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ASSESSING GEOMETRIC INTEGRITY THROUGH SPHERICAL REGRESSION TECHNIQUES

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Abstract. Traditional methods of assessing the geometric integrity of any finely engineered product requires the use of fixtures. The fit is then tested with a feeler gauge. Fixtures are expensive to construct and transport and the degree of accuracy obtained may be insufficient, depending on the tolerance specified by the procurer of the part. In this paper, we discuss an alternative scientific approach to assessing geometric quality assurance. Through the use of spherical regression techniques we are able to statistically assess the spatial integrity of geometric objects described through CAD and CMM data.

Key words and phrases: CAD, CMM, diagnostics, diamond pin, quality assurance, resolution, rotations.

1. Introduction

In many industrial settings, e.g., the automobile industry, it is necessary to assess the geometric integrity of component parts. Traditionally, the procurer of the part has issued geometric specifications and tolerances against which the supplier's product is tested. A physical mould is made of wood, plastic, etc., and an attempt is made to clamp the part into place. Tolerance is checked with a feeler gauge. Such methods are expensive, time consuming and of limited accuracy.

More recently, Computer Aided Design (CAD) has allowed the designer to create a prototype part in the form of a computer image. It is then possible to generate a file that takes the place of the traditional blueprint, or specification. The surface of the image of the part is covered with a fine mesh and both the spatial coordinates, and the coordinates of the unit normal vector are generated, at each mesh point. The resulting file is called the CAD file, and the quality assurance problem is now to test whether a sample part conforms to the CAD file, to within specified tolerances.

The device used to check the geometric integrity of a part is a Coordinate Measuring Machine (CMM). The part is held firmly in position, and points on the surface of the part are touched with the CMM probe. The spatial coordinates of the points that are touched are accurately measured and recorded. It is important to note that the coordinates so obtained are with respect to an axis system internal to the CMM. On the other hand, vectors in the CAD file are expressed relative to some coordinate system determined by the software that created the CAD file. The problem is therefore to construct a transformation between the two coordinate systems. The CMM measurements could then be transformed and checked against the CAD file.

Constructing the transformation is more difficult than might at first appear. Since the CAD file consists of only a finite number of points, the mesh points mentioned above, an arbitrarily chosen point on the part will in most instances not correspond to any CAD point. Even if such a point did appear in the CAD file, there is no practical way of identifying it. Thus it is not possible to construct a transformation by simply matching CMM points to CAD points.

A Euclidean transformation is required, i.e.,

$$x \to Ax + T,$$

where x, T are 3-vectors, and A is a 3×3 rotation matrix. The purpose of this paper is to display a method of estimating the rotation, followed by outlining diagnostic procedures that assess the fit so that a statistical determination of whether or not the part is defective can be made. An estimator of the translation parameter T is also established.

The rotation is constructed as follows. First CMM readings are used to estimate directional features of the part. By directional features, we mean e.g., unit vectors normal to small planar regions on the part, or a unit vector indicating the direction of some line, such as a trim-edge. Counterparts to these directions can be found in, or calculated from, the CAD file. Then, the methods of spherical regression, see Chang (1986), are applied to construct the rotation (plus possible reflection) that causes the least squares fit between these CMM directional data and their CAD counterparts. Rivest (1989) gives diagnostics for spherical regression, and these can be used to check the integrity of those aspects of the part represented by the directional data. Indeed, using these diagnostics for just the rotation parameter determines that a part is defective in the sense that a collection of planar regions in the part do not align with one another in the way demanded by the CAD and that it is possible to determine which plane is out of alignment. Once the rotation parameter is estimated, the translation parameter can then be estimated.

At this point an example would be of much benefit. Unfortunately, real data, with which to demonstrate the proposed methodology is unavailable to us. Consequently, we simulate data for a situation such as might arise in practice.

Our methods are most suitable for geometric figures with many planar faces and can be extended somewhat to smooth surfaces. An example of such a figure can be found in the traditional quality assurance laboratory and is called a diamond pin (see Figure 1). A diamond pin is one kind of holding fixture used in traditional quality assurance.

Plane	Area
1	ADP
2	ABP
3	BCP
4	CDP
5	ADEH
6	ABEF
7	BCFG
8	CDGH

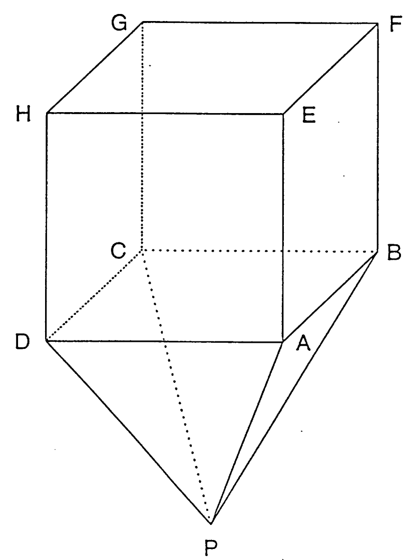

Figure 1. Diamond pin

A part whose geometric quality is to be assessed by traditional methods has to be secured in position (by holding fixtures) so that feeler gauges can be applied. Different combinations of fixtures are used to do this, depending on the physical characteristics of the part. When a diamond pin is used, a square (or sometimes rectangular) hole has been drilled in the part (during manufacture) at a precise, strategic location. The point of the diamond pin is inserted into the hole, thus aiding in locating the position of the part. In this paper, we show how our methods could be used to assess the geometric integrity of the diamond pin itself. A distorted diamond pin could lead to incorrect positioning of the part, and faulty quality assurance.

We now give a summary of what is to follow. In Section 2, we discuss measurement errors induced by the CMM, along with a method of estimating directional features of the part, from CMM data. These estimated directional features have two sources of potential error. The first is the inaccuracy inherent in the CMM, *measurement variation*, which is unavoidable in the sense that it occurs even in a perfect part. The second source of potential error is due to distortion in the part itself, *part variation*, possibly brought about by faulty manufacture. It is the presence of large part variation that we are trying to detect. Although, in practice, some margin of part variation would be allowable, this would really depend on the particular circumstances including, of course, meeting the customer-specified tolerance. Consequently, for the purposes of this paper, we will take a canonical approach and interpret a part to be within *geometric tolerance* as the bound on error brought about by the measurement variation alone. Section 3 is a review of spherical regression along with relevant diagnostics given by the penetrating paper of Rivest (1989). It is then possible to establish statistical tests to determine whether a given part meets the tolerance requirements. Section 4 attempts to illustrate the methodology on the diamond pin example. Indeed, the statistical tests appear to function quite well in not rejecting the diamond pin when no distortions are placed, while at the same time rejecting the part when we deliberately distort planar regions of the diamond pin. Some comments on other simulations are also made. In Section 5, comments related to estimating the translation parameter are made. Section 6 is a discussion section followed by an Appendix which outlines how the data is simulated.

2. Measurement of Features, CMM Resolution and the Data

Given a part to be tested it is usually possible to find various directional features that can be both identified in the CAD file, and estimated using CMM measurements. There are two types of directional features: unit vectors indicating the direction of a line in the part; and, unit vectors normal to planar regions in the part. While there are many ways the former can arise, in practice they are much less used than the latter. For this reason, we restrict our discussion to the planar situation, which is adequate for the purposes of this paper.

The procedure is therefore, to identify n planar regions on the surface of the part, $n \geq 3$. Of course, in the diamond pin example, see Figure 1, our interest would be in the eight planar regions. (We do not consider the plane $EFGH$.) The unit normal vectors to these planes in the CAD coordinate system, can be found by looking in the appropriate general region of the CAD file, for a collection of mesh points with a common normal. The corresponding normals in the CMM coordinate system are then estimated as described below, and spherical regression can be used to estimate the orthogonal transformation required to best fit the CAD points to the CMM data in a least squares sense.

2.1. CMM and CMM resolution

The demand for greater accuracy in inspection has made CMMs an integral part of modern quality assurance (Cook (1989)). While CMMs vary in size and sophistication (Overment (1991)), a component common to many CMMs is a smooth topped granite table which has been accurately leveled. A column-like piece rises vertically from the surface of the table, and slides along the table parallel to the floor. A horizontal arm is attached to the column and can slide vertically up and down. On the end of the arm is a probe, which can rotate in any direction. The part under assessment is placed on the table. When the probe touches the part, the spatial coordinates of the tip of the probe (relative to some axis system determined by the CMM) are displayed, and routed to a computer. This computer may be equiped with software to implement methods such as those described in this paper. For a recent application see Cook (1993). For comprehensive engineering and statistical reviews of CMMs as well as statistical methodologies of CAD modelling with CMM data using nonlinear regression, see Dowling et al. (1993).

Before using CMM measurements to estimate the normal vector to a plane, it is necessary to consider the nature of CMM measurement error. Typically, a CMM display consists of the three spatial coordinates it is measuring, expressed in *mm*. The three readings change independently of one another; however, only certain coordinate values are allowed for certain decimal places. Assuming that the CMM always returns the allowed coordinate value that is nearest to the true value, then the errors on the coordinates are independent of one another, and are dependent on what is called the *resolution* of the CMM, which we denote by ϵ. The unit of measurement of the resolution is known as, *micron*, where one micron is $10^{-3} mm$. Thus, in the situation where $\epsilon = 5$, the third decimal place can only take values of 0 or 5 so that the spatial coordinates have errors of plus or minus 2.5 microns. The resolution varies from one make and model type of CMM to another.

2.2. Estimating directional features

Consider a plane $\mathcal{P}$ in $\mathbb{R}^3$. One way of characterizing $\mathcal{P}$ is through it's unit normal vector, $v \in \mathbb{R}^3$, say. Now, if we take statistical measurements $x_1, \ldots, x_m$ on $\mathcal{P}$, then an estimator of v would serve as an estimator of $\mathcal{P}$. Finding the least squares estimator of v has been of interest to crystallographers, and a number of solutions have been presented, (see Schomaker et al. (1959), Blow (1960), Scheringer (1971)). A simple solution is to use the method of principal component analysis. Put

$$\overline{x} = \frac{1}{m}\sum_{i=1}^{m} x_i, \quad D = \sum_{i=1}^{m}(x_i - \overline{x})(x_i - \overline{x})^t,$$

where superscript t represents transposition. It is well known that the unit eigenvector u say, corresponding to the smallest eigenvalue of D gives the least squares estimator of v. Since u is determinated only up to sign, to ensure that u points in the right direction, i.e., is an outward normal, a point x_{m+1} can be read by the CMM, above $\mathcal{P}$. The sign of u is adjusted so that

$$u^t(x_{m+1} - \overline{x}) > 0.$$

The accuracy of the estimate u depends not only on the resolution of the CMM, but also on the number m of points on the plane, and how widely spread these points are. The greater the area covered by these points the better, but in practice one has to restrict this area to remain confident that the points are in fact planar.

2.3. CMM resolution and the data

At this point, one can see that the error structure of the data, i.e., estimates of the unit normal vectors, potentially comes from two sources: first, from measurement variation induced by the CMM resolution; second, from part variation arising from the production process. For the purposes of this paper a canonical approach will be adopted whereby tolerance will mean that aspect of the variation in the data that is due solely to CMM resolution, or in other words, measurement variation. This somewhat restrictive definition will be used in order to quantify tolerance for this paper; however, in practice some part variation will be allowed for, determined of course by customer-specified tolerance. Nevertheless, each case would demand different attention; therefore, we will restrict the definition of tolerance as above. Indeed, once tolerance bounds have been established in this regard, we would like to then be able to statistically conclude that any errors observed over and above this tolerance bound, will be the contribution of part variation. Aspects of this line of reasoning, will be quantified in the following section.

