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CONSTRUCTION OF SLICED SPACE-FILLING DESIGNS BASED ON BALANCED SLICED ORTHOGONAL ARRAYS

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Abstract: Latin hypercube designs have been widely used in computer experiments with quantitative factors. When there are both qualitative and quantitative factors in computer experiments, sliced space-filling designs have been proposed. In this article, we propose a general framework for constructing sliced space-filling designs for more flexible parameters of designs in which the whole design and each slice not only achieve maximum stratification in univariate margins, but also achieve stratification in two- or more-dimensional margins. Compared with other designs, these designs have better space-filling properties or have more columns. The construction is based on a new class of sliced orthogonal arrays, called balanced sliced orthogonal arrays, in which each slice is balanced and becomes an orthogonal array after some level-collapsing. Several approaches to constructing such balanced sliced orthogonal arrays under different level-collapsing projections are developed. Some examples are given to illustrate the construction methods.

Key words and phrases: Balanced, computer experiment, difference matrix, Latin hypercube design, sliced orthogonal array, sliced space-filling design.

1. Introduction

Latin hypercube designs (LHDs), proposed by McKay, Beckman, and Conover (1979), have been widely adopted in the design of computer experiments with quantitative factors because they spread the design points uniformly in any one-dimensional projection (Santner, Williams, and Notz (2003); Fang, Li, and Sudjianto (2006)). To improve the space-filling property of LHDs in two- or more-dimensional projections, Tang (1993) constructed LHDs, based on orthogonal arrays (OAs), that further achieve uniformity in all t-dimensional margins if an OA of strength t is employed.

For computer experiments with both qualitative and quantitative factors, Qian and Wu (2009) proposed sliced space-filling designs to deal with them. The approach starts with constructing LHDs based on sliced orthogonal arrays for quantitative factors and then partitions the LHDs into slices corresponding to different level combinations of the qualitative factors. For these sliced designs, each slice has attractive stratification in two-dimensional projections but cannot achieve one-dimensional uniformity. Later, Xu, Haaland, and Qian (2011) constructed such sliced space-filling designs based on doubly orthogonal Sudoku Latin squares that the full design and each slice achieve maximum uniformity in univariate and bivariate margins. Their constructions are restricted to the condition that the number of slices and the run size of each slice must be the same prime power. Qian (2012) constructed sliced LHDs with flexible runs in which each slice is a small LHD. However, these may not have good space-filling properties in two- or more-dimensional projections. For the details of modeling and analysis of such computer experiments, see Qian, Wu, and Wu (2008) and Han et al. (2009).

In this article, we propose a general framework for constructing sliced space-filling designs in which the full design and each slice not only achieve maximum stratification in one-dimensional margins, but also achieve stratification in two- or more-dimensional margins. Our constructions are based on a new class of sliced orthogonal arrays, called *balanced sliced orthogonal arrays* (BSOAs), which are a special type of orthogonal arrays that can be partitioned into several slices such that each slice is balanced and becomes an orthogonal array after some level-collapsing. These new sliced designs have better space-filling properties than those constructed by Qian and Wu (2009) and Qian (2012), and have more columns than those constructed by Xu, Haaland, and Qian (2011) for the same parameters.

The remainder of this article is as follows. In Section 2, some notation and definitions are introduced. Section 3 develops a general framework for constructing BSOAs under subfield projection. Section 4 constructs different types of BSOAs by elaborating on the generator matrices in the previous general framework for different parameters of designs. The construction of BSOAs under the modulus projection is given in Section 5. Section 6 presents the construction of BSOAs with a nonprime power number of levels. A general method for constructing new BSOAs from existing sliced orthogonal arrays is given in Section 7. The generation of sliced space-filling designs based on BSOAs is discussed in Section 8. Section 9 concludes with some discussions.

2. Notation and Definitions

2.1. Preliminaries

An orthogonal array, denoted by $OA(n, s^m, t)$, with n runs, m factors and strength t ($m \geq t \geq 1$) is an $n \times m$ matrix in which each column has s levels from a set of s elements, such that all possible level combinations occur equally often as rows in every $n \times t$ submatrix. An array is called *balanced* if it is an OA of strength one.

A Latin hypercube (LH) with n runs and m factors is typically an $OA(n, n^m, 1)$ in which each column is a permutation of n levels from a set S of n elements. It is called an LH over S. Usually the set S is taken to be $\{1, \ldots, n\}$. Based on an LH $\boldsymbol{L} = (l_{ij})$, following McKay, Beckman, and Conover (1979), a Latin hypercube design (LHD) with n runs and m factors in the unit cube $[0, 1)^m$ is generated by $x_{ij} = (l_{ij} - u_{ij})/n$, where the u_{ij}'s are independent uniforms on $(0, 1]$, and the n design points are given by $(x_{i1}, \ldots, x_{im})$, $i = 1, \ldots, n$. An LHD achieves maximum stratification in any one-dimensional projection when, projected onto each of the m factors, exactly one of the n design points falls within each of the n small intervals defined as $[0, 1/n), [1/n, 2/n), \ldots, [(n-1)/n, 1)$.

Let $\boldsymbol{A}$ be an $OA(n, s^m, t)$ with levels $1, \ldots, s$ and $t \geq 2$. For each column of $\boldsymbol{A}$, replace the $k = n/s$ entries of level j by a permutation of $\{(j-1)k + 1, \ldots, (j-1)k + k\}$ for $j = 1, \ldots, s$; the resulting matrix is an OA-based LH (Tang (1993)). The associated LHD achieves the stratification on the s^g grids in any g-dimensional projection for $2 \leq g \leq t$, in addition to achieving maximum stratification in any one-dimensional projection.

An $r \times c$ difference matrix (DM), denoted by $D(r, c, g)$, is an array with entries from an abelian group $\mathcal{A}$ of g elements such that every element of $\mathcal{A}$ appears equally often in the vector difference between any two columns of the array.

Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $n \times m$ and $u \times v$, respectively, matrices with entries from an abelian group $\mathcal{A}$ with the binary operation '+'. The Kronecker sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} \oplus \boldsymbol{B}$, is the $nu \times mv$ matrix $\boldsymbol{A} \oplus \boldsymbol{B} = (a_{ij} + \boldsymbol{B})$, where $a_{ij} + \boldsymbol{B}$ denotes the $u \times v$ matrix with entries $a_{ij} + b_{lk}$, $1 \leq l \leq u$ and $1 \leq k \leq v$. Throughout, the binary operation $+$ always denotes addition. A lemma of Bose and Bush (1952) shows that a larger OA can be obtained by taking the Kronecker sum of an OA and a DM.

Lemma 1. *If $\boldsymbol{A}$ is an $OA(n, s^m, t)$ with $m = t = 1$ or $t \geq 2$, and $\boldsymbol{B}$ a $\boldsymbol{D}(r, c, s)$ with entries from the same abelian group $\mathcal{A}$, then the array $\boldsymbol{H} = \boldsymbol{A} \oplus \boldsymbol{B}$ is an $OA(nr, s^{mc}, 2)$.*

2.2. Three projections

Three projections are introduced from a set F of s_1 elements to another set G of s_2 elements with s_2 dividing s_1, denoted by $s_2 | s_1$. A *projection* divides the elements of F into s_2 groups, each of size $q = s_1/s_2$, and projects any two elements of F to a same element of G if and only if they belong to the same group.

The first two projections are related to Galois fields. It is known that for any prime p and integer $u \geq 1$, there exists a Galois field $GF(p^u)$ of order p^u. The multiplicative group $GF(p^u) \setminus \{0\}$ is cyclic. Every Galois field has at least one primitive element. Throughout, the elements of any Galois field or any subset of a Galois field are arranged in lexicographical order.

