Supplementary material on proofs of "Inference for Structural Breaks in Spatial Models"

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Before proving the main theorems, we introduce some elementary lemmas first. Let $\mu_{ij} = \mu(i/n_1, j/n_2)$, $\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0])$ and $S_i^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0])$ be defined as $S_i([\boldsymbol{n}\cdot\boldsymbol{t}_0])$ by replacing Y_{ij} by $Y_{ij}-\mu_{ij}=\varepsilon_{ij}$ and μ_{ij} respectively. For example, $\bar{S}_1([\boldsymbol{n}\cdot\boldsymbol{t}_0]) = \frac{1}{k_n^2}\sum_{i=i_0}^{i_0+k_n}\sum_{j=j_0}^{j_0+k_n}\varepsilon_{ij}$ and $S_1^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]) = \frac{1}{k_n^2}\sum_{i=i_0}^{i_0+k_n}\sum_{j=j_0}^{j_0+k_n}\mu_{ij}$. Then

$$\begin{aligned} & T([\boldsymbol{n}\cdot\boldsymbol{t}_0]) \\ = & \sum_{i=1}^{4}(\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - \bar{S}_{i+1}([\boldsymbol{n}\cdot\boldsymbol{t}_0]))^2 + \sum_{i=1}^{4}(S_i^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - S_{i+1}^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]))^2 \\ & +2\sum_{i=1}^{4}(\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - \bar{S}_{i+1}([\boldsymbol{n}\cdot\boldsymbol{t}_0]))(S_i^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - S_{i+1}^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0])). \quad (S.1) \end{aligned}$$

We first establish the joint convergence of $(\bar{S}_1([\boldsymbol{n}\cdot\boldsymbol{t}_0]),\ldots,\bar{S}_4([\boldsymbol{n}\cdot\boldsymbol{t}_0]))$.

Lemma 1. *Under Conditions of Theorem 1,*

$$k_n(\bar{S}_1([\boldsymbol{n}\cdot\boldsymbol{t}_0]),\ldots,\bar{S}_4([\boldsymbol{n}\cdot\boldsymbol{t}_0])) \xrightarrow{d} \sigma\mathbf{X},$$

where $\mathbf{X} = (X_1,\ldots,X_4)$ *and* $X_1,\cdots,X_4$ *are i.i.d. standard normal variables.*

Proof. By Cramér-Wold device, it suffices to show that for any real numbers $a_1, \ldots, a_4$,

$$\sum_{i=1}^{4} a_i k_n \bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0]) \xrightarrow{d} \sigma \sum_{i=1}^{4} a_i X_i. \tag{S.2}$$

Note that when all the $a_i = 0$, (S.2) holds. Without loss of generality, we assume at least one of the $a_i \neq 0$. By Proposition 2 of El Machkouri, Volný and Wu (2013) (EVW, hereafter), we can show that as $n \to \infty$,

$$\mathrm{E}\Big(a_i k_n \bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0])\Big)^2 \longrightarrow a_i^2 \sum_{\boldsymbol{j} \in \mathbb{Z}^2} \mathrm{E}(\varepsilon_{\boldsymbol{0}} \varepsilon_{\boldsymbol{j}}) = a_i^2 \sigma^2, \quad \text{for } i = 1, \ldots, 4. \tag{S.3}$$

If $\sigma = 0$, then (S.2) follows directly by (S.3). Next, we assume $\sigma \neq 0$. Let $\mathcal{F}_m(\boldsymbol{i}) = \sigma(\eta_{\boldsymbol{j}}, ||\boldsymbol{i} - \boldsymbol{j}|| \leq m)$, $\varepsilon_{\boldsymbol{i}}(m) = \mathrm{E}(\varepsilon_{\boldsymbol{i}} | \mathcal{F}_m(\boldsymbol{i}))$ be the project of $\varepsilon_{\boldsymbol{i}}$ on $\mathcal{F}_m(\boldsymbol{i})$ and $\bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0], m)$ be defined as $\bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0])$ with $\varepsilon_{ij}(m)$ instead of ε_{ij}. Take $m = m_n$ as in Lemma 3 of EVW, then by their Proposition 3 and Lemma 2, if $m_n \to \infty$,

$$\begin{aligned} & \mathrm{E}\left[\sum_{i=1}^{4} a_i k_n \{\bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0]) - \bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0], m_n)\}\right]^2 \\ \leq \;& 16 \sum_{i=1}^{4} a_i^2 \mathrm{E}\{k_n(\bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0]) - \bar{S}_i([\boldsymbol{n} \cdot \boldsymbol{t}_0], m_n))\}^2 \to 0. \end{aligned} \tag{S.4}$$

Define

$$\begin{aligned}
\tilde{S}_1([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) &= \frac{1}{k_n^2}\sum_{i=i_0+m_n}^{i_0+k_n}\sum_{j=j_0+m_n}^{j_0+k_n}\varepsilon_{ij}(m_n),\\
\tilde{S}_2([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) &= \frac{1}{k_n^2}\sum_{i=i_0-k_n}^{i_0-m_n}\sum_{j=j_0+m_n}^{j_0+k_n}\varepsilon_{ij}(m_n),\\
\tilde{S}_3([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) &= \frac{1}{k_n^2}\sum_{i=i_0-k_n}^{i_0-m_n}\sum_{j=j_0-k_n}^{j_0-m_n}\varepsilon_{ij}(m_n),\\
\tilde{S}_4([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) &= \frac{1}{k_n^2}\sum_{i=i_0+m_n}^{i_0+k_n}\sum_{j=j_0-k_n}^{j_0-m_n}\varepsilon_{ij}(m_n).
\end{aligned}$$

