined by (1.1) bases test decisions on the values of the statistic

$$\Gamma_n := \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\lfloor nt\rfloor}}{\widehat{\gamma}_n} - 1\right|, \tag{1.2}$$

with $t_0 \in (0,1)$ and the sequential version of $\widehat{\gamma}$ defined by

$$\widehat{\gamma}_{\lfloor nt\rfloor} := \frac{1}{\sum_{j=1}^{\lfloor nt\rfloor}\mathbf{1}\{X_j > u_n\}}\sum_{j=1}^{\lfloor nt\rfloor}\log\left(\frac{X_j}{u_n}\right)\mathbf{1}\{X_j > u_n\}. \tag{1.3}$$

Likewise, a test statistic based on the Hill estimator is given by

$$\widetilde{\Gamma}_n := \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\mathrm{Hill}}(t)}{\widehat{\gamma}_{\mathrm{Hill}}(1)} - 1\right|,$$

with the sequential version of $\widehat{\gamma}_{\mathrm{Hill}}$ defined by

$$\widehat{\gamma}_{\mathrm{Hill}}(t) := \frac{1}{\lfloor k_n t\rfloor}\sum_{i=1}^{\lfloor k_n t\rfloor}\log\left(\frac{X_{\lfloor nt\rfloor:\lfloor nt\rfloor-i+1}}{X_{\lfloor nt\rfloor:\lfloor nt\rfloor-k_{\lfloor nt\rfloor}}}\right).$$

In this context, the most comprehensive theory for change-point tests is presented in Hoga (2017). The author considers a number of test statistics based on different tail index estimators, and derives their asymptotic distributions under the assumption of β-mixing data-generating processes.

In the following, we derive the asymptotic distribution of both estimators, namely, $\widehat{\gamma}_{\lfloor nt\rfloor}$ and $\widehat{\gamma}_{\mathrm{Hill}}(t)$, and the corresponding tests statistics, namely, Γ_n and $\widetilde{\Gamma}_n$, under the hypothesis of stationary time series data. For this purpose, we first prove a limit theorem for the tail empirical process of LMSV time series in two parameters. This limit theorem does not necessarily relate to the change-point context. It can therefore be considered of independent interest and, thus, as the main theoretical result of our work. Our theoretical results are accompanied by simulation studies. As an empirical application of our tests, we consider Standard & Poor's 500 daily closing index covering the period from January 2008 to December 2008, the year of the financial crisis. We identify a change in the data exactly one day after Lehman Brothers filed for bankruptcy protection, an event which is thought to have played a major role in the unfolding of the financial crisis.

1.2. Tail empirical process

In order to derive the limit distribution of the tail estimators $\widehat{\gamma}_{\lfloor nt\rfloor}$ and $\widehat{\gamma}_{\text{Hill}}(t)$, parametrized in t, and the corresponding test statistics Γ_n and $\widetilde{\Gamma}_n$, it is crucial to note that

$$\begin{aligned}\widehat{\gamma}_{\lfloor nt\rfloor} &= \frac{1}{\sum_{j=1}^{\lfloor nt\rfloor} \mathbf{1}\{X_j > u_n\}} \sum_{j=1}^{\lfloor nt\rfloor} \log\left(\frac{X_j}{u_n}\right) \mathbf{1}\{X_j > u_n\} \\ &= \frac{1}{\widetilde{T}_n(1,t)} \int_1^{\infty} s^{-1} \widetilde{T}_n(s,t) ds \,, \end{aligned} \tag{1.4}$$

where

$$\widetilde{T}_n(s,t) = \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^{\lfloor nt\rfloor} \mathbf{1}\left\{X_j > u_n s\right\}.$$

As a result, asymptotics of the considered statistics can be derived from a limit theorem for the two-parameter tail empirical process

$$e_n(s,t) := \left\{\widetilde{T}_n(s,t) - T(s,t)\right\}, \ s \in [1,\infty], \ t \in [0,1], \tag{1.5}$$

where $T\left(s,t\right)$ does not correspond to the mean of $\widetilde{T}_n(s,t)$, but rather to the limit of that mean, that is, to

$$T\left(s,t\right) := ts^{-\alpha}. \tag{1.6}$$

The tail empirical process in one parameter, namely, $e_n(s,1)$, for $s \in [1,\infty]$, has been studied in Mason (1988), Einmahl (1990), and Einmahl (1992) for independent, i.i.d. observations, in Rootzén (2009) for absolutely regular processes, and in Kulik and Soulier (2011) for LMSV time series. For the latter, the convergence of the two-parameter tail empirical process is discussed in Section 2.2.

1.3. LMSV model

A phenomenon often encountered in the context of financial time series is that the observations seem to be uncorrelated, whereas their absolute values or higher moments tend to be highly correlated. Another characteristic of financial time series is the occurrence of heavy tails. In particular, the distribution of the considered data often exhibits tails that are heavier than those of a normal distribution. These features of financial data can be covered by stochastic volatility models.

Stochastic volatility model

The LMSV model on which the theoretical results established in this article are based can be considered as a generalization of the stochastic volatility models considered in, for example, Taylor (1986). This model was first introduced by Breidt, Crato and de Lima (1998) and, independently, by Harvey (2002). Overviews of stochastic volatility models with long-range dependence and their basic properties are given in Deo et al. (2006) and Hurvich and Soulier (2009).

Stochastic volatility time series X_j, for $j \in \mathbb{N}$, are typically defined via

$$X_j = Z_j \varepsilon_j \quad \text{with} \quad Z_j = \exp\left(\frac{1}{2} Y_j\right), \tag{1.7}$$

where ε_j, for $j \in \mathbb{N}$, is a sequence of i.i.d. random variables with mean zero, and Y_j, for $j \in \mathbb{N}$, is a Gaussian process, independent of ε_j, for $j \in \mathbb{N}$.

Whereas these models are often restricted to modeling a relatively fast decay of dependence in Y_j, for $j \in \mathbb{N}$, the so-called LMSV model allows for long-range dependence. In what follows, we specify a corresponding dependence structure for Y_j, for $j \in \mathbb{N}$.

Transformed Gaussian processes

The rate of decay of the autocovariance function is crucial to the definition of the long-range dependence in time series.

Definition 1. A (second-order) stationary, real-valued time series Y_j, for $j \in \mathbb{Z}$, is called long-range dependent if its autocovariance function γ satisfies

$$\gamma_Y(k) := \operatorname{Cov}(Y_1, Y_{k+1}) \sim k^{-D} L_\gamma(k), \quad \text{as } k \to \infty,$$

with $D \in (0,1)$ for a slowly varying function L_γ. We refer to D as the long-range dependence (LRD) parameter; see Pipiras and Taqqu (2017, p.17).

