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CUMULATIVE SUM CONTROL CHARTS FOR THE COVARIANCE MATRIX

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Abstract: Cumulative sum (CUSUM) charts for controlling the covariance matrix are proposed via the projection pursuit method. Unlike traditional charts for covariance, the proposed charts can be used in a low-volume or short-run environment. It is shown that the proposed procedures are more effective than various existing ones. Their applications to monitoring a process with paired measurements are demonstrated. The CUSUM chart based on the likelihood ratio is also investigated. The performances of the two new kinds of CUSUM charts are similar. However, the likelihood ratio-based CUSUM chart requires that the size of each subgroup is larger than the dimension of the quality characteristics.

Key words and phrases: Covariance matrix, CUSUM chart, multivariable control chart, projection pursuit, Shewhart type control chart.

1. Introduction

There are many situations in which the overall quality of a product is determined by several correlated quality characteristics. Alt (1985) effectively illustrated the need for multivariate control charts in such situations. Mason, Champ, Tracy, Wierda and Young (1997) discussed implementation and interpretation issues. Various types of multivariate charts for a process mean have been suggested (see, for example, Alt (1985, 1988); Jackson (1991); Lowry, Woodall, Champ and Rigdon (1992); Lowry and Montgomery (1995); Chan and Li (1994); Ngai and Zhang (1994); Flury, Nel and Piennar (1995); Wierda (1994)) in the past decade. A brief introduction to the multivariate control problem can be found in Montgomery (1996, pp.322-330) and Ryan (1989, pp.215-227). Like the process mean, the process variability, usually summarized by a covariance matrix, is important for judging whether the process is in control. Two kinds of charts for covariance were suggested by Alt (1985). However, neither the problem of efficiently controlling the process covariance nor the comparison of these charts has been well studied.

The charts for covariance surveyed in Alt and Smith (1988) are of Shewhart type, i.e., each plotted point is based only on an independent sample (also called subgroup). The charts are easy to implement but, using only the information in an individual sample, they are insensitive to small or moderate changes of the covariance matrix. Another disadvantage is that these charts can be constructed only when each sample size is larger than the dimension of the quality characteristics. This is not always possible or practical. For example, as Ryan (1989, Chapter 6) points out, items coming off an assembly line may be produced at such a low rate that the process might have already gone out of control when a subgroup of sufficiently large size is formed. In the short-run environment, a large or moderate sample size is usually not feasible because of low-volume manufacturing. A comprehensive discussion on statistical process monitoring and control is given in a series of articles appearing in the April 1997 issue of Journal of Quality Technology.

To overcome the disadvantages mentioned, CUSUM-type charts based on the projection pursuit technique (Huber (1985)) are developed in this paper. In many cases the new charts can detect a change of covariance more than twice as fast (i.e., half of the average run length) as some existing Shewhart charts. The CUSUM chart based on the likelihood ratio (LRC) is also investigated. Like the Shewhart charts, the condition that each sample size is larger than the dimension of the quality characteristics is required in constructing the LRC. Our study shows that the projection pursuit-based CUSUM chart with reference values 1.5 and 0.5 performs better than the LRC in detecting an upward change of covariance. However, the conclusion is reversed when a certain degree of downward changes is presented in the covariance. We also show that we can adjust the reference values so that the projection pursuit-based CUSUM chart has performances similar to LRC in detecting both upward and downward changes.

The proposed charts can be used for quality control of a paired measurement system. In some situations, the precision of process measurements depends on product variability as well as measurement error. Paired measurements on each single specimen from two or more laboratories are made to account for these two sources of variation. Control charts for such a process have been investigated recently (for example, Jackson (1991) and Tracy, Youn and MaMason (1995)). The new charts can simultaneously monitor both sources of variation.

2. The New Charts

Consider a p-dimensional random vector X, which represents p quality characteristics, normally distributed with mean μ and covariance (matrix) Σ. A sample of size n, $n \geq 1$, is taken from the process over each time period. The ith sample is denoted by x_{ik}, $k = 1, \ldots, n$. The aim of a multivariate control chart is to detect the possible deviations of process mean and covariance from the nominal values (μ_0, Σ_0), assumed to be known in advance, while it gives as few "false alarms" as possible. In terms of the run length (RL) of the chart, a chart should ideally have a long RL when the process is in-control and a short RL when the process is out-of-control. The run length is defined as the number of samples to be taken before receiving an out-of-control signal. Here we speak of in-control RL and out-of-control RL, depending on whether the process is actually in- or out-of-control.

2.1. PP approach

The projection pursuit (PP) method is a powerful tool for developing this kind of multivariate chart for the mean (see Huber (1985) and Ngai and Zhang (1994)). The PP approach to control chart for covariance is based on the following (see Appendix I for proofs).

(i) $\Sigma = \Sigma_0$ if and only if $a_{\max}^\top \Sigma_0^{-1/2} X$ and $a_{\min}^\top \Sigma_0^{-1/2} X$ have unit variance, where $a_{\max}$ and $a_{\min}$ are the eigenvectors that correspond, respectively, to the largest and smallest eigenvalues of the matrix $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2}$, where $\Sigma_0^{-1/2}$ denotes the inverse of the square root of Σ_0 and $a^\top$ is the transpose of a.

(ii) $a_{\max}$ and $a_{\min}$ give the maximum and the minimum (signed) differences between the variance of $a^\top \Sigma_0^{-1/2} X$ and 1, respectively.

From (i), to test whether the covariance matrix of X deviates from the nominal Σ_0, it suffices to make univariate tests of the null hypotheses $H_{\max 0}$: "the variance of $a_{\max}^\top \Sigma_0^{-1/2} X$ is equal to 1" and $H_{\min 0}$: "the variance of $a_{\min}^\top \Sigma_0^{-1/2} X$ is equal to 1", respectively, via the projected and transformed samples $a_{\max}^\top \Sigma_0^{-1/2} x_{ik}$ and $a_{\min}^\top \Sigma_0^{-1/2} x_{ik}$, $1 \le k \le n$, $i = 1, 2, \ldots$. In practice $a_{\max}$ and $a_{\min}$ depend on Σ, are unknown, and have to be estimated. For each a, let $T_i(a)$ be a univariate statistic for testing $H_{\max 0}$, such that larger values of $T_i(a)$ give stronger evidence against $H_{\max 0}$. Then (ii) suggests that if an estimator $\hat{a}_{\max}$ gives the maximum value of $T_i(a)$, it is a natural estimator of $a_{\max}$ in the ith time period. Consequently, $T_i(\hat{a}_{\max})$ is a natural test statistic for $H_{\max 0}$ in the ith period. Similarly, we can define $\hat{a}_{\min}$ and test $H_{\min 0}$ by $T_i(\hat{a}_{\min})$.

In summary, the PP approach contains two key steps: (1) selecting a univariate control chart for variance with a test statistic T_i; (2) estimating $a_{\max}$ and $a_{\min}$ iteratively over each time period i, and calculating the values of $T_i(\hat{a}_{\max})$ and $T_i(\hat{a}_{\min})$.

There are several well-developed univariate control charts for variance. The well-known and relatively efficient ones are the CUSUM chart of Johnson and Leone (1962), and the exponentially weighted moving average (EWMA) chart of Chang and Gan (1995). We choose the Johnson-Leone CUSUM chart here (the PP extensions of the EWMA procedure is similar in principle). The remaining step is to calculate $\hat{a}_{\max}$, $\hat{a}_{\min}$, $T_i(\hat{a}_{\max})$ and $T_i(\hat{a}_{\min})$ for the Johnson-Leone chart.

Case 1. Using individual observations

The Johnson-Leone chart for variance is derived from the sequential test. Let x_i be the observation of a univariate process at the ith time period. The nominal value $\sigma_0 = 1$. Assume the process mean is zero and in-control. Let $k_l < k_u$ be two reference values. For example, we frequently use values $k_u = 1.5$ and $k_l = 0.5$. Set $SU_0 = SL_0 = 0$ and

$$SU_i = \max\{0, SU_{i-1} + x_i^2 - k_u\}, \quad SL_i = \min\{0, SL_{i-1} + x_i^2 - k_l\}, \quad i \geq 1.$$

SU_i and SL_i are called the ith CUSUM values. The Johnson-Leone chart gives an out-of-control message as soon as $SU_i > h_u$ or $SL_i < h_l$, where h_u and h_l are the respective upper and lower control limits. For simplicity, we let $h_u = h = -h_l > 0$. However, it is possible to improve the performance of the CUSUM chart by choosing some non-symmetric (h_l, h_u).