By the stationarity of $\{\varepsilon_{ij}\}$ and $m_n^{12}/k_n \to 0$ (see Lemma 3 of EVW), we have

$$\mathrm{E}\{k_n(\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) - \tilde{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n))\}^2 \le (m_n^4/k_n^2)\mathrm{E}\{\varepsilon_{11}\}^2 \to 0. \quad \text{(S.5)}$$

Thus, by (S.4) and (S.5), it suffices to show

$$\sum_{i=1}^{4} a_i k_n \tilde{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) \xrightarrow{d} \sigma \sum_{i=1}^{4} a_i X_i. \quad \text{(S.6)}$$

Since $\{\varepsilon_{ij}(m_n)\}$ is an m_n dependent field, by the central limit theory (CLT) of m dependent random field (see Theorem 2 of Heinrich (1988)), we have as $n \to \infty$,

$$k_n\{\tilde{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) \xrightarrow{d} N(0, \sigma^2),\ i = 1, \ldots, 4, \quad \text{(S.7)}$$

more details can be found in Theorem 1 of EVW. By the definition of $\varepsilon_{ij}(m_n)$, we have that $\tilde{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n)$, $i = 1, \ldots, 4$ are independent each other. Thus, from (S.7), it follows that

$$\sum_{i=1}^{4} a_i k_n \tilde{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0], m_n) \xrightarrow{d} N(0, (a_1^2+\cdots+a_4^2)\sigma^2). \tag{S.8}$$

On the other hand, since $X_1, \ldots, X_4$ are i.i.d normal variables, it follows that

$$\sigma\sum_{i=1}^{4} a_i X_i \stackrel{d}{=} N(0, (a_1^2+\cdots+a_4^2)\sigma^2).$$

Thus, by (S.8), we have (S.6) and complete the proof of the lemma. □

Proof of Theorem 1. We first show (i). By Lemma 1 and the continuous mapping theorem,

$$k_n^2\sum_{i=1}^{4}(\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - \bar{S}_{i+1}([\boldsymbol{n}\cdot\boldsymbol{t}_0]))^2 \xrightarrow{d} \sigma^2\sum_{i=1}^{4}(X_i - X_{i+1})^2. \tag{S.9}$$

Let $\boldsymbol{Z} = (Z_1, Z_2, Z_3, Z_4)^T = \Sigma(X_1, X_2, X_3, X_4)^T$, where

$$\Sigma = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\boldsymbol{Z}$ is a multivariate normal vector with mean zero and covariance $A = \Sigma I \Sigma'$ and $\sum_{i=1}^4 (X_i - X_{i+1})^2 = \sum_{i=1}^4 Z_i^2$.

On the other hand, suppose that $\boldsymbol{t}_0 = (t_1^0, t_2^0) \in B_i$ and D_i, $i = 1, \ldots, 4$ are the blocks with central $(n_1 t_1^0, n_2 t_2^0)$ and block length k_n (see Introduction section for the definition of D_i), then on $D = \bigcup_{i=1}^{4} D_i$,

$$\begin{aligned} \max_{\boldsymbol{s},\boldsymbol{t} \in D\cap(B_i\setminus\partial B_i)} \|\mu(\boldsymbol{s}) - \mu(\boldsymbol{t})\| &= \max_{\boldsymbol{s},\boldsymbol{t} \in D\cap(B_i\setminus\partial B_i)} \|f_i(\boldsymbol{s}) - f_i(\boldsymbol{t})\| \\ &\leq C\frac{k_n^{\alpha}(n_1+n_2)^{\alpha}}{n^{\alpha}}, \end{aligned} \tag{S.10}$$

where $B_i \setminus \partial B_i = \{\mathbf{s} : \mathbf{s} \in B_i, \text{ but } \mathbf{s} \notin \partial B_i\}$. Similarly, we have

$$\max_{\boldsymbol{s},\boldsymbol{t} \in D\cap(B_i^c\setminus\partial B_i)} \|\mu(\boldsymbol{s}) - \mu(\boldsymbol{t})\| \leq C\frac{k_n^{\alpha}(n_1+n_2)^{\alpha}}{n^{\alpha}}. \tag{S.11}$$

Since $k_n^{1+\alpha}(n_1+n_2)^{\alpha}/n^{\alpha} \to 0$, it follows from equations (S.10) and (S.11) that $\lim_{n\to\infty} S_i^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0])$ exists for all $i = 1, \ldots, 4$. Further, by Lemma 1,

$$\begin{aligned} & k_n^2\Big|\sum_{i=1}^{4}(\bar{S}_i([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - \bar{S}_{i+1}([\boldsymbol{n}\cdot\boldsymbol{t}_0]))(S_i^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - S_{i+1}^{\mu}([\boldsymbol{n}\cdot\boldsymbol{t}_0]) - (\mu_i - \mu_{i+1}))\Big| \\ = \;& o_p(1). \end{aligned} \tag{S.12}$$

Thus, conclusion (i) follows by equations (S.1), (S.9)–(S.12).