3. Spherical Regression and Diagnostics

Spherical regression is a procedure which statistically estimates an orientation parameter based on spherical data. The problem was originally solved by MacKenzie (1957), and the solution has been developed by Moran (1976), Stephens (1979) and Chang (1986). Rivest (1989) developed diagnostic procedures for the concentrated Fisher-von Mises distribution. Kim (1991) examined spherical regression in a decision theoretic framework and obtained Bayes estimators for the unknown rotation under general conditions. Applications of spherical regression have included: crystallography, see MacKenzie (1957); the motion of tectonic plates, see Chang (1986) and Rivest (1989); and vector cardiogram orientation, see Prentice (1986). To the best of our knowledge, this is the first application of spherical regression techniques, to an industrial setting, in particular, to quality assurance.

3.1. Estimation of rotation and diagnostics

Let u be a random vector distributed on the two dimensional unit sphere S^2 having density $f(\kappa u^t A v)$ with respect to the invariant measure on S^2. Here $v \in S^2$ is fixed and known, $\kappa > 0$ is an unknown concentration parameter and the unknown parameter of interest A, is a 3×3 rotation matrix. Let the collection of all 3×3 rotation matrices be denoted by SO(3).

Given a random sample $u_1, \ldots, u_n$, and the corresponding set $v_1, \ldots, v_n$ of design points, the objective in spherical regression is to estimate the unknown rotation A *via*,

$$\min_{A \in \mathrm{SO}(3)} n^{-1} \sum_{i=1}^{n} \|u_i - Av_i\|^2, \tag{1}$$

where $\| \cdot \|$ denotes the usual Euclidean distance.

Following MacKenzie (1957) and through a *modified* singular value decomposition, Stephens (1979) obtained a closed form solution for (1). Denote by A_{ls}, the least squares solution. Chang (1986) showed that if $A \in \mathrm{SO}(3)$ is the true parameter, then

$$A_{ls} \to A,$$

almost surely, as $n \to \infty$, provided $S = n^{-1} \sum_{i=1}^{n} v_i v_i^t$ converges to some positive definite matrix as $n \to \infty$. This latter condition is, in fact, stronger than needed in that the limit of S can allow for one 0 eigenvalue.

Extensions of this work to incorporate diagnostics similar to linear regression was made by Rivest (1989). Suppose the distribution of the u's is that of a Fisher-von Mises distribution,

$$f(\kappa u^t A v) = c(\kappa)^{-1} \exp\{\kappa u^t A v\}, \tag{2}$$

where $c(\kappa) = \kappa^{-1} \sinh \kappa$. Define

$$W_i = \begin{bmatrix} 0 & -v_{i3} & v_{i2} \\ v_{i3} & 0 & -v_{i1} \\ -v_{i2} & v_{i1} & 0 \end{bmatrix},$$

where $v_i = (v_{i1}, v_{i2}, v_{i3})^t$, $i = 1, \ldots, n$. It was shown by Rivest (1989) that

$$2n\kappa(1 - r) \to_d \chi^2_{2n-3}, \tag{3}$$

as $\kappa \to \infty$ for each fixed n, provided $\sum_{i=1}^{n} W_i^t W_i$ is nonsingular, where "$\to_d$" means convergence in distribution and $r = n^{-1} \sum_{i=1}^{n} u_i^t A_{ls} v_i$. We note that the approximation is $o_p(\kappa^{-1/2})$, hence the approximation is very good for large κ, (see Rivest (1989, page 309)). We also note that in the case of small rotations, which often is the case in practice, the least squares problem is essentially linear regression with the model matrix equal to the skew symmetric matrix W_i, (see Kane et al. (1983)).

For each fixed v_i, let $v_{i(1)}$ and $v_{i(2)}$ be mutually orthogonal unit vectors perpendicular to v_i for each $i = 1, \ldots, n$. Thus $(v_i, v_{i(1)}, v_{i(2)})$ form an orthonormal basis for $\mathbb{R}^3$. Define

$$e_i = (v_{i(1)}, v_{i(2)})^t A_{ls}^t u_i, \tag{4}$$

for $i = 1, \ldots, n$ as the *residuals* of spherical regression. We note that the motivation for defining (4) comes from ordinary linear regression. Indeed, residuals in the latter can be thought of as the orthogonal projection of the data onto the complementary subspace spanned by the design matrix and does not depend on the particular choice of basis. Note that the situation is similar in (4) in the spherical regression context. This insight was first pointed out by Rivest (1989).

We can then form the statistic

$$t_i^2 = \frac{(n - 5/2) e_i^t \Sigma_i^{-1} e_i}{2n(1-r) - e_i^t \Sigma_i^{-1} e_i}, \tag{5}$$

where $\Sigma_i = (v_{i(2)}, -v_{i(1)})^t [I - n^{-1}(I - S)^{-1}](v_{i(2)}, -v_{i(1)})$, for $i = 1, \ldots, n$. Note once again that (5) is motivated by ordinary linear regression, in that (5) is the spherical adaptation of "externally studentized residuals" (see Cook and Weisberg (1982) and Rivest (1989)). We have that

$$t_i^2 \to_d F_{2,2n-5}, \tag{6}$$

for each $i = 1, \ldots, n$ as $\kappa \to \infty$.

3.2. Testing geometric integrity

The parameter, $\kappa > 0$, records the amount of concentration of the data u, around $A^t v$, with greater concentration being determined by large values of κ. Consequently, (3) can be used to form the test,

$$H_0 : \kappa \geq \kappa_0 \quad \text{against} \quad H_1 : \kappa < \kappa_0,$$

where the κ_0 would represent the amount of allowable tolerance. By (3), a rejection region of approximate size α would be given by,

$$2n\kappa_0(1 - r) > \chi^2_{2n-3,\alpha}, \tag{7}$$

where $\chi^2_{\nu,\alpha}$ is the upper αth percentile of a chi-square distribution with ν degrees of freedom.

There are various ways of determining κ_0, depending on the situation at hand, and engineering practices. As discussed in Section 2.3 however, our approach is to base it on the CMM resolution, so that the amount of allowable tolerance is that which is induced by measurement error in the CMM. Further discussion and quantification of this point will be addressed in Section 3.3.

Another aspect involved in quality testing would be to find out whether the ith data point is an outlier, thus indicating that point to be defective, particularly if the above H_0 is rejected but the data is still concentrated. Thus (5) could be used along with (6). Indeed, let,

$$H_0: \quad i\text{th data point not defective.}$$

Then an approximate size α rejection region would be,

$$t_i^2 > F_{2,2n-5,\alpha}, \tag{8}$$

where $F_{\nu,\mu,\alpha}$ denotes the upper αth percentile of an F distribution with ν and μ degrees of freedom. Note once again that this procedure is similar to Cook's procedure for testing for outliers in ordinary linear regression, (see Rivest (1989)).

3.3. Estimating κ and the relationship to CMM resolution

An estimate for κ for a given part is found by taking CMM readings of planar readings and estimating the normals to the planes. We have found from simulations that the number of normals estimated is not as significant a factor as the resolution of the CMM, and the configuration of the CMM data points on the planes in question. In order to apply our methods, a configuration has to be decided upon, and used throughout. The most convenient configuration is a circle. We space points evenly around the circumference of a circle in the plane. The number of points measured is far less significant than the radius of the circle. It is customary to take six points around a circle, of radius $1cm$ (provided the planar region is big enough).

Lower bound estimates on κ for geometrically perfect parts are available. Consider first the error in estimating one normal. Without loss of generality, suppose that $p_1, \ldots, p_m$ are the true coordinates of $m \geq 3$ points lying on a circle of radius R. Due to the CMM measurement process, for each p_k, there is a CMM measurement error w_k attached, where w_k is bounded by $\epsilon/2$. Let v denote the unit normal of the plane determined by the points $\{p_k\}_{k=1}^m$; let u denote the unit normal of the plane found by the least squares method based on the points $\{p_k + w_k\}_{k=1}^m$; and let $\delta u = u - v$. Then a rough bound for the length of δu

(denoted as $|\delta u|$) can be found by identifying the worst possible displacement of the plane determined by the p's inside a cylinder with height ϵ, inner radius $R-\epsilon/2$ and outer radius $R+\epsilon/2$. This bound is

$$|\delta u|^2 \leq 2\left(1-\frac{R-\epsilon/2}{\sqrt{\epsilon^2/4+(R-\epsilon/2)^2}}\right).$$

With some considerable effort, the details of which can be found in Chen, Kim and Chapman (1992), one can show that,

$$\log\kappa \geq \log\Big(\frac{2n-3}{n}\Big) - \log\left\{2\left(1-\frac{R-\epsilon/2}{\sqrt{\epsilon^2/4+(R-\epsilon/2)^2}}\right)\right\}.$$

The first row of Table 1 gives these values with $R = 1cm$. We suspect that the lower bound estimates are far from being optimal and hence could be improved upon.

Alternatively, one could estimate κ through maximum likelihood. Indeed, let $s = 1-r$, where $r = n^{-1}\sum u_i^t A_{ls} v_i$. It follows from (3) that, s is distributed approximately, as $(2n\kappa)^{-1}\chi^2_{2n-3}$, when $\kappa > 0$ is large. Thus, if we observe a random sample $s_1,\ldots,s_N$, then by the usual derivations, the maximum likelihood estimator is,

$$\hat{\kappa} = \frac{2n-3}{2n\overline{s}}, \tag{9}$$

where $\overline{s}$ denotes the sample mean. Further,

$$\frac{2n-3}{2Nn^2\overline{s}^2},$$

would serve as an estimator of $\mathrm{Var}(\hat{\kappa})$. Thus the amount of variability in $\hat{\kappa}$ is very small when κ and N are large. Therefore, to get a good estimate of the magnitude of the error induced by CMM resolution, samples (of s) of size 1000 are generated for a geometrically perfect part, for a variety of scenarios obtained by varying n, and ϵ (the resolution of the CMM) with R fixed at $1cm$. Values of n are considered between $n = 3$ (the minimum number required to uniquely determine A_{ls}) and $n = 10$ (about the largest number commonly used in practice). It was found that for fixed ϵ, $\hat{\kappa}$ did not vary significantly with n. The second row of Table 1 gives a summary of the simulations run in double precision fortran. We emphasize that the usual CMM resolution encountered in practice is 5 microns; thus, a fair estimate for κ_0 when $n = 8$ is approximately e^{19}. The interpretation of the latter is simply the error likely to be encountered due exclusively to measurement variation of the CMM.

In comparing the two values, one should bear in mind that the values in Table 1 are in the natural logarithm scale. Consequently, when exponentiated, the difference between the values will be quite large. As a conservative measure, it may be suggested that the lower bound estimates be used for κ_0 in (7) instead of the maximum likelihood estimate. In the examples to follow, we will use the maximum likelihood estimates.

Finally, we end this section with the comment that if one finds neither of the above methods satisfatory, a third possibility exists in terms of estimating κ robustly. This technology is available through Ko (1992).

4. Application of Methodology to the Diamond Pin Example

This section discusses the practical aspects behind using the methodology outlined in Section 3. We use the diamond pin example of Section 1 (see Figure 1) in a simulation environment, where the simulations attempt to capture the type of situation normally encountered in practice. The way in which the data is simulated, is outlined in the Appendix.