Let x_i be the observation of a p-dimensional process at the ith time period. For simplicity, assume $\mu_0 = 0$ and $\Sigma_0 = I_p$ (p by p unit matrix). We also assume that the process mean is stable. In light of the above, to estimate $a_{\max}$ and $a_{\min}$, we first define the CUSUM values for each direction a: $SU_0^a = SL_0^a = 0$ and

$$SU_i^a = \max\{0, SU_{i-1}^a + (a^\top x_i)^2 - k_u\}, \quad SL_i^a = \min\{0, SL_{i-1}^a + (a^\top x_i)^2 - k_l\}, \quad i \geq 1.$$

According to Johnson and Leone (1962), when $a^\top x_i$ is normally distributed, $\{SU_i^a\}$ and $\{SL_i^a\}$ are the likelihood ratio statistics for testing the hypotheses, $H_{\max 0}$ and $H_{\min 0}$. Even if $a^\top x_i$ is not normally distributed, SU_i^a and SL_i^a can be still used because it can be shown that, when i is large, $T_i(a) = SU_i^a$ is approximately proportional to the difference between the variance of $a^\top X$ and the nominal value 1 when the difference exceeds $k_u - 1$ (see Appendix II); a similar conclusion holds for SL_i^a when the difference is less than $k_l - 1$. As already mentioned, it is natural to estimate $a_{\max}$ and $a_{\min}$ by $\hat{a}_{i\max}$ and $\hat{a}_{i\min}$ in which SU_i^a and SL_i^a attain the maximum and minimum, respectively.

To simplify the notation, denote $SU_i^{\hat{a}}$ and $SL_i^{\hat{a}}$ by SU_i and SL_i, respectively. Let λ_{ij}^u and λ_{ij}^l be the largest and the smallest eigenvalues of the sample matrix $x_i x_i^\top + \cdots + x_j x_j^\top$, $1 \leq j \leq i$, respectively. Define $SU_{ij} = \lambda_{ij}^u - (i - j + 1)k_u$, $SL_{ij} = \lambda_{ij}^l - (i - j + 1)k_l$, $1 \leq j \leq i$. Then it can be shown that

$$\begin{aligned} SU_i &= \max_{||a||=1} SU_i^a = \max\{0, SU_{i1}, \ldots, SU_{ii}\}, \\ SL_i &= \min_{||a||=1} SL_i^a = \min\{0, SL_{i1}, \ldots, SL_{ii}\}. \end{aligned} \tag{2.1}$$

Moreover, if $u(i)$ and $l(i)$ are such that $SU_{iu(i)} = SU_i$ and $SL_{il(i)} = SL_i$, then $\hat{a}_{i\max}$ and $\hat{a}_{i\min}$ are the eigenvectors corresponding to the largest eigenvalue of

$x_i x_i^\top + \cdots + x_{u(i)} x_{u(i)}^\top$ and the smallest eigenvalue of $x_i x_i^\top + \cdots + x_{l(i)} x_{l(i)}^\top$, respectively.

Now the CUSUM chart for covariance, to be denoted by $MCD_1(0)$ where the subscript means sample size is 1, can be defined to indicate an out-of-control message when $SU_i > h_u$ or $SL_i < h_l$. The CUSUM values $\{SU_i\}$ and $\{SL_i\}$ can be separately used to detect an upward change (an inflation) and a downward change (a shrinkage) of covariance, that is, there is a direction in which the variance of the projected process increases or decreases.

A simulation study indicates that $MCD_1(0)$ is sensitive to the process mean shift (the details are omitted here but are available from the authors). Hence we can use this chart to check whether the process mean and covariance are in-control simultaneously. The major drawback is that when $MCD_1(0)$ gives a signal, it is difficult to distinguish a mean shift from a covariance change. To control the process mean and covariance separately, a commonly used method is to form a subgroup of observations at each time period to reduce the effects of the process mean shift on the chart of covariance. The major difference between our chart and the traditional ones is that we allow small samples while maintaining the speed of covariance change detection.

Case 2. Using subgroups

Let x_{ik}, $1 \leq k \leq n$, be a subgroup of observations in the ith time period. Assume x_{ik}, $1 \leq k \leq n$, $i \geq 1$, are i.i.d. p-dimensional normal with mean μ_i and covariance Σ. Let Σ_0 be the nominal value of Σ estimated from previous samples. In this case, the principle for constructing a CUSUM control chart is the same as in Case 1, and we omit the details. This chart is denoted by $MCD_n(0)$.

2.2. Likelihood ratio approach

Roy's statistic can be used to construct a Shewhart chart, namely SR (see Appendix III for the definition). The commonly adopted method to improve the performance of a Shewhart chart is to apply the CUSUM procedure to the statistics used in that chart (for example, Roy's statistics in the SR chart). Especially, a CUSUM version of the SR chart is obtained in this way. The PP-based CUSUM chart for covariance turns out to be a procedure based on Roy's statistics of *cumulative* sample covariance matrices. The important difference between the above two CUSUM charts lies in the order of applying the CUSUM and Roy procedures. In the PP-based CUSUM chart we first calculate the cumulative sample covariance matrices and then apply Roy's procedure, while in the CUSUM version of the SR chart we first apply Roy's procedure. The principles mentioned in the last subsection support this choice. It is natural to apply the likelihood ratio procedure SA, instead of SR, to these cumulative sample covariance matrices (see Appendix III for the definition of the SA procedure). That is, for $n > p$ and $1 \le j < i$, let $S_{ij} = (i-j+1)(n-1)(-p - \log(\det(V_{ij})) + \mathrm{tr}(V_{ij}))$ where V_{ij} is defined by

$$\{(y_{j1}y_{j1}^\top + \cdots + y_{jn}y_{jn}^\top)/(n-1) + \cdots + (y_{i1}y_{i1}^\top + \cdots + y_{in}y_{in}^\top)/(n-1)\}/(i-j+1),$$

and $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the determinant and trace of a square matrix. Let $S_i = \max\{0, S_{i1}, \ldots, S_{ii}\}$. Then a new CUSUM chart, LRC_n, can be defined to indicate an out-of-control message when $S_i > h$, where h is the control limit.

Note that, according to the likelihood ratio procedure, the factor $n-1$ in S_{ij} and V_{ij} above should be n. The corresponding procedure is called LCR_n'. LCR_n' is a biased procedure in the sense that some out-of-control ARLs may be larger than the in-control ARL. So it is not surprising that the simulation in Table 2.1 of Chan and Zhang (2000; an extended version of this paper) indicates that LCR_n is significantly better than LCR_n'. We recommend using the LCR_n.

Table 2.1. Comparisons of ARL and SRL of SA, SA', LCR_n' and LCR_n (with $k_u = 1.5$, $k_l = 0.5$) charts for $n = 5$, $p = 2, 3$, where $\lambda = \lambda(\Sigma_o^{-1/2}\Sigma\Sigma_o^{-1/2})$.

$p = 2, \mu = 0$								
	SA'		SA		LCR_5'		LCR_5	
	$h = 20.7$		$h = 16$		$h = 60.$		$h = 18.6$	
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1,1)$	203	205	203	202	233	67	248	241
B_2	103	105	74.1	72.2	34.0	7.98	13.0	6.28
C_2	177	179	115	115	99.2	26.1	45.2	27.5
D_2	214	216	118	114	424	133	32.8	20.6
E_2	7.72	7.03	4.36	3.76	10.2	4.24	3.17	1.84
F_2	220	222	74.5	73.2	582	189	29.8	18.2
G_2	232	233	185	184	1232	356	102	73.7
H_2	204	206	184	179	220	61.6	97.1	70.9
J_2	7.34	6.87	5.41	4.92	6.47	1.27	2.57	0.91
K_2	9.81	9.24	4.44	3.88	11.5	4.38	3.06	1.66

$p = 3, \mu = 0$								
	SA'		SA		LCR_5'		LCR_5	
	$h = 36$		$h = 28.25$		$h = 110$		$h = 30.$	
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1,1,1)$	180	181	194	193	212	45.5	210	198
B_3	193	192	153	150	408	86.7	79.7	52.4
C_3	198	197	171	168	585	127	94.5	62.7
D_3	10.3	9.57	13.6	12.9	12.9	4.0	2.84	1.45
E_3	8.36	7.68	3.99	3.42	7.33	0.96	2.77	0.81
F_3	13.8	13.0	12.1	11.7	9.38	1.43	3.07	1.07
G_3	117	117	120	120	48.7	8.57	16.1	7.2

3. Procedure of Constructing the New Charts

First we consider the case that the nominal value Σ_0 is known, or can be estimated by the previous in-control samples. The other cases will be discussed in the next section. The procedure $MCD_1(0)$ involves the following steps.

Step 1. Determine the parameters in $MCD_1(0)$. Choose k_u and k_l (for example, $k_l = 0.5$ and $k_u = 1.5$). The parameters $h_l = -h$ and $h_u = h$ are determined by the values of in-control ARL (average run length) and SRL (standard deviation of run length), specified in advance (see Tables 5.1, 5.2). Alternatively, h is specified first and the ARL and SRL values are then determined.

Step 2. For each i, calculate the sample average $\bar{x}_i$ of x_{ik}, $1 \le k \le n$, and make the following transformation: $y_{ik} = \Sigma_0^{-1/2}(x_{ik} - \bar{x}_i)$ for $n > 1$, $y_{i1} = \Sigma_0^{-1/2}(x_{i1} - \mu_0)$ for $n = 1$.

