# RANDOMIZED RESPONSE SAMPLING WITH APPLICATIONS TO TRACKING DRUGS FOR BETTER LIFE

Shu-Ching Su, Veronica I. Salinas, Monique L. Zamora, Stephen A. Sedory and Sarjinder Singh

*Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX, USA*

## 1.    Supplementary Materials

## A.    APPENDIX-A

Proof of Equations (2.6), (2.7), (2.8) and (2.10):

The distance between the observed and true proportions is given by:

$$D = \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}(\theta_{ij} - \hat{\theta}_{ij})^2 \tag{A1}$$

or, equivalently

$$
\begin{aligned}
D ={}& \frac{1}{2}[\theta_{11} - \hat{\theta}_{11}]^2 + \frac{1}{2}[\theta_{10} - \hat{\theta}_{10}]^2 + \frac{1}{2}[\theta_{01} - \hat{\theta}_{01}]^2 + \frac{1}{2}[\theta_{00} - \hat{\theta}_{00}]^2 \\
={}& \frac{1}{2}[PT\pi_a + \pi_{ay_0} + (1-P)(1-T)\pi_{y_0} - \hat{\theta}_{11}]^2 \\
& + \frac{1}{2}[P(1-T)\pi_a + (1-P)T\pi_{y_0} - \hat{\theta}_{10}]^2 \\
& + \frac{1}{2}[P(1-T)\pi_{y_0} + (1-P)T\pi_a - \hat{\theta}_{01}]^2 \\
& + \frac{1}{2}[1 - \pi_a(P + T - PT) - \pi_{ay_0} - \pi_{y_0}(1 - PT) - \hat{\theta}_{00}]^2
\end{aligned}
\tag{A2}
$$

We consider to minimize the squared distance $D$ with respect to the three parameters of interest $\pi_a$, $\pi_{ay_0}$ and $\pi_{y_0}$. The motivation to consider the minimization of $D$ is that it leads to simple, unbiased and closed form of estimators of the three required proportions. Now we set $\frac{\partial D}{\partial \pi_a} = 0$, $\frac{\partial D}{\partial \pi_{ay_0}} = 0$, and $\frac{\partial D}{\partial \pi_{y_0}} = 0$ as follows:

(i) On setting $\frac{\partial D}{\partial \pi_a} = 0$, we have:

$$\pi_a[P^2T^2 + P^2(1-T)^2 + (1-P)^2T^2 + (P+T-PT)^2] + \pi_{ay_0}(P+T)$$

$$+ \pi_{y_0}[3P(1-P)T(1-T) + (1-PT)(P+T-PT)] \tag{A3}$$

$$= PT\hat{\theta}_{11} + P(1-T)\hat{\theta}_{10} + (1-P)T\hat{\theta}_{01} - \hat{\theta}_{00}(P+T-PT) + (P+T-PT)$$

(ii) On setting $\frac{\partial D}{\partial \pi_{ay_0}} = 0$, we have:

$$(P+T)\pi_a + 2\pi_{ay_0} + \pi_{y_0}(2-P-T) = \hat{\theta}_{11} - \hat{\theta}_{00} + 1 \tag{A4}$$

(iii)On setting $\frac{\partial D}{\partial \pi_{y_0}} = 0$, we have:

$$\pi_a[3P(1-P)T(1-T) + (P+T-PT)(1-PT)] + \pi_{ay_0}(2-P-T)$$

$$+ \pi_{y_0}[(1-P)^2(1-T)^2 + (1-P)^2T^2 + P^2(1-T)^2 + (1-PT)^2] \tag{A5}$$

$$= (1-P)(1-T)\hat{\theta}_{11} + T(1-P)\hat{\theta}_{10} + P(1-T)\hat{\theta}_{01} - (1-PT)\hat{\theta}_{00} + (1-PT)$$

Solving (A3), (A4) and (A5) for $\pi_a$, $\pi_{ay_0}$ and $\pi_{y_0}$ and by the method of moments, we have

$$\hat{\pi}_a = \frac{(P-T) - \hat{\theta}_{11}(P-T) - \hat{\theta}_{10}(4PT-3P-T) - \hat{\theta}_{01}(P+3T-4PT) - \hat{\theta}_{00}(P-T)}{4(P-T)(P+T-2PT)} \tag{A6}$$

$$\hat{\pi}_{ay_0} = \frac{1}{4(P-T)(P+T-2PT)}[(P-T)\hat{\theta}_{11}(1+2P+2T-4PT)$$

$$+\hat{\theta}_{10}(2P-1)(2T^2+2PT-P-3T)+\hat{\theta}_{01}(3P+T-2PT-2P^2)\ \ (\text{A7})$$

$$+\hat{\theta}_{00}(2P-1)(2T-1)(P-T)-(2P-1)(2T-1)(P-T)]$$

and

$$\hat{\pi}_{y_0} = \frac{(T-P)(\hat{\theta}_{11}+\hat{\theta}_{00})+(4PT-P-3T)\hat{\theta}_{10}-(4PT-3P-T)\hat{\theta}_{01}}{4(P-T)(P+T-2PT)} \qquad (\text{A8})$$