The transformed random variables $Z_j = G(Y_j)$, for $j \in \mathbb{N}$, can be considered as elements of the Hilbert space $L^2 := L^2(\mathbb{R}, \varphi(x)dx)$, that is, the space of all measurable, real-valued functions that are square-integrable with respect to the measure $\varphi(x)dx$ associated with the standard normal density function φ, equipped with the inner product

$$\langle G_1, G_2 \rangle_{L^2} := \int_{-\infty}^{\infty} G_1(x) G_2(x) \varphi(x) dx = \mathbb{E}\left[G_1(Y) G_2(Y)\right],$$

where $G_1, G_2 \in L^2(\mathbb{R}, \varphi(x)dx)$ and Y denotes a standard normally distributed random variable. In order to characterize the dependence structure of transformed Gaussian processes, we consider their expansion in Hermite polynomials.

Definition 2. For $n \geqslant 0$, the Hermite polynomial of order n is defined by

$$H_n(x) = (-1)^n \mathrm{e}^{x^2/2} \frac{d^n}{dx^n} \mathrm{e}^{-x^2/2}, \ x \in \mathbb{R}.$$

The sequence of Hermite polynomials constitutes an orthogonal basis of L^2. As a result, every $G \in L^2(\mathbb{R}, \varphi(x)dx)$ has an expansion in Hermite polynomials; that is, for $G \in L^2(\mathbb{R}, \varphi(x)dx)$ and Y following a standard normal distribution, we have

$$G(Y) \overset{L^2}{=} \sum_{r=0}^{\infty} \frac{J_r(G)}{r!} H_r(Y), \tag{1.8}$$

where $\|\cdot\|_{L^2}$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_{L^2}$.

Under the assumption that as k tends to ∞, $\gamma_Y(k)$ converges to zero at a certain rate, the asymptotically dominating term in the series (1.8) is the summand corresponding to the smallest integer r for which the Hermite coefficient $J_r(G)$ is nonzero. This index, which depends decisively on G, is called the Hermite rank.

Definition 3. Let $G \in L^2(\mathbb{R}, \varphi(x)dx)$, $\mathbb{E}\left[G(Y)\right] = 0$ for standard normally distributed Y, and $J_r(G)$, for $r \geqslant 1$, be the Hermite coefficients in the Hermite expansion of G. The smallest index $k \geqslant 1$ for which $J_k(G) \neq 0$ is called the *Hermite rank* of G; that is,

$$r := \min\left\{k \geqslant 1 : J_k(G) \neq 0\right\}.$$

Given the previous definitions, we specify the model assumptions on which the results in the following sections are based.

Definition 4. Let the data-generating process X_j, for $j \in \mathbb{N}$, satisfy

$$X_j = Z_j \varepsilon_j, \quad j \in \mathbb{N},$$

where ε_j, for $j \in \mathbb{N}$, is a sequence of i.i.d. random variables with mean zero, and Z_j, for $j \in \mathbb{N}$, is a long-range dependent transformed Gaussian process with $Z_j = \sigma(Y_j)$, for $j \in \mathbb{N}$, for some stationary, long-range dependent Gaussian process Y_j, for $j \in \mathbb{N}$, with LRD parameter D and a positive function σ. More precisely, assume that Y_j, for $j \in \mathbb{N}$, admits a linear representation with respect to an independent, standard normally distributed sequence η_k, for $k \in \mathbb{Z}$; that is,

$$Y_j = \sum_{k=1}^{\infty} c_k \eta_{j-k}, \quad j \in \mathbb{N},$$

with $\sum_{k=1}^{\infty} c_k^2 = 1$. Furthermore, suppose that (ε_j, η_j), for $j \in \mathbb{Z}$, is a sequence of i.i.d. random vectors. A sequence of random variables X_j, for $j \in \mathbb{N}$, that satisfies the previous assumption is called an LMSV time series.

Remark 1. The model assumptions generalize the preceding concepts of stochastic volatility models with long-range dependence by allowing for general transformed Gaussian sequences Z_j, for $j \in \mathbb{N}$, and dependence between the sequences $\{Y_j, j \in \mathbb{N}\}$, and $\{\varepsilon_j, j \in \mathbb{N}\}$. Instead of claiming mutual independence of Y_j, for $j \in \mathbb{N}$, and ε_j, for $j \in \mathbb{N}$, the sequence of random vectors (η_j, ε_j) is assumed to be independent. In particular, this implies that for a fixed index j, the random variables Y_j and ε_j are independent, whereas Y_j may depend on ε_i, for $i < j$. In many cases, an LMSV model incorporating this dependence structure is referred to as *LMSV with leverage*, because it allows for so-called *leverage effects* in financial time series. Not taking account of leverage, Definition 4 corresponds to the LMSV model considered in Kulik and Soulier (2011), whereas a similar model with leverage is considered in Bilayi-Biakana, Ivanoff and Kulik (2019).

It can be shown that random variables X_j, for $j \in \mathbb{N}$, satisfying Definition 4 are uncorrelated, whereas their squares inherit the dependence structure from the transformed Gaussian sequence Z_j^2, for $j \in \mathbb{N}$. Moreover, X_j, for $j \in \mathbb{N}$, inherits the tail behavior from the sequence ε_j, for $j \in \mathbb{N}$, if the marginal distribution of the random variables ε_j, for $j \in \mathbb{N}$, has a regularly varying right tail, that is, $\bar{F}_\varepsilon(x) := \mathbb{P}(\varepsilon_1 > x) = x^{-\alpha} L(x)$ for some $\alpha > 0$ and a slowly varying function L, and if $\mathbb{E}\left[\sigma^{\alpha+\delta}(Y_1)\right] < \infty$ for some $\delta > 0$. More precisely, under these assumptions, the following asymptotic equivalence holds:

$$\mathbb{P}(X_1 > x) \sim \mathbb{E}\left[\sigma^{\alpha}(Y_1)\right] \mathbb{P}(\varepsilon_1 > x), \text{ as } x \to \infty.$$

This result is known as Breiman's lemma; see Breiman (1965). It follows that Definition 4 is suited for modeling the previously described characteristic features of financial time series. In the following sections, we assume that the data-generating process X_j, for $j \in \mathbb{N}$, corresponds to an LMSV time series specified by Definition 4.

The remainder of the paper is structured as follows. In Section 2 we state the technical assumptions needed for our theoretical results. These are followed by the main theorem on the convergence of the two-parameter tail empirical process (Theorem 1). The convergence of the estimators of the tail index (Corollary

1) and the test statistics (Corollary 2) are immediate consequences. Simulation studies are presented in Section 3, and a real-data analysis is presented in Section 4. All proofs are included in the Supplementary Material. In order to establish the convergence of the two-parameter tail empirical process, we decompose it into a martingale and a long-range dependent part. The latter is dealt with in the Supplementary Material. For the former, we establish the finite-dimensional convergence using classical tools from martingale theory, and handle the tightness of the two-parameter martingale part using chaining. This is a theoretical novelty in the present context, because the methods used in related papers are not suitable (the method used in Kulik and Soulier (2011) cannot be applied to models with leverage, and the approach in Bilayi-Biakana, Ivanoff and Kulik (2019) is not suited to two-parameter processes).