Let $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$ be powers of the same prime p with integers $u_1 > u_2 \geq 1$. Let $F = GF(s_1)$ with a primitive polynomial $p_1(x)$. Any element $f(x)$ of F has the general expression $f(x) = a_0 + a_1 x + \cdots + a_{u_1-1}x^{u_1-1}$, where $a_i \in GF(p)$, $0 \leq i \leq u_1 - 1$ and $GF(p) = \{0, 1, \ldots, p-1\}$ is the residue field modulo p. Let α denote the primitive element x of F.

The first projection, denoted by ϕ, is the subfield projection proposed by Qian and Wu (2009). It can be used for the case of $u_1 = \lambda u_2$, where λ (> 1) is a positive integer. Take G to be the subfield of F with s_2 elements. Let $\beta = \alpha^{(s_1-1)/(s_2-1)}$ be the primitive element of G. Subfield theory shows that any element $f(x) \in F$ can be uniquely represented by

$$f(x) = b_0 + b_1\alpha + \cdots + b_{\lambda-1}\alpha^{\lambda-1}, \qquad b_i \in G,\ 0 \leq i \leq \lambda - 1. \tag{2.1}$$

For any $f(x)$ in (2.1), the projection $\phi(f(x))$ is defined by

$$\phi(f(x)) = b_0 + b_1\beta + \cdots + b_{\lambda-1}\beta^{\lambda-1}. \tag{2.2}$$

The second projection, denoted by φ, is the modulus projection proposed by Qian, Tang, and Wu (2009). It can be used for the case of $u_1 > u_2 \geq 1$. Let G be the $GF(s_2)$ with a primitive polynomial $p_2(x)$. For any element $f(x) = a_0 + a_1 x + \cdots + a_{u_1-1}x^{u_1-1} \in F$, if $u_2 > 1$, the projection $\varphi(f(x))$ is defined by

$$\varphi(f(x)) = f(x) \pmod{p_2(x)}. \tag{2.3}$$

If $u_2 = 1$, we take $\varphi(f(x)) = a_0$.

The third projection, denoted by ρ, was proposed by Qian, Ai, and Wu (2009). It can be used for any integers $s_1 > s_2 > 1$ with $s_2|s_1$. Here, let $F = \{0, 1, \ldots, s_1 - 1\}$ be the residue ring modulo s_1 and $G = \{0, 1, \ldots, s_2 - 1\}$ be the residue ring modulo s_2. For any $a \in F$, the projection $\rho(a)$ is defined by

$$\rho(a) = a \pmod{s_2}. \tag{2.4}$$

For a matrix $\boldsymbol{A}$, let $\boldsymbol{A}'$ denote the transpose of $\boldsymbol{A}$, $\boldsymbol{A}(i,:)$, $\boldsymbol{A}(:,j)$ and $\boldsymbol{A}(i,j)$ denote the ith row, the jth column and the (i,j)th entry, respectively, of $\boldsymbol{A}$. For any projection δ from F to G and an array $\boldsymbol{A} = (a_{ij})$ with entries a_{ij}'s from F, let $\delta(\boldsymbol{A}) = (\delta(a_{ij}))$ be the array obtained from $\boldsymbol{A}$ after its entries are collapsed according to δ. Suppose the s_2 elements of G are ordered in lexicographical order as $b_0, \ldots, b_{s_2-1}$ with $b_0 = 0$. For $j = 0, 1, \ldots, s_2 - 1$, let $\delta^{-1}(b_j) = \{a \in F | \delta(a) = b_j\}$. Define $\boldsymbol{\Gamma}$ to be the $s_2 \times q$ kernel matrix of δ (Qian and Wu (2009)), given by

$$\boldsymbol{\Gamma} = \begin{pmatrix} \delta^{-1}(b_0) \\ \delta^{-1}(b_1) \\ \vdots \\ \delta^{-1}(b_{s_2-1}) \end{pmatrix},$$

where the entries in each row are ordered in lexicographical order, and each element of F appears precisely once in $\mathbf{\Gamma}$. For the modulus projection φ, in particular, $\mathbf{\Gamma}(:,1)$ is a permutation of all elements in $F_0 = \{a_0 + a_1x + \cdots + a_{u_2-1}x^{u_2-1} | a_j \in GF(p), 0 \le j \le u_2 - 1\}$, and

$$\mathbf{\Gamma}(:,i) = \mathbf{\Gamma}(:,1) + c_i(x) \tag{2.5}$$

for $i = 1, \ldots, q$, where $c_1(x) = 0$, $c_i(x)$ is a multiple of the primitive polynomial $p_2(x)$ for $u_2 > 1$, and a polynomial in x of degree at most $u_1 - 1$ with zero constant coefficient for $u_2 = 1$.

2.3. Balanced sliced orthogonal arrays

Let $\boldsymbol{H}$ be an $OA(n_1, s_1^m, t_1)$. Suppose the n_1 rows can be partitioned into v subarrays each with n_2 rows, denoted by $\boldsymbol{H}_i$, $i = 1, \ldots, v$, and that each $\boldsymbol{H}_i$ becomes an $OA(n_2, s_2^m, t_2)$ after the s_1 levels of $\boldsymbol{H}$ are collapsed to s_2 levels according to some level-collapsing projection δ. Then $\boldsymbol{H} = (\boldsymbol{H}_1', \ldots, \boldsymbol{H}_v')'$ is called a sliced orthogonal array (SOA) of strength (t_1, t_2). This definition generalizes that in Qian and Wu (2009). An SOA $\boldsymbol{H}$ in which each slice $\boldsymbol{H}_i$ is balanced is called a balanced SOA (BSOA).

Given a BSOA $\boldsymbol{H} = (\boldsymbol{H}_1', \ldots, \boldsymbol{H}_v')'$ of strength (t_1, t_2), a sliced space-filling design $\boldsymbol{D} = (\boldsymbol{D}_1', \ldots, \boldsymbol{D}_v')'$ is generated in Section 8. Here $\boldsymbol{D}$ is an LHD based on $OA(n_1, s_1^m, t_1)$ and each slice $\boldsymbol{D}_i$ is an LHD based on $OA(n_2, s_2^m, t_2)$ for $i = 1, \ldots, v$. Such a sliced space-filling design $\boldsymbol{D}$ and each slice $\boldsymbol{D}_i$ achieves maximum stratification in any one-dimensional projection. Furthermore, $\boldsymbol{D}$ achieves better stratification in any g_1-dimensional projection for $g_1 \le t_1$, and each slice $\boldsymbol{D}_i$ achieves stratification in any g_2-dimensional projection for $g_2 \le t_2$.

3. General Framework for Constructing BSOAs with Subfield Projection

In this section we present a general framework for constructing BSOAs with the subfield projection ϕ in (2.2). Take $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$ as powers of the same prime p, where $u_1 = \lambda u_2$ with integer $\lambda > 1$. Let $F = GF(s_1)$ have the primitive element α and $G = GF(s_2)$ be a subfield of F with the primitive element $\beta = \alpha^{(s_1-1)/(s_2-1)}$. Let $\alpha_0, \ldots, \alpha_{s_1-1}$ denote the elements of F with $\alpha_0 = 0$ and $\alpha_i = \alpha^i$ for $i = 1, \ldots, s_1 - 1$.