Recall that our primary interest is in the estimation of population proportion $\pi_A$ of the sensitive characteristic $A$ in the population. Thus we suggest estimator of the required population proportion $\pi_A$ as:

$$\hat{\pi}_A = \hat{\pi}_a + \hat{\pi}_{ay_0}$$

$$=\frac{1}{4(P-T)(P+T-2PT)}[(P-T)-(2P-1)(2T-1)(P-T)$$

$$+\hat{\theta}_{11}(P-T)(1+2P-2T-4PT-1)+\hat{\theta}_{10}[(2P-1)(2T^2+2PT-P-3T)-(4PT-3P-T) \qquad (\text{A9})$$

$$+\hat{\theta}_{01}[(2T-1)(3P+T-2PT-2P^2)-(P+3T-4PT)+\hat{\theta}_{00}[(2P-1)(2T-1)(P-T)-(P-T)]$$

$$=\frac{(P-T)(\hat{\theta}_{11}-\hat{\theta}_{00})+(P+T-2)(\hat{\theta}_{01}-\hat{\theta}_{10})+(P-T)}{2(P-T)}$$

**Proof of Theorem 2.1:** Taking expected value on both sides of $\hat{\pi}_A$ , we have

$$E(\hat{\pi}_A) = E[\frac{(P-T)(\hat{\theta}_{11}-\hat{\theta}_{00})+(P+T-2)(\hat{\theta}_{01}-\hat{\theta}_{10})+(P-T)}{2(P-T)}]$$

$$=\frac{(P-T)(E(\hat{\theta}_{11})-E(\hat{\theta}_{00}))+(P+T-2)(E(\hat{\theta}_{01})-E(\hat{\theta}_{10}))+(P-T)}{2(P-T)} \qquad (\text{A10})$$

$$=\frac{(P-T)(\theta_{11}-\theta_{00})+(P+T-2)(\theta_{01}-\theta_{10})+(P-T)}{2(P-T)}$$

Note that: $\theta_{11}-\theta_{00} = \pi_a(P+T)+\pi_{y_0}(2-P-T)+2\pi_{ay_0}-1$ and $\theta_{01}-\theta_{10} = \pi_{y_0}(P-T)-\pi_a(P-T)$

So the expected value of $\hat{\pi}_A$ is given by

$$
\begin{aligned}
E(\hat{\pi}_A) &= \frac{1}{2(P-T)}[(P-T)\{\pi_a(P+T)+\pi_{y_0}(2-P-T)+2\pi_{ay_0}-1\} \\
&\quad + (P+T-2)(\pi_{y_0}-\pi_a)(P-T)+(P+T)] \\
&= \frac{2(P-T)(\pi_a+\pi_{ay_0})}{2(P-T)} \\
&= \pi_A
\end{aligned}
\tag{A11}
$$

which proves the first part of the theorem.

By using the concept or multinomial distribution, we have:

$V(\hat{\theta}_{11}) = \frac{\theta_{11}(1-\theta_{11})}{n}$, $Cov(\hat{\theta}_{11},\hat{\theta}_{10}) = \frac{-\theta_{11}\theta_{10}}{n}$, $Cov(\hat{\theta}_{10},\hat{\theta}_{01}) = \frac{-\theta_{10}\theta_{01}}{n}$, $V(\hat{\theta}_{10}) = \frac{\theta_{10}(1-\theta_{10})}{n}$,

$Cov(\hat{\theta}_{11},\hat{\theta}_{01}) = \frac{-\theta_{11}\theta_{01}}{n}$, $Cov(\hat{\theta}_{10},\hat{\theta}_{00}) = \frac{-\theta_{10}\theta_{00}}{n}$, $V(\hat{\theta}_{01}) = \frac{\theta_{01}(1-\theta_{01})}{n}$, $Cov(\hat{\theta}_{11},\hat{\theta}_{00}) = \frac{-\theta_{11}\theta_{00}}{n}$,

