Statistica Sinica **6**(1996), 471-479

LEVEL CHANGES AND TREND RESISTANCE IN $L_N(2^p4^q)$ ORTHOGONAL ARRAYS

P. C. Wang

National Central University

Abstract. Fractional factorial designs of mixed levels two and four are often carried out in industrial experiments. A convenient method to set up such designs is to use orthogonal arrays. By applying the technique of replacement to two-level orthogonal arrays, mixed-level orthogonal arrays can be established for practical applications. Treatment combinations for the experiment can be obtained by assigning factors to appropriate columns. However, when level changes in subsequent experimental runs and trend resistance of certain effects are considered, caution must be taken on replacement in order to minimize the cost of level changes and to achieve trend resistance. In this paper, level changes and the degree of trend resistance in the resulting column from the replacement are explored. When such information is available, the assignment becomes easier. Two applications are presented for illustrations.

Keywordsandphrases. Orthogonal arrays, Hadamard product, replacement, level changes, trend resistance.

1. Introduction

In industrial experiments, the most common number of levels are two, three and four. When using an orthogonal array to set up such designs, an asymmetric orthogonal array is required, that is, an array whose columns have different numbers of levels. When only factors of levels two and four are considered, the asymmetric orthogonal array needed can be obtained by applying the replacement technique to orthogonal arrays of two levels. This approach has been emphasized by authors such as Dey (1985), Wang (1990) and Wu (1989). While Wang (1990) used it to construct three series of orthogonal main-effect plans, Wu (1989) explored the maximum number of replacements in the orthogonal array of size 2^n. However, their usage of replacement ignored the fact that the observations in a factorial experiment might be highly correlated with the order of experimental runs. When this occurs, the run order is important and more consideration on replacement is required.

Randomization on experimental runs is not desired when the run order is considered. Many statisticians have discussed this point and suggested useful procedures to deal with run orders. These discussions are based on two criteria:

the trend resistance to make certain effects independent of the run order and the level change to minimize the cost of carrying out all the experimental runs. Some authors have concentrated on trend resistance. Cheng and Jacroux (1988) accomplished trend-resistant run orders for two-level factorial designs after they observed that t-factor interactions in the standard order of complete factorial designs are free of polynomial trends with $t-1$ degrees, while Bailey, Cheng and Kipnis (1992) extended this to mixed-level factorial designs. Others paid attention to both criteria. While Coster and Cheng (1988) utilized generalized foldover techiques to arrange the experimental order for the designs, with and without defining contrasts, Jacroux (1990) and Wang (1991) derived the level change properties of each column in an orthogonal array used for designing an experiment. In this article, level changes on the resulting column from replacement and its degree of trend resistance are explored. When such information is available, one can construct an appropriate asymmetric orthogonal array for one's own use. After factors are assigned to the columns of such an array to obtain experimental runs with the row order as a run order, the appropriate replacement would reduce the cost of level changes and make certain effects free of unknown trend effects.

The orthogonal array $L_N(2^p4^q)$ is defined to be an $N\times(p+q)$ matrix containing p columns of two levels and q columns of four levels such that any possible pair from any two columns appears equally often, where N is 2 to the power k for some integer k. Here we use $(1,-1)$ as two levels of 2-level factors. The orthogonal array $L_N(2^p4^q)$ can be constructed from $L_N(2^{p+3q})$ by replacement. To do this, we define the Hadamard product of two-level columns in the array to be componentwise multiplications. Any set of three columns in an orthogonal array is said to have the Hadamard property when the Hadamard product of any two columns is equal to the third. Such a set is called a Hadamard set. Each Hadamard set can be switched to a 4-level column by the replacement

$$\begin{array}{rrr} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}. \tag{1}$$

In order to arrange the run order of the experiment using $L_N(2^p4^q)$, we explore the information on level changes and trend resistance of the 4-level columns. The exploration will be given in the next section.