Let $\boldsymbol{A}_0$ be the multiplication table of F, where the rows and columns are labeled with the s_1 elements of F. It is known that $\boldsymbol{A}_0$ is a $D(s_1, s_1, s_1)$. Let $\boldsymbol{u} = (1, \alpha, \ldots, \alpha^{\lambda-1})'$. Obtain an $s_1 \times \lambda$ matrix $\boldsymbol{A}$ by taking the columns of $\boldsymbol{A}_0$ labeled with the elements of $\boldsymbol{u}$. Let $Q = \{1, \ldots, q\}$ and Q^λ be the set of all possible λ-tuples from Q, where $q = s_1/s_2$. Write $\mathbf{\Gamma}$ as the $s_2 \times q$ kernel matrix of ϕ. For any $(l_1,\ldots,l_\lambda)\in Q^\lambda$, let $\boldsymbol{C}_{(l_1,\ldots,l_\lambda)}=\boldsymbol{A}+\boldsymbol{1}_{s_1}\boldsymbol{v}'_{(l_1,\ldots,l_\lambda)}$, where $\boldsymbol{1}_{s_1}$ is the s_1-vector of ones and $\boldsymbol{v}_{(l_1,\ldots,l_\lambda)}=(\boldsymbol{\Gamma}(1,l_1),\ldots,\boldsymbol{\Gamma}(1,l_\lambda))'$. Finally, obtain an array $\boldsymbol{C}$ by the row juxtaposition of all $\boldsymbol{C}_{(l_1,\ldots,l_\lambda)}$'s. The proof of the following result is given in the Appendix.

Lemma 2. *For $u_1=\lambda u_2$ with $\lambda>1$, we have*

(i) *the matrix $\boldsymbol{C}$ is an $OA(s_1^\lambda,s_1^\lambda,\lambda)$;*

(ii) *each slice $\boldsymbol{C}_{(l_1,\ldots,l_\lambda)}$ is a balanced $D(s_1,\lambda,s_1)$ and $\phi(\boldsymbol{C}_{(l_1,\ldots,l_\lambda)})$ is an $OA(s_2^\lambda,s_2^\lambda,\lambda)$ for any $(l_1,\ldots,l_\lambda)\in Q^\lambda$.*

Let $\boldsymbol{Z}$ be a $\lambda\times m$ matrix with entries from G, such that all the columns are distinct and each column contains at least one nonzero element. Such a matrix $\boldsymbol{Z}$ is called a *generator matrix*. Two methods are proposed to construct BSOAs based on a given generator matrix $\boldsymbol{Z}$.

Method 1: Take a fixed λ-tuple $(l_1,\ldots,l_\lambda)\in Q^\lambda$. Let $\boldsymbol{H}_i=\boldsymbol{C}_{(l_1,\ldots,l_\lambda)}\boldsymbol{Z}+\alpha_i$ for $i=0,1,\ldots,s_1-1$, and $\boldsymbol{H}=(\boldsymbol{H}'_0,\ldots,\boldsymbol{H}'_{s_1-1})'$, the row juxtaposition of all $\boldsymbol{H}_i$'s. The proof of the following is given in the Appendix.

Theorem 1. *For the matrix $\boldsymbol{H}$ constructed in Method 1, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^2,s_1^m,2)$;*

(ii) *each slice $\boldsymbol{H}_i$ is an LH over F for $i=0,1,\ldots,s_1-1$;*

(iii) *if any t columns of $\boldsymbol{Z}$ are linearly independent over G, then each $\phi(\boldsymbol{H}_i)$ is an $OA(s_1,s_2^m,t)$ for $i=0,1,\ldots,s_1-1$.*

Method 2: For any λ-tuple $(l_1,\ldots,l_\lambda)\in Q^\lambda$, let $\boldsymbol{H}_{(l_1,\ldots,l_\lambda)}=\boldsymbol{C}_{(l_1,\ldots,l_\lambda)}\boldsymbol{Z}$ and $\boldsymbol{H}$ be the row juxtaposition of all $\boldsymbol{H}_{(l_1,\ldots,l_\lambda)}$'s. The following result is proved in the Appendix.

Theorem 2. *For the matrix $\boldsymbol{H}$ constructed in Method 2, we have*

(i) *each slice $\boldsymbol{H}_{(l_1,\ldots,l_\lambda)}$ is an LH over F for any $(l_1,\ldots,l_\lambda)\in Q^\lambda$;*

(ii) *if any t columns of $\boldsymbol{Z}$ are linearly independent over G, then $\boldsymbol{H}$ is an $OA(s_1^\lambda,s_1^m,t)$, and all $\phi(\boldsymbol{H}_{(l_1,\ldots,l_\lambda)})$'s are the same $OA(s_1,s_2^m,t)$ for all $(l_1,\ldots,l_\lambda)\in Q^\lambda$.*

When the generator matrix $\boldsymbol{Z}$ has full row rank over G, the matrices $\boldsymbol{H}$'s constructed in Methods 1 and 2 have an additional property, proved in the Appendix.

Theorem 3. *If $\boldsymbol{Z}$ has full row rank over G, then neither the whole matrix $\boldsymbol{H}$ constructed in Method 1 or 2, nor its each projected slice under the subfield level-collapsing ϕ, has repeated rows.*

4. Construction of BSOAs with Subfield Projection

In this section we propose several generator matrices and apply the general framework of Section 3 to construct BSOAs of different strengths with the subfield projection ϕ. The relevant notation in Section 3 is used.

4.1. Construction of BSOAs of strength (2,2)

The Rao-Hamming method in Hedayat, Sloane, and Stufken (1999) can be applied to construct the generator matrix as follows. Form a $\lambda \times m$ generator matrix $\boldsymbol{Z}_1$ with $m = (s_1 - 1)/(s_2 - 1)$ by collecting all nonzero column vectors $(z_1, \ldots, z_\lambda)'$, where $z_j \in G$ for $1 \leq j \leq \lambda$ and the first nonzero element is one. Note that any two columns of $\boldsymbol{Z}_1$ are linearly independent over G. Combined with Theorem 1, we have the following.

Theorem 4. *For the matrix $\boldsymbol{H}$ constructed in Method 1 with the generator matrix $\boldsymbol{Z}_1$, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^2, s_1^m, 2)$ with $m = (s_1 - 1)/(s_2 - 1)$;*

(ii) *each slice $\boldsymbol{H}_i$ is an LH over F and $\phi(\boldsymbol{H}_i)$ is an $OA(s_1, s_2^m, 2)$ for $i = 0, 1, \ldots, s_1 - 1$.*

When $\lambda = 2$, in particular, the BSOAs constructed in Theorem 4 have $s_2 + 1$ columns. For the same parameters, the new BSOAs constructed here have one more column than those in Xu, Haaland, and Qian (2011). Actually, from Corollary 3.21 in Hedayat, Sloane, and Stufken (1999), it is known that the number $m = (s_1 - 1)/(s_2 - 1)$ is the largest m for an $OA(s_1, s_2^m, 2)$ to exist.

Example 1. Let $p = 3, u_1 = 2$, and $u_2 = 1$, giving $s_1 = 9, s_2 = 3$, and $\lambda = 2$. Use $p_1(x) = x^2 + x + 2$ for $F = GF(9)$ with $\alpha = x$. Let G be the subfield $\{0, 1, 2\}$ of F with $\beta = 2$. The projection ϕ is $\{0, x+1, 2x+2\} \to 0$, $\{1, x+2, 2x\} \to 1$, $\{2, x, 2x+1\} \to 2$. The kernel matrix of ϕ and the generator matrix are

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & x+1 & 2x+2 \\ 1 & x+2 & 2x \\ 2 & x & 2x+1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z}_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

respectively. Take $l_1 = l_2 = 1$ in Method 1. From Theorem 4, $\boldsymbol{H}$ is an $OA(81, 9^4, 2)$ that can be partitioned into nine slices. Each $\boldsymbol{H}_i$ is an LH over F and $\phi(\boldsymbol{H}_i)$ is an $OA(9, 3^4, 2)$ for $i = 0, \ldots, 8$. For example, the transpose of $\boldsymbol{H}_0$ is

$$\begin{pmatrix} 0 & 1 & 2 & x & x+1 & x+2 & 2x & 2x+1 & 2x+2 \\ 0 & x & 2x & 2x+1 & 1 & x+1 & x+2 & 2x+2 & 2 \\ 0 & x+1 & 2x+2 & 1 & x+2 & 2x & 2 & x & 2x+1 \\ 0 & 2x+1 & x+2 & 2x+2 & x & 1 & x+1 & 2 & 2x \end{pmatrix},$$

and the transpose of $\phi(\boldsymbol{H}_0)$ is

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \end{pmatrix}.$$

If the generator matrix $\boldsymbol{Z}_1$ is applied in Method 2, the BSOAs with more slices can be obtained as in the following.