Next, we show (ii). By Theorem 1(i), the limit distribution of each block $T_n(\boldsymbol{i}k_n)$ is determined by the first term or the third term of equation (2.5) for continuous case or the structural break case respectively. Thus, the limit distribution of G_n does not depend on the second term of equation (2.5). As a result, $G_n - \mathrm{E}G_n$ has the same limit distribution as that of $\bar{G}_n - \mathrm{E}\bar{G}_n$, where $\bar{G}_n$ is defined as G_n with $Y_{\boldsymbol{s}}$ being replaced by $\varepsilon_{\boldsymbol{s}}$. Thus, it suffices to show that the conclusion holds for $\bar{G}_n$. To this end, we split the proof into two steps as follows:

First, we show that $\bar{G}_n$ can be approximated by a partial sum ($\bar{G}_n(m_n)$) of an m_n dependence process, in particular, $\bar{G}_n(m_n)$ is defined as $\bar{G}_n$ by replacing ε_{ij} with $\varepsilon_{ij}(m_n)$ (defined in Lemma 1). This is done in Lemma 2.

Second, we show $\bar{G}_n(m_n)$ has an asymptotically normal distribution by using a block technique. This is done in Lemmas 3 and 4.

Therefore, by Lemmas 2, 3 and 4, we have (ii) and complete the proof of Theorem 1. □

Lemma 2. *Suppose that $\Delta_4 < \infty$ and $m_n \to \infty$ as $n \to \infty$, then under H_0,*

$$\sqrt{n}k_n|\bar{G}_n - \bar{G}_n(m_n) - \mathrm{E}(\bar{G}_n - \bar{G}_n(m_n))| \xrightarrow{p} 0. \tag{S.13}$$

Proof. Note that

$$\begin{aligned}
& nk_n^{-2}(\bar{G}_n - \bar{G}_n(m_n)) \\
= & \sum_{i=0}^{[\frac{n_1}{k_n}]}\sum_{j=0}^{[\frac{n_2}{k_n}]}\sum_{l=1}^{4}\Big[(\bar{S}_l(ik_n, jk_n) - \bar{S}_{l+1}(ik_n, jk_n))^2 \\
& -(\bar{S}_l((ik_n, jk_n), m_n) - \bar{S}_{l+1}((ik_n, jk_n), m_n))^2\Big] \\
= & \sum_{l=1}^{4}\sum_{i=0}^{[\frac{n_1}{k_n}]}\sum_{j=0}^{[\frac{n_2}{k_n}]}[\bar{S}_l(ik_n, jk_n) + \bar{S}_l((ik_n, jk_n), m_n) - \bar{S}_{l+1}(ik_n, jk_n) - \bar{S}_{l+1}((ik_n, jk_n), m_n)] \\
& \cdot[(\bar{S}_l(ik_n, jk_n) - \bar{S}_l((ik_n, jk_n), m_n)) - (\bar{S}_{l+1}(ik_n, jk_n) - \bar{S}_{l+1}((ik_n, jk_n), m_n))] \\
= & k_n^{-4}\sum_{l=1}^{4}\sum_{\mathbf{0}\le\boldsymbol{i}\le([\frac{n_1}{k_n}],[\frac{n_2}{k_n}])}\sum_{(\boldsymbol{i}-\mathbf{1})k_n\le\boldsymbol{s},\boldsymbol{s}'\le(\boldsymbol{i}+\mathbf{1})k_n} a_{\boldsymbol{s},l}a_{\boldsymbol{s}',l}(\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n)),
\end{aligned} \tag{S.14}$$

where $a_{\boldsymbol{s},l} = 1$ if $\boldsymbol{s}$ lies in the l-th quadran with start point (ik_n, jk_n) and edge length k_n, say $I_l(i,j)$ and $a_{\boldsymbol{s},l} = -1$ if $\boldsymbol{s} \in I_{l+1}(i,j)$. As for the definition of $I_l(i,j)$, we take $I_1(i,j)$ for example, $I_1(i,j) = \{\boldsymbol{s} = (s_1, s_2) : ik_n \le s_1 \le (i+1)k_n,\, jk_n \le s_2 \le (j+1)k_n\}$. Let $\tau : \mathbb{Z} \to \mathbb{Z}^d$ be a bijection, $\mathcal{F}_i = \sigma(\varepsilon_{\tau(l)} : l \le i)$ and $P_i(\varepsilon_{\boldsymbol{t}}) = \mathrm{E}[\varepsilon_{\boldsymbol{t}}|\mathcal{F}_i] - \mathrm{E}[\varepsilon_{\boldsymbol{t}}|\mathcal{F}_{i-1}]$. Let $\widehat{\varepsilon}_{\boldsymbol{s}}^{(h)}$ and $\widehat{\varepsilon}_{\boldsymbol{s}}^{(h)}(m_n)$ be defined by replacing η_h in $\varepsilon_{\boldsymbol{s}}$ and $\varepsilon_{\boldsymbol{s}}(m_n)$ by its independent copy η'_t. For simplicity, we write $\sum_{\boldsymbol{0}\le \boldsymbol{i} \le ([\frac{n_1}{k_n}],[\frac{n_2}{k_n}])} \sum_{(\boldsymbol{i}-\boldsymbol{1})k_n \le \boldsymbol{s},\boldsymbol{s}' \le (\boldsymbol{i}+\boldsymbol{1})k_n}$ as $\sum_{\boldsymbol{i}} \sum_{\boldsymbol{s},\boldsymbol{s}'}$, denote $||X||_p = (\mathrm{E}X^p)^{1/p}$ and $||\cdot||_2$ for $p=2$. By Burkholder inequality, it follows that