Step 3. Calculate SU_i and SL_i. For each i and $1 \le j \le i$, first calculate the maximum and the minimum eigenvalues and the corresponding eigenvectors of

$$(y_{j1}y_{j1}^\top + \cdots + y_{jn}y_{jn}^\top)/(n-1) + \cdots + (y_{i1}y_{i1}^\top + \cdots + y_{in}y_{in}^\top)/(n-1), \text{ for } n > 1$$

$$y_{j1}y_{j1}^\top + \cdots + y_{i1}y_{i1}^\top, \text{ for } n = 1.$$

These are denoted λ_{ij}^u, e_{ij}^u and λ_{ij}^l, e_{ij}^l, respectively. Then we calculate $SU_{ij} = \lambda_{ij}^u - (i-j+1)k_u$, $SL_{ij} = \lambda_{ij}^l - (i-j+1)k_l$ followed by $SU_i = \max\{0, SU_{i1}, \ldots, SU_{ii}\}$, $SL_i = \min\{0, SL_{i1}, \ldots, SL_{ii}\}$. Let $u(i)$ and $l(i)$ be such that $SU_{iu(i)} = SU_i$ and $SL_{il(i)} = SL_i$.

Step 4. Check whether SU_i is above h_u and whether SL_i is below h_l. If $SU_i > h_u$, then a upward change signal of Σ in direction $e_{iu(i)}^u$ is indicated. If $SL_i < h_l$, then a downward change signal of Σ in direction $e_{il(i)}^l$ is indicated.

As in the CUSUM chart for the process mean (see, for example, Hawkins and Olwell (1998, pp.20-21)), we can give an estimate of when change occurs when SU_i or SL_i falls outside the control limits. We look backward from the period i to check the subgroups: x_{jk}, $1 \le k \le n$, $1 \le j \le i$. Similar to the CUSUM chart for the process mean, if SU_{ij} falls outside the control limits, then we predict that the change occured at a period not later than j. SU_{ij} shows the magnitude of such shift (see Appendix II). There may be several such j. It is natural to choose the most significant one, namely $u(i)$, in the sense that $SU_{iu(i)} = \max\{0, SU_{ij}, 1 \le j \le i\}$.

4. Enhancements of the New Charts

4.1. FIR CUSUM

The fast initial response (FIR) feature is useful when there are start-up problems or ineffective control actions after the previous out-of-control signal (see, Lucas and Saccucci (1990)). The aim of FIR is to reduce the RL for mean shifts that one wishes to detect without significantly reducing the in-control RL. A direct way of achieving this aim is to narrow the control limits. But simulations show that a more efficient way to improve the FIR feature of $MCD_n(0)$ is to add some time-varying constants to the CUSUM values (details are available from the authors). Here we present a method used by Ngai and Zhang (1994). Take $MCD_n(0)$ as an example. The new CUSUM values are of the forms $\{SU_i + r^{u(i)+1}h_u\}$ and $\{SL_i + r^{l(i)+1}h_l\}$, where $0 \leq r < 1$ and $u(i)$ and $v(i)$ are defined in Section 2.1. Using the same control limits of $MCD_n(0)$ and the new sequences of the CUSUM values, we can define an FIR CUSUM chart, denoted $MCD_n(r)$. For simplicity of notation, the new sequences of the CUSUM values are still denoted $\{SU_i\}$ and $\{SL_i\}$.

The spirit of this improvement can be illustrated as follows. When the process is in-control, most CUSUM values should be around zero and relatively far away from the control limits. Hence, if we add a small value to each CUSUM value, the new sequences of CUSUM values will be still within the control limits. On the contrary, when the process is out-of-control, the CUSUM sequence will tend to the control limits and then fall outside the control limits. If we add a small value to each CUSUM value, the new sequence of CUSUM values will fall outside the control limits more quickly. As a result, a shorter out-of-control RL is obtained. The reason why we choose $r^{u(i)+1}h_u$ and $r^{l(i)+1}h_l$ as the values to be put in the ith CUSUM values is the following. Note that $SU_i \leq h_u$ is equivalent to $SU_{ij} \leq h_u$, $1 \leq j \leq i$. This implies that for fixed i, the upper control limit for SU_{ij}, $1 \leq j \leq i$ is a constant. Motivated by sequential theory (for example, Siegmund (1986)), we can improve the performance of the chart by replacing the constant control limit by some exponential control limit $(1-r^{j+1})h_u$, which varies in j. Now the corresponding upper control limit for $SU_{iu(i)}$ is $(1-r^{u(i)+1})h_u$. For simplicity, we check only whether $SU_{iu(i)} > (1 - r^{u(i)+1})h_u$ (which is equivalent to $SU_{iu(i)} + r^{u(i)+1}h_u > h_u$).

4.2. No previous in-control samples

In some situations, the nominal values of (μ_0, Σ_0) are unknown at the beginning. We modify $MCD_1(r)$ by the following method: during the ith period, use the sample average and covariance $\hat{\mu}_{i-1,0}$ and $\hat{\Sigma}_{i-1,0}$ of the first $i-1$ samples to estimate μ_0 and Σ_0 if no signal appears at the first $i-1$ periods. For a singular sample covariance $\hat{\Sigma}_{i-1,0}$, $\hat{\Sigma}_{i-1,0}^{-1/2}$ is defined as the generalized inverse of $\hat{\Sigma}_{i-1,0}^{1/2}$.

5. RL Performances and Designs of the New Charts

5.1. RL performances

The performance of a control chart to detect process change when it is out-of-control is evaluated by its RL. The RLs of $MCD_n(r)$, LCR'_n and LRC_n have two properties (see Appendix IV for proofs).

(i) The distribution of RL depends only on the solutions of the equation $\det(\Sigma - \lambda\Sigma_0) = 0$, where Σ_0 is the nominal value of the process covariance when it is in-control and Σ is the real value of the process covariance.

(ii) The distributions of the RLs of $MCD_1(r)$ and $MCD_2(r)$ are the same provided they have the same parameters k_u, k_l, p, h_u and h_l, and the underlying process is normally distributed.

Property (i) also holds for the Shewhart charts SA, SA', SR and SV defined in Appendix III.

Let Σ_1 and Σ_2 be two covariance matrices such that $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ is diagonal with the same eigenvalues as $\Sigma_0^{-1/2}\Sigma_2\Sigma_0^{-1/2}$. Observe that the equations, $\det(\Sigma_1 - \lambda\Sigma_0) = 0$ and $\det(\Sigma_2 - \lambda\Sigma_0) = 0$, have the same solution in this setting. Then, by Property (i), the distributions of the RLs with Σ_1 and Σ_2 are the same. Thus, when we examine the out-of-control performance, we need only consider the situation when $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2}$ is diagonal with the ith elements being the eigenvalues of $\Sigma_o^{-1/2}\Sigma\Sigma_o^{-1/2}$. This makes the performance evaluation of $MCD_n(r)$, SA', SA, SR, SV, LCR'_n and LRC_n easier.

Although the RL distribution of $MCD_1(r)$ and $MCD_2(r)$ are the same, the assumptions behind $MCD_1(r)$ and $MCD_2(r)$ are different. In the former we assume $\mu = \mu_o$ is known and sample size is 1. In the latter we do not assume μ is known. So the process mean can be either in-control or out-of-control. But we assume the sample size is 2 in each time period. Added information is used for estimating μ.

5.2. Designs of the FIR parameter

Traditionally, we use the average RL (ARL) to summarize the main feature of an RL. However, in some situations, it may be misleading. For instance, most authors have adopted the following strategy in designing a control chart: choose the parameters in the chart so that the out-of-control ARL's are as small as possible, subject to the in-control ARL being larger than or equal to some specified number. Unfortunately, for the exponentially weighted moving average chart (EWMA) with time-varying control limits (see Lowery, Woodall, Champ and Rigdon (1992)) or $MCD_n(r)$ discussed in this paper, we can choose the parameters so that the out-of-control ARL's are extremely small (near 1) while the in-control ARL is still larger than that specified number. However, the variances of in-control RL are tending to infinity. This will result in many extremely small

RL observations even if the process is in-control (see Chan and Zhang (1997) for details). So, when we evaluate the performance of a chart, we need to calculate ARL as well as the standard deviation of RL (SRL). Furthermore, when we select the parameter r we must put some constraints on the coefficient of variation of the RL.

Recently, Chan and Zhang (1997) suggested as a constraint that the coefficient of variation of RL should be designed to be smaller than or close to 1. We use the Shewhart-type chart to illustrate the suggestion. For the Shewhart-type chart, when a process is in-control ARL $= 1/P$ and SRL $=$ ARL$\sqrt{1-P}$ (see Ryan (1989, p.144)), where P is the probability that the test statistic used in that chart will fall outside the control limits. So the coefficient of variation $SRL/ARL = \sqrt{1-P} \leq 1$. We use this constraint when we design the parameter r of $MCD_n(r)$.

