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A SYSTEMS MODEL FOR RELIABILITY STUDIES

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Abstract: In this paper a systems model is discussed which is sufficiently general to encompass most situations in which component states at time t determine the system state at time t. The model is also useful for complex systems which do not have easily identifiable components.

In terms of this model, notions of "life length" are defined and some results for standard models concerning life distributions are generalized.

Key words and phrases: Coherent structures, multi-state coherent systems, new better than used, increasing hazard rate average.

1. Introduction

The purpose of this paper is to discuss a model for representing components and systems that is sufficiently general to handle some important practical problems where neither coherent structure theory nor multi-state coherent structure theory conveniently apply. The model appears to be sufficiently general to encompass most situations in which the component states at time t determine the system state at time t.

The standard model of coherent structures regards components as being in one of but two states, "failed" or "functioning". For systems of n components, the states of the various components are usually represented together by a vector $\mathbf{x} \in \{0,1\}^n$, where $x_i = 0$ or 1 according as the ith component fails or functions, $i = 1, \ldots, n$. The *structure function* ϕ of the system maps $\{0,1\}^n$ onto $\{0,1\}$ to classify the system as "failed" or "functioning" as dictated by the component performances.

This model is often useful even when the components and system can be in a multitude of states. The reason for this is that it is often possible to *classify* the various component states as "failed" or "functioning" in such a way that a desired classification of system states is determined by component classifications.

1.1. Example

An ordinary pencil can be regarded as a system with two components (lead and eraser) both of which have many possible states. It is natural to classify the pencil as functioning if (i) some eraser remains and (ii) enough lead remains (the pencil has sufficient length) that the user can hold it without strain. Then the pencil is a series system which is classified as functioning so long as both components are appropriately classified as functioning.

1.2. Example

A combination colored pencil has one end with red lead and the other end with blue lead. This pencil is a multi-state device which is naturally classified as "functioning" when (i) some red and some blue lead remain, and (ii) the pencil has sufficient length that the user can hold it without strain. For this example, the coherent structure model is useless because there is no way to classify the components (red and blue lead) as "functioning" or "failed" in such a way that the component classifications determine the system classification.

To obtain models of greater generality it is important to differentiate between an actual representation of the state or condition of a device and a mere classification of the state. In this respect, the intent is not entirely clear for the various models of "multi-state coherent" systems which have been proposed by various authors (see, e.g., Barlow and Wu (1978), El-Neweihi, Proschan and Sethuraman (1978), Block and Savits (1982, 1984), and El-Neweihi and Proschan (1984)). These models replace the set $\{0,1\}$ of coherent structure theory by $\{0,1,\ldots,M\}$, by $[0,\infty)$, or by some other subset of $(-\infty,\infty)$; in so doing, they use the same set for all components and for the system, and the state space is linearly ordered. Consequently, these sets are best regarded as classifying rather than as representing the states. Then, multi-state systems suffer from the same problem as do coherent systems: they apply only when component classifications determine the system classification. Even the combination red and blue pencil of Example 1.2 is not very conveniently modeled as a multi-state system.

1.3. Example

For extra protection, two shields are placed in front of a radiation source, one behind the other. The system works even though there are holes in both shields just so long as two holes do not line up with the radiation source. There is no natural way to classify the shields by real numbers in such a way that component classifications determine whether or not the system is working.

Another inconvenience of the usual coherent structure model is that there is no provision made for classifying states according to more than one criterion. Multi-state systems are also limited in this respect, partly because the chosen state space is totally ordered.

1.4. Example 1.2 (cont.)

The two-colored pencil can be classified according to whether or not it is useable for writing in red (alternatively, in blue). To be so useable, it must be of sufficient length and must have lead remaining of the appropriate color.

1.5. Example

If a two component system undergoes repair as soon as one component fails, the system is a series system in the eyes of the repairman. But if the system works so long as one component works, the system is a parallel system in the eyes of the operator. Thus the structure function is not intrinsic to the system but depends upon the point of view; here, two structure functions are of simultaneous interest.

1.6. Example

Recently the space shuttle was grounded when it was discovered that the ceramic heat shield protecting a rocket nozzle had suffered severe damage on a previous flight. Even though the heat shield did its job well, the fact that it had come dangerously close to failure was a cause of considerable concern; the heat shield was unfit for use in any future flight.

It is particularly easy to consider several criteria for classifying a device when a model is used that represents the system state rather than classifies it.

2. Representation of States

A set $\mathcal{Y}$ used to represent states of a device should satisfy some natural constraints, which are discussed in this section.

Because incremental changes in state are of interest, the expression "$y - x$" should be defined for all $x, y \in \mathcal{Y}$ even though there is no reason to require $y - x \in \mathcal{Y}$. In addition, the expression "$y = x + (y - x)$" should have meaning as representing a new state y in terms of an old state x plus an incremental change. These basic requirements are met when $\mathcal{Y} \subset \mathcal{X}$ where $\mathcal{X}$ is an Abelian group under addition.

Certain pairs of points in $\mathcal{Y}$ are comparable in the sense that one state is more desirable than the other: write $x \preceq y$ to mean that the state y is at least as desirable as x. Then

$$x \preceq x \tag{2.1}$$

and

$$x \preceq y,\ y \preceq z \Rightarrow x \preceq z, \tag{2.2}$$

so that $\preceq$ is a *preorder*.