$Cov(\hat{\theta}_{01},\hat{\theta}_{00}) = \frac{-\theta_{01}\theta_{00}}{n}$ and $V(\hat{\theta}_{00}) = \frac{\theta_{00}(1-\theta_{00})}{n}$

Now by the definition of variance , we have:

$$
\begin{aligned}
V(\hat{\pi}_A) &= \frac{1}{4(P-T)^2}[(P-T)^2 V(\hat{\theta}_{11}-\hat{\theta}_{00})+(P+T-2)^2 V(\hat{\theta}_{01}-\hat{\theta}_{10}) \\
&\quad +2(P-T)(P+T-2)Cov(\hat{\theta}_{11}-\hat{\theta}_{00},\hat{\theta}_{01}-\hat{\theta}_{10})] \\
&= \frac{1}{4(P-T)^2}[(P-T)^2\{V(\hat{\theta}_{11})+V(\hat{\theta}_{00})-2Cov(\hat{\theta}_{11},\hat{\theta}_{00})\} \\
&\quad +(P+T-2)^2\{V(\hat{\theta}_{01})+V(\hat{\theta}_{10})-2Cov(\hat{\theta}_{01},\hat{\theta}_{10})\} \\
&\quad +2(P-T)(P+T-2)\{Cov(\hat{\theta}_{11},\hat{\theta}_{01})-Cov(\hat{\theta}_{11},\hat{\theta}_{10}) \\
&\quad -Cov(\hat{\theta}_{00},\hat{\theta}_{01})+Cov(\hat{\theta}_{00},\hat{\theta}_{10})\}] \\
&= \frac{1}{4(P-T)^2}[(P-T)^2\{\frac{\theta_{11}(1-\theta_{11})}{n}+\frac{\theta_{00}(1-\theta_{00})}{n}+2\frac{\theta_{11}\theta_{00}}{n}\} \\
&\quad +(P+T-2)^2\{\frac{\theta_{01}(1-\theta_{01})}{n}+\frac{\theta_{10}(1-\theta_{10})}{n}+\frac{2\theta_{10}\theta_{01}}{n}\} \\
&\quad +2(P-T)(P+T-2)\{-\frac{\theta_{11}\theta_{01}}{n}+\frac{\theta_{11}\theta_{10}}{n}+\frac{\theta_{01}\theta_{00}}{n}-\frac{\theta_{00}\theta_{10}}{n}\}]
\end{aligned}
$$

$$= \frac{1}{4n(P-T)^2}[(P-T)^2(\theta_{11}+\theta_{00}) + (\theta_{10}+\theta_{01})(P+T-2)^2 - \{(P-T)(\theta_{11}-\theta_{00})$$

$$- (P+T-2)(\theta_{10}-\theta_{01})\}^2] \tag{A12}$$

Note that:

$$\theta_{11} + \theta_{00} = \pi_a(2PT - T - P) + \pi_{y_0}(2PT - P - T) + 1$$

$$\theta_{11} - \theta_{00} = \pi_a(P+T) + \pi_{y_0}(2 - P - T) + 2\pi_{ay_0} - 1$$

$$\theta_{10} + \theta_{01} = \pi_a(P+T-2PT) + \pi_{y_0}(P+T-2PT)$$

and

$$\theta_{10} - \theta_{01} = \pi_a(P-T) - \pi_{y_0}(P-T)$$

On substituting these values in (A12), we have

$$V(\hat{\pi}_A) = \frac{1}{4n(P-T)^2}[(P-T)^2\{\pi_a(2PT-T-P) + \pi_{y_0}(2PT-P-T) + 1\}$$

$$+(P+T-2)^2\{\pi_a(P+T-2PT) + \pi_{y_0}(P+T-2PT)\}$$

$$-\{(P-T)(\pi_a(P+T) + \pi_{y_0}(2-P-T) + 2\pi_{ay_0} - 1$$

$$-(P+T-2)(\pi_a(P-T) - \pi_{y_0}(P-T))\}^2]$$

$$= \frac{1}{4n(P-T)^2}[4(1-P)(1-T)(P+T-2PT)(\pi_a + \pi_{ay_0})$$

$$+ 4(P-T)^2\pi_a(1-\pi_a) + 4(P-T)^2\pi_{ay_0}(1-\pi_{ay_0}) - 8(P-T)^2\pi_a\pi_{ay_0}] \tag{A13}$$

$$= \frac{\pi_a(1-\pi_a)}{n} + \frac{\pi_{ay_0}(1-\pi_{ay_0})}{n} - \frac{2\pi_a\pi_{ay_0}}{n} + \frac{(1-P)(1-T)(P+T-2PT)(\pi_a+\pi_{y_0})}{n(P-T)^2}$$

which proves the theorem.