2. Main Results

2.1. Assumptions

In this section, we establish the assumptions that guarantee the convergence of the two-parameter tail empirical process for LMSV time series. First, we specify the LMSV model yielding the main assumptions for the theory.

Assumption 1 (Main Assumptions)**.** *Let $X_j = Z_j\varepsilon_j$, for $j \in \mathbb{N}$, satisfy Definition 4, with $Z_j = \sigma(Y_j)$, for $j \in \mathbb{N}$, for some stationary, long-range dependent Gaussian process Y_j, for $j \in \mathbb{N}$, with autocovariance function $\gamma_Y(k) := \operatorname{Cov}(Y_1, Y_{k+1}) \sim k^{-D}L_\gamma(k)$, as $k \to \infty$, for $D \in (0,1)$, and some i.i.d. sequence ε_j, for $j \in \mathbb{N}$, with regularly varying right tail, that is, $\bar{F}_\varepsilon(x) := \mathbb{P}(\varepsilon_1 > x) = x^{-\alpha}L(x)$, for some $\alpha > 0$ and a slowly varying function L. Moreover, let r denote the Hermite rank of $\Psi(y) := \sigma^\alpha(y)$ and assume that $r < 1/D$.*

Note that for a very strong dependence (D close to zero), a large range of the Hermite ranks is allowed. For D close to one, only rank one is allowed.

In the following, we list some technical conditions that characterize the behavior of the slowly varying function L and the moments of $\sigma(Y_1)$. For this, we introduce another condition on the distribution function F_ε. This definition stems from Drees (1998c).

Definition 5 (Second-order regular variation)**.** Let $\bar{F}_\varepsilon(x) = x^{-\alpha}L(x)$ for some $\alpha > 0$ and some slowly varying function L represented by

$$L(x) = c\exp\left(\int_1^x \frac{\eta(u)}{u}du\right)$$

for some constant c and a measurable function η. Furthermore, we assume that there exists a bounded, decreasing function η^* on $[0,\infty)$, regularly varying at infinity with parameter $\rho \geqslant 0$, that is, $\eta^*(x) = x^{-\rho} L_{\eta^*}(x)$, such that

$$|\eta(s)| \leqslant C\eta^*(s)$$

for some constant C and for all $s \geqslant 0$. We say that $\bar{F}_\varepsilon$ is second-order regularly varying with tail index α and rate function η^*, and we write $\bar{F}_\varepsilon \in 2\mathrm{RV}(\alpha, \eta^*)$.

Second-order regular variation allows us to control the difference between $\bar{F}_\varepsilon$ and the function $u \mapsto u^{-\alpha}$; see Lemmas 1 and 2 in the Supplementary Material. Moreover, the specific form of L guarantees the continuity of $\bar{F}_\varepsilon$.

Assumption 2 (Technical Assumptions)**.** *Suppose the main assumptions hold. Additionally, we assume that*

(TA.1) $\bar{F}_\varepsilon \in 2\mathrm{RV}\left(\alpha, \eta^*\right)$ *and* η *is regularly varying with index* ρ*;*

(TA.2) $u_n \to \infty$, $n\bar{F}(u_n) \to \infty$, $\eta^*(u_n) = o(d_{n,r}/n + 1/\sqrt{n\bar{F}(u_n)})$*, where* $d_{n,r}$ *is defined by*

$$d_{n,r}^2 = \mathrm{Var}\left(\sum_{j=1}^{n} H_r(Y_j)\right) \sim c_r n^{2-rD} L_\gamma^r(n),\ \ c_r = \frac{2r!}{(1-Dr)(2-Dr)}; \tag{2.1}$$

(TA.3) $\mathbb{E}\left[\sigma^{\alpha+\max\{\rho,\alpha\}+\vartheta}\left(Y_1\right)\right] < \infty$ *for some* $\vartheta > 0$*;*

(TA.4) $\mathbb{E}\left[\left(\sigma\left(Y_1\right)\right)^{-1}\right] < \infty$.

Remark 2. Assumption (TA.2) handles the bias created by centering the tail empirical process, not by its mean, but rather by the limit of that mean.

Example 1. The most commonly used second-order assumption is that

$$L(x) = c\exp\left(\int_1^x \frac{\eta(u)}{u} du\right),$$

with $\eta(s) = s^{-\alpha\beta}$ for some $\beta > 0$. It then holds that $\bar{F}_\varepsilon(s) = C\left(s^{-\alpha} + \mathcal{O}(s^{-(\alpha(\beta+1))})\right)$, $s \to \infty$, for some constant $c > 0$. Furthermore, we have

$$\sup_{s \geqslant 1}\left|\frac{\bar{F}_\varepsilon(u_n s)}{\bar{F}_\varepsilon(u_n)} - s^{-\alpha}\right| = \mathcal{O}(u_n^{-\alpha\beta}).$$

In this case, (TA.2) can be replaced by the assumption $u_n^{-\alpha\beta} = o(d_{n,r}/n + 1/\sqrt{n\bar{F}(u_n)})$.

2.2. Convergence of the tail empirical process

Recall that the tail empirical process in two parameters is defined by

$$e_n(s,t) := \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^{\lfloor nt \rfloor} \mathbf{1}\left\{X_j > u_n s\right\} - ts^{-\alpha},\ s \in [1,\infty],\ t \in [0,1].$$

The following theorem establishes a characterization of its limit. In order to state this, we recall that a Hermite–Rosenblatt process of order r with a self-similarity parameter H is a stochastic process $Z_{r,H}(t)$ defined for all $t \geqslant 0$ by a multiple Wiener–Ito integral with respect to a standard Brownian motion:

$$Z_{r,H}(t) = \omega(r,H) \int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{r-1}} \left(\int_0^t \prod_{j=1}^{r} (s-x_i)_+^{H-3/2} ds\right) dB(x_m)\cdots dB(x_1),$$

where $x_+ := \max(0,x)$ and $\omega(r,H) > 0$ satisfies

$$\omega^2(r,H) = \frac{r!(2r(H-1)+1)(r(H-1)+1)}{\left(\int_0^\infty [x(x+1)]^{H-3/2} dx\right)^r};$$

see Beran et al. (2013, Sec. 3.7). In our case, $H = 1-(rD)/2$, and hence the restriction $r < 1/D$ gives $H > 1/2$. We recall that the standard Brownian motion has the self-similarity parameter $H = 1/2$. Thus, $H > 1/2$ indeed indicates the presence of long memory.