If $\{x_i\}$ and $\{y_i\}$ are convergent sequences of points in $\mathcal{Y}$, it is desirable that

$$x_i \preceq y_i, i = 1, 2, \ldots \Rightarrow \lim x_i = x \preceq y = \lim y_i, \tag{2.3}$$

but to speak of convergence it is necessary to have a topology for $\mathcal{Y}$. This motivates the assumption that $\mathcal{X}$ is a topological group. In this paper it is assumed that there is a metric d defined on $\mathcal{X}$ such that sets of the form $\{y : d(x,y) < \varepsilon\}$ form a base for the topology. In addition, it is assumed that $\mathcal{X}$ is complete (Cauchy sequences of points in $\mathcal{X}$ have limits in $\mathcal{X}$) and that $\mathcal{X}$ is separable ($\mathcal{X}$ has a countable dense subset $\mathcal{D}$); these conditions are used in Sections 5 and 6.

In summary, it is assumed that the state space $\mathcal{Y}$ has the following properties:

$\mathcal{Y} \subset \mathcal{X}$ where $\mathcal{X}$ is a separable topological Abelian group with a metric topology; in this topology, $\mathcal{X}$ is complete. (2.4)

$\mathcal{Y}$ is preordered by $\preceq$, and the ordering satisfies (2.3). (2.5)

2.1. Example 1.6 (cont.)

Consider the rocket nozzle as a cylinder C and represent the condition of the ceramic shield by a nonnegative function defined on C. If m is a function giving the original thickness of the shield, then $\mathcal{Y}$ is the set of all function ψ defined on C and satisfying

$$0 \le \psi(z) \le m(z),\ z \in C.$$

There is no practical reason for not requiring functions in $\mathcal{Y}$ to be measurable or even continuous. Clearly $\psi_1 \preceq \psi_2$ if $\psi_1(z) \le \psi_2(z)$ for all $z \in C$, and a knowledgeable engineer might want to extend this order.

In many applications, including Example 2.1, it is true that

$$x \preceq y \Rightarrow x + z \preceq y + z \quad \text{whenever} \quad x+z, y+z \in \mathcal{Y}. \tag{2.6}$$

When (2.6) holds, it provides a convenient means of extending the ordering $\preceq$ from $\mathcal{Y}$ to $\mathcal{X}$, and it provides a convenient characterization of the ordering: $x \preceq y$ if and only if $y - x \succeq 0$, i.e.,

$$x \preceq y \leftrightarrow y - x \in \mathcal{C}, \tag{2.7}$$

where $\mathcal{C} = \{z : z \succeq 0\}$. The set $\mathcal{C}$ satisfies

$$0 \in \mathcal{C} \tag{2.8}$$

and

$$u, v \in \mathcal{C} \Rightarrow u + v \in \mathcal{C}. \tag{2.9}$$

Condition (2.8) is immediate from (2.1). Condition (2.9) is true by virtue of (2.1) and (2.6): $0 \preceq u$ implies $v \preceq u + v$ and when also $0 \preceq v$, then $0 \preceq v \preceq u + v$. With (2.7), condition (2.3) holds if and only if $\mathcal{C}$ is closed.

2.2. Example

Suppose that the state of a machine which produces two distinct quantities is represented by the respective rates (x_1, x_2) of production capacity. If demand rates are (d_1, d_2), if $x_1 < d_1 < y_1$, and if $y_2 < x_2 < d_2$, it may be that $\mathbf{x} \preceq \mathbf{y}$. But if there is an ε such that $d_1 - x_1 < \varepsilon$ and $d_2 - x_2 < \varepsilon < d_2 - y_2$, then $d_1 < x_1 + \varepsilon < y_1 + \varepsilon$ and $y_2 + \varepsilon < d_2 < x_2 + \varepsilon$ so that $\mathbf{x} + (\varepsilon, \varepsilon) \not\preceq \mathbf{y} + (\varepsilon, \varepsilon)$. Here (2.6) is violated.

In many applications, including Example 2.1, there exists $\ell \in \mathcal{Y}$ such that $\ell \preceq x$ for all $x \in \mathcal{Y}$. Then there is no loss in assuming that $\ell = 0$ where 0 is the identity of the group $\mathcal{X}$, in which case $0 \preceq x$ for all $x \in \mathcal{Y}$. If (2.7) holds, it follows that

$$\mathcal{Y} \subset \mathcal{C}. \tag{2.10}$$

2.3. Example

A solar collector on the roof of a house is intended to heat domestic hot water. If the state of the system is represented by the rate of heat transfer to the water, then the rate can be negative when the system is working improperly and in fact there is no obvious "worst state" of this system.

In spite of limitations illustrated by Examples 2.2 and 2.3, conditions (2.6)-(2.10) can conveniently be exploited when they hold.

For a system of n components having respective state spaces $\mathcal{Y}_i \subset \mathcal{X}_i$, $i = 1, \ldots, n$, the most natural state space is $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$, so that a point $\mathbf{y} \in \mathcal{Y}$ is simply a vector of component states. Then, $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is an Abelian group under componentwise addition and $d(\mathbf{x},\mathbf{y}) = \|d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\|$ is a metric whenever $\|\cdot\|$ is a norm on $\mathcal{R}^n$. With componentwise ordering, (2.4) and (2.5) are satisfied.

2.4. Example 1.2 (cont.)

Represent the state of the two color pencil by a point in $\mathcal{Y} = [0, 1] \times [0, 1]$, where (x, y) gives the remaining length of red and blue lead. Here (2.7) is satisfied and $\mathcal{C} = [0, \infty)^2$.

