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A STUDY OF ASYMPTOTIC DISTRIBUTIONS OF CONCOMITANTS OF CERTAIN ORDER STATISTICS

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Abstract: Consider a random sample of size n from an absolutely continuous bivariate distribution of (X, Y). Let $X_{i:n}$ denote the ith order statistic of the X sample values and $Y_{[i:n]}$ its concomitant, the Y-value associated with $X_{i:n}$. In this article we are interested in the asymptotic behavior of some functionals of concomitants. Given two increasing integer sequences $\{r_n, n \geq 1\}$ and $\{s_n, n \geq 1\}$ with $1 \leq r_n \leq s_n \leq n$, let $V_{r_n,s_n,n} = \max(Y_{[r_n:n]}, Y_{[r_n+1:n]}, \ldots, Y_{[s_n:n]})$ and $W_{r_n,s_n,n} = \min(Y_{[r_n:n]}, Y_{[r_n+1:n]}, \ldots, Y_{[s_n:n]})$. We investigate the limiting distributions of $V_{r_n,s_n,n}$, $W_{r_n,s_n,n}$, $R_{r_n,s_n,n} = V_{r_n,s_n,n} - W_{r_n,s_n,n}$, and $M_{r_n,s_n,n} = (V_{r_n,s_n,n} + W_{r_n,s_n,n})/2$ when $\lim_{n\to\infty}(n - r_n) < \infty$ or $\lim_{n\to\infty} r_n/n = p$ for $0 < p < 1$. The statistics $R_{r_n,s_n,n}$ and $M_{r_n,s_n,n}$ can be viewed as range and midrange, respectively. Our results generalize those obtained in Nagaraja and David (1994). We also use these results to investigate the problem of locating the maximum of a nonparametric regression function as discussed in Chen, Huang and Huang (1996).

Key words and phrases: Concomitants of order statistics, regression model.

1. Introduction

In this work, some problems related to concomitants of order statistics will be investigated. Let (X_i, Y_i), $i = 1, \ldots, n$, be a random sample from an absolutely continuous bivariate distribution of (X, Y) with c.d.f. $F(x, y)$. Let $X_{r:n}$ denote the rth order statistic of the X sample values. The concomitants are obtained by arranging the data in the second component according to the ordering in the first component: the pairs are ordered by the X_i, and the Y-value associated with $X_{r:n}$ is denoted by $Y_{[r:n]}$. We call $Y_{[r:n]}$ the concomitant of the rth order statistic.

A systematic study of the concomitants including their ranks, extremes and partial sums has been undertaken only very recently. Its foundations were laid down in the papers of David and Galambos (1974) and Bhattacharya (1984), who used the term induced order statistics. Further works can be found in David, O'Connell and Yang (1977), Yang (1977) and others. The book by David (1981) has extensive materials on this subject.

Throughout the article, for any integers $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, let $f_{a_1,\ldots,a_k:n}$ denote the joint p.d.f. of the order statistics $X_{a_1:n}, \ldots, X_{a_k:n}$, and

f_U, F_U represent the p.d.f. and c.d.f. of a random variable U, respectively. Nagaraja and David (1994) investigated the finite-sample and asymptotic distributions of $V_{k,n}$ when either k is fixed or $k = [np]$ with $0 < p < 1$, where $V_{k,n} = \max(Y_{[n-k+1:n]}, \ldots, Y_{[n:n]})$. Their investigation of the distribution of $V_{k,n}$ relied on the following results for the conditional distribution:

$$f_{n-k,n-k+1,\ldots,n:n}(x_0, \ldots, x_k) = f_{n-k:n}(x_0)k!\Big\{\prod_{i=1}^{k} \frac{f_X(x_i)}{1 - F_X(x_0)}\Big\},$$

$$P(V_{k,n} \le y) = \int_{-\infty}^{\infty} [P(Y \le y|X > x)]^k f_{n-k:n}(x)dx.$$

The first identity means that, conditional on $X_{n-k:n} = x_0$, the order statistics $X_{n-k+1:n}, \ldots, X_{n:n}$ behave as the order statistics of k i.i.d. random variables with the common p.d.f. $f_X(\cdot)/[1 - F_X(x_0)]$.