We present below two examples of situations that could arise in practice and outline how the methodology is to be used. The discussion will begin with the normal vectors of both the CAD and CMM data so that we are assuming that the manufactured part has been measured by a CMM with the normal vectors

Table 1. Lower bound and simulated estimates of $\ln \kappa$

	ϵ							
n	5	10	15	20	25	30	35	40
	17.118	15.731	14.920	14.344	13.897	13.532	13.223	12.956
10	18.783	17.397	16.586	16.010	15.564	15.200	14.891	14.634
	17.098	15.712	14.900	14.324	13.876	13.512	13.204	12.936
9	18.785	17.400	16.588	16.013	15.567	15.202	14.894	14.627
	17.073	15.686	14.875	14.299	13.852	13.487	13.178	12.911
8	18.790	17.404	16.593	16.017	15.571	15.206	14.898	14.631
	17.040	15.653	14.841	14.266	13.819	13.454	13.145	12.877
7	18.774	17.388	16.577	16.000	15.555	15.191	14.882	14.615
	16.993	15.606	14.795	14.219	13.772	13.407	13.098	12.831
6	18.769	17.383	16.572	15.997	15.551	15.186	14.878	14.611
	16.924	15.537	14.726	14.150	13.703	13.338	13.029	12.762
5	18.769	17.383	16.572	15.997	15.550	15.186	14.877	14.610
	16.811	15.424	14.613	14.037	13.590	13.225	12.916	12.648
4	18.781	17.394	16.584	16.008	15.562	15.197	14.889	14.622
	16.588	15.201	14.389	13.814	13.367	13.002	12.693	12.425
3	18.744	17.358	16.547	15.972	15.525	15.161	14.852	14.585

computed as outlined in Section 2.2. In both situations, $n = 8$, $\epsilon = 5$ with the rows representing the spatial unit normal vectors obtained by CAD and CMM. In Figure 1, we identify the planar regions using the labels $1, 2, \ldots, 8$.

4.1. Example 1: No distortions

In this example, the CMM data is just a rotation of the CAD points so that no distortions other than measurement errors induced by CMM resolution are made. The data is summarized in Table 2.

By (1), the least squares estimator is

$$A_{ls} = \begin{bmatrix} 0.4242 & 0.5656 & -0.7070 \\ -0.8000 & 0.5999 & -4.43\text{E-}05 \\ 0.4242 & 0.5656 & 0.7071 \end{bmatrix}.$$

In this example, we have,

$$2n\kappa_0(1 - r) = 7.6476,$$

where by Table 1, $\ln \kappa_0 = 18.790$. Thus, the p-value in comparison with a chi-square random variable with 13 degrees of freedom, is 0.866. Thus, based on (7), we cannot reject the null hypothesis and, indeed, the manufactured part is within tolerance.

We also present the p-values associated with testing each planar region using (8). The ith datum refers to the ith row of the data. Notice that all the p-values are insignificant at any reasonable value, thus verifying the first conclusion based on the chi-square approximation.

Table 2. Example 1

	CAD			CMM	
−5.55E-17	−0.8656	−0.6000	0.7071	−1.25E-04	−0.7071
0.2656	0.4242	−0.7999	−1.15E-04	0.9999	−5.54E-05
−0.4242	0.7999	−1.11E-16	−0.9999	−5.18E-05	7.40E-05
2.42E-02	−0.8000	−0.8242	0.7070	−1.96E-05	−0.7070
0.5656	0.6000	−0.5656	3.10E-05	0.9999	3.68E-05
−0.6000	−1.0000	−0.5000	−0.9999	−0.7071	−0.7071
1.11E-16	−0.5000	−0.7071	−0.7070	−0.7071	4.05E-05
0.0000	0.7071	0.0000	5.24E-05	−2.73E-05	−1.10E-04

Table 3. Test for outliers

ith datum	1	2	3	4	5	6	7	8
t_i^2	0.390	2.306	0.697	1.671	0.969	0.255	0.982	1.156
p-value	0.688	0.155	0.523	0.242	0.416	0.780	0.471	0.357

4.2. Example 2: Distortions

In this example, in addition to the CMM data being a rotation of the CAD points a deliberate distortion in the CMM data is made. Indeed we move the point P a magnitude of 0.001 along the y-axis thereby affecting the normal vectors of the first and third planes. We note that the lines AD and CD, in Figure 1 are parallel to the x and y axes, respectively. The data is summarized in Table 4.

Again by (1), the least squares estimator is

$$A_{ls} = \begin{bmatrix} 0.4242 & 0.5656 & -0.7071 \\ -0.8000 & 0.5999 & -5.92\text{E-}05 \\ 0.4242 & 0.5657 & 0.7070 \end{bmatrix}.$$

In this example, we have,

$$2n\kappa_0(1-r) = 173.771,$$

where again by Table 1, $\ln \kappa_0 = 18.790$. Thus, the p-value is 0.000, so that based on (7), we can reject the null hypothesis and conclude that significant part variation exits.

Again, we present the p-values associated with testing each planar region using (8). We note that for this example, the latter is more meaningful since the overall test is declaring significant part variation. Notice that excluding the first and third datum, the remaining p-values are insignificant at any reasonable value. Thus given rejection of the part being within tolerance from the chi-square approximation, the individual t_i^2 indicates that the source of the part variation comes from possibly the first and third planar regions of the diamond pin.

Table 4. Example 2

	CAD			CMM	
−5.55E-17	−0.8656	−0.6000	0.7075	−2.29E-05	−0.7075
0.2656	0.4242	−0.7999	2.36E-05	0.9999	−4.39E-05
−0.4242	0.7999	−1.11E-16	−0.9999	6.46E-05	1.11E-04
2.42E-02	−0.8000	−0.8242	0.7073	2.50E-04	−0.7074
0.5656	0.6000	−0.5656	1.18E-04	0.9999	−8.33E-05
−0.6000	−1.0000	−0.5000	−0.9999	−0.7066	−0.7068
1.11E-16	−0.5000	−0.7071	−0.7066	−0.7067	−4.69E-05
0.0000	0.7071	0.0000	3.07E-05	−2.79E-05	−3.20E-05

Table 5. Test for outliers

ith datum	1	2	3	4	5	6	7	8
t_i^2	4.492	0.591	4.110	1.863	0.056	0.001	0.011	0.004
p-value	0.044	0.574	0.054	0.210	0.946	0.999	0.989	0.997

4.3. Comments

In general, the test based on (8) did an excellent job of determining significant part variation, and the test based on (9) did a reasonable job of locating the source of the variation (i.e. determining which plane or planes have been displaced from the ideal). In Example 2 above, we had deliberately displaced the first and third planar regions.

Extensive simulations have been performed and some of the results can be found in Chapman and Kim (1992). Invariably we found that the overall chi-squared approximation (8) did very well in declaring whether or not a part is within tolerance as specified in Section 2.3. In most instances, (9) was also effective in detecting distorted planar regions. However, with regard to (9), unanticipated results sometimes occured.

We feel there are two reasons for this. The first reason is due to the problem of multiple comparisons. Even if (8) declares the part to be within tolerance, the fact that we are individually comparing each planar region according to some size α could lead to a declaration that one or more of the planar regions is distorted, when this is not the case. This is the same kind of situation as occurs in ordinary linear regression. Second, we must remember that (8) and (9) are based on approximations that are asymptotic in κ. If (8) rejects the part (because the errors are not sufficiently concentrated), we still have to assume concentrated errors to apply (9). If the part is rejected by (8) due to gross error (because the concentration parameter is small), then the asymptotics on which (9) is based may fail. As a recommendation, conclusions should be based on (8), with a cautious examination of (9) to look for distorted regions.

5. Estimating the Translation

Our focus hitherto, has been solely on the rotation parameter; hence our concern is only with regard to distortion in planar regions. Of course, to get an overall assessment, in particular, with respect to things such as height, width, length, distinguished markings, etc., we also need to estimate the translation vector.

Since our discussion is with respect to planar regions, we will keep it that way; however, one can also use other features such as directed lines to get at the translation vector. Indeed, if n planes are of interest, let $\{(v_i, p_i) : i = 1, \ldots, n\}$ describe the CAD planes and $\{(u_i, x_i) : i = 1, \ldots, n\}$ describe the corresponding CMM planes. Here v, u are normal vectors to the CAD, CMM planes respectively, and, p, x are points on the corresponding planes for CAD, CMM respectively. We note that, in the case of the latter, no identication between the p's and the x's are assumed except that they are points on the corresponding CAD and CMM planes respectively.

For the translation vector $T \in \mathbb{R}^3$, we note that

$$u_i^t(Ap_i - T) = u_i^t x_i, \tag{10}$$

even though $Ap_i - T$ and x_i do not have to correspond to the same point for $i = 1, \ldots, n$. Let $U^t = (u_1, \ldots, u_n)$ and $M^t = (u_1^t A_{ls} p_1 - u_1^t x_1, \ldots, u_n^t A_{ls} p_n - u_n^t x_n)$, where A_{ls} is the least square fit of the rotation parameter. Then (10) determines the system $UT = M$, where this system consists of n equations in 3 unknowns. Consequently, if we assume U is of rank 3, then a least squares estimator of T can be obtained as,

$$\hat{T} = (U^t U)^{-1} U^t M. \tag{11}$$

The difficulty in analysing the distribution of this estimator is that the entries U are derived from estimates of the "true" normal vector. Furthermore, the M vector also has corresponding entries and, in addition, has the estimator A_{ls} of the unknown rotation A. Although, the above bears a remarkable resemblance to how the constant term in linear regression is determined, the distributional theory of (11) needs to be further examined before any statistical use for it can be determined. In that regard, one promising way of attacking the problem is to envision the problem in a multivariate framework and assume that CMM points are samples from a three dimensional multivariate normal distribution with mean $Av+T$ and covariance $\sigma^2\mathbf{I}$. Such is the case of what is called Procrustes analysis, (see Bingham, Chang and Richards (1992)). Some developments along the lines of what is needed have been made and could potentially be used for the problem at hand.

6. Discussion

We note that the methodology employed in this paper is most useful when the part in question has distinct features such as; flat regions, edges, points, etc. This is because in the case of planar regions, estimation of the rotation parameter can be achieved because the normal vector to the plane does not change with location. In practice nonlinear CAD designs, such as B-splines, can arise. This creates difficulties due to the fact that the normal vector to the surface will change according to location. Consequently we would not be able to immediately apply the techniques of this paper without some modifications. One possibility is to assume local flatness in some radius of each point relative to some scale. This would essentially reduce the problem to that of planar features. An alternative strategy would be to take advantage of possible symmetries. In particular, one could estimate normal vectors at local extrema between the CAD and CMM points.

We have also assumed throughout this paper that the CAD points remain fixed. From a practical point of view this assumption is valid since the CAD points are computer generated whereas the CMM points are taken from actual measurements. Nevertheless, some investigation into variable CAD points should be addressed and is open for further investigation.

We would also like to make some comments on the Fisher-von Mises distribution assumption. Indeed (3) and (6) depend on $f(\kappa u^t Av)$ being Fisher-von Mises in as much as the distributional theory is only worked out for that case. However, it is felt that much of Rivest's results should extend to a broader class of densities based on appropriate smoothness conditions and the rotational symmetry of $f(\cdot)$, since the main technique is to employ second-order analysis along the lines of Watson (1983). Consequently, even if the Fisher-von Mises assumption is incorrect, we feel that the methodology will remain intact and much of the needed results can be obtained intuitively. Certainly the simulations we have performed seem to bear this out.

Nevertheless, as a defence for the Fisher-von Mises distribution, because of the scale involved in the concentration parameter, we feel that it is not an inappropriate assumption. Indeed, just from rudimentary plots of simulated data, it appears as though the data clumps exponentially around the preferred direction. This assumption can of course be tested using a goodness of fit test for the Fisher-von Mises distribution proposed by Rivest (1986).