3. Evaluation of States

The structure function ϕ of a coherent structure can be regarded as an indicator of whether or not the system states meets an appropriate criterion. Then ϕ, quite naturally, takes the values 0 or 1 and it classifies the states as either "good" or "bad".

More generally, let $U \subset \mathcal{Y}$ be the set of "good" states as judged by some criterion. Because the ordering $\preceq$ reflects desirability it must be that any state better than a good state is also good, i.e.,

$$x, y \in \mathcal{Y},\ x \preceq y \text{ and } x \in U \ \Rightarrow\ y \in U. \tag{3.1}$$

When (3.1) is satisfied, U is said to be an *upper set.* The complement L of U with respect to $\mathcal{Y}$ is a *lower set* (i.e., $x \in L, y \preceq x \Rightarrow y \in L$) and L contains all "bad" states. Trivialities are avoided by assuming that $U \cap \mathcal{Y}$ and $L \cap \mathcal{Y}$ are nonempty so that both "good" and "bad" states exist.

When (2.6) holds and the ordering $\preceq$ has been extended to $\mathcal{X}$, it may be convenient to replace (3.1) by

$$x \in U \text{ and } x \preceq y \Rightarrow y \in U \tag{3.1$'$}$$

whether or not $y \in \mathcal{Y}$, and to replace L by the complement of U with respect to $\mathcal{X}$. When this alternative is available, it has no significance apart from possible convenience.

Since U is an upper set, the indicator function

$$\phi_U(x) = \begin{cases} 1, & \text{if } \mathcal{X} \in U, \\ 0, & \text{if } \mathcal{X} \in L, \end{cases}$$

is non-decreasing.

3.1. Example 2.4 (cont.)

The two color pencil works so long as there is some lead of each color and the pencil is not too short. Here

$$U = \{(x, y) : x > 0, y > 0 \,\text{and}\, x + y > a\}.$$

3.2. Example 2.1 (cont.)

Failure of the ceramic heat shield might be regarded as occuring if its thickness anywhere reaches zero, and then the indicator function of the good states is given by

$$\phi(\psi) = \begin{cases} 1, & \text{if } \psi(z) > 0 \text{ for all } z \in C, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

But also of interest is the function ϕ^*, where

$$\phi^*(\psi) = \begin{cases} 0, & \text{if } \psi \text{ is a condition bad enough to ground the shuttle,} \\ 1, & \text{otherwise.} \end{cases} \tag{3.2}$$

Alternatively, ϕ and ϕ^* can be replaced by $\tilde{\phi}$ where

$$\tilde{\phi}(\psi) = \begin{cases} 0, & \text{if } \phi(\psi) = 0, \\ 1, & \text{if } \phi(\psi) = 1,\ \phi^*(\psi) = 0, \\ 2, & \text{if } \phi^*(\psi) = 1. \end{cases}$$

In general, a function ϕ taking finitely many values can quite naturally replace finitely many indicators of lower sets when these sets are nested. However, Example 1.4 motivates consideration of two lower sets which can be represented in the notation of Example 2.4 as $L_R = \{(x, y) : x + y < a \quad \text{or} \quad x = 0\}$, $L_B = \{(x, y) : x + y < a \text{ or } y = 0\}$; these sets are *not* nested.

3.3. Example

In an aircraft that can fly on any k of its n engines, a curious passenger will want to know not just that the aircraft can fly, but also how many engines are operable. If $\mathbf{x} = (x_1, \ldots, x_n)$ where $x_i = 0$ or 1 according as the ith engine is failed or not, then the function of interest,

$$\phi(\mathbf{x}) = \sum_1^n x_i,$$

takes on the values $0, 1, \ldots, n$.

Note here that the multi-state coherent system would be somewhat forced because the set $\{0, 1, \ldots, n\}$ of appropriate system classifications is not appropriate for classifying components.

3.4. Example 2.1 (cont.)

For the rocket nozzle heat shield,

$$\phi(\psi) = \min_{z \in C} \psi(z) \tag{3.3}$$

might be of interest, and this takes on values in a finite interval $[0, a]$.

Just as in the case of indicator function, several multi-valued functions evaluating points in $\mathcal{Y}$ may be of simultaneous interest.

3.5. Example

A continuous length of pipe is installed for the purpose of conveying water from the point A to the point B. Represent the state of the system by a function ψ defined on a cyliner C (the pipe) which gives at each point of C the burst pressure of the pipe. If the pipe has a hole at $z \in C$ then $\psi(z) = 0$.

The state of the pipe might be evaluated using any or all of these functions:

$\phi_1(\psi) =$ 1 or 0 according as the pipe can or cannot convey sufficient water to meet demand at B,

$\phi_2(\psi) =$ maximum rate at which the pipe can convey water,

$\phi_3(\psi) =$ fraction of water introduced at A (under a given pressure) which arrives at B.

Although this system has a complex state space, it has no natural components.

Of course one could combine several functions used to evaluate the state of a system and use a single vector-valued function. In general there appears to be no natural constraint on an evaluating function ϕ except that it take values in a partially ordered space and be order preserving, i.e., $x \preceq y \Rightarrow \phi(x) \leq \phi(y)$.