There is another problem in evaluating the performance of the proposed charts. Ideally, we should calculate all of the out-of-control ARL and SRL when we evaluate the performance of $MCD_n(r)$ and LRC_n. However, it is impossible to carry out this task using the Monte Carlo simulation, as there are many out-of-control cases. A commonly used approach is to select some typical out-of-control matrices. Recall in Subsection 5.1 that for $MCD_n(r)$ and LRC_n with the nominal Σ_0, we need only to select some typical out-of-control matrices from all the Σ with $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2}$ being diagonal. For each $p = 2$ we select nine typical out-of-control matrices: $\Sigma = \Sigma_0^{1/2}T\Sigma_0^{1/2}$, $\lambda(T) = B_2, C_2, D_2, E_2, F_2, G_2, H_2, J_2, K_2$, where $\lambda(T)$ stands for the vector of the eigenvalues of T and

$$\begin{aligned}
&B_2 = (1.5, 0.5), \quad C_2 = (1.25, 0.75), \quad D_2 = (1.5, 1.1),\\
&E_2 = (4.3, 1.), \quad F_2 = (1.5, 1.2), \quad G_2 = (1.1, 1.2),\\
&H_2 = (0.9, 1.2), \quad J_2 = (0.1, 2.5), \quad \text{and} \quad K_2 = (2.5, 3.5).
\end{aligned}$$

B_2, D_2, F_2, C_2, G_2, and H_2 represent some typical moderate or small changes; E_2, J_2 and K_2 are the examples of large changes. Similarly, for $p = 3$, we select the typical out-of-control matrices Σ with $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2}$ having the vectors of the eigenvalues B_3, C_3, D_3, E_3, F_3, and G_3, respectively, where

$$\begin{aligned}
&B_3 = (1.5, 1., 1.), \quad C_3 = (1.2, 1.3, 1.), \quad D_3 = (4.5, 3., 2.),\\
&E_3 = (0.1, 0.2, 0.5), \quad F_3 = (0.1, 2.5, 0.5), \quad \text{and} \quad G_3 = (0.5, 1.5, 1.).
\end{aligned}$$

For each typical case, the out-of-control ARL and SRL are calculated by Monte Carlo simulation. The replicate number in all of these Monte Carlo simulations is $6,000$ or $12,000$.

In the following cases, shown in Table 5.1, we first select FIR parameter r for $MCD_n(r)$. Then h is calculated by some pre-determined in-control ARL and

SRL (corresponding to Σ_0). The out-of-control ARL and SRL are then simulated (corresponding to B_2, C_2, D_2, and E_2). For illustration purposes, the in-control ARL and SRL are chosen to cover various r and h values within certain ranges.

Table 5.1. ARL and SDR values of $MCD_n(r)$ with $k_u = 1.5$, $k_l = 0.5$, $p = 2$ and $\mu = 0$ when the process is in-control (Σ_0) and out-of-control (Σ), where

$$\lambda = \lambda(\Sigma_o^{-1/2} \Sigma \Sigma_o^{-1/2}).$$

$n = 1$ or $n = 2$								
(r, h)	(0.0,	11.8)	(0.3,	11.85)	(0.6,	12)	(0.8,	12.68)
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1,1)$	129	121	130	123	130	128	130	152
B_2	44.8	38.1	44.4	38.1	40.9	37.7	32.5	37.7
C_2	86.7	80.9	87.0	81.8	85.7	85.9	78.5	92.4
D_2	35.1	31.1	34.7	31.3	32.3	31.8	26.8	33.0
E_2	5.82	4.19	5.61	4.13	4.84	3.98	3.74	3.51

$n = 5$								
(r, h)	(0.0,	3.5)	(0.4,	3.52)	(0.5,	3.53)	(0.6,	3.54)
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1,1)$	104	99	105	101	104	102	102	104
B_2	18.7	15.6	17.8	15.4	17.1	15.3	15.8	14.9
C_2	49.5	46.7	48.9	47.3	48.2	47.7	46.7	48.1
D_2	15.7	13.3	14.9	13.2	14.3	13.2	13.3	13.2
E_2	2.18	1.25	2.01	1.19	1.92	1.14	1.79	1.07

$n = 10$								
(r, h)	(0.0,	1.7)	(0.3,	1.72)	(0.45,	1.73)	(0.6,	1.9)
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1,1)$	124	122	131	129	132	132	192	202
B_2	11.8	9.50	11.6	9.60	11.0	9.6	11.1	10.5
C_2	40.1	38.0	40.6	39.1	40.4	39.8	48.6	50.2
D_2	10.4	8.80	10.2	8.90	9.80	8.90	10.0	10.0
E_2	1.36	0.61	1.32	0.58	1.27	0.53	1.22	0.50

Case 1. $n = 1$ or $n = 2$.

For $p = 2$, $k_u = 1.5$ and $k_l = 0.5$, the ARL and SRL of $MCD_1(r)$ with $r = 0, 0.3, 0.6, 0.8$ are shown in Table 5.1. It suggests that for $p = 2$, $r = 0.6$ has a better ability to detect the process covariance change among $MCD_1(r)$, $0 \leq r < 1$, subject to the condition that the coefficient of variation of the RL is close to 1. If we prefer a chart with a small coefficient of variation, $r = 0$ is a good choice. Note that, as pointed out in Subsection 5.1 (ii), the above result also holds for $n = 2$.

Case 2. $n = 5$.

For $p = 2$, $k_u = 1.5$ and $k_l = 0.5$, the ARL and SRL of $MCD_5(r)$ with $r = 0.0, 0.4, 0.5, 0.6$ are given in Table 5.1. Making a comparison of these numerical results, we suggest that any value between 0.4 and 0.6 is a reasonable choice for r.

Case 3. $n = 10$.

For $p = 2$, $k_u = 1.5$ and $k_l = 0.5$ the ARL and SRL of $MCD_{10}(r)$ with $r = 0.0$, 0.3, 0.45 and 0.6 are shown in Table 5.1. Any value between 0.3 and 0.45 seems a reasonable choice for r.

For $p = 3$ and 4, the results are similar and are not shown here.

Table 5.2. The in-control ARL and SRL of $MCD_1(0.6)$, $MCD_2(0.6)$, $MCD_1(0)$, $MCD_2(0)$, $MCD_5(0)$ and $MCD_{10}(0)$ with $k_u = 1.5$, $k_l = 0.5$ for $p = 2, 3, 4$ and various control limits h.

$p=2$			$p=3$			$p=4$		
$MCD_1(0.6)$ or $MCD_2(0.6)$								
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
10*	45	26	13*	42	26	15*	37	26
12	130	128	15	109	109	20	156	159
15	298	289	18	231	229	22	242	242
17	506	489	20	385	382	23	308	306
18	642	566	22	590	538	25	476	456
$MCD_1(0)$ or $MCD_2(0)$								
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
12	139	133	18	246	232	22	262	249
15	309	303	20	397	370	23	326	306
17	521	479	22	610	535	25	498	452
$MCD_5(0)$								
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
3.0*	43	25	4.0*	47	25	5.0*	45	24
3.5	106	104	4.5	131	129	5.5	105	100
4.0	182	178	5.3	302	301	6.0	164	162
4.3	249	244	5.5	373	365	6.5	257	251
4.5	308	300	6.0	599	535	7.0	411	386
$MCD_{10}(0)$								
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
1.2*	32	24	1.7*	34	24	2.3*	42	25
1.7	123	121	2.2	122	121	2.5	89	87
2.0	250	252	2.5	239	237	3.0	252	248
2.2	404	389	2.7	374	370	3.2	381	371
2.5	760	626	3.0	684	585	3.3	465	441

Note: $'*'$—the underlying RL has been truncated by 71.

In summary, for fixed n, p and control limits, as r tends to 1, the ARL of the chart decreases while the SRL of the chart increases. For fixed n and p, it is possible to adjust $0 \leq r < 1$ and control limits so that the in-control ARL is not less than some specified level, the out-of-control ARL is as small as possible and the coefficient of variation of RL is below or close to 1.

In-control ARL and SRL are two important quantities in selecting a chart. The in-control ARL and SRL of $MCD_1(0.6)$, $MCD_1(0)$, $MCD_2(0)$, $MCD_2(0.6)$, $MCD_5(0)$ and $MCD_{10}(0)$ under various control limits h are listed in Tables 5.2. A similar table for LRC_5 and LRC_{10} is Table 5.3 of Chan and Zhang (2000).

Table 5.3. The in-control ARL and SRL of LRC_n for $(p, n) = (2, 5)$, $(3, 5)$, $(4, 5)$ and $(2, 10)$, and for various control limits h.