Theorem 1. *Let X_j, for $j \in \mathbb{N}$, be a stationary time series with a marginal tail distribution function $\bar{F}$. Moreover, assume that Assumptions* 1 *and* 2 *hold.*

1. *If $n/d_{n,r} = o(\sqrt{n\bar{F}(u_n)})$, then as $n \to \infty$,*

$$\frac{n}{d_{n,r}} e_n(s,t) \Rightarrow \frac{s^{-\alpha}}{\mathbb{E}\left[\sigma^\alpha(Y_1)\right]} \frac{J_r(\Psi)}{r!} Z_{r,H}(t), \tag{2.2}$$

where $\Psi(y) = \sigma^\alpha(y)$, r is the Hermite rank of Ψ, $Z_{r,H}$ is an r th-order Hermite process, $H = 1-(rD)/2$, and $d_{n,r}^2$ is defined in (2.1).

2. *If $\sqrt{n\bar{F}(u_n)} = o(n/d_{n,r})$, then as $n \to \infty$,*

$$\sqrt{n\bar{F}(u_n)} e_n(s,t) \Rightarrow W(s^{-\alpha},t), \tag{2.3}$$

where W denotes a standard Brownian sheet.

The convergence holds in a two-parameter Skorohod space, that is, $\Rightarrow$ denotes weak convergence in $D\left([1,\infty]\times[0,1]\right)$.

The dichotomy of the limiting process is explained by the decomposition of the tail empirical process into the sum of a martingale and a partial sum of long-range dependent random variables, which can be viewed as a special case of Doob's decomposition; see the Supplementary Material. If $n/d_{n,r} = o(\sqrt{n\bar{F}(u_n)})$, the martingale part in the decomposition becomes negligible, such that the limiting process arises from the convergence of the long-range dependent part. If $\sqrt{n\bar{F}(u_n)} = o(n/d_{n,r})$, the long-range dependent part in the decomposition becomes negligible, such that the limiting process arises from the convergence of the martingale part. The same decomposition is employed in Kulik and Soulier (2011), Betken and Kulik (2019), and Bilayi-Biakana, Ivanoff and Kulik (2019).

The assumption $\sqrt{n\bar{F}(u_n)} = o(n/d_{n,r})$ yields the "standard" convergence (2.3), which in turn implies the "standard" convergence for the change-point statistics studied below. This is important for data applications. Indeed, the quantiles of (the functionals of) the limiting process in (2.3) are rather easy to simulate under the null hypothesis of no change. Furthermore, under this limiting regime, the validity of the i.i.d. bootstrap can be conjectured. On the other hand, the limiting process in (2.2) is much harder to simulate. Indeed, first, one has to know the Hermite rank and the αth moment of the *unobservable* process Y_j. However, even if we know this, a simulation of the Hermite–Rosenblatt process is not an easy task. As such, it is important from a statistical point of view to be able, if possible, to work under a regime that guarantees the validity of (2.3). This is how we approach the simulation studies.

Ignoring the slowly varying components, $\sqrt{n\bar{F}(u_n)} = o\left(n/d_{n,r}\right)$ means that $n^{3/2-rD} = o(u_n^{\alpha/2})$. For a given long-memory parameter D and the Hermite rank r, this induces restrictions on the thresholds u_n for the "standard" convergence (2.3) to hold. The stronger the memory (i.e., the smaller D is), the larger is the needed threshold. Intuitively, under strong dependence, we can use only extreme observations to remove the effect of long memory.

Furthermore, if the rates $n/d_{n,r}$ and $\sqrt{n\bar{F}(u_n)}$ are asymptotically equivalent (up to a constant), it can be conjectured that a limiting process is a linear combination of (uncorrelated, but not independent) limiting processes that appear on the right-hand side of both (2.2) and (2.3).

2.3. Convergence of the tail estimators

Recall that the considered tail index estimators of $\gamma = 1/\alpha$ are defined by

$$\widehat{\gamma}_{\lfloor nt \rfloor} := \frac{1}{\sum_{j=1}^{\lfloor nt \rfloor} \mathbf{1}\{X_j > u_n\}} \sum_{j=1}^{\lfloor nt \rfloor} \log\left(\frac{X_j}{u_n}\right) \mathbf{1}\{X_j > u_n\}$$

and

$$\widehat{\gamma}_{\mathrm{Hill}}(t) := \frac{1}{\lfloor k_n t \rfloor} \sum_{i=1}^{\lfloor k_n t \rfloor} \log \left(\frac{X_{\lfloor nt \rfloor : \lfloor nt \rfloor - i + 1}}{X_{\lfloor nt \rfloor : \lfloor nt \rfloor - k_{\lfloor nt \rfloor}}} \right),$$

where k_n and u_n are related by $k_n = \lfloor n\bar{F}(u_n) \rfloor$, such that $k_n \to \infty$ and $k_n/n \to 0$.

Based on Theorem 1, the limiting distributions of $\widehat{\gamma}_{\lfloor nt \rfloor}$ and $\widehat{\gamma}_{\mathrm{Hill}}(t)$ can be established in $D[t_0, 1]$, for any $t_0 \in (0, 1)$.

Corollary 1. *Let X_j, for $j \in \mathbb{N}$, be a stationary time series with a marginal tail distribution function $\bar{F}$. Moreover, assume that Assumptions 1 and 2 hold.*

1. *If $n/d_{n,r} = o\left(\sqrt{n\bar{F}(u_n)}\right)$, then as $n \to \infty$,*

$$\frac{n}{d_{n,r}} t \left(\widehat{\gamma}_{\lfloor nt \rfloor} - \gamma \right) \Rightarrow 0 \quad \text{and} \quad \frac{n}{d_{n,r}} t \left(\widehat{\gamma}_{\mathrm{Hill}}(t) - \gamma \right) \Rightarrow 0$$

in $D[t_0, 1]$, for all $t_0 \in (0, 1)$.

2. *If $\sqrt{n\bar{F}(u_n)} = o\left(n/d_{n,r}\right)$, then as $n \to \infty$,*

$$\sqrt{n\bar{F}(u_n)} t \left(\widehat{\gamma}_{\lfloor nt \rfloor} - \gamma \right) \Rightarrow \int_1^\infty s^{-1} W\left(s^{-\alpha}, t\right) ds - \alpha^{-1} W\left(1, t\right) \tag{2.4}$$

$$\sqrt{k_n} t \left(\widehat{\gamma}_{\mathrm{Hill}}(t) - \gamma \right) \Rightarrow \int_1^\infty s^{-1} W\left(s^{-\alpha}, t\right) ds - \alpha^{-1} W\left(1, t\right) \tag{2.5}$$

in $D[t_0, 1]$, for all $t_0 \in (0, 1)$.

Remark 3.

1. The zero limit in the first part of Corollary 1 stems from the degenerate nature of the limiting process in (2.2). Indeed, the limiting process is random in t and deterministic in s.

2. Following Kulik and Soulier (2011), we conjecture that the proper scaling in the first case is $a_n = \sqrt{n\bar{F}(u_n)}$, yielding the same limit as in the second case. However, because this is beyond the scope of this study, we do not consider the corresponding argument in detail.