**Proof of Remark 2.2:**

The maximum likelihood estimates of $\pi_a$ , $\pi_{ay_0}$ and $\pi_{y_0}$ are obtained by solving the three non-linear equations given by:

$$\frac{\hat{\theta}_{11}PT}{\theta_{11}} + \frac{\hat{\theta}_{10}P(1-T)}{\theta_{10}} + \frac{\hat{\theta}_{01}(1-P)T}{\theta_{01}} - \frac{\hat{\theta}_{00}(P+T-PT)}{\theta_{00}} = 0 \tag{A14}$$

$$\frac{\hat{\theta}_{11}}{\theta_{11}} - \frac{\hat{\theta}_{00}}{\theta_{00}} = 0 \tag{A15}$$

$$\frac{\hat{\theta}_{11}(1-P)(1-T)}{\theta_{11}} + \frac{\hat{\theta}_{10}(1-P)T}{\theta_{10}} + \frac{\hat{\theta}_{01}P(1-T)}{\theta_{01}} - \frac{\hat{\theta}_{00}(1-PT)}{\theta_{00}} = 0 \tag{A16}$$

From equation (A15), we have:

$$\hat{\theta}_{11}\theta_{00} = \hat{\theta}_{00}\theta_{11}$$

or

$$\pi_a[PT\hat{\theta}_{00} + (P+T-PT)\hat{\theta}_{11}] + \pi_{y_0}[(1-P)(1-T)\hat{\theta}_{00} + (1-PT)\hat{\theta}_{11}] + \pi_{ay_0}(\hat{\theta}_{00} + \hat{\theta}_{11}) = \hat{\theta}_{11} \tag{A17}$$

Subtracting equation (A16) from equation (A14),we have:

$$\frac{\hat{\theta}_{10}}{\theta_{10}} = \frac{\hat{\theta}_{01}}{\theta_{01}} \tag{A18}$$

or

$$\pi_a[T(1-P)\hat{\theta}_{10} - P(1-T)\hat{\theta}_{01}] + \pi_{y_0}[P(1-T)\hat{\theta}_{10} - (1-P)T\hat{\theta}_{01}] = 0 \tag{A19}$$

Using (A14) and (A18) in (A16), we have:

$$\frac{\hat{\theta}_{11}PT}{\theta_{11}} + \frac{\hat{\theta}_{10}P(1-T)}{\theta_{10}} + \frac{\hat{\theta}_{01}(1-P)T}{\theta_{01}} - \frac{\hat{\theta}_{00}(P+T-PT)}{\theta_{00}} = 0$$

or

$$\frac{\hat{\theta}_{11}}{\theta_{11}} = \frac{\hat{\theta}_{10}}{\theta_{10}}$$

or

$$\hat{\theta}_{11}[P(1-T)\pi_a + (1-P)T\pi_{y_0}] = \hat{\theta}_{10}[PT\pi_a + \pi_{ay_0} + (1-P)(1-T)\pi_{y_0}]$$

or

$$[P(1-T)\hat{\theta}_{11} - PT\hat{\theta}_{10}]\hat{\pi}_a + [(1-P)T\hat{\theta}_{11} - (1-P)(1-T)\hat{\theta}_{10}]\hat{\pi}_{y_0} - \hat{\theta}_{10}\pi_{ay_0} = 0 \qquad (A20)$$

From (A17), (A19) and (A20), we have

$$\begin{bmatrix} (P+T-PT)\hat{\theta}_{11} + PT\hat{\theta}_{00} & (1-P)(1-T)\hat{\theta}_{00} + (1-PT)\hat{\theta}_{11} & (\hat{\theta}_{11}+\hat{\theta}_{00}) \\ (1-P)T\hat{\theta}_{10} - P(1-T)\hat{\theta}_{01} & P(1-T)\hat{\theta}_{10} - (1-P)T\hat{\theta}_{01} & 0 \\ P(1-T)\hat{\theta}_{11} - PT\hat{\theta}_{10} & (1-P)T\hat{\theta}_{11} - (1-P)(1-T)\hat{\theta}_{10} & -\hat{\theta}_{10} \end{bmatrix} \begin{bmatrix} \pi_a \\ \pi_{y_0} \\ \pi_{ay_0} \end{bmatrix} \begin{bmatrix} \hat{\theta}_{11} \\ 0 \\ 0 \end{bmatrix}$$

$$(A21)$$

which proves that the maximum likelihood estimates obtained by solving the three non-linear equations remain the same as the least square distance estimates.