$(p,n) = (2,5)$			$(p,n) = (2,10)$			$(p,n) = (3,5)$			$(p,n) = (4,5)$		
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
16.8	133	126	16.0	157	151	28.8	184	180	55.8	175	174
17.6	174	165	16.8	210	205	29.6	225	217	56.6	194	192
18.4	234	229	17.6	284	280	30.4	280	275	57.4	217	213
18.6	252	244	18.4	379	365	30.8	313	307	60.0	293	282
19.2	308	301	19.2	507	469	31.2	347	340			
20.0	415	391	20.0	679	574	32.0	419	399			

6. Applications to Paired Measurements

Grubbs's model is used for assessing bias and precision of paired measurement systems (see Blackwood and Bradley (1991)). We employ this model to account for the bias and variation of observed measurements of a process. For simplicity, we consider only the systems with two devices being used to measure a process. Then Grubbs's model is of the form

$$x_{i1} = \mu + d_i + \alpha_1 + \epsilon_{i1}, \quad x_{i2} = \mu + d_i + \alpha_2 + \epsilon_{i2}, \quad i \geq 1, \tag{6.1}$$

where x_{ij} is the observed measurement when the ith specimen is measured with the jth device, $j = 1, 2$, μ is the hidden (or true) process mean, d_i is the true random deviation of the process from μ for the ith specimen, α_j is the bias for device j, and ϵ_{ij} is the random error for the ith specimen when measured by device j. In practice the variation observed from a process is due to the hidden process variability and the error of the measurement device. If we use only a single device to measure a process, it is impossible to account for the sources when an out-of-control signal appears. We show that if two devices are used, a synthetic chart can be plotted for the variations of two devices as well as for the process variance. A similar chart can also be plotted for the relative precision—the ratio between the hidden process variance and the total observed variance. To this end, we first formulate the above model in terms of the multivariate statistical analysis. Let $\mu_i = (\mu + \alpha_1, \mu + \alpha_2)^\top$, $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2})^\top$, $dd_i = (d_i, d_i)^\top$ and $x_i = (x_{i1}, x_{i2})^\top$. Then model (6.1) is equivalent to

$$x_i = \mu_i + dd_i + \epsilon_i, \quad i \geq 1.$$

Assume that, given dd_i, the expectations of ϵ_i and $\epsilon_{i1}\epsilon_{i2}$ are zero. Let σ_d^2, $\sigma_{\epsilon_1}^2$ and $\sigma_{\epsilon_2}^2$ are the variances of the process and the devices. Under these assumptions, the covariance matrix of x_i is

$$\begin{pmatrix} \sigma_d^2 + \sigma_{\epsilon_1}^2 & \sigma_d^2 \\ \sigma_d^2 & \sigma_d^2 + \sigma_{\epsilon_2}^2 \end{pmatrix}$$

and the relative precision of the process is $\sigma_d^2 / \sqrt{(\sigma_d^2 + \sigma_{e_1}^2)(\sigma_d^2 + \sigma_{e_2}^2)}$.

The covariance of x_i changes if the process variance or one of device variances changes. Hence, to control the process variance or the measurement variance of the devices, it is sufficient to control the covariance of the multivariate variable x_i. Similarly, to control the relative precision, it suffices to control the correlation of paired measurements.

Example 6.1. This example involves a data set used by Tracy, Young and Mason (1995, p.374). The data set contains 19 paired measurements from a petrochemical industry. The first 15 observations are taken from a stable process and used to estimate the in-control μ_0 and Σ_0. The results are

$$\hat{\mu}_0 = (7.09, 7.113) \quad \text{and} \quad \hat{\Sigma}_0 = \begin{pmatrix} 0.1498 & 0.0334 \\ 0.0334 & 0.0241 \end{pmatrix}.$$

Tracy, Young and Mason (1995) applied the partial T^2-Shewhart control chart for the process mean to this data set and demonstrated that observations 18 and 19 indicate that the process mean is out-of-control (the referee pointed out that the partial T^2 statistics are not independent and F-distributed as Tracy, Young and Mason claimed). They showed that obervations 18 and 19 are located at opposite regions (see Figure 3 in Tracy, Young and Mason (1995)). In our opinion the locations of observations 18 and 19 imply that the process covariance, not the process mean, is changed. For, if the process mean is changed and the covariance is in-control, consecutive observations should be not far from each other. Here we use $MCD_2(0.6)$ to support our view. Paired measurements 16 through 19 are used to check whether the hidden process variance and the device variances are in control. To reduce the effect of the process mean, these four paired measurements are grouped into two samples. Then these two samples are monitored by $MCD_2(0.6)$. From Table 5.2, with $k_u = 1.5$, $k_l = 0.5$ and $h = 12$, the in-control (ARL, SRL) is $(130, 128)$. The CUSUM values are $SU_1 = 6.88$, $SL_1 = -4.83$, $SU_2 = 107.4$ and $SL_2 = -4.95$. SU_2 falls outside the upper control limit. Therefore, an out-of-control signal appears at the second subgroup. The other CUSUM values are within the control limits. Thus both the hidden process variance and the variances of measurement system are taken as in-control at the first subgroup.

7. Illustrative Examples

The following two examples show that $MCD_1(r)$ can be used when both μ_0 and Σ_0 are known or can be estimated from the previous samples. $MCD_1(r)$ can detect both changes in the process mean and covariance. However, when we obtain an out-of-control signal, it is difficult to determine whether the signal is due to change of the process covariance or the process mean.

Example 7.1. We begin with 28 samples of size 1 (see Table 7.1 of Chan and Zhang (2000)). The first two samples are drawn from a three-dimensional normal $N(0, I_3)$, while the remains of samples are drawn from a three-dimensional normal $N(0, G)$ with

$$G = \begin{pmatrix} 1.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{pmatrix}.$$

We use $MCD_1(0)$ and the FIR chart $MCD_1(r)$ to detect covariance change in this data set.

First, we use $MCD_1(0)$.

Step 1. Select suitable parameters for $MCD_1(0)$. We choose $k_u = 1.5$, $k_l = 0.5$ and $h = 15$. The in-control (ARL,SRL) is then $(118, 109)$, obtained by simulation.

Step 2. Calculate y_i by multiplying each sample by $\Sigma_0^{-1/2}$. In this example $x_i = y_i$.

Step 3. Calculate the CUSUM values, SU_i and SL_i, and $u(i)$ and $l(i)$.

Step 4. Check whether SU_i or SL_i falls outside the control limits. The CUSUM values are plotted on the chart in Figure 7.1. They show that $MCD_1(0)$ gives an out-of-control signal at the 6th sample. Note that $u(6)$ can be used to estimate when the covariance change occurs.

Now we use $MCD_1(r)$ to monitor the same data set. Using $k_u = 1.5$, $k_l = 0.5$, $p = 3$, $n = 1$, $r = 0.6$ and $h = 15$ in Table 5.2, the in-control

(ARL,SRL) is $(109, 109)$. $MCD_1(0.6)$ gives an inflated out-of-control signal at the 3rd sample. So, compared with $MCD_1(0)$, $MCD_1(0.6)$ has a faster response to an initial change.

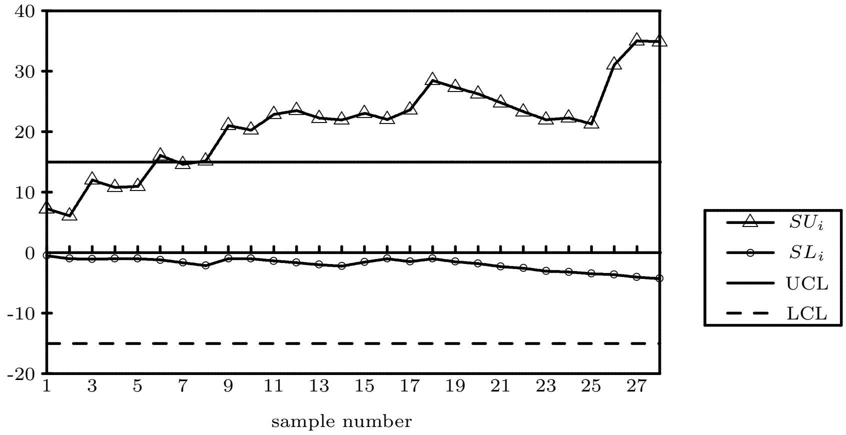

Figure 7.1. Control Chart for Example 7.1.

In the short-run production environment, to detect possible covariance change before the end of the production process, we use the maximum number of observations to truncate the RL of $MCD_n(r)$. Example 7.2 illustrates such a situation. It shows that, unlike other charts, the proposed chart can be used even where the sample size is smaller than the dimension of process distribution and the process mean varies between samples.

Table 7.1. Simulated data used in Example 7.1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
−2.90552	0.17469	2.17416	−0.46873	1.50455	2.84555	−0.06947
0.51099	−0.92729	−1.74290	−1.29043	1.29883	−0.57591	1.15380
0.27008	−1.48665	−0.61098	2.35554	−0.85250	1.01441	−2.02818
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
−1.34906	−1.70090	0.02583	1.72567	1.66900	−0.76182	−0.05649
0.36748	2.47792	1.17191	−0.95384	−0.60798	0.13484	−1.81073
−1.15512	−0.88689	−1.02823	0.50284	−0.91250	1.11752	0.97035
x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}
−1.64530	−1.30068	−1.69236	−1.33948	−0.91634	−0.85476	0.48865
0.96462	−0.81295	9.26316	−3.10570	0.06917	0.74236	−0.50608
3.08148	−0.91224	1.10580	2.75213	0.54215	−0.50952	0.01627
x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}
−1.24435	−1.32192	1.67164	0.57368	−0.85195	1.13292	−0.13808
0.29925	0.18265	0.80070	−1.49815	−1.98915	1.68993	0.64309
−0.18781	0.32869	−0.45742	0.25999	2.56061	−1.36947	−1.09751

Example 7.2. Suppose the maximum number of samples is 71. The first two samples are from a three-dimensional normal $N(0, I_3)$ and the rest are from a three-dimensional normal with mean $(1, -2, -3)^\top$ and covariance

$$C = \begin{pmatrix} 2.00 & -0.25 & -0.25 \\ -0.25 & 1.00 & -0.25 \\ -0.25 & -0.25 & 2.00 \end{pmatrix}.$$

We use $MCD_3(0.6)$ truncated by 71 to check whether a change has occurred in the process covariance. Let $k_u = 1.5$, $k_l = 0.5$, and $h = 13$. According to Table 5.2, the in-control ARL is 42 (< 71). Calculate and plot the CUSUM values on the chart. An inflated out-of-control signal appears at the 4 sample. In practice we may not want to stop the process immediately when an out-of-control signal appears in order to obtain some additional samples to estimate the magnitude of the change. As an example, here we stop the process after we obtain 10 addtional out-of-control samples.