Theorem 5. *For the matrix $\boldsymbol{H}$ constructed in Method 2 with the generator matrix $\boldsymbol{Z}_1$, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^\lambda, s_1^m, 2)$ with $m = (s_1 - 1)/(s_2 - 1)$;*

(ii) *each slice $\boldsymbol{H}_{(l_1,\ldots,l_\lambda)}$ is an LH over F and all $\phi(\boldsymbol{H}_{(l_1,\ldots,l_\lambda)})$'s are the same $OA(s_1, s_2^m, 2)$ for all $(l_1, \ldots, l_\lambda) \in Q^\lambda$.*

Example 2. Let $p = 2, u_1 = 3$, and $u_2 = 1$, giving $s_1 = 8, s_2 = 2$, and $\lambda = 3$. Use $p_1(x) = x^3 + x + 1$ for $F = GF(8)$ with $\alpha = x$. Let G be the subfield $\{0, 1\}$ of F with $\beta = 1$. The kernel matrix of ϕ and the generator matrix are given by

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & x+1 & x^2+1 & x^2+x \\ 1 & x & x^2 & x^2+x+1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z}_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad (4.1)$$

respectively. From Theorem 5, $\boldsymbol{H}$ is known to be an $OA(512, 8^7, 2)$ that can be partitioned into 64 slices. For any $(l_1, l_2, l_3) \in Q^3$, the slice $\boldsymbol{H}_{(l_1,l_2,l_3)}$ is an LH over F. All $\phi(\boldsymbol{H}_{(l_1,l_2,l_3)})$'s are the same $OA(8, 2^7, 2)$ for all $(l_1, l_2, l_3) \in Q^3$.

4.2. Construction of BSOAs of strength $(2, \lambda)$

Motivated by Bush's method in Hedayat, Sloane, and Stufken (1999), the generator matrix is constructed as follows. For ease in presentation, let $\boldsymbol{e}_i$ be the λ-vector whose ith element is one and other elements are zero, $\boldsymbol{I}_k$ be the $k \times k$ identity matrix, and

$$\boldsymbol{W}_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta & \beta^2 & \cdots & \beta^{s_2-1} \\ \vdots & \vdots & & \vdots \\ \beta^{k-2} & \beta^{2(k-2)} & \cdots & \beta^{(s_2-1)(k-2)} \\ \beta^{k-1} & \beta^{2(k-1)} & \cdots & \beta^{(s_2-1)(k-1)} \end{pmatrix}.$$

According to the values of s_2 and λ, form a generator matrix as

$$\boldsymbol{Z}_2 = \begin{cases} (\boldsymbol{I}_\lambda, \boldsymbol{1}_\lambda) & \text{for } \lambda \geq s_2, \\ (\boldsymbol{I}_3, \boldsymbol{W}_3) & \text{for } \lambda = 3 \text{ and } s_2 \text{ is even}, \\ (\boldsymbol{W}_3', \boldsymbol{I}_{s_2-1}) & \text{for } \lambda = s_2 - 1 \text{ and } s_2 \text{ is even}, \\ (\boldsymbol{e}_1, \boldsymbol{e}_\lambda, \boldsymbol{W}_\lambda) & \text{otherwise}. \end{cases} \tag{4.2}$$

Then the following lemma is obtained, with proof in the Appendix.

Lemma 3. *Any $\lambda \times \lambda$ submatrix of the generator matrix $\boldsymbol{Z}_2$ in (4.2) has full rank over G.*

Combining Lemma 3 and Theorem 1, the following result is obtained.

Theorem 6. *For the matrix $\boldsymbol{H}$ constructed in Method 1 with the generator matrix $\boldsymbol{Z}_2$ in (4.2), we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^2, s_1^m, 2)$, where m is the number of columns of $\boldsymbol{Z}_2$;*

(ii) *each slice $\boldsymbol{H}_i$ is an LH over F and $\phi(\boldsymbol{H}_i)$ is an $OA(s_1, s_2^m, \lambda)$ for $i = 0, 1, \ldots, s_1 - 1$.*

From Corollaries 2.22 and 3.9 in Hedayat, Sloane, and Stufken (1999), it is known that the number $m = s_2 + 2$ is the largest m for an $OA(s_1, s_2^m, 3)$ to exist when s_2 is even, and the number $m = \lambda + 1$ is the largest m for an $OA(s_1, s_2^m, \lambda)$ to exist when $\lambda \geq s_2$.

Example 3. Let $p = 2, u_1 = 3$, and $u_2 = 1$, giving $s_1 = 8, s_2 = 2$, and $\lambda = 3$. Use $p_1(x) = x^3 + x + 1$ for $F = GF(8)$ with $\alpha = x$. Let G be the subfield $\{0, 1\}$ of F with $\beta = 1$. The kernel matrix $\boldsymbol{\Gamma}$ is given in (4.1) and the generator matrix is $\boldsymbol{Z}_2 = (\boldsymbol{I}_3, \boldsymbol{1}_3) = (\boldsymbol{I}_3, \boldsymbol{W}_3)$. Take $l_1 = l_2 = l_3 = 1$ in Method 1. From Theorem 6, $\boldsymbol{H}$ is an $OA(64, 8^4, 2)$ that can be partitioned into eight slices. Each slice $\boldsymbol{H}_i$ is an LH over F and $\phi(\boldsymbol{H}_i)$ is an $OA(8, 2^4, 3)$ for $i = 0, 1, \ldots, 7$.

4.3. Construction of BSOAs of strength (λ, λ)

This construction combines Method 2 with the generator matrix $\boldsymbol{Z}_2$ in (4.2). From Lemma 3 and Theorem 2, we have the following.

Theorem 7. *For the matrix $\boldsymbol{H}$ constructed in Method 2 with the generator matrix $\boldsymbol{Z}_2$ in (4.2), we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^\lambda, s_1^m, \lambda)$, where m is the number of columns of $\boldsymbol{Z}_2$;*

(ii) *each slice $\boldsymbol{H}_{(l_1,\ldots,l_\lambda)}$ is an LH over F and all $\phi(\boldsymbol{H}_{(l_1,\ldots,l_\lambda)})$'s are the same $OA(s_1, s_2^m, \lambda)$ for all $(l_1, \ldots, l_\lambda) \in Q^\lambda$.*

Example 4. Let $p = 2, u_1 = 3$, and $u_2 = 1$, giving $s_1 = 8, s_2 = 2$, and $\lambda = 3$. Use $p_1(x) = x^3 + x + 1$ for $F = GF(8)$ with $\alpha = x$. Let G be the subfield $\{0, 1\}$ of F with $\beta = 1$. The kernel matrix $\mathbf{\Gamma}$ of ϕ is given in (4.1) and the generator matrix is $\boldsymbol{Z}_2 = (\boldsymbol{I}_3, \mathbf{1}_3) = (\boldsymbol{I}_3, \boldsymbol{W}_3)$. From Theorem 7, $\boldsymbol{H}$ is an $OA(512, 8^4, 3)$ that can be partitioned into 64 slices. Each slice $\boldsymbol{H}_{(l_1,l_2,l_3)}$ is an LH over F and all $\phi(\boldsymbol{H}_{(l_1,l_2,l_3)})$'s are the same $OA(8, 2^4, 3)$ for all $(l_1, l_2, l_3) \in Q^3$.