2. Main Results

A standard procedure to set up an orthogonal array of size $N=2^k$ for some integer k is to utilize k standard $N\times 1$ columns $X_1=(1,1,\ldots,1,-1,-1,\ldots,-1)^T$,

$X_2=(1,\dots,1,-1,\dots,-1,1,\dots,1,-1,\dots,-1)^T,\dots,X_k=(1,-1,1,-1,\dots,1,-1)^T$. Then the matrix $(X_1,X_2,X_1X_2,X_3,X_1X_3,X_2X_3,X_1X_2X_3,X_4,\dots,X_1X_2X_3\dots X_k)$ is an $L_N(2^{N-1})$ orthogonal array. The columns in this array can be grouped into $(2^k-1)/3$ or $(2^k-5)/3$ Hadamard sets depending on even or odd k (see Wu (1989)). It can be observed that vector X_i changes level 2^i-1 times. The ordering of the standard columns $X_1,\dots,X_k$ will be retained throughout the paper. Let $\ell(A)$ denote the number of level changes in a vector A with convention $\ell(\phi)=0$. The following lemma from Jacroux (1990) or Wang (1991) is needed.

Lemma 1. *For $i_1>i_2>\cdots>i_t$, $\ell(X_{i_1}X_{i_2}\dots X_{i_t})=\ell(X_{i_1})-\ell(X_{i_2})+\cdots+(-1)^{t-1}\ell(X_{i_t})=2^{i_1}-2^{i_2}+\cdots+(-1)^{t-1}2^{i_t}-\delta$, where $\delta=0$ if t is even, $=1$ otherwise.*

For any column $C=X_{i_1}X_{i_2}\dots X_{i_t}$, we say C contains columns X_{i_1}, $X_{i_2},\dots$, and X_{i_t}. By the definitions of the $X_i's$ and Lemma 1, we have the following lemma:

Lemma 2. *Let $A=X_{i_1}X_{i_2}\dots X_{i_t}$ and $B=X_{j_1}X_{j_2}\dots X_{j_s}$, where $i_1>i_2>\cdots>i_t$ and $j_1>j_2>\cdots>j_s$ for some s and t. Then*
(i) *if $i_1>j_1$, $\ell(A)>\ell(B)$, and*
(ii) *if $i_1=j_1$, $i_2=j_2,\dots,i_{r-1}=j_{r-1}$ and $i_r>j_r$ then $\ell(A)>\ell(B)$ when r is odd and $\ell(A)<\ell(B)$ otherwise.*
Conversely,
(iii) *if $\ell(A)>\ell(B)$, then one of* (a) *$i_1>j_1$,* (b) *$i_1=j_1$, $i_2=j_2,\dots,i_{r-1}=j_{r-1}$ and $i_r>j_r$ for some odd r, or* (c) *$i_1=j_1$, $i_2=j_2,\dots,i_{u-1}=j_{u-1}$ and $i_u<j_u$ for some even u, is true.*

Proof. Proofs of the three parts are similar. We just prove the first. For $t>1$,

$$\begin{aligned}\ell(A)-\ell(B)&=\ell(X_{i_1})-\ell(X_{j_1})-\ell(X_{i_2})+\ell(X_{i_3}\dots X_{i_t})+\ell(X_{j_2}\dots X_{j_s})\\&\ge\ell(X_{i_1})-\ell(X_{j_1})-\ell(X_{i_2})\\&=2^{i_1}-2^{j_1}-2^{i_2}+1>0\end{aligned}$$

since neither i_2 nor j_1 can be larger than i_1-1. It is trivial for $t=1$.

Corollary. *$\ell(X_j)\ge\ell(X_{i_1}\dots X_{i_t})$ if and only if $j\ge\max\{i_1,i_2,\dots,i_t\}$.*

Theorem 1. *Let A, B and C be three columns in an orthogonal array forming a Hadamard set. Then $\ell(A)\le\ell(B)+\ell(C)$.*

Proof. When A changes its levels, one of B and C has to change also.

Note that when A is just a standard column with $\ell(A)>\max\{\ell(B),\,\ell(C)\}$, we have $\ell(A)=\ell(B)+\ell(C)$. This can be shown by direct calculations.

Theorem 2. *Let A, B and C be any three columns in $L_N(2^{N-1})$ with the Hadamard property and* $\max\{\ell(B),\, \ell(C)\} < \ell(A)$. *Assume* $A = X_{i_1}X_{i_2}\dots X_{i_t}$ *with* $i_1 > i_2 > \cdots > i_t$ *and* $t > 1$.

(i) *If* $i_1 > i_2 + 1$, *then* $\ell(B) + \ell(C) \le \ell(A) + 2\ell(X_{i_2}\dots X_{i_t})$, *with equality when* $B = AX_j$ *and* $C = X_j$ *for* $i_2 < j < i_1$.