8. Comparison with Other Charts

In this section, using Monte Carlo simulation, the performance of the proposed charts is compared with that of others. Definitions of alternative charts are given in Appendix III. The replicate number in this study is $6,000$ or $12,000$, as before.

1. SR chart—the Shewhart chart based on Roy's maximum and minimum eigenvalues of sample variances (see Anderson (1984, p.328)).

2. SA chart—the Shewhart chart based on Anderson's test of covariance (see Alt (1988, p.344)).

3. SV chart—the Shewhart chart based the square root of Hotelling's generalized variance (see Alt (1988, p.349)).

Note that the sample size $n > p$ is required in all cases. We take $n = 5$ and $p = 2$, 3 in this study.

Table 8.1 shows the results of the comparison of these charts with $MCD_5(0)$ (with $k_u = 1.5$ and $k_l = 0.5$) and LRC_5 for $p = 2$, 3 when $n = 5$. It is clear from these comparisons that the SA and SR charts are more effective than the SV chart in detecting process covariance change. Compared with the SA, SR and SV charts, in many cases $MCD_5(0)$ and LRC_5 can detect moderate or small covariance change more than twice as fast. The SR chart is slightly better than $MCD_5(0)$ and LRC_5 in detecting the relatively larger covariance change E_2. It is found that $MCD_5(0)$ with $k_u = 1.5$ and $k_l = 0.5$ can perform better than LRC_5 when only upward changes exist (for example, F_2 and K_2), and worse than LRC_5 when a certain amount of downward change happens (for example, B_2 and J_2). It is clear that, by adjusting the reference values, we can make MCD_n more sensitive to some pre-specified changes at the cost of effectiveness for detecting some other changes.

Table 8.1. Comparisons of ARL and SRL of $MCD_n(0)$, LRC_n, SR, SA and SV (with $k_u = 1.5$, $k_l = 0.5$) charts for $n = 5$, $p = 2, 3$, where

$$\lambda = \lambda(\Sigma_o^{-1/2}\Sigma\Sigma_o^{-1/2}).$$

$p = 2, \mu = 0$										
	$MCD_5(0)$		SR		SA		SV		LRC	
	$h = 4.3$		$h = 3.11$		$h = 16$		$h = 3.66$		$h = 18.6$	
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1, 1)$	247	244	205	204	203	202	211	212	248	241
B_2	26.1	21.3	55.1	55.0	74.0	73.6	651	565	13.0	6.28
C_2	81.9	78.1	121	121	153	153	269	265	45.2	27.5
D_2	20.0	16.9	31.8	30.9	116	115	41.5	40.8	32.8	20.6
E_2	2.40	1.36	2.44	1.90	4.35	3.78	5.82	5.36	3.17	1.84
F_2	17.8	14.9	27.5	26.6	110	109	32.9	32.2	29.8	18.2
G_2	59.0	55.2	74.4	73.2	181	181	80.5	79.8	102	73.7
H_2	93.9	90.2	119	117	181	180	158	158	97.1	70.9
J_2	5.26	2.89	8.44	7.89	5.41	4.87	>211		2.57	0.91
K_2	2.32	1.25	2.21	1.64	4.41	3.90	2.56	2.04	3.06	1.66

$p = 3, \mu = 0$										
	$MCD_5(0)$		SR		SA		SV		LRC	
	$h = 4.95$		$h = 3.73$		$h = 28.7$		$h = 4.05$		$h = 29.8$	
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1, 1)$	209	206	206	206	205	205	197	196	203	189
B_3	32.5	29.3	53.6	52.9	181	181	75.0	74.4	78.3	51.6
C_3	35.9	32.4	58.6	58.0	187	187	69.0	68.2	90.7	59.9
D_3	1.87	0.94	1.69	1.08	4.81	4.31	2.34	1.77	2.80	1.44
E_3	12.4	0.60	> 206		12.9	12.6	> 197		2.75	0.82
F_3	5.77	3.16	11.4	11.0	11.5	10.9	>197		3.06	1.07
G_3	24.1	17.7	62.7	62.4	117	117	423	409	16.0	7.07

As pointed out in Hawkins and Olwell (1997, pp.87, 144-145), in one dimensional cases the sample variance of each subgroup has a χ^2 distribution (which belongs to the gamma family). By applying the likelihood ratio procedure to the gamma distribution, they showed that if the Johnson-Leone CUSUM chart is used to monitor for a change in variance from the in-control standardized variance 1 to a larger variance σ_u^2, or to a smaller variance σ_l^2, the optimal reference values should be $k_u = \sigma_u^2 \log(\sigma_u^2)/(\sigma_u^2 - 1)$, $k_l = \sigma_l^2 \log(\sigma_l^2)/(\sigma_l^2 - 1)$. For example, if $\sigma_u^2 = 1.5$ and $\sigma_l^2 = 0.5$, we recover the frequently used reference values $k_u = 1.5$ and $k_l = 0.5$ approximately. However, in multivariate cases, neither the largest nor the smallest eigenvalues of the sample covariance matrix of each subgroup follows the χ^2 distribution (see Muirhead (1982, pp.420-425)). This suggests that it may be possible to improve the performance of $MCD_n(0)$ and SR by choosing suitable k_u and k_l or pair values of (h_l, h_u). Unfortunately it is time-consuming if we try to directly optimize $MCD_n(0)$ with respect to k_u and k_l. Here we find a relatively fast way to adjust the reference values to substantially improve the performance of $MCD_5(0)$ in detecting downward changes. Although $MCD_5(0)$ with $k_u = 1.5$ and $k_l = 0.5$ is very fast for detecting the upward changes, it is very slow when some downward changes happen. For example, out-of-control ARLs for some downward changes can be larger than the in-control ARL. This means that, like LCR'_5, $MCD_5(0)$ with $k_u = 1.5$ and $k_l = 0.5$ is biased. The simulation in Table 8.2 of Chan and Zhang (2000) shows that we can choose the reference values so the resulting $MCD_5(0)$ is unbiased, that is, the out-of-control ARLs are less than the in-control ARL. First, we set $k_u = 1.5$, $k_l = 0.5$ and $h = h_o$. Then we choose $c_o > 0$ such that $ARL(c_o) = \max_{c>0} ARL(c)$, where $ARL(c)$ is the out-of-control ARL when $\Sigma = c\Sigma_o$. Now if $k_u = 1.5/c_o$, $k_l = 0.5/c_o$ and $h = h_o/c_o$, then the corresponding $ARL(c)$ for the new $MCD_n(0)$ attains its maximum at $c = 1$. Table 8.2 of Chan and Zhang (2000) shows that the performance of $MCD_5(0)$, with the adjusted reference values, is now very similar to that of $LCR_5(0)$ and can detect upward changes as well as downward changes.

Table 8.2. Comparisons of ARL and SRL of $MCD_n(0)$, LRC_n, SR, SA and SV charts for $n = 5$, $p = 2, 3$, where $\lambda = \lambda(\Sigma_o^{-1/2}\Sigma\Sigma_o^{-1/2})$.