From Lemma 3, we know that the rows of $\boldsymbol{Z}_2$ in (4.2) are linearly independent over G. According to Theorem 3, neither the BSOA obtained in this section nor its each projected slice under the subfield projection ϕ has repeated rows.

5. Construction of BSOAs with Modulus Projection

In this section, we construct BSOAs of strength $(2, 2)$ with the modulus projection φ in (2.3). Take $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$ as powers of the same prime p, where $u_1 \geq 2u_2 - 1$ and $u_1 \neq u_2$. Let $F = GF(s_1)$ have the primitive polynomial $p_1(x)$ and $G = GF(s_2)$ the primitive polynomial $p_2(x)$. Since s_1 is not a power of s_2, we can construct BSOAs with different parameters from those constructed in Section 4.

Write $\mathbf{\Gamma}$ as the $s_2 \times q$ kernel matrix of φ, where $q = s_1/s_2$. For $i, j = 1, \ldots, q$, let $\boldsymbol{B}_{ij} = \mathbf{\Gamma}(:, i)\, \mathbf{\Gamma}(:, j)'$. For $j = 1, \ldots, q$, obtain an array $\boldsymbol{B}_j$ by juxtaposing all the rows of $\boldsymbol{B}_{1j}, \ldots, \boldsymbol{B}_{qj}$, $\boldsymbol{B}_j = (\boldsymbol{B}'_{1j}, \ldots, \boldsymbol{B}'_{qj})'$. The proof of the following is given in Appendix.

Lemma 4. *For $\boldsymbol{B}_{ij}$'s and $\boldsymbol{B}_j$'s constructed above, we have*

(i) *each matrix $\boldsymbol{B}_j$ is a $D(s_1, s_2, s_1)$ for $j = 1, \ldots, q$;*
(ii) *each matrix $\varphi(\boldsymbol{B}_{ij})$ is a $D(s_2, s_2, s_2)$ for $i, j = 1, \ldots, q$.*

Qian and Wu (2009) call the matrix $\boldsymbol{B}_j$ in Lemma 4 a sliced difference matrix (SDM), a DM that can be partitioned into several slices and each slice becomes a DM after some level-collapsing.

Let $\boldsymbol{A}$ be an $OA(n, s_1^m, t)$ with $m = t = 1$ or $t \geq 2$, taking levels from F. For a fixed $1 \leq j \leq q$, put

$$\boldsymbol{H}_j = \boldsymbol{A} \oplus \boldsymbol{B}_j \quad \text{and} \quad \boldsymbol{H}_{ij} = \boldsymbol{A} \oplus \boldsymbol{B}_{ij},$$

for $i = 1, \ldots, q$. Obviously, $\boldsymbol{H}_j = (\boldsymbol{H}'_{1j}, \ldots, \boldsymbol{H}'_{qj})'$. Here $\varphi(\boldsymbol{A})$ is an $OA(n, s_2^m, t)$ with $m = t = 1$ or $t \geq 2$. By using Lemma 1 and Lemma 4, we have the following.

Theorem 8. *For the matrix $\boldsymbol{H}_j$ constructed above, we have*

(i) *the matrix $\boldsymbol{H}_j$ is an $OA(ns_1, s_1^{ms_2}, 2)$;*
(ii) *each slice $\boldsymbol{H}_{ij}$ is balanced and $\varphi(\boldsymbol{H}_{ij})$ is an $OA(ns_2, s_2^{ms_2}, 2)$ for $i = 1, \ldots, q$.*

Example 5. Let $p = 2, u_1 = 3$, and $u_2 = 2$, giving $s_1 = 8$ and $s_2 = 4$. Use $p_1(x) = x^3 + x + 1$ for $F = GF(8)$ and $p_2(x) = x^2 + x + 1$ for $G = GF(4)$. The modulus projection φ is $\{0, x^2 + x + 1\} \to 0$, $\{1, x^2 + x\} \to 1$, $\{x, x^2 + 1\} \to x$, $\{x + 1, x^2\} \to x + 1$, and the kernel matrix $\mathbf{\Gamma}$ of φ in transpose is

$$\begin{pmatrix} 0 & 1 & x & x+1 \\ x^2+x+1 & x^2+x & x^2+1 & x^2 \end{pmatrix}. \tag{5.1}$$

From Lemma 4, the matrix $\boldsymbol{B}_1 = (\boldsymbol{B}'_{11}, \boldsymbol{B}'_{21})'$ is a $D(8, 4, 8)$, while $\varphi(\boldsymbol{B}_{11})$ and $\varphi(\boldsymbol{B}_{21})$ are $D(4, 4, 4)$. Let $\boldsymbol{A}$ be an $OA(8, 8^1, 1)$. From Theorem 8, $\boldsymbol{H}_1$ is an $OA(64, 8^4, 2)$, each slice $\boldsymbol{H}_{i1}$ is balanced and $\varphi(\boldsymbol{H}_{i1})$ is an $OA(32, 4^4, 2)$ for $i = 1, 2$.

Theorem 8 works for $u_1 = \lambda u_2$ $(\lambda > 1)$, but the BSOAs constructed by applying Theorem 8 are much larger than those in Section 4 for the same number of slices.

When $u_1 = 2u_2 - 1$ and $u_2 \neq 1$, in particular, a different construction is provided in the following to yield new BSOAs with more slices than those in Theorem 8. For $i, j = 1, \ldots, q$, let $\boldsymbol{H}_{ij} = \mathbf{\Gamma}(:, i) \oplus \boldsymbol{B}_{j2}$. Take $\boldsymbol{H}$ to be the row juxtaposition of all $\boldsymbol{H}_{ij}$'s. A detailed proof of the following is given in the Appendix.

Theorem 9. *If $u_1 = 2u_2 - 1$ and $u_2 \neq 1$, for the matrix $\boldsymbol{H}$ constructed above we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^2, s_1^{s_2}, 2)$;*

(ii) *each slice $\boldsymbol{H}_{ij}$ is balanced and $\varphi(\boldsymbol{H}_{ij})$ is an $OA(s_2^2, s_2^{s_2}, 2)$ for $i, j = 1, \ldots, q$.*

Example 6. Let $p = 2, u_1 = 3$, and $u_2 = 2$, giving $s_1 = 8$ and $s_2 = 4$. Use $p_1(x) = x^3 + x + 1$ for $F = GF(8)$ and $p_2(x) = x^2 + x + 1$ for $G = GF(4)$. The kernel matrix $\mathbf{\Gamma}$ of φ is given in (5.1). From Theorem 9, $\boldsymbol{H}$ is an $OA(64, 8^4, 2)$ that can be partitioned into four slices. Each slice $\boldsymbol{H}_{ij}$ is balanced and $\varphi(\boldsymbol{H}_{ij})$ is an $OA(16, 4^4, 2)$ for $i, j = 1, 2$.

6. Construction of BSOAs with a Nonprime Power Number of Levels

In this section, we construct BSOAs with a nonprime power number of levels by using the projection ρ in (2.4). Take $s_1 > s_2 > 1$ with $s_2 | s_1$ and $t \geq 2$. Let $F = \{0, 1, \ldots, s_1 - 1\}$ be the residue ring modulo s_1 and $G = \{0, 1, \ldots, s_2 - 1\}$ be the residue ring modulo s_2.

First construct an OA of strength t with levels from F as follows. Let $\boldsymbol{A}$ be an $s_1^t \times t$ array that has each of the s_1^t possible t-tuples from F as a row. Obtain a new column vector $\boldsymbol{b}$ by adding the entries in each row of $\boldsymbol{A}$. Then $\boldsymbol{H} = (\boldsymbol{A}, \boldsymbol{b})$ is an $OA(s_1^t, s_1^{t+1}, t)$ (Hedayat, Sloane, and Stufken (1999)).