4. Ordered State Spaces vs Multi-valued Structure Functions

Most generalizations of coherent structures have emphasized a structure function ϕ taking more than two values, but the purpose of this paper is to focus on the natural order of the set $\mathcal{Y}$ of system states. There are circumstances in which the two viewpoints coincide: A system with state space $\mathcal{Y}$ ordered by $\preceq$ can be characterized by a multi-valued structure function ϕ if $\phi : \mathcal{Y} \to \mathcal{R}$ satisfies

$$x \preceq y \text{ if and only if } \phi(x) \leq \phi(y) \text{ whenever } x, y \in \mathcal{Y}. \tag{4.1}$$

But (4.1) trivially implies that the ordering $\preceq$ satisfies not only (2.1) and (2.2) but also

$$x \preceq y \quad \text{or} \quad y \preceq x, \tag{4.2}$$

and (4.2) is a condition too strong to often hold in practice. Moreover even (2.1), (2.2) and (4.2) together do not imply the existance of a real valued function ϕ satisfying (4.1) (see Kuratowski (1966), p.26).

If the structure function ϕ is not required to be real-valued but is allowed to take values in a partially ordered space, say $(\mathcal{Z}, \leq)$, then it is always possible to arrange for the existance of a function ϕ satisfying (4.1); indeed, with $(\mathcal{Z}, \leq)$ = $(\mathcal{Y}, \preceq)$, the identity function is an example. Such a function ϕ does little but shift attention from the ordering $\preceq$ to $\leq$ and one may as well start with $\mathcal{Z}$ in place of $\mathcal{Y}$ as the system state space.

In Example 3.2, the function defined by (3.1) can be evaluated if the value of the function defined by (3.3) is known. It would be nice to have one "informative" order-preserving function ϕ, the values of which determine the values of all other order preserving functions defined on $\mathcal{Y}$. Such a function ϕ would serve as a "multi-valued structure function" in the sense of previous authors.

4.1. Proposition. *Suppose that* $(\mathcal{Y}, \preceq)$ *is a pre-ordered space,* i.e., $\preceq$ *satisfies* (2.1) *and* (2.2). *Let* $(\mathcal{Z}, \leq)$ *be another pre-ordered space and suppose that* $\phi : \mathcal{Y} \to \mathcal{Z}$ *satisfies* (4.1). *If* $(\mathcal{W}, \leq_*)$ *is a pre-ordered space and* $\psi : \mathcal{Y} \to \mathcal{W}$ *is order preserving, then there exists a function* $f : \mathcal{Z} \to \mathcal{W}$ *such that* $\psi(x) = f(\phi(x))$ *for all* $x \in \mathcal{Y}$.

Proof. It is convenient to replace $\mathcal{Y}, \mathcal{Z}$ and $\mathcal{W}$ by spaces of equivalent classes determined by the respective pre-orders so that in effect the orderings all satisfy $x \leq y$ and $y \leq x \Rightarrow x = y$. Then (4.1) guarantees that ϕ has an inverse and $f = \psi(\phi^{-1})$.

If such a ϕ satisfying (4.1) exists, one would probably want to replace $\mathcal{Y}$ by $\mathcal{Z}$ for representing the system states.

5. Random States and Time Evaluation of a System

To regard the state of a device as random, it is necessary to have a σ-field of subsets of $\mathcal{X}$ (or $\mathcal{Y}$) so that the notion of measurability is meaningful. Because there is a topology for $\mathcal{X}$, it is convenient to use the σ-field of Borel subsets of $\mathcal{X}$.

If the device operates over time, then the state $X(t)$ of the device at time t is ordinarily a random variable and $\{X(t), t \geq 0\}$ is a stochastic process taking values in $\mathcal{Y}$.

Suppose that the initial state m of the device is fixed, i.e.,

$$P\{X(0) = m\} = 1 \text{ for some } m \in \mathcal{Y}. \tag{5.1}$$

In case 0 is taken to be the worst possible state, it follows that $m \succeq 0$. In most applications,

$$x \in \mathcal{Y} \Rightarrow x \preceq m,$$

so that the device will never be in a better state than it is at time $t = 0$, i.e.,

$$P\{X(t) \preceq m \text{ for all } t \geq 0\} = 1. \tag{5.2}$$

To avoid measure theoretic problems, assume that

$$\text{sample paths are right continuous.} \tag{5.3}$$

In the absence of repair it is often possible to determine on physical grounds that

$$P\{X(s) \succeq X(t) \text{ for all } s \leq t\} = 1. \tag{5.4}$$

Of course (5.1) and (5.4) imply (5.2).

Under the conditions (2.6) and (5.1)-(5.4), $Y(t) = m - X(t)$ can be regarded as the degradation by time t, and the process $Y = \{Y(t), t \geq 0\}$ satisfies (5.3) as well as

$$P\{Y(0) = 0\} = 1, \tag{5.1$'$}$$

$$P\{Y(t) \succeq 0 \text{ for all } t \geq 0\} = 1, \tag{5.2$'$}$$

$$P\{Y(s) \preceq Y(t) \text{ for all } s \leq t\} = 1. \tag{5.3$'$}$$

Because other statements about X can be similarly translated to statements about Y and conversely, there is no need to study both processes. See, e.g., Proposition 5.3 below.

In most studies of reliability, the concepts of "time to failure" and "life distribution" play an important role. In the context of the model discussed in this paper, these concepts are preserved with the aid of the following notation and definition.

5.1. Notation

For any measurable set $V \subset \mathcal{X}$, let

$$T_V = \inf\{t : X(t) \in V\} \quad (T_V = \infty \text{ if } \{t : X(t) \in V\} = \emptyset). \tag{5.5}$$

For $\mathcal{F}$ the class of distributions with increasing hazard rate average and $\mathcal{X} = \mathcal{R}$, the following definition is essentially due to Ross (1979). Various extensions have been proposed, e.g., by Block and Savits (1981) and Marshall and Shaked (1986b).