Alternatively, it can be easily seen that the Cramer-Rao lower bound of variance of the maximum likelihood estimate:

$$\hat{\pi}_A^{mle} = \hat{\pi}_a^{mle} + \hat{\pi}_{ay_0}^{mle} \qquad (A22)$$

is given by:

$$V(\hat{\pi}_A^{mle}) \geq \frac{(G^2 - BC) + (F^2 - AC) + 2(CD - FG)}{n(AG^2 + BF^2 + CD^2 - ABC - 2DFG)} \qquad (A23)$$

where

$A = [\frac{P^2 T^2}{\theta_{11}} + \frac{P^2(1-T)^2}{\theta_{10}} + \frac{(1-P)^2 T^2}{\theta_{01}} + \frac{(P+T-PT)^2}{\theta_{00}}]$; $B = [\frac{1}{\theta_{11}} + \frac{1}{\theta_{00}}]$;

$C = [\frac{(1-P)^2(1-T)^2}{\theta_{11}} + \frac{(1-P)^2 T^2}{\theta_{10}} + \frac{P^2(1-T)^2}{\theta_{01}} + \frac{(1-PT)^2}{\theta_{00}}]$; $D = [\frac{PT}{\theta_{11}} + \frac{(P+T-PT)}{\theta_{00}}]$;

$F = [P(1-P)T(1-T)(\frac{1}{\theta_{11}} + \frac{1}{\theta_{10}} + \frac{1}{\theta_{01}}) + \frac{(P+T-PT)(1-PT)}{\theta_{00}}]$; and $G = [\frac{(1-P)(1-T)}{\theta_{11}} + \frac{(1-PT)}{\theta_{00}}]$.

PROGRAM–I : SAS MACRO USED IN THE SIMULATION STUDY

```
%MACRO SHU_CHING_SU( III , P_in , T_in , PIa_in , PIy_in , PIay_in );

DATA DATA2&III ;

P = &P_in ;

T = &T_in ;

PIa = &PIa_in ;

PIy = &PIy_in ;

PIay = &PIay_in ;

TH11C = P*T*PIa + PIay +(1−P)*(1−T)*PIy ;

TH10C = P*(1−T)*PIa + (1−P)*T*PIy ;

TH01C = (1−P)*T*PIa + P*(1−T)*PIy ;

TH00C = 1−PIa*(P+T−P*T)−PIay−PIy*(1−P*T);

PI_A = PIa + PIay ;

PI_Y = PIy + PIay ;

TH11OS = (P+T−1)*PI_A + (1−P)*(1−T);

TH10OS = (P–T)*PI_A + T*(1−P);

TH01OS = (T–P)*PI_A + P*(1−T);

TH00OS = (1−P–T)*PI_A + P*T;

PAYY = P*T*PI_A/TH11OS;

PAYN = P*(1−T)*PI_A/TH10OS;

PANY = (1−P)*T*PI_A/TH01OS;

PANN = (1−P)*(1−T)*PI_A/TH00OS;

PROTOS = MAX(PAYY, PAYN, PANY, PANN);

PAYYC = (P*T*PIa+PIay)/TH11C;
```

PAYNC = P*(1−T)*PIa/TH10C;

PANYC = (1−P)*T*PIa/TH01C;

PANNC = PIa*(1−P)*(1−T)/TH00C;

PROTOC = MAX(PAYYC, PAYNC, PANYC, PANNC);

PROTR = PROTOS/PROTOC;

THG1 = P * PI_A + (1−P)* PI_Y;

THG2 = T * PI_A + (1−T) * PI_Y;

PYG1 = (P+(1−P)*PI_Y)*PI_A/THG1;

PNG1 = PI_A*(1−P)*(1−PY_Y)/(1−THG1);

PYG2 = (T+(1−T)*PI_Y)*PI_A/THG2;

PNG2 = PI_A*(1−T)*(1−PY_Y)/(1−THG2);

PROTG = MAX(PYG1, PNG1, PYG2, PNG2);

RPG = PROTG/PROTOC;

TERM1 = THG1*(1−THG1);

TERM2 = THG2*(1−THG2);

VARG2 = ( (1−T)*SQRT(TERM1) + (1−P)*SQRT(TERM2) )**2/(P−T)**2;

COMPO1 = TH11C*(1−TH11C) + TH00C*(1−TH00C)+2*TH11C*TH00C;