3. The limit in (2.4) and (2.5) corresponds to $\gamma B(t)$, for $t \in [0, 1]$, where B is a standard Brownian motion.

2.4. Asymptotic distribution of the test statistics

Recall that the considered test statistics for the change-point problem (H, A) are defined by

$$\Gamma_n := \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\lfloor nt\rfloor}}{\widehat{\gamma}_n}-1\right| \text{ and } \widetilde{\Gamma}_n := \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\mathrm{Hill}}(t)}{\widehat{\gamma}_{\mathrm{Hill}}(1)}-1\right| .$$

Using the convergence obtained in Corollary 1, we derive the asymptotic distribution of the test statistics.

Corollary 2. *Let X_j, for $j \in \mathbb{N}$, be a stationary time series with a marginal tail distribution function $\bar{F}$. Moreover, assume that Assumptions 1 and 2 hold. If $\sqrt{n\bar{F}(u_n)} = o\left(n/d_{n,r}\right)$, then, for all $t_0 \in (0,1)$, as $n \to \infty$,*

$$\sqrt{n\bar{F}(u_n)} \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\lfloor nt\rfloor}}{\widehat{\gamma}_n}-1\right| \Rightarrow \sup_{t\in[t_0,1]} \left|B(t)-tB(1)\right|,$$

$$\sqrt{k_n} \sup_{t\in[t_0,1]} t\left|\frac{\widehat{\gamma}_{\mathrm{Hill}}(t)}{\widehat{\gamma}_{\mathrm{Hill}}(1)}-1\right| \Rightarrow \sup_{t\in[t_0,1]} \left|B(t)-tB(1)\right|,$$

where $B(t)$, for $t \in [0,1]$, denotes a standard Brownian motion.

3. Simulations

The following specifications apply to all of our simulations:

$$X_j = \sigma(Y_j)\varepsilon_j, \quad j \geqslant 1 , \tag{3.1}$$

where

- ε_j, for $j \geqslant 1$, is an i.i.d. sequence of Pareto-distributed random variables generated by the function `rgpd` (`fExtremes` package in `R`);
- Y_j, for $j \geqslant 1$, is a fractional Gaussian noise sequence generated by the function `simFGN0` (`longmemo` package in `R`) with Hurst parameter H;
- $\sigma(y) = \exp(y)$.

Under the alternative, we insert a change of height h at location $k = \lfloor n\tau\rfloor$ by simulating independent and identically Pareto-distributed observations ε_j, for $j \geqslant 1$, with ε_j, for $j = 1, \ldots, k$, having the tail index $\alpha_1 = \cdots = \alpha_k = \alpha$, and ε_j, for $j = k+1, \ldots, n$, having the tail index $\alpha_{k+1} = \cdots = \alpha_n = \alpha + h$. According to Breiman's lemma, this induces a change in the tail index of the observations $X_1, \ldots, X_n$.

We base the test decisions on the statistic $\widetilde{\Gamma}_n := \max_{1\leqslant k\leqslant n-1} \Gamma_{k,n}$, where

$$\widetilde{\Gamma}_{k,n} = \frac{k}{n}\left|\frac{\widehat{\gamma}_{\mathrm{Hill}}\,(k/n)}{\widehat{\gamma}_{\mathrm{Hill}}(1)} - 1\right| \text{ with } \widehat{\gamma}_{\mathrm{Hill}}\,(t) = \frac{1}{\lfloor k_n t\rfloor}\sum_{i=1}^{\lfloor k_n t\rfloor} \log\left(\frac{X_{\lfloor nt\rfloor:\lfloor nt\rfloor-i+1}}{X_{\lfloor nt\rfloor:\lfloor nt\rfloor-\lfloor k_n t\rfloor}}\right), \tag{3.2}$$

and we choose a significance level of 5%.

For the computation of the test statistic, the choice of k_n is a delicate matter. Hall (1982) shows that the optimal choice of k_n depends on the tail behavior of the data-generating process. Owing to this circularity, DuMouchel (1983) suggests choosing k_n proportional to the sample size. As noted in Quintos, Fan and Phillips (2001), a corresponding choice of k_n has been shown to perform well in simulations and is widely used by practitioners. Hence, we choose k_n as p percent of the sample size n, where $p = 10\%$ or $p = 20\%$. This is a standard choice in the context of high quantile estimation.

The power of the testing procedures is analyzed by considering different choices for the height of the level shift, denoted by h, and the location of the change point, denoted by τ. In the tables, the columns that are superscribed by $h = 0$ correspond to the frequency of a type-1 error, that is, the rejection rate under the hypothesis.

Both the Hurst parameter and the tail index, seem to have a significant effect on the rejection rates of the change-point test. An increase in dependence, that is, an increase of the Hurst parameter H, leads to an increase in the number of rejections. On the one hand, this leads to an increase in power. On the other hand, it results in a larger deviation of the empirical size from the significance level. However an increase in the tail thickness, that is, a decrease of the tail parameter α, results in improved performance, in that the empirical power increases, whereas the empirical size draws closer to the level of significance. Indeed, if the tail is thicker, more observations are informative about the tail, such that tail changes become easier to detect. Moreover, the empirical power of the test is higher for changes to heavier tails; that is, the test tends to better detect changes with a negative change-point height h.

Considering small values of H and α, that is, for heavy-tailed time series with unincisive long-range dependence, the simulation results concur with the expected behavior of change-point tests. First, an increasing sample size corresponds to an improvement of the finite-sample performance; that is, the empirical size approaches the level of significance and the empirical power increases. Second, the empirical power of the testing procedures increases when the height of

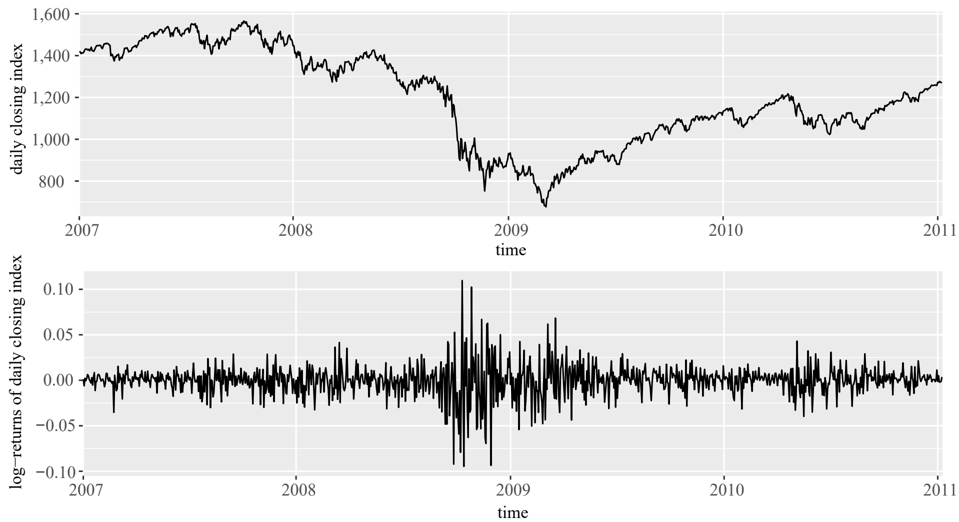

Figure 1. Daily closing index of S&P's 500 and its log-returns from January 2007 to December 2010.

the level shift increases. Third, the empirical power is higher for breakpoints located in the middle of the sample than it is for change-point locations that lie close to the boundary of the testing region. A comparison of the rejection rates in Tables 1 and 3 reveals that an increase in the tail thickness is better detected in the presence of late changes, that is, when $\tau = 0.75$, than it is in the presence of early changes, that is, when $\tau = 0.25$.