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Appendix

The data for the simulations and the examples are created in the following way:

- n planar regions are identified on a part ($n \geq 3$);
- the corresponding unit normal vectors $v_1, \ldots, v_n$ are located in the CAD data file;
- six points are measured with a CMM on each of the n planes. Each set of six points is spaced around a circle of radius $1cm$.

The steps taken in the simulation are:

Step 1. Let

$$v_0 = (0,0,1), \quad \theta_i = \pi i/3 + w_i, \quad y_i = (\cos\theta_i, \sin\theta_i, 0) \quad (1 \leq i \leq 6),$$

where w_i is a uniform variate in the range $(-0.05\pi, +0.05\pi)$. The purpose of the w_i's is to simulate the inaccuracy that arises when the CMM operator judges by eye that the points are evenly spaced around the circle.

Step 2. Rotation matrices R_j $(1 \leq j \leq n)$ are selected with the property that R_1v_0, R_2v_0, R_3v_0 are mutually orthogonal. Put

$$v_j = R_jv_0 \quad (1 \leq j \leq n).$$

The above orthogonality requirement is included to avoid the degenerate situation where all the normals are nearly in the same straight line.

Step 3. An arbitrary rotation R is applied to all the data. Then

$$\tilde{u}_j = Rv_j \quad (1 \leq j \leq n)$$

are the 'true' CMM normals, and if we put

$$x_{ji} = RR_jy_i \quad (1 \leq j \leq n, 1 \leq i \leq 6),$$

the vectors $x_{j1}, x_{j2}, \ldots, x_{j6}$ are the CMM points that would be used to estimate $\tilde{u}$ *via* the method of Section 2.

The above data simulates a part that is geometrically perfect, and has been measured without any error, i.e., the CMM has $\epsilon = 0$. The algorithms of Sections 2 and 3 would lead to estimators

$$u_j = \tilde{u}_j \quad (1 \leq j \leq n),$$

and a rotation matrix $A_{ls} = R$.

Different scenarios are considered, by varying n and the resolution ϵ, of the CMM. The basic idea of the study is to simulate measurement errors on the CMM readings x_{ji} consistent with the error structure discussed in Section 2.1, so that the κ induced by the resolution of the CMM would be the allowable tolerance κ_0. By taking many iterations, κ would be estimated for each scenario. Once κ is estimated for parts that are geometrically perfect, the estimate $\hat{\kappa}$, would serve as the tolerance level, i.e., $\kappa_0 = \hat{\kappa}$. We then simulate "defective" parts by displacing one or more CMM normals by some small angle ϕ. This is achieved by setting

$$x_{ji} = RR_j^{\phi} y_i \quad (1 \leq i \leq 6)$$

for one or more values of j, where R_j^{ϕ} is R_j composed with a rotation through an angle ϕ about an arbitrary axis. The diagnostics can now be tested, for various angles ϕ.

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COMMENT : AN INDUSTRY VIEW OF COORDINATE MEASUREMENT DATA ANALYSIS

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Chapman, Chen and Kim (CCK) have offered an application of spherical regression techniques to the problem of locating a geometric object in space using coordinate measurements of surface points on that object. This "localization" of geometric objects is a necessary tool for many applications involving machine vision, robotics, and coordinate measurement. While localization involves fitting a statistical model to obtain parameter estimates, it has traditionally been treated in the engineering literature only as an optimization problem. Thus, the CCK work is a welcome contribution to the new body of literature on the application of statistics to coordinate measurement.

My intent here is provide an industry perspective on the analysis of coordinate measurement data for manufactured parts, and to discuss the CCK methodology in that context. As a statistician in industry, I am more concerned with the processes that manufacture parts than with the individual parts. So, in analyzing coordinate measurement data I must go beyond conformance-to-tolerance to provide information that is useful for assessing and improving the performance of manufacturing processes. The CCK work, with its focus on how to localize a very high-precision part described by a linear geometric model and detect very small levels of part shape variation, is not directed toward that larger goal. Still, elements of their approach are applicable to a number of problems that arise in the context of obtaining and analyzing coordinate data from manufacturing processes.

1. Nominal Shape and Form Error

1.1. Nominal shape

Much of the design work in industry today is handled with Computer-Aided Design (CAD) systems. A CAD model for a part is a detailed and complicated representation of the part geometry, but for our purposes it is sufficient to denote the model by $\mathcal{M}(\boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{\phi})$, where the parameter vectors $\boldsymbol{t}$ and $\boldsymbol{\theta}$ describe the orientation of the geometry relative to some application-specific coordinate system, and the parameter vector $\boldsymbol{\phi}$ represents the shape of the part. The six orientation parameters consist of the 3×1 translation vector $\boldsymbol{t}$, and the 3×1 vector $\boldsymbol{\theta}$ of Euler angles for the rotations. These angles are used to construct the rotation matrix $\boldsymbol{A}$ in the usual manner (see, e.g., Kane et al. (1983)). The application-specific coordinate system is typically determined by the way the part is to be used, for example, the "car position" system used in automotive applications. I use the vector $\boldsymbol{\phi}$ as a shorthand for the shape parameters, which are actually a large collection of lines, B-spline surfaces, and other elements.

For each part to be manufactured, there exists a CAD model describing the ideal shape of the part, which I will refer to as the *nominal* shape. That CAD model will properly locate the nominal shape within the application-specific coordinate system. Thus, the orientation parameters are all zero, and we can denote the model by $\mathcal{M}(\mathbf{0}, \mathbf{0}, \boldsymbol{\phi}^*)$, where $\mathbf{0}$ is the null vector, and $\boldsymbol{\phi}^*$ represents the nominal shape parameters.

Figure 1 displays a simple part (Part F0) — a hollow, straight extrusion with square cross-section — that I will use as an illustration. The shape description for this model would include parameters for the shape of the cross-section (in the x-y plane), as well as parameters for the profile of the extrusion (along the z-axis).

Z

Y

X

Figure 1. Rendering of the Straight Extrusion with Square Cross-Section (Part F0), including the locations of the 24 measurement points.

1.2. Form error

Once a part is manufactured, attention is focused on how the shape of the actual manufactured part deviates from the design intent. This deviation is characterized by the *formerror*. The form error for a surface point p, denoted f_p, is defined as the distance between the surface of the actual part and the surface of the nominal part at that point. In practice, the assessment of conformance-to-tolerance for a particular part is made by comparing the form errors to customer specifications. Typically those specifications are expressed in terms of a profile, or form, tolerance. A *formtolerance* defines a tolerance zone within which a region or feature of the part must lie. Different tolerance zones may be specified for different regions or features on the part. Form tolerances are typically based on the so-called Taylor's principle; that is, for a part to conform to tolerance, the part surface must lie within an envelope bounded by a similar perfect geometry that is offset, in both positive and negative directions, from the nominal surface. See Figure 2 for an illustration of a tolerance zone for the exterior surface of the square cross-section of the F0 extrusion. For a particular region $\mathcal{R}$, the tolerance zone puts limits $\pm L_{\mathcal{R}}$ on the form errors for points in that region (symmetry is assumed for simplicity). The region is said to be within tolerance if all of the form errors in that region are within the limits, that is, if $\max_{p \in \mathcal{R}} \mid f_p \mid \leq L_{\mathcal{R}}$.

Figure 2. Illustration of form tolerance zone for the square cross-section of extrusion F0.

This approach to assessing conformance-to-tolerance is different than the CCK approach, where the allowable "tolerance" is determined by the inherent CMM measurement variation, and represented by κ_0. In most manufacturing applications we require that the measurement error be small relative to the manufacturing process variation. Thus, it would not be particularly useful to test the hypothesis proposed by CCK. However, if we were to select κ_0 to reflect the requirement imposed by the tolerance zone, perhaps by letting κ_0 be equal to some multiple of $L_{\mathcal{R}}$, then their testing procedure would be more consistent with current industry practice.

2. Determining the Form Errors

2.1. Dimensional measurement

In order to determine the form errors, we must characterize the actual shape of a manufactured part. To do that, dimensional measurements are taken by a Coordinate Measuring Machine (CMM) or similar device. A CMM is a computer-controlled device that uses a contact probe mounted on the end of a robotic arm to record data. The CMM tracks the (x, y, z) location of the probe tip as it moves in three-dimensional space, and coordinate measurements are recorded when the probe makes contact with the part. The measurement process begins with the operator mounting the part onto a holding fixture within the CMM. Then, a *referenceframe* is established that aligns the physical orientation of the part with the application-specific coordinate system of the nominal CAD model (e.g., car position). This allows the CMM to "see" the part as it appears in the CAD system. The probe then measures the locations of m designated surface points $\boldsymbol{n}_i$ along prescribed approach vectors $\boldsymbol{v}_i$ $(i = 1, \ldots, m)$. The $\boldsymbol{n}_i$ are points on the surface of $\mathcal{M}(\mathbf{0}, \mathbf{0}, \boldsymbol{\phi}^*)$, and the unit-vectors $\boldsymbol{v}_i$ are typically normal to the surface of $\mathcal{M}(\mathbf{0}, \mathbf{0}, \boldsymbol{\phi}^*)$ at those points. When the probe contacts the part while trying to locate $\boldsymbol{n}_i$ along the vector $\boldsymbol{v}_i$, the CMM records the point of contact as the observed surface point coordinate $\boldsymbol{a}_i$. Figure 1 shows the $m = 24$ surface points to be measured on the straight F0 extrusion. Note that the measurements are taken in "slices" through the part, so that all of the points fall onto three cross-sections (two at the ends, one in the middle). Each cross-section has eight measurement points.

Because CMM operation is based on the CAD model, the crucial step in the measurement process is the establishment of the reference frame. Generally, the reference frame is established using the 3'2'1 *Method*, in which the part is located in space by (1) establishing a plane with *three* measured points to fix one translation and two rotation parameters, (2) establishing a line perpendicular to that plane with *two* measured points to fix another translation and rotation parameter, and finally (3) measuring *one* point to fix the final translation parameter. The surfaces where these reference points lie are often selected because they are relatively flat, and tend to be robust to manufacturing errors. This 3-2-1 method for "soft" fixturing of parts for CMM measurement is a carryover from the methods used to position parts with hard fixtures. To the statistician, this method does not appear to be very robust, because a minimum number of points are used and slight errors in any of the six points can greatly affect the resulting reference frame. One application of the CCK methodology would be to improve the methods by which the reference frame is established. By taking additional measurements on a few nearly planar datum surfaces of the part, the CCK method could be applied and would provide not only an improved estimate of the necessary alignment, but also diagnostics concerning the adequacy of that alignment.

The measurement error variation for the coordinate measurement process will be influenced by many factors, including environment control, referencing techniques, fixture design and fixturing practices. The CCK view that measurement error will be near the machine precision may be valid for high-precision parts, but for many parts the actual variation in the measurement process will be much larger than the inherent variation of the CMM. See Hulting (1992) for further discussion of this point.

2.2. Estimation of form errors

For the ith measured surface point, the form error is typically estimated by the *as’measureddistancefromnominal*, $\delta_i = \mathrm{sign}(\boldsymbol{d}_i)\times(\boldsymbol{d}_i'\boldsymbol{d}_i)^{1/2}$, where $\boldsymbol{d}_i = \boldsymbol{p}(\boldsymbol{a}_i;\mathbf{0},\mathbf{0},\boldsymbol{\phi}^*) - \boldsymbol{a}_i$, $\boldsymbol{p}(\boldsymbol{a}_i;\boldsymbol{t},\boldsymbol{\theta},\boldsymbol{\phi})$ is the point on the surface of the model $\mathcal{M}(\boldsymbol{a}_i;\boldsymbol{t},\boldsymbol{\theta},\boldsymbol{\phi})$ that is closest to the measured point $\boldsymbol{a}_i$, and $\mathrm{sign}(\cdot)$ is a sign function that indicates the direction of the deviation $\boldsymbol{d}$ relative to the surface normal vector. That is, δ_i is simply the shortest distance between the observed point $\boldsymbol{a}_i$ and the surface of the nominal model $\mathcal{M}(\mathbf{0},\mathbf{0},\boldsymbol{\phi}^*)$. Note that if the approach vectors $\boldsymbol{v}_i$ are normal to the surface of $\mathcal{M}(\mathbf{0},\mathbf{0},\boldsymbol{\phi}^*)$, then $\boldsymbol{p}(\boldsymbol{a}_i;\mathbf{0},\mathbf{0},\boldsymbol{\phi}^*)$ will be equal to $\boldsymbol{n}_i$.