$$
\begin{aligned}
& \mathrm{E}\Big(\sum_{\boldsymbol{i}} \sum_{\boldsymbol{s},\boldsymbol{s}'} a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} \{(\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n)) \\
& -\mathrm{E}[(\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n))]\}\Big)^2 \\
= \;& \mathrm{E}\Big\{ \sum_{h=-\infty}^{\infty} \sum_{\boldsymbol{i}} \sum_{\boldsymbol{s},\boldsymbol{s}'} P_h\{a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} (\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n))\}\Big\}^2 \\
\le \;& 4 \sum_{h=-\infty}^{\infty} \mathrm{E}\Big\{ \sum_{\boldsymbol{i}} \sum_{\boldsymbol{s},\boldsymbol{s}'} P_h\{a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} (\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n))\}\Big\}^2 \\
\le \;& 4 \sum_{h=-\infty}^{\infty} \Big\{ \sum_{\boldsymbol{i}} \Big\| \sum_{\boldsymbol{s},\boldsymbol{s}'} P_h\{a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} (\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n))\}\Big\|\Big\}^2 \\
\le \;& 8 \sum_{h=-\infty}^{\infty} \Big(\sum_{\boldsymbol{i}} \sum_{\boldsymbol{s}} \|\mathrm{E}\{\sum_{\boldsymbol{s}'} a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} (\varepsilon_{\boldsymbol{s}} + \varepsilon_{\boldsymbol{s}}(m_n) - \widehat{\varepsilon}_{\boldsymbol{s}}^{(h)} - \widehat{\varepsilon}_{\boldsymbol{s}}^{(h)}(m_n))(\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n))|\mathcal{F}_h\}\|\Big)^2 \\
& +8 \sum_{h=-\infty}^{\infty} \Big(\sum_{\boldsymbol{i}} \sum_{\boldsymbol{s}'} \|\mathrm{E}\{\sum_{\boldsymbol{s}} a_{\boldsymbol{s},l} a_{\boldsymbol{s}',l} (\varepsilon_{\boldsymbol{s}'} - \varepsilon_{\boldsymbol{s}'}(m_n) - \widehat{\varepsilon}_{\boldsymbol{s}'}^{(h)} + \widehat{\varepsilon}_{\boldsymbol{s}'}^{(h)}(m_n))(\widehat{\varepsilon}_{\boldsymbol{s}}^{(h)} + \widehat{\varepsilon}_{\boldsymbol{s}}^{(h)}(m_n))|\mathcal{F}_h\}\|\Big)^2 \\
=: \;& \Pi_{1n} + \Pi_{2n}. \qquad \text{(S.15)}
\end{aligned}
$$

For Π_{1n}, similar to (S.4), by Proposition 3 and Lemma 2 of EVW, we can show that as $m_n \to \infty$,

$$\begin{aligned}\Pi_{1n} &\le 8\sum_{h=-\infty}^{\infty}\Big\{\sum_{\boldsymbol{i}}\sum_{\boldsymbol{s}}\{||P_h(\varepsilon_{\boldsymbol{S}}+\varepsilon_{\boldsymbol{S}}(m_n))||_4\}\Big(\Big\|\sum_{\boldsymbol{S}'}(\varepsilon_{\boldsymbol{S}'}-\varepsilon_{\boldsymbol{S}'}(m_n))\Big\|_4\Big)\Big\}^2\\
&\le \Big(\Delta_4\Big\|\sum_{\boldsymbol{S}'}(\varepsilon_{\boldsymbol{S}'}-\varepsilon_{\boldsymbol{S}'}(m_n))\Big\|_4\Big)\Big\{\sum_{\boldsymbol{i},\boldsymbol{s}}\sum_{h=-\infty}^{\infty}\{||P_h(\varepsilon_{\boldsymbol{S}}+\varepsilon_{\boldsymbol{S}}(m_n))||_4\}\Big(\Big\|\sum_{\boldsymbol{S}'}(\varepsilon_{\boldsymbol{S}'}-\varepsilon_{\boldsymbol{S}'}(m_n))\Big\|_4\Big)\Big\}\\
&= o(nk_n^2\Delta_4^2)=o(nk_n^2).\end{aligned}$$

Similarly, we have

$$\Pi_{2n}=o(nk_n^2).$$

Thus, by (S.14) and (S.15), we have

$$\mathrm{E}\{(\bar{G}_n-\bar{G}_n(m_n))-\mathrm{E}(\bar{G}_n-\bar{G}_n(m_n))\}^2=o(n^{-1}k_n^{-2}).$$

This completes the proof of Lemma 2. □

Next, we use a block technique to show the asymptotic normality of $\bar{G}_n(m_n)$. Let

$$\begin{aligned}G_{n,1}(m_n) &= \frac{k_n^2}{n}\sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]}\sum_{j=0}^{[\frac{n_2}{(l_n+4)k_n}]}\sum_{p=i(l_n+4)}^{[i(l_n+4)+l_n]}\sum_{q=j(l_n+4)}^{[j(l_n+4)+l_n]}\bar{T}_n(pk_n,qk_n,m_n)\\
G_{n,2}(m_n) &= \frac{k_n^2}{n}\sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]}\sum_{j=0}^{[\frac{n_2}{(l_n+4)k_n}]}\sum_{p=i(l_n+4)}^{[i(l_n+4)+l_n]}\sum_{q=j(l_n+4)+l_n}^{(j+1)(l_n+4)}\bar{T}_n(pk_n,qk_n,m_n)\\
G_{n,3}(m_n) &= \frac{k_n^2}{n}\sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]}\sum_{j=01}^{[\frac{n_2}{(l_n+4)k_n}]}\sum_{p=i(l_n+4)+l_n}^{(i+1)(l_n+4)}\sum_{q=j(l_n+4)}^{[j(l_n+4)+l_n]}\bar{T}_n(pk_n,qk_n,m_n)\\
G_{n,4}(m_n) &= \frac{k_n^2}{n}\sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]}\sum_{j=0}^{[\frac{n_2}{(l_n+4)k_n}]}\sum_{p=i(l_n+4)+l_n}^{(i+1)(l_n+4)}\sum_{q=j(l_n+4)+l_n}^{(j+1)(l_n+4)}\bar{T}_n(pk_n,qk_n,m_n),\end{aligned}$$