Let $Q = \{1, \ldots, q\}$ with $q = s_1/s_2$ and $\mathbf{\Gamma}$ be the kernel matrix of ρ. For any $(l_1, \ldots, l_t) \in Q^t$, take $\boldsymbol{H}_{(l_1,\ldots,l_t)}$ to be the $s_2^t \times (t+1)$ submatrix of $\boldsymbol{H}$ consisting of all the rows $(h_1, \ldots, h_t, h_{t+1})$'s, where h_i is an element of $\mathbf{\Gamma}(:, l_i)$ for $i = 1, \ldots, t$ and h_{t+1} is the sum of the first t elements. Then divide all the t-tuples of Q^t into q^{t-1} groups, each of size q, in such a way that two t-tuples $(l_1, \ldots, l_t)$ and $(l'_1, \ldots, l'_t)$ are in the same group if and only if there is a fixed $a \in Q$ such that $l'_i - l_i = a \pmod q$ for all $1 \le i \le t$. Let $\boldsymbol{u}_{i1}, \ldots, \boldsymbol{u}_{iq}$ be all the t-tuples in the ith group, for $i = 1, \ldots, q^{t-1}$. Obtain an array $\boldsymbol{H}_i$ by juxtaposing all the rows of $\boldsymbol{H}_{\boldsymbol{u}_{i1}}, \ldots, \boldsymbol{H}_{\boldsymbol{u}_{iq}}$, $\boldsymbol{H}_i = (\boldsymbol{H}'_{\boldsymbol{u}_{i1}}, \ldots, \boldsymbol{H}'_{\boldsymbol{u}_{iq}})'$. Clearly, the $\boldsymbol{H}_i$'s form a row partition of $\boldsymbol{H}$. The proof of the following is given in Appendix.

Theorem 10. *For the matrix $\boldsymbol{H}$ constructed above, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^t, s_1^{t+1}, t)$;*

(ii) *each slice $\boldsymbol{H}_i$ is balanced and $\rho(\boldsymbol{H}_i)$ is an $OA(s_2^t q, s_2^{t+1}, t)$ for $i = 1, \ldots, q^{t-1}$.*

Example 7. Let $s_1 = 6$, $s_2 = 3$, and $t = 3$. Use $F = \{0, 1, \ldots, 5\}$ and $G = \{0, 1, 2\}$. The projection ρ is given as $\{0, 3\} \to 0$, $\{1, 4\} \to 1$, $\{2, 5\} \to 2$, and the kernel matrix $\mathbf{\Gamma}$ of ρ in transpose is

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}.$$

Let $\boldsymbol{H}$ be an $OA(216, 6^4, 3)$, where the last column is the sum of the first three columns over F. Then $\boldsymbol{H}_1 = (\boldsymbol{H}'_{(1,1,1)}, \boldsymbol{H}'_{(2,2,2)})'$, $\boldsymbol{H}_2 = (\boldsymbol{H}'_{(1,1,2)}, \boldsymbol{H}'_{(2,2,1)})'$, $\boldsymbol{H}_3 = (\boldsymbol{H}'_{(1,2,1)}, \boldsymbol{H}'_{(2,1,2)})'$, and $\boldsymbol{H}_4 = (\boldsymbol{H}'_{(1,2,2)}, \boldsymbol{H}'_{(2,1,1)})'$ form a partition of $\boldsymbol{H}$. From Theorem 10, each $\boldsymbol{H}_i$ is balanced and $\rho(\boldsymbol{H}_i)$ is an $OA(54, 3^4, 3)$ for $i = 1, \ldots, 4$.

When $s_2^t = \tau s_1$ with the integer $\tau \ge 1$, we present another construction which yields new BSOAs with more slices than those in Theorem 10. Let $\boldsymbol{A}_0$ be an $OA(s_2^t, s_2^{t+1}, t)$ with entries from G. For each column of $\boldsymbol{A}_0$, replace the s_2^{t-1} entries of level i with the elements of $\mathbf{\Gamma}(i+1, :)$ in such a way that each element of $\mathbf{\Gamma}(i+1, :)$ appears τ times, for $i = 0, 1, \ldots, s_2 - 1$. After such replacement, denote by $\boldsymbol{A}$ the resulting matrix. For any $(l_1, \ldots, l_t) \in Q^t$, let $\boldsymbol{v}_{(l_1,\ldots,l_t)}$ be a $(t+1)$-vector, where the first t elements are $\mathbf{\Gamma}(1, l_1), \ldots, \mathbf{\Gamma}(1, l_t)$ and the last one is the sum of the first t elements over F, and $\boldsymbol{H}_{(l_1,\ldots,l_t)} = \boldsymbol{A} + \mathbf{1}_{s_2^t} \boldsymbol{v}'_{(l_1,\ldots,l_t)}$. Finally, obtain an array $\boldsymbol{H}$ by the row juxtaposition of all $\boldsymbol{H}_{(l_1,\ldots,l_t)}$'s. Then the following result is obtained, with proof in the Appendix.

Theorem 11. *If $s_2^t = \tau s_1$ with $\tau \ge 1$, for the matrix $\boldsymbol{H}$ constructed above, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(s_1^t, s_1^{t+1}, t)$;*

(ii) *each slice $\boldsymbol{H}_{(l_1,\ldots,l_t)}$ is balanced and $\rho(\boldsymbol{H}_{(l_1,\ldots,l_t)})$ is an $OA(s_2^t, s_2^{t+1}, t)$ for any $(l_1, \ldots, l_t) \in Q^t$.*

Example 8. Let $s_1 = 18$, $s_2 = 6$, and $t = 2$. Use $F = \{0, 1, \ldots, 17\}$ and $G = \{0, 1, \ldots, 5\}$. From Theorem 11, $\boldsymbol{H}$ is an $OA(324, 18^3, 2)$ that can be partitioned into nine slices. Each slice $\boldsymbol{H}_{(l_1,l_2)}$ is balanced and $\rho(\boldsymbol{H}_{(l_1,l_2)})$ is an $OA(36, 6^3, 2)$ for any $(l_1, l_2) \in Q^2$.

7. Obtain new BSOAs from Existing SOAs

Now we discuss a procedure for constructing new BSOAs from existing SOAs by taking the Kronecker sum of an SOA and a DM. Suppose $\boldsymbol{A} = (\boldsymbol{A}'_1, \ldots, \boldsymbol{A}'_v)'$ is an $OA(n_1, s_1^m, t)$ with $m = t = 1$ or $t = 2$, taking levels from an abelian group F, and that there is a level-collapsing projection δ such that each $\delta(\boldsymbol{A}_i)$ is an $OA(n_2, s_2^m, t)$ with entries from an abelian group G, for $i = 1, \ldots, v$. Let $\boldsymbol{B}$ be a $D(r, c, s_1)$ with entries from F. Put $\boldsymbol{H} = \boldsymbol{A} \oplus \boldsymbol{B}$ and $\boldsymbol{H}_i = \boldsymbol{A}_i \oplus \boldsymbol{B}$, for $i = 1, \ldots, v$. Obviously, $\boldsymbol{H} = (\boldsymbol{H}'_1, \ldots, \boldsymbol{H}'_v)'$.

Theorem 12. *If each $\boldsymbol{A}_i$ is balanced or $\boldsymbol{B}$ is balanced, for the matrix $\boldsymbol{H}$ constructed above, we have*

(i) *the matrix $\boldsymbol{H}$ is an $OA(n_1 r, s_1^{mc}, 2)$;*

(ii) *each slice $\boldsymbol{H}_i$ is balanced and $\delta(\boldsymbol{H}_i)$ is an $OA(n_2 r, s_2^{mc}, 2)$ for $i = 1, \ldots, v$.*

This theorem can be readily proved by following Lemma 1 and the definition of δ. If $\boldsymbol{B}$ is a balanced $D(r, c, s_1)$, all the SOAs constructed by Qian and Wu (2009) can be used as the matrix $\boldsymbol{A}$ in Theorem 12. Otherwise, by subtracting the first column from all the columns of $\boldsymbol{B}$ and deleting the first column, the remaining matrix is a balanced $D(r, c-1, s_1)$.