COMPO2 = TH01C*(1−TH01C) + TH10C*(1−TH10C)+2*TH10C*TH01C;

COMPO3 = −TH11C*TH01C +TH11C*TH10C + TH00C *TH01C −TH00C*TH10C;

VARC = ( (P−T)**2*COMPO1 +(P+T−2)**2*COMPO2 + 2*(P−T)*(P+T−2)*COMPO3)

/(4*(P−T)**2);

THW = P*PI_A +(1−P)*(1−PI_A);

VARW = THW*(1−THW)/(2*P−1)**2;

VAROS = ( (P+T−1)**2*(P*T+(1−P)*(1−T))+(P−T)**2*(T*(1−P)+P*(1−T)))

```
/(4*((P+T-1)**2+(P-T)**2)**2)-(2*PI_A-1)**2/4;

A = P**2*T**2/TH11C +P**2*(1-T)**2/TH10C+(1-P)**2*T**2

/TH01C+(P+T-P*T)**2/TH00C;

B = 1/TH11C + 1/TH00C;

C = (1-P)**2*(1-T)**2/TH11C + (1-P)**2*T**2/TH10C  + P**2*(1-T)**2/TH01C

+(1-P*T)**2/TH00C;

D = P*T/TH11C + (P+T-P*T)/TH00C;

F = P*(1-P)*T*(1-T)*(1/TH11C+1/TH10C+1/TH01C)+(P+T-P*T)*(1-P*T)/TH00C;

G = (1-P)*(1-T)/TH11C + (1-P*T)/TH00C;

DENO = A*G**2 + B*F**2 + C*D**2 - A*B*C - 2*D*F*G;

NEOM = F**2 + G**2 -B*C-A*C+2*C*D-2*F*G;

LOWERB = NEOM/DENO;

REW = VARW/VARC;

REOS = VAROS/VARC;

RE1 = VARG2/VARC;

RE2 = VARG2/LOWERB;

KEEP P T PI_A PI_Y PIa PIy PIay RE1 RE2 REW REOS PROTR RPG;

PROC PRINT DATA=DATA2&III;

RUN;

%MEND SHU_CHING_SU(III, P_in, T_in, PIa_in, PIy_in, PIay_in);

RUN;

%SHU_CHING_SU(1, 0.6, 0.35, 0.05, 0.7, 0.02);

%SHU_CHING_SU(2, 0.7, 0.35, 0.05, 0.7, 0.02);

%SHU_CHING_SU(3, 0.6, 0.45, 0.05, 0.7, 0.02);
```

```
%SHU_CHING_SU(4, 0.7, 0.45, 0.05, 0.7, 0.02);

RUN;

DATA DATA3;

SET DATA21 DATA22 DATA23 DATA24;

PROC PRINT DATA= DATA3;

PROC EXPORT DATA=DATA3 OUTFILE='c:\SASDATAFILES\OUT11.xls' DBMS=EXCEL REPLACE;

RUN;


PROGRAM-II

SAS CODES FOR SOLVING NON-LINEAR EQUATIIONS;

DATA DATA1;

TH11H = 11/127;

TH10H = 8/127;

TH01H = 6/127;

TH00H = 102/127;

P = 0.686;

T = 0.314;

RUN;

PROC MODEL DATA=DATA1 OUT=DATA2 ;

EQ.A  =   TH11H * P * T/( P*T * X + Z + (1-P)*(1-T)* Y  )

              + TH10H*P*(1-T)/( P*(1-T)*X + (1-P)*T*Y  )

          + TH01H*(1-P)*T/( (1-P)*T*X + P*(1-T)*Y )

          - TH00H*(P+T-P*T)/(1-X*(P+T-P*T)-Z-Y*(1-P*T) ) ;

EQ.B = TH11H/(P*T * X + Z + (1-P)*(1-T)* Y   )
```

$$- \text{TH00H}/(\ 1-X*(P+T-P*T)-Z-Y*(1-P*T)\ )\ ;$$

$$\text{EQ.C}\ =\ \text{TH11H}*(1-P)*(1-T)/(\ P*T\ *\ X\ +\ Z\ +\ (1-P)*(1-T)*\ Y\ )$$

$$+\ \text{TH10H}*(1-P)*T/(\ P*(1-T)*X\ +\ (1-P)*T*Y\ )$$

$$+\ \text{TH01H}*P*(1-T)/(\ (1-P)*T*X\ +\ P*(1-T)*Y\ )$$

$$-\ \text{TH00H}*(1-P*T)/(\ 1-X*(P+T-P*T)-Z-Y*(1-P*T)\ )\ ;$$

SOLVE X Y Z;

RUN;

DATA DATA3;

SET DATA2;

PIA=X+Z;

KEEP PIA X Y Z;

PROC PRINT DATA=DATA3;

RUN;


PROGRAM-III:SAS codes used for real data estimates.