In Figure 3 we display simulated measurements for 50 of the straight F0 extrusions shown in Figure 1. I have used simulated data in order to illustrate the behavior I have observed in a variety of situations; in particular, I have slightly exaggerated certain features of the data to clarify the concepts presented here. The as-measured distances from nominal from the 50 parts are summarized by boxplots for each measurement point. The implied tolerance limits on the form errors are drawn as solid lines, and are tighter for the two end cross-sections ($\pm 0.5mm$) than for the center cross-section ($\pm 1.5mm$). This summary of the process data could be used to draw conclusions about process performance. Ideally, all of the boxplots would be centered near zero and exhibit low levels of variation. Here, the systematic departures from zero suggest that the parts may be bowed (center cross-section has larger deviations than ends), and that all of the cross-sections may be misshapen. There is also a significant amount of variability at each of the data points.

2.3. Improved estimation of form error

While these as-measured distances are commonly used in industry to estimate form error, their use ignores a potentially significant source of error in the measurements. $Alignment error$, which is the discrepancy between the coordinate system used by the CMM to measure the part and the coordinate system of the nominal CAD model, is caused by imperfect part referencing and can be significant. Ideally, alignment error can be corrected by applying the correct rigid body transformation to make the two coordinate systems coincident. However, we do not know that correct transformation.

This problem has long been recognized in the engineering literature, where the method of $localization$, or "best-fitting," was developed. Essentially, the measured data is used to estimate the transformation, the transformation is applied, and new estimates of the form error are calculated. The transformation is estimated by finding the translation vector $\hat{\boldsymbol{t}}$ and the rotation angles $\hat{\boldsymbol{\theta}}$ that minimize some function of the distances $\epsilon_i = \text{sign}(\boldsymbol{e}_i) \times (\boldsymbol{e}_i' \boldsymbol{e}_i)^{1/2}$, where $\boldsymbol{e}_i = \boldsymbol{p}(\boldsymbol{a}_i; \boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{\phi}^*) - \boldsymbol{a}_i$. That is, we seek the new orientation of the CAD model that provides the "best-fit" between the measured data and the nominal shape. Typically, the function $\sum_{i=1}^m \epsilon_i^2$ is minimized to obtain least-squares estimates of the parameters; the localization problem can then be viewed as fitting a nonlinear regression model. The best-fit distances $\hat{\epsilon}_i$ become the new estimates of the form errors.

Note that in my description of localization I have transformed the part model to match the data. This is consistent with the usual statistical formulation of fitting models to data, and is also used by CCK. It is worth noting that some engineers, and some localization software, prefer to transform the data to match the model. The use of orthogonal distances insures that the results are identical.

To illustrate the effect of the localization, I have plotted the $\hat{\epsilon}_i$ from a least-squares localization analysis of the simulated extrusion data in Figure 4 (I used software developed at Alcoa to perform the analysis). Note how our perception of part quality has changed from the previous figure. In particular, we now find that the ends of the part are much closer to target, and although there is a slight bend in the part, it is well within tolerance. However, the deviations for the end cross-sections are, on average, slightly larger than zero, which suggests that the cross-sections are enlarged.

2.4. Statistical issues

Again, the CCK methodology is directed at the the least-squares localization problem for a specific class of parts and for certain data-collection schemes. While many commercial packages are available for performing localization analyses,

these packages typically view localization as only an optimization problem. Clearly, localization is model-fitting and it should be considered from a statistical viewpoint. That is, confidence intervals for parameters and/or inference about the form error estimates ϵ_i should be obtained. This is the main contribution of the CCK methodology; it puts the localization problem into a statistical framework and provides the necessary inferences. This recognition of the role of statistics in this engineering context is more widespread, as evidenced by the small, but growing, body of literature on this topic (see Kurfess and Banks (1990), Chen and Chen (1992), Dowling et al. (1994)). Localization can also be a very computationally-intensive task when implemented in a general fashion. CCK have demonstrated how to take advantage of the planar geometric model to reduce that computational burden.

From the viewpoint of fitting models to data, an often-overlooked consideration is the selection of the measurement "design," that is, the selection of the $\boldsymbol{n}_i$ and $\boldsymbol{v}_i$ ($i = 1, \ldots, m$). For industry, this is a very important issue because of the relatively high cost of CMM measurements. More work of the impact of point selection on the form error estimates is required (see Dowling et al. (1993) for more details).

3. Interpreting Part Shape

Plots of the localized distances $\hat{\epsilon}_i$, such as those of Figures 3 and 4, provide a characterization of part shape that is usually adequate for assessing conformance-to-tolerance and process capability. However, such displays are generally inadequate for providing information for process improvement. The reason is that plots of the individual $\hat{\epsilon}_i$ tend to focus attention on individual point locations and ignore shape deviations that affect several regions of the part (e.g., bending of an extrusion).

For the simple part presented here it is easy to look at Figure 4 and draw conclusions about the overall shape. The same would be true of the diamond pin example in CCK. However, for more complex shapes, this is a very difficult task. Thus we need to identify tools for extracting relevant shape information from the data. For planar geometric models, the t_i^2 statistics presented by CCK provide diagnostics for identifying planar regions of the part that may be in error, and similar diagnostics need to be developed for more general geometries as well. The main limitation is that these diagnostics cannot characterize the type of error. Consider the CCK example of Section 4.2. In that case the diagnostics detect an error and identify suspect planar regions, but they do not tell us whether the error is due to a local deformation in one of the surfaces (thus affecting the surface normal estimate), or due to a more global problem that has caused the planes to be misaligned with one another. Thus, we must also have visualization tools that link the statistical information in the $\hat{\epsilon}_i$ to the geometrical information in $\mathcal{M}(\hat{\boldsymbol{t}}, \hat{\boldsymbol{\theta}}, \boldsymbol{\phi}^*)$ (see Hulting (1993), Hu (1994)). Additionally, we need to develop succinct summaries of the data that provide meaningful shape descriptions to the process engineers.

Extreme care must be exercised when using a visualization tool to link patterns in the $\hat{\epsilon}_i$ to the geometry, because patterns in the $\hat{\epsilon}_i$ can reflect measurement problems rather than true deviations from nominal shape. Recall that CMM measurement programs are written to measure parts with the nominal shape. For example, the CMM program for the points on the straight extrusion (Figure 1) will use probe approaches that are normal to the nominal shape. However, if the extrusion is not actually straight, the probe path may not be normal to the actual shape. This is depicted in Figure 5 (a top view of the extrusion in the x-z plane), where bending in the extrusion will cause the measurement slices to incorrectly characterize the cross-section.

Figure 5. An illustration of the effect of part shape errors on the measurement process.

Even though no manufactured part will have exactly the nominal shape, both the CMM measurement programs and the localization analyses implicitly assume that these parts do have the nominal shape. We can overcome this deficiency by explicitly recognizing the departures from nominal shape and fitting a realistic geometric model to the data. To do this we allow some of the shape parameters $\boldsymbol{\phi}$ to vary along with the orientation parameters during localization. We refer to this wider class of models as *manufacturedpartmodels* (MPM), a term coined by Yu Wang.

To illustrate the idea of an MPM in two-dimensions, consider the four panels of Figure 6. The top panel shows the nominal shape for our straight extrusion (the x-z plane). The second panel shows a collection of measurements that may have been taken from an actual part (these points do not follow the layout depicted in Figure 1). A localization analysis would fit the nominal shape to the data and produced the result shown in the third panel. We can see the lack of fit pattern in the data, and how it suggests that the extrusion may be bent slightly. An MPM analysis would recognize the possibility of a bend, and allow the shape parameter that controls the bend to vary. The result is the fit obtained in the bottom panel.

Figure 6. An illustration of the MPM concept.

To fit an MPM by least squares we find the translation vector $\hat{\boldsymbol{t}}$, the rotation angles $\hat{\boldsymbol{\theta}}$, and the shape parameters $\hat{\boldsymbol{\phi}}$ that that minimize $\sum_{i=1}^{m} \gamma_i^2$, where $\gamma_i = \text{sign}(\boldsymbol{g}_i) \times (\boldsymbol{g}_i'\boldsymbol{g}_i)^{1/2}$, and $\boldsymbol{g}_i = \boldsymbol{p}(\boldsymbol{a}_i; \boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{\phi}) - \boldsymbol{a}_i$. While the model residuals $\hat{\gamma}$ do not have a clear interpretation in terms of the form error, the estimated shape parameters $\hat{\boldsymbol{\phi}}$ provide succinct, and meaningful, shape information for the process engineers.

Figure 7 displays the $\hat{\gamma}_i$ values after performing an MPM analysis of the simulated extrusion data. The model only includes a single shape parameter, which is the bend parameter described by Figure 6. The MPM describes the data quite well, and in particular we now see that the apparently enlarged cross-sections were an artifact of measurement.

From the industry point of view, the extraction of meaningful shape descriptions that can be related to process variables is extremely important. The MPM methodology, and current work on the data visualization, appear to promising areas for further research.

4. Conclusion

The ideas presented here were developed through my interaction with engineers (in particular, I would like to acknowledge the contributions of Paul Fussell and Yu Wang). Continued collaboration between the two disciplines is necessary for statistical thinking to become part of the process of collecting and interpreting coordinate measurement data. For certain situations, CCK have developed the theoretical basis for statistical inference in localization analyses. The next step is to work with the engineering community to find suitable applications for their methodology.

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COMMENT: SOME LOCAL LINEAR MODELS FOR THE ASSESSMENT OF GEOMETRIC INTEGRITY

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The authors are to be congratulated for a very stimulating paper. Applying statistical methods for directional data to the assessment of geometric integrity is a novel idea that is raising challenging statistical issues.

One can view the proposed procedure for assessing geometric integrity as an investigation of the propagation of errors related to both CMM measurements and a possible lack of conformity of the part. The impact of these errors are investigated using two methods. A bound for the maximal impact of the CMM errors on each u_i is first derived, then approximation to the errors' contribution to the estimated rotation are obtained with Rivest's (1989) first order expansion. This result can be reexpressed as a perturbation analysis. Let ε_{i1} and ε_{i2} represent the impact of the errors on $A^t u_i$, then

$$A^t u_i = v_i\{1-(\varepsilon_{i1}^2+\varepsilon_{i2}^2)/2\} + \varepsilon_{i1} v_{i(1)} + \varepsilon_{i2} v_{i(2)} + o(\kappa^{-1}), \qquad \text{(D1)}$$

for $i = 1,\ldots,n$, where v_i, $v_{i(1)}$ and $v_{i(2)}$ are CAD vectors defined in (4) and A

is the rotation relating CAD directions to CMM directions. The contribution of the ε_{ij}'s to the least squares rotation A_{ls} can be written using the following notation:

- Let X be the $2n \times 3$ matrix whose rows $2i-1$ and $2i$ are equal to $v_{i(1)}^t$ and $-v_{i(2)}^t$,
- Let ε be the $2n \times 1$ vector of the ε_{ij}'s,
- Let $a = (a_1, a_2, a_3)^t = (X'X)^{-1}X'\varepsilon$ and

$$\delta A = \begin{pmatrix} o & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix};$$

then $A_{ls} = A(l + \delta A) + o(\kappa^{-1/2})$. Thus, the impact of the perturbations $\{\varepsilon_{ij}\}$ on A_{ls} is, to a first degree of approximation, equal to the vector of the regression of ε on X. The statistic $2n(1-r)$ defined in Section 3.1 is the sum of the squared residuals for the above regression.