(ii) *If* $i_1 = i_2 + 1$, *and* $t \ge 3$, *then* $\ell(B) + \ell(C) \le \ell(A) + 2\ell(X_{i_4}\dots X_{i_t})$, *with equality when* $t = 3$, $B = X_{i_1}X_{i_2}$ *and* $C = X_{i_3}$.

Proof. (i) Since A contains X_{i_1}, exactly one of B and C has to contain X_{i_1}, say B. Let E be a column formed by the Hadamard product of standard columns in B with indices strictly between i_1 and i_2. Remove all the standard columns with indices greater than i_2 from A and B respectively to obtain columns D and F; then we have $A = X_{i_1}D$, $B = X_{i_1}EF$ and $C = AB = DEF$.

Now

$$\ell(C) = \ell(E) \pm \ell(DF), \quad \ell(B) = \ell(X_{i_1}) - \ell(E) \mp \ell(F)$$

and so

$$\begin{aligned} \ell(B) + \ell(C) &= \ell(X_{i_1}) \pm \ell(DF) \mp \ell(F) \\ &= \ell(A) + \ell(D) \pm (\ell(DF) - \ell(F)) \\ &\le \ell(A) + 2\ell(D), \qquad\qquad \text{[Theorem 1]} \end{aligned}$$

where $(+,-)$ or $(-,+)$ depends on the number of standard columns forming E.

(ii) Without loss of generality, assume X_{i_1} in B. By $\ell(B) < \ell(A)$ and Lemma 2, if B contains X_{i_1}, then B also needs to contain X_{i_2}. Let $D = X_{i_4}\dots X_{i_t}$; then $B = X_{i_1}X_{i_2}X_{i_3}CD$. To complete the proof, we consider two cases: (a) C containing X_{i_3} and (b) B containing X_{i_3}. In each case, we compute the level changes of B and C using different formulas.

Case (a):

$$\begin{aligned} \ell(B) + \ell(C) &= \ell(X_{i_1}X_{i_2}) + \ell(X_{i_3}CD) + \ell(X_{i_3}) - \ell(X_{i_3}C) \\ &= \ell(A) + \ell(D) + \ell(X_{i_3}CD) - \ell(X_{i_3}C) \\ &\le \ell(A) + 2\ell(D). \qquad\qquad \text{[Theorem 1]} \end{aligned}$$

Case (b):

$$\begin{aligned} \ell(B) + \ell(C) &= \ell(X_{i_1}X_{i_2}) + \ell(X_{i_3}) - \ell(CD) + \ell(C) \\ &= \ell(A) + \ell(D) - \ell(CD) + \ell(C) \\ &\le \ell(A) + 2\ell(D). \qquad\qquad \text{[Theorem 1]} \end{aligned}$$

The equality assertion is obvious by direct computations.

Now assume that A, B and C are three columns in $L_N(2^{N-1})$ with $A = BC$. Replacing three columns (B, C, A) using the replacement scheme (1) with a 4-level column R, we obtain an $L_N(4^1 2^{N-4})$ array. Since each level change in R requires two across A, B and C,

$$\ell(R) = (\ell(A) + \ell(B) + \ell(C))/2. \tag{2}$$

In fact, the result (2) is true for the replacement on any three columns in an orthogonal array having the Hadamard property. A lower bound and an upper bound for the resulting column R from the replacement (1) can be obtained based on the above theorems. For example, $\ell(R)$ is less than or equal to the sum of any two of $\ell(A)$, $\ell(B)$ and $\ell(C)$, and greater than or equal to maximum level changes among columns A, B and C. This information is useful for choosing appropriate Hadamard sets for replacement when the cost of level changes is considered.

In order to investigate the degree of trend resistance of the resulting column from the replacement, we need to define trend resistance. Let $P_j(V, \omega) = i$ if symbol ω occurs the j-th time in the i-th component of vector V, where V is an $N \times 1$ vector in $L_N(2^p4^q)$, ω is any symbol in V, i is between 1 and N and j is any positive integer less than or equal to $N/2$ or $N/4$. For example, $P_2(V, -1) = 4$ and $P_3(V, 1) = 6$ for $V = (1, -1, 1, -1, -1, 1, -1, 1)^t$ in $L_8(2^7)$. Any column K in an $L_N(2^p4^q)$ orthogonal array is q-trend resistant, denoted by $P(K) = q$, if $\sum_{j=1}^{v} P_j(K, \omega)^x$ is independent of symbols in the column with $v = N/2$ or $N/4$ for $x = 0, 1, \ldots, q$, but not for $x = q + 1$. In this case the degree of trend resistance for the column is q.