where $\bar{T}_n(pk_n, qk_n, m_n)$ is defined as $\bar{T}_n(pk_n, qk_n)$ by replacing $\varepsilon_{\boldsymbol{S}}$ by $\varepsilon_{\boldsymbol{S}}(m_n)$ and l_n is a constant sequence tending to infinity. Then $\bar{G}_n(m_n) = \sum_{i=1}^4 G_{n,i}(m_n)$. The next lemmas consider the limit behaviors of the $G_{n,i}(m_n)$, $i = 1, \ldots, 4$.

Lemma 3. *When $\Delta_p < \infty$ for some $p \geq 4$, then*

$$\sqrt{n}k_n[G_{n,i}(m_n) - \mathrm{E}G_{n,i}(m_n)] \xrightarrow{p} 0, \quad \textit{for } i = 2, 3, 4.$$

Proof. The proofs for $G_{n,i}(m_n)$, $i = 2, 3, 4$ are similar, we only give $G_{n,2}(m_n)$ in details. Let $\xi_{ij} = \sum_{p=i(l_n+4)}^{[i(l_n+4)+l_n]} \sum_{q=j(l_n+4)+l_n}^{(j+1)(l_n+4)} T_n(pk_n, qk_n, m_n)$. Since $\{\varepsilon_{\boldsymbol{S}}(m_n)\}$ is a stationary m_n-dependent sequence with $m_n = o(k_n)$, it follows that $\{\xi_{ij}\}$ is an independent sequence. Thus,

$$\begin{aligned}
\mathrm{E}\{\sqrt{n}k_n(G_{n,2}(m_n) - \mathrm{E}G_{n,2}(m_n))\}^2 &= \frac{k_n^6}{n} \sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]} \sum_{j=0}^{[\frac{n_2}{(l_n+4)k_n}]} \mathrm{E}(\xi_{ij} - \mathrm{E}\xi_{ij})^2 \\
&= \frac{k_n^4}{(l_n+4)^2} \mathrm{E}(\xi_{00} - \mathrm{E}\xi_{00})^2. \qquad \text{(S.16)}
\end{aligned}$$

Let $I_l(i,j)$ be defined as in Lemma 2 and $I_5(i,j) = I_1(i,j)$ and l_n satisfy

$l_n/4 \in \mathbb{Z}$. Then

$$
\begin{aligned}
& k_n^4(\xi_{00} - \mathrm{E}\xi_{00}) \\
= \; & k_n^4 \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} [T_n(pk_n, qk_n, m_n) - \mathrm{E}T_n(pk_n, qk_n, m_n)] \\
= \; & \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} \sum_{l=1}^{4} \Big[\big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) - \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 \\
& - \mathrm{E}\big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) - \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 \Big] \\
= \; & \sum_{l=1}^{4} \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} \Big[\Big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \Big)^2 - \mathrm{E}\Big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \Big)^2 \\
& + \Big(\sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \Big)^2 - \mathrm{E}\Big(\sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \Big)^2 \Big] \\
& - 2 \sum_{l=1}^{4} \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} \Big[\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \\
& - \mathrm{E}\big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big) \Big] \\
= \; & 8 \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} \Big[\big(\sum_{\boldsymbol{s} \in I_1(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 - \mathrm{E}\big(\sum_{\boldsymbol{s} \in I_1(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 \Big] \\
& - 2 \sum_{l=1}^{4} \sum_{p=0}^{l_n} \sum_{q=l_n}^{l_n+4} \Big[\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \\
& - \mathrm{E}\big(\sum_{\boldsymbol{s} \in I_l(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \sum_{\boldsymbol{s} \in I_{l+1}(p,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big) \Big] \\
=: \; & 8X_{n1} - 2X_{n2}.
\end{aligned}
$$

We write X_{n1} as the sum of independent blocks:

$$
X_{n1} = \sum_{q=l_n}^{l_n+4} \sum_{i=0}^{3} \sum_{j=0}^{l_n/4} \Big[\big(\sum_{\boldsymbol{s} \in I_1(4j+i,\, q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 - \mathrm{E}\big(\sum_{\boldsymbol{s} \in I_1(4j+i,q)} \varepsilon_{\boldsymbol{s}}(m_n) \big)^2 \Big].
$$