5.2. Definition. Let $\mathcal{F}$ be a class of univariate distribution functions and let $X = \{X(t), t \geq 0\}$ be a stochastic process taking values in $\mathcal{Y}$. Then

(i) X is said to be an *upper $\mathcal{F}$-process* if T_U has a distribution in $\mathcal{F}$ for all closed upper sets U such that $P\{T_U < \infty\} > 0$,

(ii) X is said to be a *lower $\mathcal{F}$-process* if T_L has a distribution in $\mathcal{F}$ for all closed lower sets L such that $P\{T_L < \infty\} > 0$.

In the remainder of this section, it is assumed without further mention that (2.4) and (2.5) are satisfied.

5.3. Proposition. *Suppose that $\preceq$ satisfies* (2.6). *If X is an upper* (*lower*) *$\mathcal{F}$ process, $m \in \mathcal{X}$ and $Y(t) = m - X(t)$, then Y is a lower* (*upper*) *$\mathcal{F}$ process.*

Proof. Suppose X is an upper $\mathcal{F}$ process and L is a closed lower set. Then

$$\inf\{t : m - X(t) \in L\} = \inf\{t : X(t) \in m - L\},$$

and since $m - L$ is a closed upper set, $\{t : m - X(t) \in L\}$ has a distribution in $\mathcal{F}$. If X is a lower $\mathcal{F}$ process the proof is similar.

Upper NBU (*new better than used*) processes have been discussed by Marshall and Shaked (1986b), and upper IHRA (*increasing hazard rate average*) processes have been discussed by Shaked and Shanthikumar (1987). A number of examples can be found in these papers, where it is usually assumed that $\mathcal{X}$ is a separable Banach space, (2.6) holds, and C is a closed convex cone.

In this context, Marshall and Shaked (1986b) prove that if Y_1 and Y_2 are independent upper NBU processes on $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, which satisfy (5.1′), (5.2′), (5.3) and (5.4), then (Y_1, Y_2) is an upper NBU process on $\mathcal{Y}_1 \times \mathcal{Y}_2$. The remainder of this section is devoted to showing that the corresponding result is true for IHRA processes. As shown in Corollary 5.7, this result generalizes the well known theorem that if $T_1, \ldots, T_n$ are independent IHRA random variables and τ is a coherent life function of order n, then $\tau(T_1, \ldots, T_n)$ has an IHRA distribution.

5.4. Lemma. *If X satisfies* (5.3) *and* (5.4), *then for all closed lower sets L,*

$$T_L > t \Leftrightarrow X(t) \notin L. \tag{5.6}$$

Proof. Suppose $T_L > t$. Then $X(s) \notin L$ for all $s < t$ and in particular $X(t) \notin L$. On the other hand, if $X(t) \notin L$, then by (5.4) $X(s) \notin L$ for all $s \le t$ and so $T_L \ge t$. Suppose $T_L = t$; then there exist $t_1 \ge t_2 \ge \cdots$ such that $X(t_i) \in L$ and $t_i \to t$. By right continuity (5.3), $\lim_{i\to\infty} X(t_i) = X(t)$, and since L is closed, this means $X(t) \in L$, a contradiction. Consequently $T_L \ne t$ so that $T_L > t$.

In the following it is convenient to use the notation

$$P_u(A) = P\{X(u) \in A\} \tag{5.7}$$

for all Borel subsets A of $\mathcal{X}$. For the case $\mathcal{X} = \mathcal{R}^n$ and $\mathcal{C} = [0, \infty)^n$, the following theorem is due to Block and Savits (1981).

5.5. Theorem. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ satisfy* (2.4) *and* (2.5) *and that X is a process which satisfies* (5.1)-(5.4). *Then X is a lower* IHRA *process if and only if*

$$Eh(X(t)) \le E^{1/\alpha} h^\alpha(X(\alpha t)) \tag{5.8}$$

for all $\alpha \in (0, 1]$, $t \ge 0$ and all Borel measurable nonnegative nondecreasing functions h such that the expectations exist.

Proof. Suppose that (5.8) holds. Let L be a closed lower set and let h be the indicator function of L^c. Then

$$P\{X(t) \notin L\} = Eh(X(t)) \le E^{1/\alpha} h^\alpha(X(\alpha t)) = [P\{X(\alpha t) \notin L\}]^{1/\alpha}.$$

By Lemma 5.4 this is equivalent to

$$P\{T_L > t\} \le [P\{T_L > \alpha t\}]^{1/\alpha};$$

thus X is an IHRA process.

Suppose that X is an IHRA process and let $h = I_U$ be the indicator function of an open upper set U. Then

$$\begin{aligned} Eh(X(t)) &= P\{X(t) \in U\} = P\{T_{U^c} > t\} \le [P\{T_{U^c} > \alpha t\}]^{1/\alpha} \\ &= [P\{X(\alpha t) \in U\}]^{1/\alpha} = E^{1/\alpha} h^\alpha(X(\alpha t)). \end{aligned}$$

Thus (5.8) holds when h is the indicator function of an open upper set.