4. Data

Analyses of financial time series, such as stock market prices, usually focus on log-returns rather than of the observed data. As an example, we consider the log-returns of the daily closing indices of Standard & Poor's 500 (S&P 500) defined by

$$L_t := \log\left(\frac{P_t}{P_{t-1}}\right),$$

where P_t denotes the value of the index on day t in the period January 2007 to December 2010; see Figure 1. The data set was obtained from Yahoo Finance, and consists of $n = 1{,}014$ observations.

Comparing the plots of the sample autocorrelation function of the log-returns and the sample autocorrelation function of their absolute values in Figure 2, we observe a phenomenon often encountered in the context of financial data: the log-returns of the index appear to be uncorrelated, whereas the absolute log-returns

Table 1. Rejection rates (%) of the change-point test based on the statistic $\widetilde{\Gamma}_n$, $k_n = \lfloor np \rfloor$, for LMSV time series (Pareto-distributed ε_j, for $j \geqslant 1$) of length n with a Hurst parameter H, tail index α, and a shift in the mean of height h after a proportion $\tau = 0.25$. The calculations are based on 5,000 simulation runs.

			$\alpha = 4$					$\alpha = 3$					$\alpha = 2$				
	p	n	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$
$H = 0.6$	0.1	300	10.1	9.4	9.2	8.9	8.8	11.8	9.2	9.1	9.0	8.8	14.5	10.5	9.7	9.9	9.2
		500	8.8	7.5	7.6	7.0	7.1	9.9	8.3	8.1	7.3	6.9	20.3	10.5	7.8	7.2	7.5
		1,000	6.8	6.6	6.2	5.6	5.3	9.4	6.7	5.7	6.1	5.2	37.3	10.6	6.2	6.1	7.5
	0.2	300	7.0	6.5	6.7	6.1	7.1	7.5	7.6	7.3	6.4	6.9	15.6	8.9	7.2	7.4	7.3
		500	6.9	5.9	5.5	5.8	5.7	7.4	6.0	5.5	5.8	5.8	26.2	8.4	6.0	6.2	7.0
		1,000	4.8	4.7	4.8	4.6	4.9	7.7	5.1	4.2	4.6	4.6	52.1	9.6	4.7	5.5	7.2
$H = 0.7$	0.1	300	11.4	11.3	10.5	10.2	11.3	12.2	12.1	10.6	9.8	10.5	16.8	12.5	11.4	10.2	10.4
		500	10.4	9.7	9.7	9.2	9.2	12.5	10.3	9.3	8.8	9.1	20.6	11.0	9.6	9.6	9.1
		1,000	9.5	7.9	7.8	7.4	7.6	11.6	8.2	8.1	6.8	7.7	39.3	11.6	7.7	7.6	9.1
	0.2	300	9.1	9.2	9.1	8.9	8.4	10.0	8.9	8.2	8.3	8.6	18.5	10.0	8.0	8.5	9.7
		500	8.1	7.7	8.1	7.4	7.5	8.8	9.0	7.7	7.6	7.8	28.5	10.4	7.3	7.7	8.2
		1,000	7.3	7.1	6.8	6.4	6.7	10.3	7.1	6.8	6.6	6.5	57.1	11.4	6.8	6.6	8.8
$H = 0.8$	0.1	300	15.0	14.6	14.7	14.6	14.2	15.8	14.1	14.3	14.1	12.5	18.6	13.4	13.0	14.1	12.6
		500	14.3	14.6	13.6	13.3	13.9	15.4	13.2	13.2	11.4	12.7	21.6	13.5	10.8	12.3	13.7
		1,000	13.6	13.3	13.0	12.6	12.9	16.4	13.5	12.9	11.7	11.5	43.4	14.5	10.7	10.8	12.8
	0.2	300	13.9	13.9	13.7	13.2	13.3	13.4	12.5	12.6	12.7	13.0	20.3	12.5	10.7	11.9	10.9
		500	13.8	13.3	12.4	12.3	13.0	14.1	12.6	12.7	12.6	13.7	32.6	13.5	10.1	11.1	12.1
		1,000	14.2	12.9	12.9	13.1	13.6	16.9	13.3	12.9	11.4	12.1	62.5	15.7	10.0	11.1	12.9
$H = 0.9$	0.1	300	19.9	19.3	19.0	19.6	19.3	18.8	17.6	18.9	18.9	18.9	19.2	16.8	16.6	16.5	17.7
		500	20.6	20.7	20.9	20.6	21.1	19.0	19.1	18.4	18.3	18.6	25.2	16.0	15.2	16.4	17.5
		1,000	23.1	22.1	22.7	22.2	23.4	24.4	20.2	19.5	19.9	19.2	50.9	20.2	15.6	17.0	20.0
	0.2	300	21.0	21.7	20.8	20.0	20.7	19.5	17.7	18.7	18.7	19.3	26.1	16.3	14.0	15.5	17.9
		500	22.7	23.7	22.9	22.8	22.7	22.1	19.6	20.1	19.2	20.2	41.4	18.2	15.3	16.4	19.7
		1,000	26.7	26.3	26.2	25.8	27.4	28.7	23.2	22.7	23.7	23.9	75.5	24.3	17.5	20.0	22.2

Table 2. Rejection rates (%) of the change-point test based on the statistic $\widetilde{\Gamma}_n$, $k_n = \lfloor np \rfloor$, for LMSV time series (Pareto-distributed ε_j, for $j \geqslant 1$) of length n with a Hurst parameter H, tail index α, and a shift in the mean of height h after a proportion $\tau = 0.5$. The calculations are based on 5,000 simulation runs.