Since $m_n = o(k_n)$, it is easy to see that for any given i and q,

$$\Big\{(\sum_{\boldsymbol{S}\in I_1(4j+i,\,q)} \varepsilon_{\boldsymbol{S}}(m_n))^2 - \mathrm{E}(\sum_{\boldsymbol{S}\in I_1(p,q)} \varepsilon_{\boldsymbol{S}}(m_n))^2\}$$

is an independent sequence. This gives that

$$\begin{aligned}
\mathrm{E}(X_{n1})^2 &\le 20^2 \sum_{j=0}^{l_n/4} \mathrm{E}\Big[(\sum_{\boldsymbol{S}\in I_1(4j+i,\,q)} \varepsilon_{\boldsymbol{S}}(m_n))^2 - \mathrm{E}(\sum_{\boldsymbol{S}\in I_1(4j+i,q)} \varepsilon_{\boldsymbol{S}}(m_n))^2\Big]^2 \\
&\le 100 l_n \mathrm{E}\Big[\sum_{\boldsymbol{S}\in I_1(4j+i,\,q)} \varepsilon_{\boldsymbol{S}}(m_n))\Big]^4 \\
&= 100 l_n \sum_{\boldsymbol{S}_1,\cdots,\boldsymbol{S}_4\in I_1(4j+i,\,q)} \sum_{h,l\in\mathbb{Z}} \mathrm{E}[P_l(\varepsilon_{\boldsymbol{S}_1}(m_n))P_l(\varepsilon_{\boldsymbol{S}_2}(m_n))P_h(\varepsilon_{\boldsymbol{S}_3}(m_n))P_h(\varepsilon_{\boldsymbol{S}_4}(m_n))] \\
&= O(l_n k_n^4).
\end{aligned}$$

Similarly, we can show $\mathrm{E}(X_{n2})^2 = O(l_n k_n^4)$. Consequently,

$$\mathrm{E}[k_n^4(\xi_{00} - \mathrm{E}\xi_{00})]^2 \le 64\mathrm{E}(X_{n1})^2 + 4\mathrm{E}(X_{n2})^2 = O(l_n k_n^4). \tag{S.17}$$

Thus, by (S.16), we have

$$\mathrm{E}\{\sqrt{n}k_n(G_{n,2}(m_n) - \mathrm{E}G_{n,2}(m_n))\}^2 = O(1/l_n) = o(1), \tag{S.18}$$

and complete the proof of Lemma 3. □

Lemma 4. *When $\Delta_p < \infty$ for some $p > 4$ and $m_n = o(k_n)$, then there exists a constant $\sigma_0 > 0$ such that*

$$\sqrt{n}k_n[G_{n,1}(m_n) - \mathrm{E}G_{n,1}(m_n)] \xrightarrow{d} N(0, \sigma_0^2).$$

Proof. Let $\eta_{ij} = \sum_{p=i(l_n+4)}^{[i(l_n+4)+l_n]} \sum_{q=j(l_n+4)}^{[j(l_n+4)+l_n]} T_n(pk_n, qk_n, m_n)$. Since $\{\varepsilon_{\boldsymbol{S}}(m_n)$ is an m_n-dependent stationary process and $m_n = o(k_n)$, it follows that $\{\eta_{ij}\}$

is an independent sequence and

$$\begin{aligned}&\sqrt{n}k_n[G_{n,1}(m_n)-\mathrm{E}G_{n,1}(m_n)]\\ =\ &\frac{(l_n+4)k_n}{\sqrt{n}}\sum_{i=0}^{[\frac{n_1}{(l_n+4)k_n}]}\sum_{j=0}^{[\frac{n_2}{(l_n+4)k_n}]}(l_n+4)^{-1}k_n^2(\eta_{ij}-\mathrm{E}\eta_{ij}).\end{aligned}$$

By the Lindeberg Central Limit Theorem, it suffices to show that

$$\mathrm{E}[(l_n+4)^{-1}k_n^2(\eta_{00}-\mathrm{E}\eta_{00})]^2<\infty \tag{S.19}$$

and for any $\varepsilon>0$,

$$\mathrm{E}\{[(l_n+4)^{-1}k_n^2(\eta_{00}-\mathrm{E}\eta_{00})]^2I(|\eta_{00}-\mathrm{E}\eta_{00}|>\varepsilon\sqrt{n}/k_n^3)\}\to 0. \tag{S.20}$$

By Hölder inequality, it follows that

$$\begin{aligned}&\mathrm{E}\{[(l_n+4)^{-1}k_n^2(\eta_{00}-\mathrm{E}\eta_{00})]^2I(|\eta_{00}-\mathrm{E}\eta_{00}|>\varepsilon\sqrt{n}/k_n^3)\} \qquad \text{(S.21)}\\ \leq\ &(l_n+4)^{-2}k_n^4\{\mathrm{E}|\eta_{00}-\mathrm{E}\eta_{00}|^{p/2}\}^{4/p}\{P(|\eta_{00}-\mathrm{E}\eta_{00}|>\varepsilon\sqrt{n}/k_n^3)\}^{1-4/p}.\end{aligned}$$

Similar to ξ_{00}, we have

$$\begin{aligned}&k_n^4(\eta_{00}-\mathrm{E}\eta_{00})\\ =\ &8\sum_{p,q=0}^{l_n}\Big[(\sum_{\boldsymbol{S}\in I_1(p,q)}\varepsilon_{\boldsymbol{S}}(m_n))^2-\mathrm{E}(\sum_{\boldsymbol{S}\in I_1(p,q)}\varepsilon_{\boldsymbol{S}}(m_n))^2\Big]\\ &-2\sum_{l=1}^{4}\sum_{p,q=0}^{l_n}\Big[\sum_{\boldsymbol{S}\in I_l(p,q)}\varepsilon_{\boldsymbol{S}}(m_n)\sum_{\boldsymbol{S}\in I_{l+1}(p,q)}\varepsilon_{\boldsymbol{S}}(m_n)-\mathrm{E}(\sum_{\boldsymbol{S}\in I_l(p,q)}\varepsilon_{\boldsymbol{S}}(m_n)\sum_{\boldsymbol{S}\in I_{l+1}(p,q)}\varepsilon_{\boldsymbol{S}}(m_n))\Big]\\ =:\ &8\zeta_{n1}-2\zeta_{n2}.\end{aligned}$$