	$p = 2, \mu = 0$									
	$MCD_5(0)$		SR		SA		SV		LRC_5	
	$h = 2.34/0.72$		$h_u = 3.2$		$h = 16.2$		$h = 3.66$		$h = 18.6$	
	$k_u = 1.5/0.72$		$h_l = 0.493$							
	$k_l = 0.5/0.72$									
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
(1.0, 1.0)	247	244	219	209	219	219	211	211	248	240
(0.2, 0.2)	6.47	0.57	23.9	23.2	10.5	10.0	>2000		2.91	0.91
(0.4, 0.8)	10.9	2.83	96.1	95.1	90.2	89.7	>1981		11.0	4.70
(0.8, 0.6)	22.2	11.5	140	139	141	141	> 1731		24.5	12.7
(1.0, 0.6)	24.2	13.3	156	154	153	154	> 1269		28.8	15.1
(1.6, 0.6)	18.1	11.6	85.0	83.7	78.9	78.3	246	244	15.3	8.16
(2.0, 0.6)	11.2	8.04	36.2	35.6	36.8	36.4	110	110	9.76	5.36
(1.0, 0.8)	99.9	88.3	199	193	201	198	506	471	93.3	65.2
(1.4, 1.2)	45.2	43.9	115	113	139	139	39.6	38.7	39.1	24.5
(2.0, 1.2)	11.0	9.79	30.0	29.6	42.7	41.6	16.5	16.1	12.3	7.19
(1.6, 1.6)	13.9	12.7	40.8	40.3	60.6	59.4	14.4	14.0	14.6	8.33
(1.8, 1.6)	10.5	9.22	29.1	28.4	44.5	43.6	11.4	11.0	11.6	6.63
(2.0, 1.6)	8.06	6.85	21.2	20.9	32.3	31.6	9.35	8.97	9.51	5.40
(1.8, 1.8)	8.29	7.18	22.4	21.7	33.9	33.1	9.16	8.72	9.69	5.43
(2.0, 1.8)	6.83	5.62	17.2	16.8	26.1	25.4	7.68	7.22	8.22	4.61
(2.0, 2.0)	5.74	4.62	13.8	13.5	20.9	20.2	6.49	6.01	7.16	4.00

Table 8.2. (continued)

	$p=3, \mu=0$									
	$MCD_5(0)$ $h=2.34/0.75$ $k_u=1.5/0.75$ $k_l=0.5/0.75$		SR $h_u=4.4$ $h_l=0.49944$		SA $h=28.7$		SV $h=4.05$		LRC_5 $h=29.8$	
λ	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL	ARL	SRL
$(1.0, 1.0, 1.0)$	209	206	213	205	205	205	197	196	203	189
$(0.2, 0.2, 0.2)$	7.74	0.54	45.9	45.8	12.4	12.0	>2000		2.74	0.84
$(0.2, 0.6, 1.4)$	8.19	1.19	83.8	83.5	52.6	52.5	>1988		5.77	1.93
$(0.2, 1.4, 2.0)$	6.40	2.53	49.7	48.8	32.8	32.4	893	673	5.04	1.88
$(0.4, 0.8, 0.8)$	12.3	2.68	132	32	115	115	>1905		13.1	5.15
$(0.4, 0.8, 1.2)$	12.2	2.90	139	138	115	114	> 1560		13.2	5.24
$(0.6, 1.8, 2.0)$	7.29	5.29	43.1	42.2	50.2	49.3	36.2	35.9	9.02	4.38
$(0.6, 2.0, 2.0)$	6.21	4.52	34.0	33.5	41.7	40.9	29.7	28.7	8.00	3.91
$(0.8, 0.8, 2.0)$	12.9	10.9	78.1	76.3	82.7	82.0	107	107	15.0	7.89
$(0.8, 1.0, 1.4)$	45.6	39.0	189	184	165	167	147	147	42.1	23.2
$(1.2, 1.6, 1.8)$	9.81	8.40	64.1	63.3	89.9	89.7	16.5	15.9	16.2	8.53
$(1.2, 1.8, 2.0)$	6.60	5.39	37.2	36.5	59.6	59.0	12.3	11.8	11.4	5.97
$(1.4, 1.8, 1.8)$	7.39	6.08	44.1	43.4	71.0	70.3	11.3	10.7	12.8	6.72
$(1.6, 1.6, 1.6)$	9.51	8.24	63.2	62.6	94.3	92.5	13.1	12.6	16.7	8.73
$(1.6, 1.8, 2.0)$	5.68	4.43	29.6	28.9	53.0	52.8	8.46	7.96	10.1	5.24
$(1.8, 1.8, 2.0)$	5.16	3.95	25.5	25.2	47.1	47.0	7.38	6.88	9.29	4.75
$(1.8, 2.0, 2.0)$	4.61	3.41	21.1	20.8	39.8	39.0	6.54	6.04	8.28	4.21

9. Conclusions

We have proposed CUSUM charts $MCD_n(r)$ and LRC_n for monitoring the change of the covariance matrix of a multivariate normal process using projection pursuit and the likelihood ratio, respectively. $MCD_n(r)$ is a natural extension of the CUSUM chart of Johnson and Leone (1962) for variance, while LRC_n is not.

The distribution of the run lengths of the control charts $MCD_n(r)$ and LRC_n depend on the nominal value Σ_0 and the actual Σ of the process only, specifically through the eigenvalues of the equation $\det(\Sigma - \lambda\Sigma_0) = 0$.

Unlike the Shewhart-type or likelihood ratio-based charts, where the subgroup size n must be at least equal to the dimension p, the PP based CUSUM charts can be used for any size $n \geq 1$. In Table 9.1 of Chan and Zhang (2000), some out-of-control ARLs of $MCD_2(0)$ with $p = 3$ show that $MCD_2(0)$ still has a good performance when the sample size is less than the dimension of the quality charateristics.

Simulation studies on the ARL and SRL show that a considerable improvement on the other three charts, in terms of faster detection of the covariance change, can be achieved if $MCD_n(r)$ (or LRC_n) is used. Calculation is, however, more complicated.

Table 9.1. The out-of-control ARL and SRL of $MCD_2(0)$ with $p = 3, \mu = 0$, $h = 10/0.74$, $k_u = 1.5/0.74$, $k_l = 0.5/0.74$ and the in-control (ARL,SRL)= $(170, 161)$ where $\lambda = \lambda(\Sigma_o^{-1/2}\Sigma\Sigma_o^{-1/2})$.

λ	ARL	SRL	λ	ARL	SRL	λ	ARL	SRL
$(0.2, 0.2, 0.2)$	24.9	1.49	$(0.2, 0.8, 1.4)$	26.0	5.89	$(0.4, 0.4, 1.4)$	33.4	9.44
$(0.2, 0.4, 0.6)$	27.3	2.58	$(0.2, 1.2, 1.4)$	24.7	7.05	$(0.4, 0.6, 2.0)$	25.7	14.4
$(0.2, 0.4, 1.4)$	26.1	5.18	$(0.2, 1.4, 1.4)$	23.4	7.95	$(0.4, 0.8, 1.4)$	37.7	14.0
$(0.2, 0.4, 2.0)$	21.2	8.42	$(0.2, 1.4, 2.0)$	18.0	9.32	$(0.4, 0.8, 2.0)$	25.4	14.8
$(0.2, 0.6, 0.6)$	27.6	2.80	$(0.2, 1.6, 1.6)$	20.1	9.06	$(0.4, 1.0, 1.0)$	41.7	12.3
$(0.2, 0.6, 1.6)$	24.7	6.94	$(0.2, 2.0, 2.0)$	13.9	8.81	$(0.4, 1.0, 2.0)$	24.4	14.9
$(0.2, 0.6, 2.0)$	21.0	8.67	$(0.4, 0.4, 0.6)$	35.0	6.14	$(0.4, 1.2, 1.4)$	34.7	15.3
$(0.2, 0.8, 0.8)$	27.7	3.03	$(0.4, 0.4, 0.8)$	35.7	6.48	$(0.4, 1.2, 1.6)$	30.6	15.7
$(0.2, 0.8, 1.2)$	27.1	4.34	$(0.4, 0.4, 1.2)$	35.1	7.92	$(0.4, 1.2, 2.0)$	22.9	14.6
$(0.6, 0.6, 0.8)$	59.2	22.9	$(0.6, 1.6, 2.0)$	19.6	16.0	$(0.8, 2.0, 2.0)$	14.2	11.9
$(0.6, 0.6, 1.6)$	43.7	25.1	$(0.8, 0.8, 0.8)$	126	87.0	$(1.0, 1.0, 1.8)$	34.1	31.0
$(0.6, 0.6, 1.8)$	35.8	22.6	$(0.8, 0.8, 1.0)$	146	111	$(1.0, 1.0, 2.0)$	26.0	22.9
$(0.6, 0.8, 1.0)$	78.0	42.2	$(0.8.0.8, 1.6)$	57.2	47.8	$(1.0, 1.2, 1.6)$	40.0	37.7
$(0.6, 0.8, 1.4)$	60.6	37.9	$(0.8, 0.8, 2.0)$	30.2	25.6	$(1.0, 1.2, 2.0)$	23.3	20.6
$(0.6, 1.0, 2.0)$	28.0	21.7	$(0.8, 1.2, 1.6)$	44.6	40.3	$(1.0, 1.8, 2.0)$	15.3	13.0
$(1.6, 1.6, 1.8)$	14.8	12.5	$(1.2, 1.2, 1.4)$	45.3	42.3	$(1.6, 1.6, 2.0)$	13.0	10.7
$(1.2, 1.2, 1.6)$	33.7	30.9	$(1.6, 1.8, 1.8)$	13.1	11.0	$(1.2, 1.2, 1.8)$	26.1	23.2
$(1.8, 1.8, 2.0)$	10.7	8.61	$(1.2, 1.4, 1.8)$	22.6	20.0	$(1.8, 2.0, 2.0)$	9.76	7.75
$(1.4, 1.4, 1.6)$	23.7	21.2	$(1.4, 1.4, 1.8)$	19.5	16.9	$(1.4, 1.4, 2.0)$	16.4	13.9

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Appendix I
Proofs of (i) and (ii) in Section 2

To prove (i), we note that the variances of $a_{\max}^\top \Sigma_0^{-1/2} X$ and $a_{\min}^\top \Sigma_0^{-1/2} X$ are

$$a_{\max}^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a_{\max} = \max_{||a||=1} a^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a$$

and

$$a_{\min}^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a_{\min} = \min_{||a||=1} a^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a,$$

respectively. Hence, the variances of $a_{\max}^\top \Sigma_0^{-1/2} X$ and $a_{\min}^\top \Sigma_0^{-1/2} X$ are equal to 1 if and only if $\max_{||a||=1} a^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a = \min_{||a||=1} a^\top \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} a = 1$, which holds if and only if $\Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2}$ is an identity matrix, that is, $\Sigma_0 = \Sigma$.