			$\alpha = 4$					$\alpha = 3$					$\alpha = 2$				
	p	n	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$
$H=0.6$	0.1	300	10.5	9.9	10.1	9.3	9.3	13.8	10.3	8.7	8.7	8.5	41.8	14.5	9.7	8.6	8.6
		500	9.5	8.4	7.5	7.6	7.7	13.5	8.9	7.5	6.7	6.7	60.2	14.8	7.8	8.3	6.9
		1,000	8.2	6.4	5.8	5.9	4.9	15.8	8.6	6.6	5.0	5.5	85.3	19.5	6.2	7.2	7.3
	0.2	300	8.3	7.4	6.3	7.1	6.4	10.1	8.0	6.7	7.2	6.2	48.5	11.2	7.2	7.3	7.1
		500	6.8	5.7	6.4	5.4	5.1	10.8	6.6	5.7	5.8	5.5	68.7	12.3	6.0	6.1	7.1
		1,000	6.6	4.7	4.6	4.4	4.2	14.3	5.8	4.4	4.6	4.8	91.1	18.5	4.7	5.3	7.5
$H=0.7$	0.1	300	12.5	11.4	10.9	10.8	10.1	15.3	12.0	10.6	10.0	10.3	44.3	15.9	11.4	10.0	10.2
		500	11.9	10.5	9.2	9.2	8.5	16.1	11.2	9.4	8.9	8.0	62.6	16.6	9.6	8.2	9.1
		1,000	10.5	8.8	7.4	7.7	7.0	19.1	10.1	7.4	7.3	7.0	86.6	21.3	7.7	7.7	8.6
	0.2	300	10.2	9.2	8.6	8.2	8.7	13.1	10.0	8.5	8.8	8.3	50.2	13.1	8.0	8.5	9.0
		500	10.0	8.6	7.1	8.6	7.6	13.7	7.9	7.4	6.8	7.3	70.2	14.7	7.3	7.3	8.1
		1,000	9.2	6.7	7.1	7.2	7.1	17.2	7.4	6.2	6.4	6.4	91.9	21.1	6.8	7.7	9.0
$H=0.8$	0.1	300	15.8	15.2	14.6	12.9	13.0	17.8	14.7	13.2	13.7	13.1	47.7	18.6	13.0	13.5	12.4
		500	16.3	13.7	13.6	13.4	12.6	20.7	15.0	13.0	12.2	11.0	66.2	19.7	10.8	11.9	11.2
		1,000	16.5	14.6	13.2	12.1	11.4	25.2	14.5	13.3	11.1	10.9	88.1	25.4	10.7	11.3	11.5
	0.2	300	15.0	12.9	12.7	13.5	13.2	17.3	14.3	12.0	12.0	11.0	56.0	17.4	10.7	11.2	11.8
		500	15.1	13.6	12.5	12.7	12.0	19.7	14.4	12.4	10.5	12.3	74.3	19.2	10.1	10.9	11.8
		1,000	16.8	14.0	13.5	12.7	12.0	25.4	15.2	11.9	11.2	11.5	93.5	26.9	10.0	11.7	14.2
$H=0.9$	0.1	300	22.2	19.6	19.6	20.6	18.3	23.5	19.1	17.7	17.5	17.2	53.0	21.5	16.6	15.1	16.2
		500	22.4	21.0	20.1	20.3	18.2	26.4	20.3	17.7	17.3	16.7	71.5	24.4	15.2	15.1	15.9
		1,000	26.3	22.9	22.8	21.1	21.3	36.0	23.5	19.8	18.4	18.5	92.4	32.2	15.6	16.6	18.5
	0.2	300	22.4	22.3	20.0	19.9	19.9	25.9	19.5	18.3	17.5	18.2	64.6	21.5	14.0	16.2	16.1
		500	24.3	23.4	21.9	22.0	21.6	29.7	23.0	20.4	20.2	19.8	82.1	27.5	15.3	15.8	18.0
		1,000	30.8	28.0	27.1	25.5	26.5	39.7	27.1	22.7	23.2	22.7	96.6	37.2	17.5	19.2	23.0

Table 3. Rejection rates (%) of the change-point test based on the statistic $\widetilde{\Gamma}_n$, $k_n = \lfloor np \rfloor$, for LMSV time series (Pareto-distributed ε_j, for $j \geqslant 1$) of length n with a Hurst parameter H, tail index α, and a shift in the mean of height h after a proportion $\tau = 0.75$. The calculations are based on 5,000 simulation runs.

			$\alpha = 4$					$\alpha = 3$					$\alpha = 2$				
	p	n	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$	$h=-1$	$h=-0.5$	$h=0$	$h=0.5$	$h=1$
$H=0.6$	0.1	300	10.7	9.4	9.1	9.0	8.8	13.5	9.8	8.4	8.6	8.3	45.8	13.4	9.4	7.7	8.0
		500	8.5	7.7	6.7	7.2	6.3	12.5	8.7	7.3	7.4	7.1	61.3	14.0	7.6	7.4	7.2
		1,000	6.6	6.2	5.6	5.2	5.6	14.6	7.4	6.3	5.4	4.8	83.5	16.5	6.6	5.9	5.8
	0.2	300	7.3	7.3	6.2	7.3	6.2	9.6	7.1	6.1	6.0	6.3	45.2	10.6	6.5	6.2	5.9
		500	6.8	5.8	5.5	5.6	5.2	9.6	6.7	5.8	5.0	5.2	63.3	11.5	6.4	5.1	5.6
		1,000	5.8	5.3	4.6	4.5	3.4	11.3	5.6	4.4	4.7	4.9	87.3	15.0	5.1	5.0	5.8
$H=0.7$	0.1	300	12.7	11.0	12.2	11.4	10.6	15.7	11.5	10.2	10.3	9.9	47.2	15.5	10.0	9.8	9.8
		500	11.3	9.9	9.3	9.2	9.5	14.8	9.6	10.3	9.6	8.6	60.9	14.7	9.0	8.1	8.7
		1,000	9.7	8.9	8.0	7.6	7.2	16.5	9.8	8.1	6.8	6.9	83.6	18.6	7.6	7.7	7.1
	0.2	300	10.1	10.5	8.3	8.3	8.4	12.0	9.9	9.1	8.6	8.6	47.2	11.9	8.1	7.9	7.6
		500	9.5	8.4	7.5	7.8	7.3	12.6	8.1	7.1	7.1	8.1	64.4	14.1	7.8	7.4	6.8
		1,000	8.8	7.0	7.1	6.3	6.2	15.0	8.4	6.3	6.3	6.7	87.4	16.4	6.2	6.2	6.7
$H=0.8$	0.1	300	16.1	14.5	14.9	12.9	14.9	17.9	15.4	13.5	14.0	13.4	49.7	17.4	13.5	12.1	11.7
		500	16.1	14.5	14.1	13.7	12.2	19.6	14.7	13.5	11.9	11.5	64.8	18.9	12.3	10.7	11.0
		1,000	15.5	14.3	13.3	12.9	12.1	21.8	15.0	11.8	11.4	10.4	84.2	22.3	11.0	10.3	10.6
	0.2	300	15.9	12.8	13.6	13.3	12.9	16.0	13.0	13.0	12.8	11.3	50.3	16.0	10.8	9.9	10.0
		500	14.7	13.4	12.9	12.3	12.4	19.3	13.6	11.9	11.8	11.7	67.5	17.5	10.1	10.0	9.7
		1,000	15.7	14.1	12.9	12.6	12.1	22.2	13.9	12.4	11.6	11.5	88.2	21.8	9.9	10.0	10.8
$H=0.9$	0.1	300	21.6	20.4	18.6	19.3	19.1	24.4	20.1	17.4	17.4	17.8	55.8	21.5	15.7	14.9	14.5
		500	22.4	21.4	20.9	20.0	18.8	25.9	19.5	18.8	17.5	16.7	70.4	23.4	15.3	14.8	14.7
		1,000	25.5	22.7	21.7	22.1	21.6	31.8	22.0	19.2	18.2	18.7	88.2	29.1	14.5	15.3	15.0
	0.2	300	22.6	21.0	19.1	19.8	20.6	24.0	19.5	18.2	17.8	17.7	61.2	21.7	14.6	13.5	14.2
		500	24.0	23.1	23.4	22.1	21.2	28.0	21.3	20.2	19.1	18.7	74.7	25.0	15.2	15.4	15.7
		1,000	29.7	26.3	26.9	25.9	27.2	34.9	25.1	22.7	22.7	23.6	91.5	33.5	17.6	16.9	18.9