Here we give one detailed construction with the modulus projection φ by using Theorem 12. Take $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$, where $u_1 > u_2 \geq 1$. Let $F = GF(s_1)$ have a primitive polynomial $p_1(x)$ and $G = GF(s_2)$ a primitive polynomial $p_2(x)$. Let $\boldsymbol{\Gamma}$ be the $s_2 \times q$ kernel matrix of φ, where $q = s_1/s_2$. For $i = 1, \ldots, q$, let $\boldsymbol{A}_i = \boldsymbol{\Gamma}(:, i)$. Obtain an SOA $\boldsymbol{A}$ with only one column by the row juxtaposition of all $\boldsymbol{A}_i$'s, $\boldsymbol{A} = (\boldsymbol{A}'_1, \ldots, \boldsymbol{A}'_q)'$. Take $\boldsymbol{B}$ to be the columns of the multiplication table of F labeled with all the nonzero elements of F, which is a balanced $D(s_1, s_1 - 1, s_1)$. Put $\boldsymbol{H} = \boldsymbol{A} \oplus \boldsymbol{B}$ and $\boldsymbol{H}_i = \boldsymbol{A}_i \oplus \boldsymbol{B}$ for $i = 1, \ldots, q$. From Theorem 12, $\boldsymbol{H}$ is an $OA(s_1^2, s_1^{s_1-1}, 2)$, each $\boldsymbol{H}_i$ is balanced, and $\varphi(\boldsymbol{H}_i)$ is an $OA(s_1 s_2, s_2^{s_1-1}, 2)$ for $i = 1, \ldots, q$.

8. Generation of Sliced Space-Filling Designs Based on BSOAs

In this section, we use the BSOAs constructed in the previous sections to generate sliced space-filling designs. The randomization approach is different from those of Tang (1993), Qian and Wu (2009), and Qian and Ai (2010).

Suppose $\boldsymbol{H} = (\boldsymbol{H}'_1, \ldots, \boldsymbol{H}'_v)'$ is a BSOA, where $\boldsymbol{H}$ is an $OA(n_1, s_1^m, t_1)$, each $\boldsymbol{H}_i$ is balanced and becomes an $OA(n_2, s_2^m, t_2)$ after the s_1 levels of $\boldsymbol{H}$

are collapsed to s_2 levels according to some level-collapsing projection δ. This randomization approach proceeds as follows.

Step 1. The projection δ divides the s_1 levels of $\boldsymbol{H}$ into s_2 groups, each of size $q = s_1/s_2$, and two levels are in the same group if and only if they are projected to the same one. Arbitrarily label the s_2 groups as groups $1, 2, \ldots, s_2$, and then relabel the q levels within the lth group as a random permutation of $\{(l-1)q+1, \ldots, (l-1)q+q\}$ for $l = 1, \ldots, s_2$. Now the levels of $\boldsymbol{H}$ are $1, \ldots, s_1$.

Step 2. Let $w = n_1/s_1$ and $e = n_2/s_1$. For $l = 1, \ldots, s_1$, let $\boldsymbol{M}_l$ be the $e \times v$ matrix given by

$$\begin{pmatrix} (l-1)w+1 & (l-1)w+2 & \cdots & (l-1)w+v \\ (l-1)w+v+1 & (l-1)w+v+2 & \cdots & (l-1)w+2v \\ \vdots & \vdots & \vdots & \vdots \\ (l-1)w+(e-1)v+1 & (l-1)w+(e-1)v+2 & \cdots & (l-1)w+w \end{pmatrix}.$$

For $k = 1, \ldots, m$, by randomly shuffling the entries in each row of $\boldsymbol{M}_l$ and then randomly shuffling the entries in each column, obtain a new matrix $\boldsymbol{M}_{lk}$. Replace the e entries of level l in the kth column of $\boldsymbol{H}_i$ with the e elements of $\boldsymbol{M}_{lk}(:, i)$ for $i = 1, \ldots, v$. After such replacement is done for all the columns of $\boldsymbol{H}$, let $\boldsymbol{L} = (\boldsymbol{L}_1', \ldots, \boldsymbol{L}_v')'$ be the resulting matrix, where $\boldsymbol{L}_i$ is the submatrix of $\boldsymbol{L}$ corresponding to $\boldsymbol{H}_i$ for $i = 1, \ldots, v$.

Step 3. Generate an $n_1 \times m$ matrix $\boldsymbol{D} = (d_{ij})$ by letting $d_{ij} = (l_{ij} - u_{ij})/n_1$, where l_{ij} is the (i, j)th entry of $\boldsymbol{L}$ and u_{ij}'s are independent random variables with uniform distributions on $(0, 1]$. Denote by $\boldsymbol{D}_i$ the submatrix of $\boldsymbol{D}$ corresponding to $\boldsymbol{L}_i$ for $i = 1, \ldots, v$.

From Step 2, it can be seen that $\boldsymbol{L}$ is the matrix obtained by replacing the w entries of level l in each column of $\boldsymbol{H}$ with a permutation of $\{(l-1)w + 1, \ldots, (l-1)w + w\}$ for $l = 1, \ldots, s_1$. Thus, $\boldsymbol{L}$ is an LH based on $OA(n_1, s_1^m, t_1)$. Similarly, it can be shown that for $i = 1, \ldots, v$, each $\boldsymbol{L}_i$ becomes an LH based on $OA(n_2, s_2^m, t_2)$ after the level z of $\boldsymbol{L}$ is collapsed to $\lceil z/v \rceil$ for $z = 1, \ldots, n_1$, where $\lceil a \rceil$ is the smallest integer not less than a.

Theorem 13. *For the design $\boldsymbol{D} = (\boldsymbol{D}_1', \ldots, \boldsymbol{D}_v')'$ obtained above, we have*

(i) *the design $\boldsymbol{D}$ and each slice $\boldsymbol{D}_i$ achieve maximum stratification in any one-dimensional projection;*

(ii) *when projected onto any g $(\leq t_1)$ dimensions, the design $\boldsymbol{D}$ achieves the stratification on the s_1^g grids;*

Table 1. The matrix $\boldsymbol{H}$ in Example 9.

$\boldsymbol{H}_{11}$				$\boldsymbol{H}_{12}$				$\boldsymbol{H}_{21}$				$\boldsymbol{H}_{22}$			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
1	1	1	1	7	8	4	3	2	2	2	2	8	7	3	4
2	4	6	8	8	5	7	6	1	3	5	7	7	6	8	5
6	2	3	7	4	7	2	5	5	1	4	8	3	8	1	6
5	3	8	2	3	6	5	4	6	4	7	1	4	5	6	3
3	3	3	3	5	6	2	1	4	4	4	4	6	5	1	2
4	2	8	6	6	7	5	8	3	1	7	5	5	8	6	7
8	4	1	5	2	5	4	7	7	3	2	6	1	6	3	8
7	1	6	4	1	8	7	2	8	2	5	3	2	7	8	1
5	5	5	5	3	4	8	7	6	6	6	6	4	3	7	8
6	8	2	4	4	1	3	2	5	7	1	3	3	2	4	1
2	6	7	3	8	3	6	1	1	5	8	4	7	4	5	2
1	7	4	6	7	2	1	8	2	8	3	5	8	1	2	7
7	7	7	7	1	2	6	5	8	8	8	8	2	1	5	6
8	6	4	2	2	3	1	4	7	5	3	1	1	4	2	3
4	8	5	1	6	1	8	3	3	7	6	2	5	2	7	4
3	5	2	8	5	4	3	6	4	6	1	7	6	3	4	5