The proof of (ii) is similar.

Appendix II

Consider SU_i^a. We want to prove that as $v \to \infty$, $SU_v^a/v \to E(a^\top X)^2 - k_u$ in probability when $E(a^\top X)^2 > k_u$. In fact, if we define $w_0 = 0$, $w_m = \sum_{j=1}^m ((a^\top x_j)^2 - k_u)$, $m \geq 1$, then $SU_m^a = \max\{0, w_m - w_{m-1}, w_m - w_{m-2}, \ldots, w_1 - w_0\}$. Suppose $E(a^\top x_i)^2 = 1$, $i = 1, \ldots, m_0$, and $E(a^\top x_i)^2 = \sigma_a^2$, $i = m_0+1, \ldots$, with $\sigma_a^2 > k_u$. Then $w_i = \sum_{j=m_0+1}^i ((a^\top x_j)^2 - k_u) + \sum_{j=1}^{m_0} ((a^\top x_j)^2 - k_u)$. As $v \to \infty$, we have $w_i/v = (\sigma_a^2 - k_u) i/v + o_p(1)$ where $o_p(1)$ is uniform for $1 \leq i \leq v$ (see Pollard (1984, p.106)). Hence for large v, $SU_v^a/v = \max\{0, (\sigma_a^2 - k_u)(1 - 1/v), \ldots, (\sigma_a^2 - k_u)(1 - v/v)\} + o_p(1) = \sigma_a^2 - k_u + o_p(1)$ when $\sigma_a^2 > k_u$. The proof is completed.

Appendix III

Here we give the definitions of the SR, SA and SV charts. As in Section 3, we first make a transformation of each x_{ij} into y_{ij} by multiplying by $\Sigma_0^{-1/2}$, $1 \leq j \leq i$. Let $s^2(y_i) = \sum_{j=1}^n (y_{ij} - \overline{y}_i)(y_{ij} - \overline{y}_i)^\top/(n-1)$ be the sample covariance of the ith transformed sample, where $\overline{y}_i = \sum_{j=1}^n y_{ij}/n$.

SR Chart. Let SR_{ui} and SR_{li} denote the maximum and minimum eigenvalues of $s^2(y_i)$, C_{ur} and C_{lr} be the upper and lower control limits, and k_u and k_l be the upper and lower reference values. We choose $k_u = 1.5$ and $k_l = 0.5$ as we did for $MCD_5(0)$. Then, the SR chart is defined to indicate an out-of-control signal when $SR_{ui} - k_u > C_{ur}$ or $SR_{li} - k_l < C_{lr}$. Here, we choose $C_{ur} = -C_{lr} = h$, and h is a positive constant.

SA Chart. Suppose $n > p$. Let

$$SA_i = (n-1)(-p - \log(\det(s^2(y_i))) + \mathrm{tr}(s^2(y_i))),$$
$$SA_i' = n(-p - \log(\det((n-1)s^2(y_i)/n)) + \mathrm{tr}((n-1)s^2(y_i)/n)),$$

where $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the determinant and trace of a square matrix. Then the SA chart is defined to give an out-of-control signal when $SA_i > h$. Here, without confusion, h denotes the upper control limit of the SA chart. Similarly, the SA' chart is defined by using SA_i'.

SV Chart. Suppose $n > p$. Let

$$SV_i = (\sqrt{\det(s^2(y_i))} - b_3)/\sqrt{b_1 - b_3^2},$$

where $b_1 = (n-1)^{-p}\prod_{k=1}^{p}(n-k)$, $b_3 = (2/(n-1))^{p/2}\Gamma(n/2)/\Gamma((n-p)/2)$, and $\Gamma(\cdot)$ is the Gamma function. Then, the SV chart is defined to indicate an out-of-control signal when SV_i falls outside the upper and lower control limits C_{ur} and C_{lr}. Usually we choose $C_{ur} = -C_{lr} = h$, where h is a positive constant.

Appendix IV

Property (i) directly follows from the definitions of $MCD_n(r)$ and LRC_n.

To prove Property (ii), we first let $n = 2$ and $y_k^* = (x_{k1} - x_{k2})/\sqrt{2}$, $k = 1, 2, \ldots$. Then,

$$(y_{j1}y_{j1}^\top + \cdots + y_{jn}y_{jn}^\top)/(n-1) + \cdots + (y_{i1}y_{i1}^\top + \cdots + y_{in}y_{in}^\top)/(n-1) = y_j^* y_j^{*\top} + \cdots + y_i^* y_i^{*\top}.$$

Note that, under the i.i.d. and normal assumption, $\{y_k^*, \quad k \geq 1\}$ has the same distributions as $\{x_{k1} - \mu_o, \quad k \geq 1\}$. So the distributions of the statistics SU_i and SL_i are invariant when we replace $x_{k1} - \mu_o$ by y_{kn}. Thus, the RL distributions of $MCD_1(0)$ and $MCD_2(0)$ are the same.

References

Alt, F. B. (1985). Multivariate quality control. In *Encyclopedia of Statistical Sciences*, Vol. 1 (Edited by S. Kotz and N. L. Johnson). Wiley, New York.

Alt, F. B. and Smith, N. D. (1988). Multivariate process control. In *Handbook of Statistics.* Vol. 7 (Edited by P. R. Krishnaiah and C. R. Rao). North-Holland, Amsterdam.

Anderson, T. W. (1984). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

Blackwood, L. G. and Bradley, E. L. (1991). An Omnibus tests for comparing two measuring devices. *J. Qual. Tech.* **23**, 12-16.

Chan, L. K. and Li, G. Y. (1994). A Multivariate control chart for detecting linear trends. *Comm. Statist. Simul. Comput.* **23**, 997-1012.

Chan, L. K. and Zhang, J. (1997). Some issues on the designs of CUSUM and EWMA charts. Technical Report. Department of Management Sciences, City University of Hong Kong.

Chan, L. K. and Zhang, J. (2000). Cumulative sum control charts for covariance matrix. Report 2000-4. EURANDOM, the Nertherlands (http://www.eurandom.tue.nl).

Chang, T. C. and Gan, F. F. (1995). A Cumulative sum control chart for monitoring process variance. *J. Qual. Tech.* **27**, 109-119.

Flury, D., Nel, D. G. and Piennar, I. (1995). Simultaneous detection of shift in means and variances. *J. Amer. Statist. Assoc.* **90**, 1474-1481.

Hawkins, D. M. and Olwell, D. H. (1998). *Cumulative Sum Charts and Charting for Quality Improvement.* Springer, New York.

Huber, P. (1985). Projection pursuit. *Ann. Statist.* **13**, 435-475.

Jackson, J. E. (1991). *A User's Guide to Principal Components.* Wiley, New York.

Johnson, N. L. and Leone, F. C. (1962). Cumulative sum control charts. *Indust. Qual. Control* **18**, 15-21; **19**, 22-36.

Lowry, C. A. and Montgomery, D. C. (1995). A review of multivariate control charts. *IIE Trans.* **27**, 800-810.

Lowry, C. A., Woodall, W. H., Champ, C. H. and Rigdon, S. E. (1992). A multivariate exponentially weighted moving average control chart. *Technometrics* **34**, 46-53.

Mason, R. L., Champ, C. W., Tracy, N. D., Wierda, S. J. and Young, J. C. (1997). Assessment of multivariate process control techniques. *J. Qual. Tech.* **29**, 140-143.

Montgomery, D. C. (1991). *Introduction to Statistical Quality Control.* 2nd edition. Wiley, New York.

Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory.* Wiley, New York.

Ngai, H. M. and Zhang, J. (1994). Multivariate cumulative sum control charts based on projection pursuit. *Statist. Sinica* **11**, 747-766.

Pollard, D. (1984). *Convergence of Stochastic Processes.* Springer, New York.

Ryan, T. R. (1989). *Statistical Methods for Quality Improvement.* Wiley, New York.

Siegmund, D. (1985). *Sequential Analysis: Tests and Confidence Intervals.* Springer, New York.

Tracy, N. L., Young, J. C. and Mason, R. L. (1995). A bivariate control chart for paired measurements. *J. Qual. Tech.* **27**, 370-376.

Wierda, S. J. (1994). Multivariate statistical process control: recent results and directions for future research. *Statist. Neerlandica* **48**, 161-169.

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