Write ζ_{n1} as the sums of independent blocks, i.e.,

$$\zeta_{n1}=\sum_{b_1,b_2=0}^{3}\sum_{a_1,a_2=0}^{l_n/4}\Big[(\sum_{\boldsymbol{S}\in I_1(4a_1+b_1,\,4a_2+b_2)}\varepsilon_{\boldsymbol{S}}(m_n))^2-\mathrm{E}(\sum_{\boldsymbol{S}\in I_1(4a_1+b_1,4a_2+b_2)}\varepsilon_{\boldsymbol{S}}(m_n))^2\Big].$$

Since $\{(\sum_{\boldsymbol{s}\in I_1(4a_1+b_1,\,4a_2+b_2)}\varepsilon_{\boldsymbol{s}}(m_n))^2 - \mathrm{E}(\sum_{\boldsymbol{s}\in I_1(4a_1+b_1,4a_2+b_2)}\varepsilon_{\boldsymbol{s}}(m_n))^2\}$ is a sequence of independent blocks for any given (b_1, b_2), it follows from Burkholder inequality that

$$
\begin{aligned}
&[\mathrm{E}|\zeta_{n1}|^{p/2}]^{4/p} \\
\leq\ & 32p\sum_{a_1,a_2=0}^{l_n/4}\Big[\mathrm{E}\Big(\sum_{\boldsymbol{s}\in I_1(4a_1+b_1,\,4a_2+b_2)}\varepsilon_{\boldsymbol{s}}(m_n)\Big)^2 - \mathrm{E}\Big(\sum_{\boldsymbol{s}\in I_1(4a_1+b_1,4a_2+b_2)}\varepsilon_{\boldsymbol{s}}(m_n)\Big)^2\Big]^2 \\
\leq\ & 2pl_n^2\mathrm{E}\Big[\sum_{\boldsymbol{s}\in I_1(4a_1+b_1,\,4a_2+b_2)}\varepsilon_{\boldsymbol{s}}(m_n)\Big]^4 = O(l_n^2k_n^4). \qquad \text{(S.22)}
\end{aligned}
$$

Similarly, we can show that $[\mathrm{E}|\zeta_{n2}|^{p/2}]^{4/p} = O(l_n^2k_n^4)$. Thus,

$$
\begin{aligned}
\{\mathrm{E}|(\eta_{00}-\mathrm{E}\eta_{00})|^{p/2}\}^{4/p} &= k_n^{-8}[\mathrm{E}|k_n^4(\eta_{00}-\mathrm{E}\eta_{00})|^{p/2}]^{4/p} \\
&= O(l_n^2k_n^{-4}). \qquad \text{(S.23)}
\end{aligned}
$$

This combining with Hölder inequality implies that

$$
\begin{aligned}
P(|\eta_{00}-\mathrm{E}\eta_{00}| > \varepsilon\sqrt{n}/k_n^3) &\leq (\varepsilon\sqrt{n}/k_n^3)^{-p/2}\mathrm{E}|\eta_{00}-\mathrm{E}\eta_{00}|^{p/2} \\
&= O\{(\varepsilon\sqrt{n}k_n^{-3})^{-p/2}(l_n^2k_n^{-4})^{p/4}\} \\
&= O\{(n^{-1/2}k_nl_n)^{p/2}\}.
\end{aligned}
$$

Thus, by virtue of (S.21) and (S.23) and taking $l_n = o(n^{1/2}k_n^{-1})$, we have

$$
\begin{aligned}
&\mathrm{E}\{[(l_n+4)^{-1}k_n^2(\eta_{00}-\mathrm{E}\eta_{00})]^2 I(|\eta_{00}-\mathrm{E}\eta_{00}| > \varepsilon\sqrt{n}/k_n^3)\} \\
=\ & O\{(n^{-1/2}k_nl_n)^{p/2}\} = o(1).
\end{aligned}
$$

Therefore, (S.20) holds. Equation (S.19) follows directly from (S.23) and Lemma 4 is proved. □

Lemma 5. *Let* $U_{n\boldsymbol{j}} = b_n k_n[T_n^*(A_n(\mathcal{I}_{\boldsymbol{j}})) - \mathrm{E}^* T_n^*(A_n(\mathcal{I}_{\boldsymbol{j}}))]$. *Suppose that* b_n *and* k_n *satisfy the condition of Theorem 2. Then for any* $\varepsilon > 0$,

$$P^*\{|U_{n\boldsymbol{j}}| > \varepsilon\sqrt{|\mathcal{J}_n|}\} \to 0, \text{ in probability.}$$

Proof. Observe that

$$P^*\{|U_{n\boldsymbol{j}}| > \varepsilon\sqrt{|\mathcal{J}_n|}\} = \frac{1}{|\mathcal{I}_n|}\sum_{\boldsymbol{j}\in\mathcal{I}_n} I(|b_n k_n[T_n(A_n(\boldsymbol{j})) - \mathrm{E}^* T_n^*(A_n(\mathcal{I}_{\boldsymbol{j}}))]| > \varepsilon\sqrt{|\mathcal{J}_n|}).$$