This perturbation analysis sheds some light on the distributional assumptions in the paper. All the tests proposed in Sections 3.1 and 3.2 pertain to the regression of ε on X. They are valid as long as the perturbations $(\varepsilon_{i1}, \varepsilon_{i2})^t$ are independent normal deviates with the same variance. A special case when this holds true is when the distribution of u_i is Fisher-von Mises with a large concentration parameter (Watson (1984)). Thus, as pointed out in Section 6, the Fisher-von Mises assumption is not needed for the inference to be valid. Note also that distributional features can be ascertained with the residuals $\{e_i\}$ defined by (4). These residuals are not uniquely defined; any rotation of $v_{i(1)}$ and $v_{i(2)}$ around v_i yields a different residual. Normality diagnostics should therefore be built with the residuals squared lengths $e_i^t e_i$ which are uniquely defined. If the ε_{ij}'s are normally distributed, $e_i^t e_i$ follows approximately an exponential distribution. Exponential goodness of fit procedures (D'Agostino and Stephens (1986)) could be used to test the normality assumption.

This discussion explores some of the avenues opened up by Chen, Chapman and Kim. First it proposes a model for the various errors affecting CMM measurements. Then it derives, using a local regression model, first order approximations of the CMM measurement error impact on the estimated unit vectors u_i's. This leads to new statistical methods for the assessment of geometric integrity.

How are errors transmitted to unit vector u_i?

Since the CMM measurements are the data to be analyzed, it is useful to have a model describing the action of the various sources of variation on these measurements. Such a model expresses the CMM coordinates x_{ki} of the kth point recorded in planar region i as

$$x_{ki} = T_i + y_{ki}(Av_{i(1)} + \tau_{i1}Av_i) + z_{ki}(Av_{i(2)} + \tau_{i2}Av_i) + e_{ki}, \qquad \text{(D2)}$$

where:

- T_i is the coordinate vector of a point in planar region i;
- $Av_{i(1)} + \tau_{i1}Av_i$ and $Av_{i(2)} + \tau_{i2}Av_i$ is a basis of the planar region under study, $(Av_{i(1)}, Av_{i(2)})$ is the basis according to CAD specifications while $(\tau_{i1}Av_i, \tau_{i2}Av_i)$ represents a possible departure from these specifications; in a part without distortion, one would have $\tau_{i1} = \tau_{i2} = 0$ for each i;
- y_{ki} and z_{ki} are the coordinates of x_{ki} in the ith planar region;
- e_{ki} is the vector of CMM measurement errors for the kth reading.

Using Chapman, Chen and Kim terminology, the e_{ki}'s represent *measurement* variation while τ_{i1} and τ_{i2} stand for *part* variation. The e_{ki}'s and the τ_{ij}'s are assumed to be small or $O(\kappa^{-1/2})$.

To investigate how the errors in Model (D2) are transmitted to unit vector u_i, one can derive a first order approximation to the perturbation ε_{i1} and ε_{i2} appearing in the expression for u_i given in (D1) in terms of the components of Model (D2). This problem was first investigated by Chapman (1994). The following presentation relies on a local linear model underlying the calculation of unit vector u_i. Since u_i is the eigenvector corresponding to the smallest eigenvalue of D_i, defined in Section 2.2, one can approximate ε_{i1} and ε_{i2} by minimizing

$$\begin{aligned}
&\sum_{k=1}^{m}[(x_{ki} - \overline{x}_i)^t u_i]^2 \\
&= \sum_{k=1}^{m}\{[(y_{ki} - \overline{y}_i)(Av_{i(1)} + \tau_{i1}Av_i) + (z_{ki} - \overline{z}_i)(Av_{i(2)} + \tau_{i1}Av_i) + e_{ki} - \overline{e}_i]^t \\
&\qquad\qquad\qquad \cdot [Av_i + \varepsilon_{i1}Av_{i(1)} + \varepsilon_{i2}Av_{i(2)}]\}^2 + o(\kappa^{-1}) \\
&= \sum_{k=1}^{m}[v_i^t A^t(e_{ki} - \overline{e}_i) + (\varepsilon_{i1} + \tau_{i1})(y_{ki} - \overline{y}_i) + (\varepsilon_{i2} + \tau_{i2})(z_{ki} - \overline{z}_i)]^2 + o(\kappa^{-1}).
\end{aligned}$$

The values of $(\varepsilon_{i1} + \tau_{i1}, \varepsilon_{i2} + \tau_{i2})$ minimizing this expression are equal, up to a sign change, to the least squares coefficient of y and x in the regression of $\{v^t A^t e_{ki}\}$ on $\{1, y_{ki}, z_{ki}\}$. It also worth noting that $\{(x_{ki} - x_i)^t u_i\}$ are the approximate residuals for this regression and that the smallest eigenvalue, say λ_i, of D_i is approximately equal to the sum of the squared residuals. Furthermore, following the argument in the theorem in Rivest (1989), one shows that the difference between the true value of $(\varepsilon_{i1} + \tau_{i1}, \varepsilon_{i2} + \tau_{i2})$ and the approximation obtained through the above regression is $o(\kappa^{-1/2})$. Thus

$$\begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \end{pmatrix} = -\begin{pmatrix} \tau_{i1} \\ \tau_{i2} \end{pmatrix} - \left(Z_i^t Z_i\right)^{-1} \sum_{k=1}^{m} \begin{pmatrix} y_{ki} - \overline{y}_i \\ z_{ki} - \overline{z}_i \end{pmatrix} v_i^t A^t(e_{ki} - \overline{e}_i) + o(\kappa^{-1/2}), \quad \text{(D3)}$$

where Z_i is the $m \times 2$ matrix whose kth row is given by $\{y_{ki} - \overline{y}_i, z_{ki} - \overline{z}_i\}$. In Equation (D3) the part variation of planar region i shows up, as was to be expected, in vector u_i together with some function of the the CMM measurements errors.

Analysis of variance procedures for assessing geometric integrity

In this section we assume that the CMM measurement errors are approximately normally distributed, i.e. that the $v_i^t A^t e_{ki}$'s are independent normal deviates with variance equal to $1/\kappa$. By using the classical theory of linear models one can obtain the following first order approximation to the joint distribution of ε_{i1}, ε_{i2} and λ_i.

Theorem 1. *If, for* $i = 1, \ldots, n$ *and* $k = 1, \ldots, m$, $v_i^t A^t e_{ki}$ *are independent normal deviates with variance* $1/\kappa$, *then for any* i, *as* κ *goes to infinity, the following results hold.*

(i) $\kappa^{1/2}(\varepsilon_{i1}, \varepsilon_{i2})^t$ *is asymptotically distributed as a* $N_2(-\kappa^{1/2}(\tau_{i1}, \tau_{i2})^t, (Z_i^t Z_i)^{-1})$,

(ii) $\kappa\lambda_i$ *is asymptotically distributed as a* χ^2 *with* $m-3$ *degrees of freedom,*

(iii) $(\varepsilon_{i1}, \varepsilon_{i2})^t$ *and* λ_i *are asymptotically independent.*

If the three components of e_{ki} *are independent normal deviates with variance* $1/\kappa$, *then the assumption of the theorem is true since*

$$\mathrm{Var}(v_i^t A^t e_{ki}) = v_i^t A^t I A v_i / \kappa \; = \; \kappa^{-1}.$$

Theorem 1 permits one to adapt some standard analysis of variance techniques to the assessment of geometric integrity.

In Section 3.3 the authors propose to place the m measurements for the calculation of u_i on a circle of radius R. This amounts to taking $(y_{ki}, z_{ki}) = R[\cos(2\pi k/m), \sin(2\pi k/m)]$. For such coordinates, $\overline{y}_i = \overline{z}_i = 0$ and $Z_i^t Z_i = R^2 m I/2$. To check, a posteriori, that the measurements have really been taken on a circle of radius R, one could compare the theoretical value of $Z_i^t Z_i$ with an estimator derived from Theorem 1. This estimator is $(u_{i(1)}, u_{i(2)})^t D_i (u_{i(1)}, u_{i(2)})$ where $u_{i(1)}$ and $u_{i(2)}$ are the eigenvectors corresponding to the largest and the second largest eigenvalues of D_i.

In Model (D2), the geometric integrity of a part is characterized by the hypothesis $H_0 : \tau_{i1} = \tau_{i2} = 0$ for $i = 1, \ldots, n$. A test for H_0 is constructed as in a one-way classification where the planar regions are the treatments and where the CMM measurements are the repetitions within treatments. Within region variability is measured by $\sum \lambda_i$ while $2n(1-r)$ characterizes between region variability. Recall that $2n(1-r)$ is the sum of the squared residuals for the regression of ε on X introduced at the beginning of this discussion. Taking m measurements on a circle of radius R yields perturbations ε_{ij} whose approximate variance is $2/(\kappa m R^2)$. Thus, under H_0, $\kappa R^2 mn(1-r)$ is approximately distributed as χ^2 with $(2n-3)$ degrees of freedom. The comparison of the between region and the within region variation is summarized in the ANOVA table presented in Table 1.

Table 1. Analysis of variance table for assessing geometric integrity

Source of variation	df	SS	EMS	F-statistic
part variation	$2n-3$	$R^2mn(1-r)$	$\kappa^{-1}+\frac{R^2m}{2}\sigma_\tau^2$	$\frac{R^2[mn(1-r)]/(2n-3)}{\sum\lambda_i/[n(m-3)]}$
measurement variation	$n(m-3)$	$\sum_{i=1}^{n}\lambda_i$	κ^{-1}	

A large value of the F-statistic in Table 1 indicates that the between region variation is larger than what would be expected if the part met the CAD specifications. This F test is an alternative to the chi-square test proposed in Section 3.2. If the F-test is significant then it is possible to estimate the proportion of variance accounted for by part variation. Assuming, that in Model (D2), the τ_{ij}'s are independent normal deviates with zero mean and variance equal to σ_τ^2, the expectation of the part variation mean square is easily found to be equal to $\kappa^{-1}+R^2m\sigma_\tau^2/2$. Thus σ_τ^2 can be estimated as in a standard random effect ANOVA model.

The F-test of Table 1 and the chi-square test of Section 3.2 are not unbiased. They are insensitive to alternatives for which the $2n\times 1$ vector of the τ_{ij}'s is non null and belongs to the column space of X. This makes the proposed tolerance test unsuitable for parts with few, say 2 or 3, planar regions since the test is then biased for most alternative hypotheses.

The normality of the $v_i^tA^te_{ki}$ is crucial for the validity of the test presented in Table 1. This assumption can be tested by applying standard regression diagnostics to the residuals $\{(x_{ki}-\overline{x}_i)^tu_i\}$ for the linear models underlying the calculation of u_i.

Suppose that one were interested in testing the integrity of planar region i. One would, for instance, like to detect whether region i contains some undesirable curvature. When κ is known, possibly from the CMM specifications, planar region i can be declared distorted at level α if $\kappa\lambda_i$ is larger than the $100(1-\alpha)$th percentile of the chi-square distribution with $m-3$ degrees of freedom. If $\text{Var}(v_i^tA^te_{ki})$ is not known a test could still be constructed by comparing λ_i to the λ's for the other planar regions with a variance test such as Hartley's or Cochran's.