Now we define $V_{r_n,s_n,n} = \max(Y_{[r_n:n]}, Y_{[r_n+1:n]}, \ldots, Y_{[s_n:n]})$, for any two integer sequences $\{r_n, n \ge 1\}$, $\{s_n, n \ge 1\}$, $1 \le r_n \le s_n \le n$, r_n, s_n and $s_n - r_n + 1$ are increasing in n. Note that $V_{k,n}$ is $V_{r_n,s_n,n}$ with $r_n = n - k + 1$ and $s_n = n$. When $r_n = n - k + 1$ and $s_n = n - k + k_1$, we define $V^{\star}_{k,k_1,n} = \max(Y_{[n-k+1:n]}, Y_{[n-k+2:n]}, \ldots, Y_{[n-k+k_1:n]})$, where $1 \le k_1 \le k \le n$. To study $V_{r_n,s_n,n}$, we employ the same idea as in Nagaraja and David (1994) by observing, for $s_n \le n - 1$,

$$\begin{aligned} &f_{r_n-1,r_n,\ldots,s_n+1:n}(x_0, \ldots, x_{s_n-r_n+1}, x_0^{'}) \\ &= f_{r_n-1,s_n+1:n}(x_0, x_0^{'}) \cdot (s_n - r_n + 1)! \cdot \Big\{\prod_{i=1}^{s_n-r_n+1} \frac{f_X(x_i)}{F_X(x_0^{'}) - F_X(x_0)}\Big\}. \end{aligned}$$

The above result means that, conditional on $(X_{r_n-1:n}, X_{s_n+1:n}) = (x_0, x_0^{'})$, $X_{r_n:n}$, $\ldots, X_{s_n:n}$ behave as the order statistics of $s_n - r_n + 1$ i.i.d. random variables with p.d.f. $f_X(\cdot)/[F_X(x_0^{'}) - F_X(x_0)]$. Therefore, the finite-sample and limiting distributions of $V_{r_n,s_n,n}$ can be obtained by employing arguments in Nagaraja and David (1994) when either $\lim_{n\to\infty}(n - r_n) < \infty$ or $0 < \lim_{n\to\infty} r_n/n \le \lim_{n\to\infty} s_n/n < 1$. To facilitate the discussion, we will consider $\lim_{n\to\infty}(n-r_n) < \infty$ and $\lim_{n\to\infty}(n - r_n) = \infty$ separately. For convenience, they will be referred to as the extreme and quantile cases, respectively.

This article is organized as follows. Sections 2 and 3 are primarily concerned with the asymptotic distribution of $V_{r_n,s_n,n}$. We attempt to give conditions on F_X or F_Y which guarantee that $V_{r_n,s_n,n}$ has a limiting distribution as the sample size tends to infinity. We also give a recipe for finding appropriate normalizing constants of $V_{r_n,s_n,n}$. In particular, the extreme case is discussed in Section 2 and the quantile case is considered in Section 3. In Section 4, we derive asymptotic distributions of two summary statistics $R_{r_n,s_n,n} = V_{r_n,s_n,n} - W_{r_n,s_n,n}$ and $M_{r_n,s_n,n} =$

$(V_{r_n,s_n,n}+W_{r_n,s_n,n})/2$, where $W_{r_n,s_n,n} = \min(Y_{[r_n:n]}, Y_{[r_n+1:n]}, \ldots, Y_{[s_n:n]})$. (Again we define $W^{\star}_{k,k_1,n} = \min\{Y_{[n-k+1:n]}, Y_{[n-k+2:n]}, \ldots, Y_{[n-k+k_1:n]}\}$.) Note that $R_{r_n,s_n,n}$ and $M_{r_n,s_n,n}$ can be viewed as the range and midrange, respectively, of a random sample. Finally, we discuss possible asymptotic distributions of $V_{r_n,s_n,n}$ in Section 5 when (X, Y) satisfies a measurement error model $X = Y + \epsilon$, which is closely related to the extreme case mentioned earlier. This investigation is motivated by the problem of locating the maximum of a nonparametric regression function as found in Chen, Huang and Huang (1996).

Estimation of the maximum of a regression function has been considered extensively in the literature. A lot of work has been done on stochastic approximation schemes, which include the Kiefer-Wolfowitz procedure (Kiefer and Wolfowitz (1952)) and the response surface method (Box and Wilson (1951)). These methods have been applied extensively to on-line identification and adaptive control of stochastic systems. The method discussed in this paper is quite different in nature from those just mentioned. Literature on non-sequential and sequential methods can be found in Chen, Huang and Huang (1996).