Let $h = I_U$ be the indicator function of a measurable upper set and let $L = U^c$. Since $\mathcal{X}$ is a complete separable metric space it follows (Ash (1972, p.180)) that for fixed $t > 0$, $\alpha \in (0,1)$ and $\varepsilon > 0$ there exists a compact set $C_\varepsilon \subset L$ such that

$$P_u(C_\varepsilon) \geq P_u(L) - \varepsilon \quad \text{for} \quad u = \alpha t \quad \text{or} \quad u = t.$$

Let $C_\varepsilon^- = \{x : x \preceq c \text{ for some } c \in C_\varepsilon\}$. Then $C_\varepsilon \subset C_\varepsilon^- \subset L$ and

$$P_u(C_\varepsilon^-) \geq P_u(L) - \varepsilon \quad \text{for} \quad u = \alpha t \quad \text{or} \quad u = t.$$

To see that C_ε^- is closed, let $y_1, y_2, \ldots$ be a sequence of points in C_ε^- which converge to y and let $c_1, c_2, \ldots$ be a sequence of points in C_ε such that $c_i \succeq y_i, i = 1, 2, \ldots$. Since C_ε is compact, $\{c_i\}_{i=1}^\infty$ has a convergent subsequence $\{c_{j_i}\}_{i=1}^\infty$ converging to $c \in C_\varepsilon$. Since $c_{j_k} \succeq y_{j_k}, k = 1, 2, \ldots$, it follows from (2.3) that $c \succeq y$ or $y \in C_\varepsilon^-$. Thus C_ε^- is closed.

Let $U_\varepsilon = (C_\varepsilon^-)^c$ so that $U \subset U_\varepsilon$. Then for $u = \alpha t$ or $u = t$,

$$P_u(U) = 1 - P_u(L) \geq 1 - P_u(C_\varepsilon^-) - \varepsilon = P_u(U_\varepsilon) - \varepsilon.$$

Thus,

$$[P_{\alpha t}(U)]^{1/\alpha} \geq [P_{\alpha t}(U_\varepsilon) - \varepsilon]^{1/\alpha} \geq \{[P_t(U_\varepsilon)]^\alpha - \varepsilon\}^{1/\alpha} \geq \{[P_t(U)]^\alpha - \varepsilon\}^{1/\alpha}.$$

Now let $\varepsilon \to 0$ to conclude that

$$[P_{\alpha t}(U)]^{1/\alpha} \geq P_t(U).$$

This proves (5.8) for indicator functions of measurable upper sets.

Now, let h be a function of the form $h = \sum_{j=1}^m a_j h_j$, where $a_j \geq 0$ and h_j is the indicator function of a measurable upper set $U_j, j = 1, \ldots, m$. For such an h,

$$\begin{aligned} Eh(X(t)) &= \sum_{j=1}^m a_j Eh_j(X(t)) = \sum_{j=1}^m a_j P_t(U_j) \leq \sum_{j=1}^m a_j [P_{\alpha t}(U_j)]^{1/\alpha} \\ &\leq \Big[\sum_{j=1}^m a_j^\alpha P_{\alpha t}(U_j)\Big]^{1/\alpha} = E^{1/\alpha} h^\alpha(X(\alpha t)). \end{aligned}$$

To complete the proof, use the Lebesgue monotone convergence theorem.

5.6. Theorem. *Suppose that for $i = 1, 2$, $\mathcal{X}_i$ and $\mathcal{Y}_i$ satisfy* (2.4) *and* (2.5) *for the ordering $\preceq_i$. Let $X_i = \{X_i(t), t \geq 0\}$ be a lower* IHRA *process on $\mathcal{Y}_i$ which satisfies* (5.1), (5.3) *and* (5.4), *and suppose that X_1, X_2 are independent. Then*

(X_1, X_2) *is a lower* IHRA *process on* $\mathcal{Y}_1 \times \mathcal{Y}_2 \subset \mathcal{X}_1 \times \mathcal{X}_2$ *which satisfies* (5.1), (5.3) *and* (5.4) *with the product topology and componentwise ordering.*

Proof. If h is a Borel measurable nonnegative increasing function defined on $\mathcal{X}_1 \times \mathcal{X}_2$, then because X_1 and X_2 are independent, it follows from Theorem 5.5 that

$$\begin{aligned} Eh(X_1(t), X_2(t)) &= E_1[E_2(h(X_1(t), X_2(t))] \leq E_1[E_2 h^\alpha(X_1(t), X_2(\alpha t))]^{1/\alpha} \\ &\leq \{E_1 E_2 h^\alpha(X_1(\alpha t), X_2(\alpha t))\}^{1/\alpha}. \end{aligned}$$

5.7. Corollary. *If* $T_1, \ldots, T_n$ *are independent* IHRA *random variables and* τ *is the life function of a coherent system, then* $\tau(T_1, \ldots, T_n)$ *has an* IHRA *distribution.*

Proof. Let $\mathcal{X}_i = \mathcal{R}$, $\mathcal{C}_i = [0, \infty)$, and

$$X_i(t) = \begin{cases} 1, & \text{if } t < T_i, \\ 0, & \text{if } t \geq T_i, \end{cases}$$

$i = 1, \ldots, n$. Because T_i is IHRA, X_i is an IHRA process, $i = 1, \ldots, n$. Let $L = \{\mathbf{x} : \tau(\mathbf{x}) \leq 0\}$. Because τ is increasing, L is a lower subset of $\mathcal{R}^n$. By Theorem 5.6, $T_U = \inf\{t : (X_1(t), \ldots, X_n(t)) \in L\}$ has an IHRA distribution. But τ extends the corresponding structure function ϕ and

$$T_U = \inf\{t : \phi(X_1(t), \ldots, X_n(t)) = 0\} = \tau(T_1, \ldots, T_n).$$

Note. Here and in the following, "coherent life functions" are as defined by Esary and Marshall (1970). Such functions can have irrelevant components.