Estimating geometric characteristics through lines and edges

An analysis similar to that presented in Table 1 can be carried out when the directional characteristics of a part are estimated using lines and edges. A model for the kth measurement taken in the ith linear component would then be given by

$$x_{ki} = T_i + y_{ki}(Av_i + \tau_{i1}Av_{i(1)} + \tau_{i2}Av_{i(2)}) + e_{ki}, \qquad k = 1, \ldots, m, \qquad \text{(D4)}$$

where $Av_i + \tau_{i1}Av_{i(1)} + \tau_{i2}Av_{i(2)}$ is the true direction of linear component i (v_i is the CAD specification while τ_{i1} and τ_{i2} represent part variation) and y_{ki} is the coordinate of the kth measurement taken on linear component i. To estimate Av_i one can take u_i equal to the eigenvector corresponding to the *largest* eigenvalue of the sums of squares and cross-products matrix D_i for the ith linear component. Vector u_i can be written as in (D1) where the perturbations ε_{i1} and ε_{i2} can be expressed in terms of the components of Model (D4). Approximations to ε_{i1} and ε_{i2} are derived by minimizing $\mathrm{tr}(D_i) - u_i^t D_i u_i$. Straightforward calculations show that this expression is equal to

$$\sum_{k=1}^{m}[(y_{ki} - \overline{y}_i)(\tau_{i1} - \varepsilon_{i1}) + (e_{ki} - \overline{e}_i)^t Av_{i(1)}]^2$$
$$+ \sum_{k=1}^{m}[(y_{ki} - \overline{y}_i)(\tau_{i2} - \varepsilon_{i2}) + (e_{ki} - \overline{e}_i)^t Av_{i(2)}]^2 + o(\kappa^{-1}).$$

Thus, for $j = 1, 2, \tau_{ij} - \varepsilon_{ij}$ is approximately equal to the slope of the regression of $e_{ki}^t Av_{i(j)}$ on y_{ki}, i.e.,

$$\varepsilon_{ij} = \tau_{ij} - \Big(SS_{iy}\Big)^{-1} \sum_{k=1}^{m}(y_{ki} - \overline{y}_i)(e_{ki} - \overline{e}_i)^t Av_{i(j)} + o(\kappa^{-1/2}), \text{where } SS_{iy} = \sum_{k=1}^{m}(y_{ki} - \overline{y}_i)^2.$$

The sum of the smallest two eigenvalues of D_i, $\lambda_{i2} + \lambda_{i3}$, represents the combined sum of the squared residuals for the two regressions while the sum of the two squared residuals for observations k is $(x_{ki} - \overline{x}_i)^t(x_{ki} - \overline{x}_i) - [u_i^t(x_{ki} - \overline{x}_i)]^2$. Theorem 1 is readily generalized to an experimental set-up where geometrical characteristics are determined with linear components.

Theorem 2. *If, for* $i = 1, \ldots, n$ *and* $k = 1, \ldots, m$, $v_{i(1)}^t A^t e_{ki}$ *and* $v_{i(2)}^t A^t e_{ki}$ *are independent normal deviates with variance* $1/k$, *then for any* i, *as* κ *goes to infinity, the following results hold.*
(i) $\kappa^{1/2}(\varepsilon_{i1}, \varepsilon_{i2})^t$ *is asymptotically distributed as a* $N_2(\kappa^{1/2}(\tau_{i1}, \tau_{i2})^t, I/SS_{iy})$,
(ii) $\kappa(\lambda_{i2} + \lambda_{i3})$ *is asymptotically distributed as a* χ^2 *with* $2m - 4$ *degrees of freedom.*
(iii) $(\varepsilon_{i1}, \varepsilon_{i2})^t$ *and* $(\lambda_{i2} + \lambda_{i3})$ *are asymptotically independent* ▷

An analysis of variance table similar to that presented in Table 1 can therefore be constructed for Model (D4). Techniques need to be developed for comparing the two methods for estimating directional characteristics, either planar regions or linear components.

Estimating the translation

To estimate the translation vector in (10), would it make sense to interchange CMM and CAD data? Equation (10) would then become $v_i^t(A^t x_i + T) = v_i^t p_i$ while in Formula (11) for $\hat{T}$, U^t would be replaced by $V^t = (v_1, \ldots, v_n)$, and the component of vector M would be $\{v_i^t p_i - v_i^t A_{ls}^t x_i\}$. With this approach, V is fixed and the only random components of $\hat{T}$ are x_i and A_{ls}.

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COMMENT

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Using coordinate measuring machines (CMM's) for dimensional measurement has become very popular in industry due to their flexibility, accuracy, and ease of automation. Their applications involve various activities ranging from tolerance verification, problem diagnostics, and quality assurance. Dowling et al. (1993a) review and discuss statistical issues involving CMMs. As pointed out there, a great deal of research has been done in the fields of mechanical and industrial engineering and very work has been done in the statistical field. However, there are many interesting statistical problems related to CMM measurements. The work by the present authors is one of the earlier statistical papers in this area. Hopefully, their paper and others in CMM will stimulate more statisticians to work in this area. In the following discussion, I first comment on some specific issues of the paper, then discuss some general issues related to CMM.

1. Specific Comments on the Paper

I feel uncomfortable with the approach of determining the tolerance specification (requirement) based on measurement error. In practice, the tolerance specification is usually specified by the customer or the designer based on the functionality or interchangebility of a part, which does not depend on measurement error. Of course, the size of measurement error is a concern for tolerance verification (verifying if a specific part conforms to the tolerance specification). A measuring device will be ineffective if the measurement error size of the device is as large as the size of the tolerance specification. A general guideline given in DeGarmo et al. (1988) is that the size of measurement error should be one tenth that of tolerance specification.

In this paper the authors focus on the problem of assessing the geometric integrity of a part. It is equivalent to verifying the angularity (including perpendicularity and parallelism) of the planes and ignoring the surface roughness issue. This is justifiable in certain manufacturing processes where the surface roughness is negligible compared to the flatness tolerance specification. When the surface roughness is large, the tolerance verification problem becomes more complicated since the requirements on the angularity and the surface roughness have to be satisfied simultaneously. Puncochar (1990) classifies this tolerance requirement as orientation tolerance and provides detailed examples on the specification. The problem of verifying the orientation tolerance for complex geometry is very challenging. The same problem has received much attention in geometric dimensioning problems involving other measuring devices.

2. General Comments on CMM Problems

2.1. Design issues of CMM measurements

Relatively little work has been done on issues related to choosing appropriate design methods for CMM measurements. Below we review and discuss this problem and relate it to some statistical literature. We focus our discussion on the problem of sampling CMM measurements from a surface.

To sample data from a surface, the probe of a CMM machine is programmed to measure the vertical height on a part at a set of pre-determined (x, y) locations in the horizontal plane. The design problem here will be to determine how many of these measurements should be taken and where these points should be located. Suppose the true relationship between the vertical height and the (x, y) locations of a particular surface is:

$$Z = f(\theta, x, y, \delta) + \epsilon, \tag{2.1}$$

where Z is the actual measurement of the vertical height, θ is a vector of fixed unknown parameters, and δ and ϵ represent the manufacturing error and measurement error respectively. Note that f describes the true relationship between the vertical height and the (x, y) locations for a particular part. Z differs from $f(\theta, x, y, \delta)$ due to the measurement error ϵ. The f function can be different for different parts due to the manufacturing error structure. We assume here that (x, y) locations are not subject to error. In general, they may also be subject to measurement error, which may further complicate the problem. (See Dowling et al. (1993b) for further detail of the measurement error structure.)

Three design strategies have been commonly used in the CMM literature: uniform (equidistant), simple random sampling, and stratified random sampling. Dowling et al. (1993a) discuss these three design strategies and review the literature that compares them. In general, the uniform and stratified sampling methods are better than the simple random sampling because they ensure a better coverage of the entire design region. Other than these strategies, Liang and Woo (1994a & b) propose an optimal design strategy based on the optimal discrepancy sequence (ODS) (see Niederreiter (1978)). They showed by simulations and theoretical arguments that the ODS designs give a much smaller mean square error than the uniform and simple random sampling methods on estimating average surface roughness.

Although most of these works were done in the fields of industrial and mechanical engineering, their methods are closely related to statistical methods. In statistical design literature, the design strategies are always driven by design objectives.

A common statistical design objective is to minimize the mean squared error of the estimate of a parameter of interest, such as the mean or variance of the response. In the context of CMM problems, the parameters of interest are the measures of surface roughness. As described in Liang and Woo (1994a & b), there are three basic categories of surface roughness measures: statistical descriptors, extreme-value descriptors, and texture descriptors. An example of statistical descriptors is the average roughness which is defined to be the expected absolute deviation of the response from the mean surface. An example of extreme-value descriptors is the maximum peak-to-valley height of the surface. This measure is equivalent to the true deviation range defined in Dowling et al. (1993a) and is defined to be the minimum distance between any two ideal features that bound the entire surface of interest. Examples of texture descriptors can be found in Thomas (1981).

Note that these measures are summary measures over the entire surface and thus are independent of the (x, y) locations. They are defined for each part and their values are fixed for particular parts. According to different purposes, we can formulate different estimation problems. For the purpose of inspection, we are interested in estimating the surface roughness of a particular part based on finite sampling measurements of the part. One may use this estimate to decide if the part should be accepted or rejected. In this case the parameter of interest is the true roughness measure of a particular part. Much research has been done on estimating a particular roughness measure, namely, the true deviation range. Dowling et al. (1993a) provide a detail review of this work. For the purpose of characterizing manufacturing processes, we are interested in knowing the average surface roughness of all the parts produced from a particular manufacturing process. Liang and Woo (1994a & b) discussed the justification of using the average surface roughness for characterizing manufacturing processes. In this case the parameter of interest is the average value of the true roughness measures of all the parts produced from a manufacturing process.

A common design objective for these estimation problems is to minimize the root mean squared error (RMSE) of the estimate of the parameter of interest. Liang and Woo (1994a & b) showed that the RMSE of the estimate of the average roughness (defined earlier) under the ODS designs is much smaller than that under the uniform and simple random sampling designs. This result is consistent with the results in the statistical literature.

McKay et al. (1979) introduced the latin hypercube sampling (LHS) design for sampling from high dimensional vectors. A characteristic of the LHS design is that it ensures a dense and even coverage of the entire range when the design is projected onto a single dimension. As shown in Stein (1987) and Owen (1992), the LHS design effectively eliminates the first order effects of the surface and thus reduces the variance of the estimate. Tang (1993) and Owen (1992) extend the LHS design to the orthogonal array (OA)-based LHS designs (called U-designs) to further filter out higher order terms. These designs are very similar to the ODS designs suggested in Liang and Woo (1994a &b).

In conclusion, the LHS designs, the ODS designs, and the related designs seem to be quite promising for sampling CMM measurements from a surface. However, although some theoretical justification and simulation studies of these designs have been done, they are somewhat limited and not directly related to the specific problems. More research is needed to establish the theoretical foundation of these designs in the context of CMM problems. An interesting specific problem is to investigate the relationship between the LHS and the ODS designs. Owen (1994) has applied some discrepancy sequences to extend the OA-based LHS designs to gain further properties.

To characterize a manufacturing process, sometimes it is not enough to rely on a single summary measure, such as the surface roughness measures. An alternative is to study the entire surfaces of multiple parts to characterize and understand the systematic and random error structure of the process. To study the entire surface, the prediction error of the entire surface becomes an important measure for experimental design.