(iii) *when projected onto any g $(\leq t_2)$ dimensions, each slice $\boldsymbol{D}_i$ achieves the stratification on the s_2^g grids, for $i = 1, \ldots, v$.*

Example 9. Consider the BSOA $\boldsymbol{H} = (\boldsymbol{H}_{11}', \boldsymbol{H}_{12}', \boldsymbol{H}_{21}', \boldsymbol{H}_{22}')'$ constructed in Example 6, where $\boldsymbol{H}$ is an $OA(64, 8^4, 2)$, each slice $\boldsymbol{H}_{ij}$ is balanced and $\varphi(\boldsymbol{H}_{ij})$ is an $OA(16, 4^4, 2)$ for $i, j = 1, 2$. According to Step 1 of the randomization approach, we first relabel the levels 0 and $x^2 + x + 1$ as 1 and 2, the levels 1 and $x^2 + x$ as 3 and 4, the levels x and $x^2 + 1$ as 5 and 6, and the levels $x + 1$ and x^2 as 7 and 8, respectively. Table 1 presents the array $\boldsymbol{H}$ and the four slices after such relabeling is carried out. Next, we use this $\boldsymbol{H}$ to generate an LHD, denoted by $\boldsymbol{D}^{new}$, according to Steps 2 and 3. The four columns of $\boldsymbol{D}^{new}$ are represented by the factors x_1, x_2, x_3, and x_4, respectively. Theorem 13 shows that for the design $\boldsymbol{D}^{new}$, each of the 64 intervals of the form $[(i-1)/64, i/64)$ $(i = 1, \ldots, 64)$ contains exactly one point when projected onto any univariate margin, and each of the 64 grids of the form $[(i-1)/8, i/8) \times [(j-1)/8, j/8)$ $(i, j = 1, \ldots, 8)$ contains exactly one point when projected onto any bivariate margin. Furthermore, for any slice of $\boldsymbol{D}^{new}$, each of the 16 intervals of the form $[(i-1)/16, i/16)$ $(i = 1, \ldots, 16)$ contains exactly one point when projected onto any univariate margin, and each of the 16 grids of the form $[(i-1)/4, i/4) \times [(j-1)/4, j/4)$ $(i, j = 1, \ldots, 4)$ contains exactly one point when projected onto any bivariate margin.

Now consider the other two sliced space-filling designs with the same parameters. The first one is given by Qian and Wu (2009), constructed by randomizing

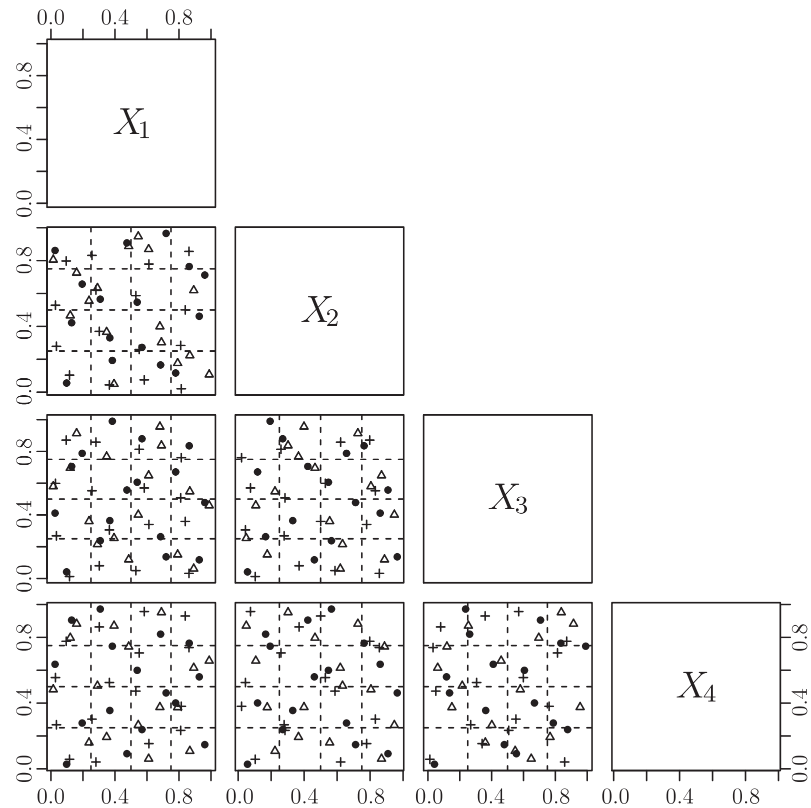

Figure 1. Bivariate projections of $\boldsymbol{D}_1^{new}$, $\boldsymbol{D}_1^{QW}$ and $\boldsymbol{D}_1^{Q}$ corresponding to the symbols $\bullet$, $+$ and $\triangle$, respectively, in Example 9.

an $SOA(64, 8^5, 2)$ with four slices. For comparison, we denote by $\boldsymbol{D}^{QW}$ the subdesign consisting of the first four columns of the design. The second design is a sliced LHD, denoted by $\boldsymbol{D}^{Q}$, constructed by using the method in Qian (2012). Figure 1 depicts the bivariate projections of the first slices of $\boldsymbol{D}^{new}, \boldsymbol{D}^{QW}$, and $\boldsymbol{D}^{Q}$. The three slices are denoted by $\boldsymbol{D}_1^{new}, \boldsymbol{D}_1^{QW}$, and $\boldsymbol{D}_1^{Q}$, respectively. It is shown that $\boldsymbol{D}_1^{new}$ and $\boldsymbol{D}_1^{Q}$ achieve maximum stratification in any univariate margin, and $\boldsymbol{D}_1^{new}$ and $\boldsymbol{D}_1^{QW}$ achieve maximum stratification in any bivariate margin. Similar results can be obtained for any other slice. Thus, $\boldsymbol{D}_1^{new}$ and also $\boldsymbol{D}^{new}$ are preferred.

9. Discussions

This article proposes different methods for constructing sliced space-filling designs for more flexible numbers of runs and slices in which the whole design and each slice not only achieve maximum stratification in univariate margins, but also achieve stratification in two- or more-dimensional margins. The construction of these new designs is based on balanced sliced orthogonal arrays (BSOAs). BSOAs are different from the SOAs of Qian and Wu (2009) in that each slice is balanced. These designs are intended for designing computer experiments with both qualitative and quantitative factors or with multiple models. They are also suitable for designing computer experiments with two codes of different levels of accuracy, as the whole design and any slice constitute a nested space-filling design (Qian, Tang, and Wu (2009), and Qian, Ai, and Wu (2009)).

The naive method to construct a BSOA is as follows. One can begin with an $OA(n, s^m, t)$ and obtain some isomorphic OAs by permuting its levels. Then a BSOA can be constructed by simply juxtaposing these isomorphic OAs. Note that no level projection is needed in such a BSOA. The sliced space-filling design based on this BSOA can also be generated by following the three steps in Section 8. It is known that the whole design and each slice can achieve stratification on the s^g grids in all g-dimensional projections for all $g \leq t$. Recall that for the sliced designs constructed in this paper, the whole design can achieve stratification on smaller grids than each slice does. That is to say, the whole design can spread the design points more uniformly on the design space. So for the same parameters, the newly constructed designs are superior to those generated through this naive method.

Since it is possible to generate a great many of sliced space-filling designs based on a given BSOA, we can further use the distance, correlation, or other optimal criteria to find optimal designs. Moreover, some methods of Qian, Ai, and Wu (2009) can be adapted to construct BSOAs with mixed levels, which allow the factors to have different levels of uniformity.

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