6. Multivariate Properties

As was mentioned in the introduction, it is often desirable to classify the state of a device as "good" or "bad" according to several criteria. Particularly when one classification represents "functioning" or "failed" and other classifications are used to warn of imminent failure, joint distributions of times of entrance into corresponding lower sets are of interest. These considerations motivate the following multivariate version of Definition 5.2.

6.1. Definition. Let $\mathcal{F}$ be a class of multivariate distributions, and let $X = \{X(t), t \geq 0\}$ be a stochastic process taking values in $\mathcal{Y}$. Then

(i) X is said to be an *upper $\mathcal{F}$-process* if $T_{U_1}, \ldots, T_{U_m}$ have a joint-distribution in $\mathcal{F}$ for all finite collections $U_1, \ldots, U_m$ of closed upper sets for which $P\{T_{U_i} < \infty\} > 0, i = 1, \ldots, m$;

(ii) X is said to be a *lower $\mathcal{F}$-process* if $T_{L_1}, \ldots, T_{L_m}$ have a joint-distribution in $\mathcal{F}$ for all finite collections $L_1, \ldots, L_m$ of closed lower sets such that $P\{T_{L_i} < \infty\} > 0, i = 1, \ldots, m$.

As in the one dimensional case there is a duality between upper and lower $\mathcal{F}$-processes and here, only lower $\mathcal{F}$-processes are discussed.

In the following, $(X_1, \ldots, X_m) \in \mathcal{F}$ means that the joint distribution of $X_1, \ldots, X_n$ is in $\mathcal{F}$.

6.2. Theorem. *Let $\mathcal{F}$ be a class of univariate distributions. Then* (i) *X is a lower $\mathcal{F}$-process if and only if* (ii) *for every finite collection $L_1, \ldots, L_m$ of closed lower sets and every coherent life function τ of order m, $\tau(T_{L_1}, \ldots, T_{L_m}) \in \mathcal{F}$.*

Proof. Because $\tau(T_{L_1}, \ldots, T_{L_m}) = T_{L_i}$ is a coherent life function, the fact that (ii) implies (i) is trivial. Suppose (i) and let τ have the minimal path representation $\tau(t_1, \ldots, t_m) = \max_{1 \le \ell \le p} \min_{j \in P_\ell} t_j$. Then $L = \bigcap_{1 \le \ell \le p} \bigcup_{j \in P_\ell} L_j$ is a closed lower set such that $T_L = \tau(T_{L_1}, \ldots, T_{L_m})$.

Theorem 6.2 is given by Marshall and Shaked (1986b) for upper NBU processes where the proof is the same.

Theorem 6.2 shows that even Definition 5.2 has multivariate implications because the joint distribution of $T_{L_1}, \ldots, T_{L_m}$ must have the property therein described. The class of joint distributions for which $\tau(T_1, \ldots, T_m) \in \mathcal{F}$ for all coherent life functions τ of order m is denoted by $C_3(\mathcal{F})$ by Marshall and Shaked (1986a). If $X \in \mathcal{F}$, then it is always true that $(X, \ldots, X) \in C_3(\mathcal{F})$, but vectors with independent components in $\mathcal{F}$ are in $C_3(\mathcal{F})$ only for families $\mathcal{F}$ of distributions "closed under the formation of coherent systems".

Following Marshall and Shaked (1986a), write $(T_1, \ldots, T_m) \in C_4(\mathcal{F})$ to mean that for all increasing homogeneous functions $g : [0, \infty)^m \to [0, \infty)$, $g(T_1, \ldots, T_m) \in \mathcal{F}$.

Upper C_4(IHRA)-processes are said to be "strongly IFRA" by Shaked and Shanthikumar (1987). They provide examples of such processes.

It is of interest to see that under some circumstances there is a counterpart to Theorem 5.6 for lower C_4(IHRA)-processes and lower C_4(NBU)-processes. This result requires the following preliminary result.

Let $\Lambda_x = \{z : z \preceq x\}$, $\Upsilon_x = \{z : z \succeq x\}$ and for any set A, denote the interior of A by A^0.

6.3. Lemma. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ satisfy* (2.4) *and* (2.5), *and suppose that for all $x \in \mathcal{Y}$,*

$$x \text{ belongs to the closure of } \Upsilon_x^0. \qquad (6.1)$$

Let $\mathcal{F}$ be a class of multivariate distributions such that

$$\mathcal{F} \text{ is closed under weak limits,} \tag{6.2}$$

and

$$\text{if } U = (U_1, \ldots, U_m) \in \mathcal{F},\ V = (V_1, \ldots, V_m) \in \mathcal{F} \text{ where } U \text{ and } V \text{ are independent, then } (\min(U_1, V_1), \ldots, \min(U_m, V_m)) \in \mathcal{F}. \tag{6.3}$$

Let X be a process satisfying (5.1), (5.3) *and* (5.4). *Then, the following are equivalent:*

$$T_{L_1}, \ldots, T_{L_m} \text{ have a joint distribution in } \mathcal{F} \text{ for all finite collections } L_1, \ldots, L_m \text{ of closed lower sets;} \tag{6.4}$$

$$T_{L_1}, \ldots, T_{L_m} \text{ have a joint distribution in } \mathcal{F} \text{ for all finite collections } L_1, \ldots, L_m \text{ of closed lower sets having the form } \Lambda_{x_k} \text{ for some } x_1, \ldots, x_m \in \mathcal{Y}. \tag{6.5}$$