Theorem 3. *Assume that columns A, B and C in $L_N(2^{N-1})$ form a Hadamard set and are r-, s- and t-trend resistant repectively. Let R be the 4-level column resulting from the replacement of (A, B, C). Then $P(R) = \min\{r, s, t\}$.*

Proof. The result is obvious because columns A, B and C are orthogonal contrasts of column R.

The degree of trend resistance of any column in the $L_N(2^{N-1})$ orthogonal array depends on the number of different standard columns forming the column. In fact, it is equal to the number minus one (see Cheng and Jacroux (1988) or Wang (1991)). It follows that the degree of trend resistance for the resulting 4-level column from the replacement is one less than the minimum of the numbers of standard columns forming three columns in a Hadamard set.

3. Applications

In this section we present two examples in the $L(2^p4)$ orthogonal array. Here we merely illustrate the usefulness of our results in the previous section. We do not intend to establish explicit rules to gain the best plans in the general array

$L(2^p4^q)$. The issue is really difficult even in the case $q = 1$ or 2. This has been investigated, using our results, by Wang and Chen (1994).

Table 1. An ordered $L_{16}(2^{15})$ orthogonal array

	column number														
run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
3	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
4	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
5	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
6	1	−1	−1	−1	−1	1	1	−1	−1	1	1	1	1	−1	−1
7	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1
8	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
9	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
10	−1	−1	1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1
11	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1
12	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1
13	−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
14	−1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1
15	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
16	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
	1	1	2	2	1	1	3	3	1	1	2	2	1	1	4
*		2		3	2	3		4	3	2	3	4	2	4	
					3				4	3	4		4		
										4					

* The last row indicates the indices of standard columns forming the corresponding columns.

Example 1. Suppose an experiment of one 4-level factor and eight 2-level factors is considered with level-change costs. Assuming a constant cost, c, we can use an ordered $L_{16}(2^{15})$ orthogonal array given in Table 1 for the arrangement of the experiment,where the numbers under any column in the table are the indices of standard columns forming the column. Note that the number of level changes in the i-th column of Table 1 is equal to i. There are many Hadamard sets for the replacement to obtain a 4-level column. After a Hadamard set is picked, we can just choose the first eight available columns in the ordered $L_{16}(2^{15})$ for the eight 2-level factors and assign factors to the columns accordingly. This would achieve an experimental plan. Such plans represented by the Hadamard set are listed in Table 2, with the total number of level changes in the fourth column. The least cost is 53c. Note that the last row can be shown based on our results.

To see this, check the first eleven columns in $L_{16}(2^{15})$. The numbers of level changes in these columns are less than or equal to 11. Using the results in (ii) of Theorem 2 and formula (2), we can prove that the resulting columns from the replacement of Hadamard sets in these columns change levels less than eleven times. This implies that the total number of level changes in each plan is greater than or equal to $66 - 11 = 55$. The other rows in the table can be found by direct computations using our results. Three plans with the least cost on level changes are observed in Table 2. However, if the main-effect of the 4-level factor is required to be one-trend resistant, there is only one choice, that is, the plan with Hadamard set $\{6, 10, 12\}$. This is obtained based on the second column of Table 2.

Table 2. Level changes of plans

plans	degree of trend resistance of R	level changes of R	total level changes
(1,14,15)	0	15	59
(2,13,15)	0	15	58
(3,12,15)	0	15	57
(4,11,15)	0	15	56
(5,10,15)	0	15	55
(6, 9,15)	0	15	55
(7, 8,15)	0	15	55
(3,13,14)	0	15	57
(2,12,14)	1	14	57
(5,11,14)	1	15	55
(4,10,14)	1	14	55
(7, 9,14)	0	15	54
(6, 8,14)	1	14	55
(1,12,13)	0	13	57
(6,11,13)	1	15	54
(7,10,13)	0	15	53
(4, 9,13)	1	13	55
(5, 8,13)	1	13	55
(7,11,12)	0	15	53
(6,10,12)	1	14	53
(5, 9,12)	1	13	54
(4, 8,12)	1	12	55
others	0 or 1	≤ 11	≥ 55