Box and Draper (1987) discuss some strategies for various design objectives, such as minimization of variance, bias, or mean square error of the predicted response. They assume that the underlying true model is linear and recommend some design strategies for the cases when the fitted model is not true (e.g., fitting a first order linear model when the second order linear model is true). Their design objective is to minimize the average mean squared error of the predicted response over the entire design region, which is called the integrated mean squared error (IMSE). They decompose the IMSE into two components: average squared bias (ASB) and average variance (AV). The bias and variance components can be interpreted as being caused by the systematic error and random error of the fitted model respectively. They point out that in some cases the ASB component can be reduced by spreading the design points evenly over the design region and the AV component can be reduced by placing the points at the extreme points of the region. Thus these two criteria may conflict with each other. In practice, an experimenter needs to make trade-offs between the two criteria based on his knowledge of the size of the systematic and random error.

These design strategies can be useful in the context of CMM problems. As described in Model (2.1), there are two stages of error in the CMM height measurements Z. First, Z deviates from the true surface height due to measurement error which may include both systematic and random error depending on the measuring equipment and environment. Second, the true surface deviates from the ideal feature (e.g., a plane) due to manufacturing error which may also include both the systematic and random error from manufacturing (see Dowling et al. (1993b) for a detailed discussion of error structure). As pointed out in Dowling et al. (1993a), most model fitting methods in the CMM literature attempt to fit a model to the ideal feature of the surface. This is similar to the situation discussed in Box and Draper (1987) where the ideal feature (a lower order model) is fitted to the data while the true surface can be described by a higher order model. Note, however, that the lower and higher order models here are not limited to linear models.

For fitting non-linear models to the surface, the IMSE design objective can still be applied except that the computation will be much more complex. Sacks et al. (1989) developed optimal design strategies to minimize IMSE for modeling computer experiments using kriging models. These optimal design strategies are appropriate when the true surface can be well approximated by the assumed model.

2.2. Other issues

As described in Dowling et al. (1993a), a great deal of research has been done in estimating the true deviation range of a particular part. Most of this research is of theoretical interest and the practical impact is limited. In order to use these estimates for inspection, decision rules have to be developed (e.g. how to compare the estimates with the tolerance specification to decide when to reject a part). In addition, the risk of these decision rules have to be assessed to estimate the chance of misclassification of good parts and bad parts. Kurfess and Banks (1990) have developed a sequential hypothesis testing approach for this purpose.

For tolerance verification (inspection) of a particular part, the true deviation range is only one of many surface roughness measures (as mentioned earlier). This measure has been getting so much attention because it was specified in the standards (see ANSI Y14.5M-1982). As pointed out in Dowling et al. (1993a), these standards were developed based on gauging technology. Currently, studying groups have been formed to establish standards for evaluating the performance of CMMs.

As emphasized in modern quality control methods, inspection is only a passive tool to improve product quality. A more effective way to improve quality is to characterize the process and to understand the process error structure by combining the statistical modeling of the CMM data and the engineering knowledge of the process. Hulting (1993) gave a good example of this for characterizing an extrusion process. More research is needed for developing general methods and strategies along this direction.

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REJOINDER

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The discussants raise both technical issues (i.e. pertaining to practical quality assurance questions) and matters of statistical theory. We will deal with the technical issues first, because the theoretical advances made by Rivest in his discussion allow some of these technical issues to be addressed, at least partially.

1. Soft-Gauging Procedures

Hulting's comments are particularly perceptive in view of what motivated our paper: the construction of a quality assurance software package. The advent of CAD data brought new demands to geometric quality assurance, and accentuated the need to replace traditional hard fixtures with soft-gauging techniques. The first requirement is a transformation

$$x \to Ax + T \quad (A \in \mathrm{SO}(3),\, T \in \mathbb{R}^3), \tag{1}$$

that relates CAD and CMM data, i.e. that establishes a *referenceframe*, in Hulting's terminology. Rather than using only the time-honoured 3,2,1 method, the transformation can be more broadly based. Any combination of planes, lines and identifiable points that completely fix the position of the body can be used to construct the transformation. By an identifiable point we mean a point associated with the body, such as the centre of a circular hold drilled in the surface of the body, the coordinates of which can be identified in the CAD data file. The rotation matrix is estimated from the directional data (lines, normals to planes) as in the paper, and the translation is estimated as follows.

Let $\{(v_i, p_i); 1 \le i \le n\}, \{(u_i, x_i); 1 \le i \le n\}$ describe CAD and corresponding CMM planes, $\{(v_i', p_i'); 1 \le i \le n'\}, \{(u_i', x_i'); 1 \le i \le n'\}$ describe CAD and corresponding CMM lines (i.e. v_i', u_i' are unit vectors in the direction of the lines, and p_i', x_i' are points on the lines, $\{p_i''; 1 \le i \le n''\}, \{x_i''; 1 \le i \le n''\}$ be identifiable CAD points and their CMM counterparts. Then, following Rivest's suggesion to transform from CMM to CAD, we have

$$v_i^t(A^t x_i + T) = v_i^t p_i, \quad 1 \le i \le n, \tag{2}$$

$$v_i \times (A^t x_i + T) = v_i \times p_i, \quad 1 \le i \le n', \tag{3}$$

$$(A^t x_i + T) = p_i, \quad 1 \le i \le n'', \tag{4}$$

where $\times$ denotes cross product. This yields $n + 2n' + 3n''$ equations from which the coordinates of T can be estimated (by least squares) provided an appropriate combination of planes, lines and identifiable points has been chosen.

The second requirement is to refine this transformation, in order to minimize *alignmenterror*, by the method of *localization*. A new transformation is constructed as described by Hulting, Section 2.3. A Nelder-Mead algorithm is used (favouring robustness over speed), so that, as Hulting states, the problem is treated as one of optimization.

Our goal is to put this procedure in its proper statistical setting. As a start we consider only the transformation $x \to Ax + T$, and restrict our attention to the rotation matrix A. This means considering directional data alone. We use distributional theory obtained by Chang (1986), Rivest (1989) to develop a quality assurance capability that we hope may be useful in certain circumstances. Hulting's suggestion that our methods may provide diagnostics concerning adequacy of alignment of reference frames is thus highly germane.

2. Measurement Error Variation

Our assumption that measurement error can be equated with CMM resolution is, as Hulting suggests, naive. Furthermore, the values of CMM resolution used in our simulation are outmoded: modern machines have finer resolutions, in the range of 0.1 microns (Sakai (1990)).

Given the importance of CMMs in quality control, initial verification and periodic reverification of the precision and accuracy of these machines is essential. In view of this, national standards for performance evaluation (e.g. ANSI (1985)) have been developed. There are three main sources of CMM error (Paggs (1990)). The first, and according to Pahk and Burdekin (1990) the most significant, is systematic inaccuracy due to physical defects in the construction of the CMM (e.g. misalignment of axes, axes not straight). The other sources of error are faults in the probe system, and environmental factors such as temperature, vibration and dust (Matsumiya (1990)). One method of CMM verification is the parametric calibration technique. The CMM is represented by a geometric model and rigid body kinematics is applied. Distinct error components are identified and measured with, e.g. a laser interferometer. A volumetric error map is constructed, and the CMM software can be compensated appropriately. (Pahk and Burdekin (1990)).

Manufacturers specifying the precision of a given CMM via its $individual axis measuring accuracy$, or U_1-value. There is a U_1-value for each of the three coordinates, and all three values are often (but not always) the same. A U_1-value is obtained by placing a one-dimensional artifact of known length (i.e. whose length L can be traced to some standard) along the appropriate axis, and repeatedly measuring the length. The U_1-value is the half-width of a 99 percent confidence interval for the length, derived from the sample variance. It is usually of the form $a + bL$, where a, b are constants. Hulting (1992) notes the inherent multivariate nature of CMM data, and observes that such one-dimensional summaries of this data result in lost information. He suggests multivariate modeling of coordinate data, via a random effects linear model that focuses on between-operator and within-operator errors.

Our methodology rests on the assumption that (in Rivest's notation) the $(\epsilon_{i1}, \epsilon_{i2})^t$ have the same variance $1 \leq i \leq n$. Given that CMM precision varies with location, and that it is subject to operator and environmental error, it is unlikely that this assumption is valid.

3. Rivest's Linear Model

Rivest introduces a model (D2) that allows for part variation. He investigates the transmission of CMM errors through the estimation process for the ith planar normal, by minimizing the quadratic form $u_i^t D_i u_i$. Asymptotically (as the CMM measurement error $\to 0$), this can be viewed as a regression problem, and the asymptotic distribution of the estimator of u_i is obtained (D3). An analysis of variance procedure is described that allows the apportionment of part variation and measurement variation. This represents a substantial advance in the theory, and allows the possibility of introducing tolerancing into our diagnostics.

Both Hulting and Tsui note that our diagnostics test a component part against perfection (i.e. allow for zero tolerance), and this is unlikely to be of use in practice. Consider first a situation in which the planar regions are assumed to be flat, so the only issue is whether the planes are aligned correctly with respect to one another. Then, tolerance regarding the alignment of the planes could be specified via the choice of vectors $v_{i(1)}^t, -v_{i(2)}^t$ (which must be chosen so that $\{v_i, v_{i(1)}, v_{i(2)}\}$ is an orthonormal basis) and the magnitude of the values τ_{i1}, τ_{i2} of Rivest's discussion. Let τ denote the $2n$-vector with $2i-1$, $2i$ coordinates τ_{i1}, τ_{i2}. The test of Section 3.2 becomes to test $2n\kappa(1-r)$ against a non-central χ^2_{2n-3} distribution, with non-centrality factor

$$\frac{1}{2} \cdot \tau^t (I - X(X^tX)^{-1}X^t)\tau.$$

How to choose the $v_{i(1)}, v_{i(2)}, \tau_{i1}, \tau_{i2}$ is not clear. It will depend, of course, on the purposes for which the part is intended and the degree of precision required. Discussion with engineers and practitioners is necessary, but even then the problem remains a challenging one.

When the flatness of the planar regions is also an issue, then Rivest's analysis of variance procedure could be useful in apportioning variation within planes and between planes. Note that Rivest's Theorem 1 applies for any configuration of CMM points in the plane (not only for points evenly spaced around a circle, as in the paper, and Chapman (1995)). Thus any test for planarity based on $\kappa\lambda_i$ could accomodate the different sampling schemes reviewed in Tsui's discussion.

4. Outstanding Problems

In conclusion we would like to highlight two problems raised in the comments. The first is the question of distributional assumptions. A fundamental assumption that underlies our methodology is that $(\epsilon_{i1}, \epsilon_{i2})^t$ are normally distributed, with equal variance. While normality may not be too strong an assumption, a close study of CMM performance shows that equality of variance may not hold. Furthermore, there is a need to deal with directional data of all sorts: planar normals as in the paper, directions of lines (as in Rivest's comment), joins of identifiable points, etc. There is a need to extend the results of Rivest (1989) to the context where the $(\epsilon_{i1}, \epsilon_{i2})^t$ are not necessarily of equal variance, and thus develop a more realistic version of the diagnostic of Section 3.2.

The second problem is to introduce geometric tolerancing into our methodology. ANSI tolerancing procedures (ANSI (1982)) are formulated with reference to traditional hard gauging techniques (at least implicitly), and offer little assistance. Cooperation is required between engineers and practitioners, and statisticians, to develop tolerancing criteria that are appropriate for soft gauging, and statistically sound. This remark applies to geometric quality assurance in general, and our methods in particular.

We feel that both these problems are sufficiently challenging to be worthy of study, and have the potential for significant practical application.

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