Proof. Trivially, (6.4) implies (6.5). Suppose that (6.5) holds, let $L_1, \ldots, L_m$ be nonempty closed lower sets and let

$$L_{i,k} = \{z : d(z, y) < 1/k \text{ for some } y \in L_i\},\ i = 1, \ldots, m,\ k = 1, 2, \ldots.$$

Because $\mathcal{X}$ is separable, there is a countable dense subset $\mathcal{D}$ of $\mathcal{X}$. Let $L_{i,k} \cap \mathcal{D} = \{d_{1,i,k}, d_{2,i,k}, \ldots\}$, $i = 1, \ldots, m$, $k = 1, 2, \ldots$. For typographical reasons write $\Lambda_{\ell,i,k}$ in place of $\Lambda_{d_{\ell,i,k}}$. Then $\cup_{\ell=1}^{\infty} \Lambda_{\ell,i,k} \subset L_{i,k}$. To see also that $L_{i,k} \subset \cup_{\ell=1}^{\infty} \Lambda_{\ell,i,k}$ let $x \in L_{i,k}$. Since $L_{i,k}$ is open, it follows that $\Upsilon_x^0 \cap L_{i,k}$ is open. By (6.1), x is in the closure of Υ_x^0. Since x is also in the interior of $L_{i,k}$, this means that $\Upsilon_x^0 \cap L_{i,k} \neq \phi$. Thus, there exists $d \in \Upsilon_x^0 \cap L_{ik} \cap \mathcal{D}$; then $d \in \Upsilon_x$, that is $x \in \Lambda_d$ and $d \in L_{ik} \cap \mathcal{D}$. Consequently $L_{i,k} \subset \cup_{\ell=1}^{\infty} \Lambda_{\ell,i,k}$, and so $L_{ik} = \cup_{\ell=1}^{\infty} \Lambda_{\ell,i,k}$.

Let $L_{i,k}^{(j)} = \cup_{\ell=1}^{j} \Lambda_{\ell,i,k}$. Then $T_{L_{i,k}^{(j)}} = \min_{1 \leq \ell \leq j} T_{\Lambda_{\ell,i,k}}$. By hypothesis (6.5), $T_{L_{i,k}^{(j)}}$, $i = 1, \ldots, m$, have a joint distribution in $\mathcal{F}$. Since $\mathcal{F}$ is closed under weak limits it follows that $\lim_{j \to \infty} T_{L_{i,k}^{(j)}}$, $i = 1, \ldots, m$, have a joint distribution in $\mathcal{F}$. But $\lim_{j \to \infty} T_{L_{i,k}^{(j)}} = T_{L_{i,k}}$ (Blumenthal and Getoor (1968, p.53)). Since $\lim_{k \to \infty} T_{L_{i,k}} = T_{L_i}$, $i = 1, \ldots, m$, it follows that T_{L_i}, $i = 1, \ldots, m$, have a joint distribution in $\mathcal{F}$.

A special case of the above lemma forms a part of Theorem 3.1 of Marshall and Shaked (1986b). They make use of the fact that if $\mathcal{X}$ is a linear space and $\preceq$ satisfies (2.7) where $\mathcal{C}$ is a convex cone with nonempty interior, then (6.1) is satisfied.

6.4. Theorem. *Suppose that for $i = 1, 2, \mathcal{X}_i, \mathcal{Y}_i$ and $\preceq_i$ satisfy* (2.4), (2.5) *and* (6.1); *suppose also that X_i is a process on $\mathcal{Y}_i$ which satisfies* (5.1), (5.3) *and* (5.4). *Let $\mathcal{F}$ be a class of multivariate distributions satisfying* (6.2), (6.3) *and* *if* $U = (U_1, \ldots, U_m) \in \mathcal{F}$, $V = (V_1, \ldots, V_m) \in \mathcal{F}$ *where* U *and* V *are independent, then* $(\max(U_1, V_1), \ldots, \max(U_m, V_m)) \in \mathcal{F}$. (6.6)

If X_i *is a lower* $\mathcal{F}$*-process,* $i = 1, 2$, *and* X_1, X_2 *are independent, then* $X = (X_1, X_2)$ *is a lower* $\mathcal{F}$*-process.*

Proof. For any closed lower subset L of $\mathcal{Y}_1 \times \mathcal{Y}_2$, let T_L be given by (5.5), and similarly for any closed lower subset $L^{(i)}$ of $\mathcal{Y}_i$, let $T_{L^{(i)}}$ be given by (5.5) with X_i in place of X, $i = 1, 2$. According to Lemma 6.3, it is sufficient to show that $(T_{L_1}, \ldots, T_{L_m}) \in \mathcal{F}$ whenever each L_j has the form $L_j^{(1)} \times L_j^{(2)}$ with

$$L_j^{(i)} = \Lambda_{u_j^{(i)}}, \; i = 1, 2.$$

But $T_{L_j} = \max(T_{L_j^{(1)}} T_{L_j^{(2)}})$. Because of (6.6), it follows that X is a lower $\mathcal{F}$-process.

Classes $\mathcal{F}$ of multivariate distributions which satisfy the conditions of Theorem 6.4 include the classes, G_3(IHRA), G_3(NBU), G_4(IHRA), G_4(NBU), C_4(IHRA) and C_4(NBU) of Marshall and Shaked (1986a).

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