The results in the previous section give us ranges for the number of level changes in R. For example, the resulting columns R obtained from the first seven plans change levels $\ell(X_4)$=15 times, while the numbers of level changes in those from the next six plans are between $\ell(X_1X_4)=14$ and $\ell(X_4X_1)+2\ell(X_1)=16$. This can be confirmed by checking the third column of Table 2. The first seven plans in the table can be excluded more easily from the choice of good plans although the numbers of their level changes can be directly computed. The Hadamard sets in these plans all contain column 15 and so the resulting columns from the replacement change levels 15 times. Other than column 15, we have to pick one column from columns 1 to 7 and one from columns 8 to 14 to form the Hadamard set. This means that the eight columns for 2-level factors in these seven plans need to contain six columns in the first set of columns (columns 1 to 7) and two in the second (columns 8 to 14). This implies that the total number of level changes for each of these plans is greater than or equal to $15+(1+2+3+4+5+6)+(8+9) = 53$. However, columns 7 and 8 should be picked simultaneously to form the Hadamard set and so the total number of level changes is never equal to 53. We exclude these plans because their costs on level changes are greater than 53c. Now, based on the result (i) in Theorem 2, the resulting 4-level columns in the next six plans change levels either 14 or 15 times. Following a discussion similiar to the first seven plans (for example, columns 7 and 9 should be excluded simultaneously), we can also exclude these plans with X_1X_4, but not X_4 in their Hadamard set. This leaves us to choose the best plans with costs 53c from the remaining nine plans.

Example 2. First-order trend effects are considered in an experiment having one 4-level factor and six 2-level factors. The cost for level changes of the 4-level factor is much higher than for those of 2-level factors. Note that the factor must be assigned to the columns formed by at least two standard columns in order to achieve one-trend resistant main-effects. Since the 4-level factor is most expensive, we must choose the columns with least level changes for it. The three columns for the replacement to gain least costs are X_1X_2, X_2X_3 and X_1X_3. The 4-level factor will change its levels six times in the experiment carried out in the row order when it is assigned to the resulting column. Then $X_1X_2X_3$, X_3X_4, $X_1X_3X_4$, $X_1X_2X_3X_4$, $X_2X_3X_4$, X_2X_4 are picked for the assignment of 2-level factors to achieve minimum costs and one-trend resistant main-effect.

Acknowledgement

I am grateful to the editor, the associate editor, and two referees for their constructive comments, which have led to major improvements in the presentation. This research was partially supported by National Science Council grant NSC81-0208-M008-20.

References

Bailey, R. A., Cheng C.-S. and Kipnis, P. (1992). Construction of trend-resistant factorial designs. *Statist. Sinica* **2**, 393-411.

Cheng, C.-S. and Jacroux, M. (1988). The construction of trend-free run orders of two-level factorial designs. *J. Amer. Statist. Assoc.* **83**, 1152-1158.

Coster, D. C. and Cheng, C.-S. (1988). Minimum cost trend-free run orders of fractional factorial designs. *Ann. Statist.* **16**, 1188-1205.

Dey, A. (1985). *Orthogonal Fractional Factorial Designs.* John Wiley, New Delhi.

Jacroux, M. (1990). Methods for constructing trend-resistant run orders of 2-level factorial experiments. *Statistical Design and Analysis of Industrial Experiments.* (Edited by S. Ghosh), 441-457, Dekker, New York.

Wang, P. C. (1990). On the constructions of some orthogonal main-effect plans. *Sankhyā Ser. B* **52**, 319-323.

Wang, P. C. (1991). Symbol changes and trend resistance in orthogonal plans of symmetric factorials. *Sankhyā Ser. B* **53**, 297-303.

Wang, P. C. and Chen, M. W. (l995). Minimum cost run orders in 4^m2^n factorial designs. Submitted.

Wu, C. F. J. (1989). Construction of 2^m4^n designs via a grouping scheme. *Ann. Statist.* **17**, 1880-1885.

Graduate Institute of Statistics, National Central University, Chungli 32054, Taiwan.

(Received September 1993; accepted April 1995)