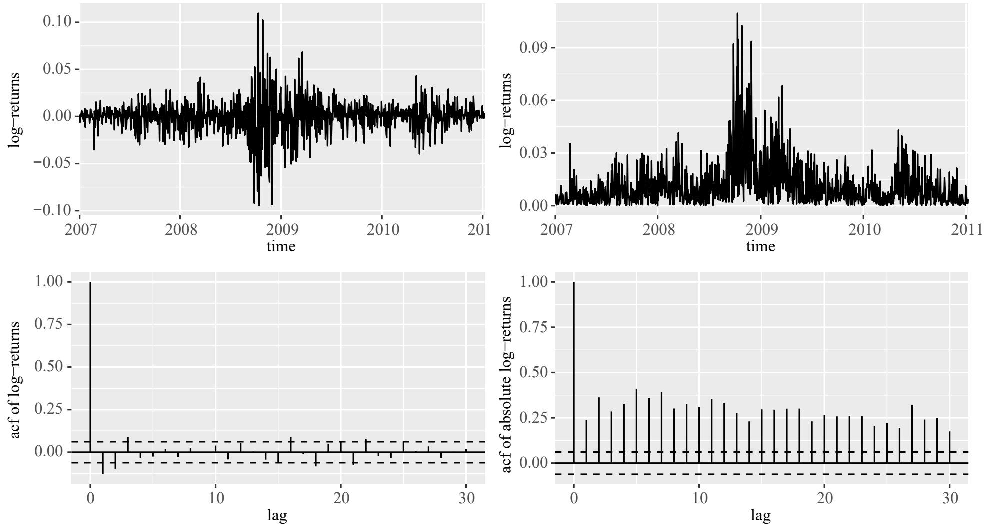

Figure 2. Sample autocorrelation of the log-returns and absolute log-returns of S&P's 500 daily closing index from January 2007 to December 2010. The two dashed horizontal lines mark the bounds for the pointwise 95% confidence interval of the autocorrelations under the assumption of data generated by white noise.

tend to be highly correlated.

Furthermore, the plot in Figure 1 shows that the considered time series exhibits *volatility clustering*, meaning that large price changes, that is, log-returns with relatively large absolute values, tend to cluster. This indicates that observations are not independent across time, although the absence of linear autocorrelation suggests that the dependence is nonlinear; see Cont (2005).

Another characteristic of financial time series is the occurrence of heavy tails. In particular, probability distributions of log-returns often exhibit tails that are heavier than those of a normal distribution. For the S&P 500 data, this property is highlighted by the QQ plot in Figure 3.

The previously described features of financial data are all covered by the LMSV model considered in our study.

In view of the fact that the LMSV model captures the properties of the log-returns of S&P's 500 daily closing index, we analyze the data with respect to a change in the tail index.

We base the test decision on the statistic defined in (3.2). We choose $k_n = \lfloor np \rfloor$; that is, p defines the proportion of the data on which the estimation of the tail index is based. Choosing $p = 0.1$, the value of the test statistic corresponds to $\widetilde{\Gamma}_n = 1.48207$. The 95%-quantile of the limit distribution $\sup_{t\in[0,1]} |B(t) - tB(1)|$

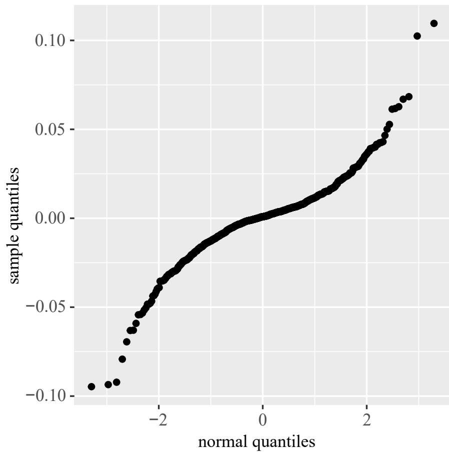

Figure 3. QQ plot for the log-returns of S&P's 500 daily closing index from January 2007 to December 2010.

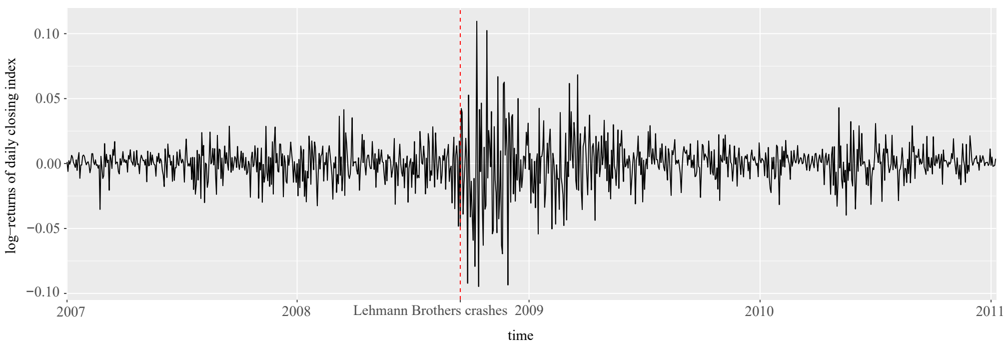

Figure 4. Log-returns of the daily closing index of S&P's 500 from January 2007 to December 2010. The red, dashed line indicates the estimated change-point location.

is equal to 1.3463348. Therefore, choosing the critical value for the hypothesis test correspondingly, the value of $\widetilde{\Gamma}_n$ indicates a change-point in the tail index at a 5% level of significance.

A natural estimate for the change-point location is given by the point in time k, where $\Gamma_{k,n}$ attains its maximum. For the considered data, this point corresponds to September 16, 2008; that is, one day after September 15, 2008, when Lehman Brothers filed for bankruptcy protection; see Figure 4.

Supplementary Material

The online Supplementary Material contains all the proofs for this study.

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