DATA DATA1;

NS = 127;

TH11H = 11/NS;

TH10H = 8/NS;

TH01H = 6/NS;

TH00H = 102/NS;

P = 0.686;

T = 0.314;

DATA DATA2;

```
SET DATA1;

PIA0 = ((P–T)–TH11H*(P–T)–TH10H*(4*P*T–3*P–T)–TH01H*(P+3*T–4*P*T)–TH00H*(P–T))

/(4*(P–T)*(P+T–2*P*T));

PIAY0 =( (P–T)*TH11H*(1+2*P+2*T–4*P*T) + TH10H*(2*P–1)* (2*T**2+2*P*T–P–3*T)

+ TH01H*(2*T–1)*(3*P+T–2*P*T–2*P**2) +TH00H*(2*P–1)*(2*T–1)*(P–T)  –(2*P–1)*(2*T–1)*(P–T)  )

/(4*(P–T)*(P+T–2*P*T));

PIY0 = ((T–P)*(TH11H+TH00H)+(4*P*T–P–3*T)*TH10H–(4*P*T–3*P–T)*TH01H+(P–T))

/(4*(P–T)*(P+T–2*P*T));

VPIAHAT = PIA0*(1–PIA0)/(NS–1)+PIAY0*(1–PIAY0)/(NS–1)–2*PIA0*PIAY0/NS

+(1–P)*(1–T)*(P+T–2*P*T)*(PIA0+PIY0)/(NS*(P–T)**2);

PIA_HAT = PIA0 + PIAY0;

DATA DATA3;

SET DATA2;

LL  = PIA_HAT – 1.96 *SQRT(VPIAHAT);

UL  = PIA_HAT + 1.96 *SQRT(VPIAHAT);

SE_A = SQRT(VPIAHAT);

KEEP PIA_HAT LL UL SE_A;

PROC PRINT DATA = DATA3;

RUN;

DATA DATA4;

SET DATA2;

BB_EST = 0.1339;

VAR_BB_EST = BB_EST*(1–BB_EST)/(NS–1);

ZCAL = (PIA_HAT–BB_EST)/SQRT(VAR_BB_EST+VPIAHAT);
```

KEEP ZCAL;

PROC PRINT DATA = DATA4;

RUN;

Table 1: Simulated values of different measures of protection and efficiency for different values of parameters with $P = 0.686$ and $T = 0.314$.