Thus, it is enough to show that for all $\boldsymbol{j}$, as $n \to \infty$,

$$P\Big\{b_n k_n|T_n(A_n(\boldsymbol{j})) - \mathrm{E}^* T_n^*(A_n(\mathcal{I}_{\boldsymbol{j}}))| > \varepsilon\sqrt{|\mathcal{J}_n|}\Big\} \to 0. \tag{S.24}$$

By the definition of $T_n(A_n(j))$, we have

$$\begin{aligned}
& P\Big\{b_n k_n|T_n(A_n(\boldsymbol{j})) - \mathrm{E}^* T_n^*(A_n(\mathcal{I}_{\boldsymbol{j}}))| > \varepsilon\sqrt{|\mathcal{J}_n|}\Big\} \\
\leq\ & P\Big\{\sum_{\boldsymbol{i}:[(\boldsymbol{i}-\mathbf{1})k_n,(\boldsymbol{i}+\mathbf{1})k_n]\subseteq A_n(\boldsymbol{j})} |(T_n(\boldsymbol{i}k_n) - \mathrm{E}(T_n(\boldsymbol{i}k_n))| > \varepsilon b_n\sqrt{|\mathcal{J}_n|}/(3k_n^3)\Big\} \\
& + P\Big\{\frac{1}{|\mathcal{I}_n|}\sum_{\boldsymbol{j}\in\mathcal{I}_n}\sum_{\boldsymbol{i}:[(\boldsymbol{i}-\mathbf{1})k_n,(\boldsymbol{i}+\mathbf{1})k_n]\subseteq A_n(\boldsymbol{j})} |(T_n(\boldsymbol{i}k_n) - \mathrm{E}(T_n(\boldsymbol{i}k_n))| > \frac{\varepsilon b_n\sqrt{|\mathcal{J}_n|}}{3k_n^3}\Big\} \\
& + P\Big\{|\sum_{\boldsymbol{i}:A_n(\boldsymbol{j})} \mathrm{E}(T_n(\boldsymbol{i}k_n) - \frac{1}{|\mathcal{I}_n|}\sum_{\boldsymbol{j}\in\mathcal{I}_n}\sum_{\boldsymbol{i}:A_n(\boldsymbol{j})} \mathrm{E}(T_n(\boldsymbol{i}k_n))| > \frac{\varepsilon b_n\sqrt{|\mathcal{J}_n|}}{3k_n^3}\Big\} \\
=:\ & BT_{n1} + BT_{n2} + BT_{n3},
\end{aligned} \tag{S.25}$$

where $\sum_{\boldsymbol{i}:A_n(\boldsymbol{j})}$ denotes the sum: $\sum_{\boldsymbol{i}:[(\boldsymbol{i}-\mathbf{1})k_n,(\boldsymbol{i}+\mathbf{1})k_n]\subseteq A_n(\boldsymbol{j})}$. Since $b_n = o(\min(n_1, n_2))$, it is easy to get that $BT_{n3} \to 0$. Using the same arguments as in the proofs of G_n (see Lemmas 2–4), we have

$$\mathrm{Var}\Big(\sum_{\boldsymbol{i}} T_n(\boldsymbol{i}k_n)\Big) = O(b_n^2 k_n^{-6}),$$

which combining with condition $b_n = o(\min(n_1, n_2))$ implies that

$$BT_{n1} \leq 9\varepsilon^{-2} n^{-1} k_n^6 \mathrm{Var}\Big(\sum_{\boldsymbol{i}:A_n(\boldsymbol{j})} T_n(\boldsymbol{i}k_n)\Big) = O(n^{-1}b_n^2) = o(1). \tag{S.26}$$

Similarly,

$$BT_{n2} = O(n^{-1}b_n^2) = o(1). \tag{S.27}$$

Thus, by (S.25), we have (S.24) as desired. □

Proof of Theorem 2. Given $Y_{\boldsymbol{s}}$, $\boldsymbol{s} \in I_n$, $\sqrt{|\mathcal{J}_n| b_n^2} k_n (G_n^* - E^* G_n^*)$ is a sum of i.i.d. variables $U_{n\boldsymbol{j}} = \{b_n k_n |\mathcal{J}_n|^{-1/2} T_n^*(B_n(\boldsymbol{I}_{\boldsymbol{j}})) - \mathrm{E}^*[b_n k_n |\mathcal{J}_n|^{-1/2} T_n^*(B_n(\boldsymbol{I}_{\boldsymbol{j}}))]\}$. By Lindeberg-Fell central limit theory, it is enough to show the Lindeberg condition $\mathrm{E}^*[U_{n\boldsymbol{j}}^2 I(|U_{n\boldsymbol{j}}| > \varepsilon)] \to 0$ in probability. Since $\mathrm{E}\{\mathrm{E}^*(U_{n\boldsymbol{j}}^2)\} = \mathrm{var}\{b_n |\mathcal{J}_n|^{-1/2} T_n(B_n(\mathbf{1})\} < \infty$, it follows that $\mathrm{E}^*[U_{n\boldsymbol{j}}^2 I(|U_{n\boldsymbol{j}}| > \varepsilon)] = \mathrm{E}^*[U_{n\boldsymbol{j}}^2 I(|U_{n\boldsymbol{j}}| > M)] + \mathrm{E}^*[U_{n\boldsymbol{j}}^2 I(\varepsilon < |U_{ni}| \leq M)] = o_p(1)$ by Lemma 5 and taking $M \to \infty$. Thus, the Lindeberg condition holds and Theorem 2 is proved. □