| $\pi_a$ | $\pi_{y_0}$ | $\pi_{ay_0}$ | $\pi_A$ | $\pi_Y$ | RP(OS) | RP(G) | RE(OS) | RE(1) | RE(2) |
|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 0.70 | 0.02 | 0.07 | 0.72 | 1.0396 | 0.9162 | 2.2227 | 1.1560 | 2.1225 |
| 0.05 | 0.75 | 0.02 | 0.07 | 0.77 | 1.0949 | 0.9284 | 2.0955 | 1.0898 | 2.0401 |
| 0.10 | 0.60 | 0.05 | 0.15 | 0.65 | 1.0319 | 0.9816 | 2.2524 | 1.2114 | 2.1660 |
| 0.10 | 0.65 | 0.05 | 0.15 | 0.70 | 1.0798 | 0.9944 | 2.1264 | 1.1437 | 2.0802 |
| 0.15 | 0.65 | 0.05 | 0.20 | 0.70 | 1.0380 | 0.9682 | 1.9758 | 1.0800 | 1.9573 |
| 0.15 | 0.65 | 0.10 | 0.25 | 0.75 | 1.1715 | 1.0799 | 1.9459 | 1.0775 | 1.9459 |
| 0.20 | 0.60 | 0.05 | 0.25 | 0.65 | 1.0000 | 0.9648 | 1.9459 | 1.0775 | 1.9281 |
| 0.20 | 0.65 | 0.10 | 0.30 | 0.75 | 1.1290 | 1.0530 | 1.8343 | 1.0262 | 1.8301 |
| 0.20 | 0.65 | 0.15 | 0.35 | 0.80 | 1.2101 | 1.1223 | 1.8194 | 1.0257 | 1.7826 |
| 0.25 | 0.60 | 0.05 | 0.30 | 0.65 | 1.0094 | 0.9790 | 1.8343 | 1.0262 | 1.8302 |
| 0.25 | 0.60 | 0.10 | 0.35 | 0.70 | 1.0819 | 1.0360 | 1.8194 | 1.0257 | 1.8154 |
| 0.25 | 0.52 | 0.10 | 0.35 | 0.62 | 1.0336 | 1.0179 | 1.9611 | 1.1056 | 1.9596 |
| 0.25 | 0.60 | 0.10 | 0.35 | 0.70 | 1.0819 | 1.0360 | 1.8194 | 1.0257 | 1.8154 |
| 0.25 | 0.60 | 0.15 | 0.40 | 0.75 | 1.1435 | 1.0864 | 1.8091 | 1.0254 | 1.7732 |
| 0.25 | 0.60 | 0.20 | 0.45 | 0.80 | 1.1965 | 1.1320 | 1.8030 | 1.0252 | 1.7040 |
| 0.30 | 0.56 | 0.05 | 0.35 | 0.61 | 1.0014 | 0.9899 | 1.8031 | 1.0165 | 1.8006 |
| 0.30 | 0.54 | 0.10 | 0.40 | 0.64 | 1.0478 | 1.0278 | 1.8254 | 1.0346 | 1.8228 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.30 | 0.52 | 0.10 | 0.40 | 0.62 | 1.0372 | 1.0237 | 1.8588 | 1.0536 | 1.8582 |
| 0.30 | 0.52 | 0.15 | 0.45 | 0.67 | 1.0853 | 1.0593 | 1.8522 | 1.0532 | 1.8290 |
| 0.30 | 0.50 | 0.15 | 0.45 | 0.65 | 1.0741 | 1.0538 | 1.8865 | 1.0727 | 1.8702 |
| 0.30 | 0.51 | 0.20 | 0.50 | 0.71 | 1.1213 | 1.0891 | 1.8670 | 1.0627 | 1.7956 |
| 0.30 | 0.51 | 0.25 | 0.55 | 0.76 | 1.1517 | 1.1139 | 1.8692 | 1.0628 | 1.7132 |
| 0.35 | 0.51 | 0.10 | 0.45 | 0.61 | 1.0392 | 1.0304 | 1.7872 | 1.0162 | 1.7816 |
| 0.35 | 0.52 | 0.10 | 0.45 | 0.62 | 1.0440 | 1.0324 | 1.7716 | 1.0073 | 1.7641 |
| 0.35 | 0.35 | 0.15 | 0.50 | 0.50 | 1.0000 | 1.0196 | 2.0761 | 1.1817 | 2.0761 |
| 0.35 | 0.35 | 0.20 | 0.55 | 0.55 | 1.0325 | 1.0386 | 2.0792 | 1.1822 | 2.0612 |
| 0.35 | 0.50 | 0.10 | 0.45 | 0.60 | 1.0344 | 1.0285 | 1.8030 | 1.0252 | 1.7991 |
| 0.35 | 0.50 | 0.15 | 0.50 | 0.65 | 1.0742 | 1.0570 | 1.8010 | 1.0251 | 1.7660 |
| 0.35 | 0.35 | 0.20 | 0.55 | 0.55 | 1.0325 | 1.0386 | 2.0792 | 1.1822 | 2.0612 |
| 0.35 | 0.35 | 0.25 | 0.60 | 0.60 | 1.0613 | 1.0576 | 2.0886 | 1.1838 | 2.0162 |
| 0.35 | 0.35 | 0.30 | 0.65 | 0.65 | 1.0791 | 1.0689 | 2.1046 | 1.1865 | 1.9394 |
| 0.40 | 0.47 | 0.05 | 0.45 | 0.52 | 0.9921 | 1.0091 | 1.7716 | 1.0073 | 1.7703 |
| 0.40 | 0.47 | 0.10 | 0.50 | 0.57 | 1.0303 | 1.0325 | 1.7697 | 1.0073 | 1.7623 |
| 0.40 | 0.43 | 0.15 | 0.55 | 0.58 | 1.0459 | 1.0457 | 1.8355 | 1.0437 | 1.8088 |
| 0.40 | 0.43 | 0.20 | 0.60 | 0.63 | 1.0751 | 1.0661 | 1.8419 | 1.0440 | 1.7575 |
| 0.50 | 0.35 | 0.10 | 0.60 | 0.45 | 1.0061 | 1.0294 | 1.8091 | 1.0254 | 1.8051 |
| 0.50 | 0.35 | 0.15 | 0.65 | 0.50 | 1.0304 | 1.0422 | 1.8194 | 1.0257 | 1.7836 |