2. Asymptotic Distribution of $V_{r_n,s_n,n}$ in the Extreme Case

In this section, we first give an expression for the c.d.f. of $V_{r_n,s_n,n}$, and then we investigate the limiting distribution of $V_{r_n,s_n,n}$ in the extreme case which is summarized as Result 1. Since results for $W_{r_n,s_n,n}$ can be obtained easily by suitably modifying the conditions for $V_{r_n,s_n,n}$, we will only give detailed results for $V_{r_n,s_n,n}$ throughout.

To begin with, we need some notation. Let $\alpha(F) = \inf\{x : F(x) > 0\}$ and $\omega(F) = \sup\{x : F(x) < 1\}$. Denote the pth quantile of F by $\xi_F(p)$. The symbol $\to_d$ represents convergence in distribution.

We now derive the c.d.f. of $V_{r_n,s_n,n}$. Note that, conditional on the values of the order statistics, the concomitants are independent (see Bhattacharya (1984)). We then have

$$\begin{aligned} P(V_{r_n,s_n,n} \le y) &= P(Y_{[r_n:n]} \le y, \ldots, Y_{[s_n:n]} \le y) \\ &= \iint_{x_0' > x_0} \left\{F_3(y|x_0, x_0')\right\}^{s_n - r_n + 1} \cdot f_{r_n-1,s_n+1:n}(x_0, x_0')dx_0dx_0'. \end{aligned} \tag{1}$$

Here $F_3(y|x, x') = P(Y \le y | x \le X \le x')$ when $x \le x'$. When X and Y are independent, $F_3(y|x_0, x_0') = F_Y(y)$ and $P(V_{r_n,s_n,n} \le y) = F_Y^{s_n-r_n+1}(y)$. Otherwise, $V_{r_n,s_n,n}$ will depend on $(X_{r_n-1:n}, X_{s_n+1:n})$.

To proceed, we need the definition of domain of attraction. Assume that F belongs to the domain of (maximum) attraction of G (in short, $F \in D(G)(max)$), that is, $\lim_{n\to\infty} F^n(a_n + b_n x) = G(x)$ for some normalizing constants $a_n \in R$

and $b_n > 0$. It is well known that G must be one of the three extreme value c.d.f.'s: $\Phi_\alpha(x) = \exp(-x^{-\alpha})I(x > 0)$, $\Psi_\alpha(x) = \exp(-(-x)^{-\alpha})I(x < 0)$, or $\Lambda(x) = \exp(-e^{-x})$, where $I(\cdot)$ is the indicator function. A convenient reference for the well-known von Mises conditions to guarantee that $F \in D(G)(max)$ for each of the three cases is Resnick (1987, Propositions 1.15-1.17).

If $\lim_{n\to\infty}(n-r_n) = c$ where c is a positive integer, we may assume $n-r_n \equiv c$ for $n \geq 1$ without loss of generality. Also, we assume $s_n - r_n + 1 \equiv c^*$, where c^* is a positive integer. From Lemma 1 of Nagaraja and David (1994) [also see Theorem 7 of Falk (1987)], we have the following lemma.

Lemma 1. *For a random variable X, if the von Mises conditions are satisfied ($F_X \in D(G)(max)$), then there exist constants a_n, $b_n > 0$, such that the p.d.f. of $((X_{n-r_{k+1}+1:n} - a_n)/b_n, (X_{n-r_k+1:n} - a_n)/b_n, (X_{n-r_{k-1}+1:n} - a_n)/b_n, \ldots, (X_{n-r_1+1:n} - a_n)/b_n)$ converges to $g_{r_{k+1},r_k,r_{k-1},\ldots,r_1}$, where for $w_{r_{k+1}} < w_{r_k} < w_{r_{k-1}} < \cdots < w_{r_1}$,*

$$g_{r_{k+1},r_k,r_{k-1},\ldots,r_1}(w_{r_{k+1}}, w_{r_k}, w_{r_{k-1}}, \ldots, w_{r_1})$$
$$= g(w_{r_{k+1}})\prod_{i=1}^{k}\frac{g(w_{r_i})}{G(w_{r_i})}\cdot\frac{(-\log G(w_{r_1}))^{r_1-1}}{(r_1-1)!}\prod_{j=2}^{k+1}\frac{[\log G(w_{r_{j-1}})-\log G(w_{r_j})]^{r_j-r_{j-1}-1}}{(r_j-r_{j-1}-1)!}.$$

By Lemma 1 and (1), we can easily get the following result which is an extension of Result 1 of Nagaraja and David (1994).

Result 1. Suppose the condition of Lemma 1 holds and assume that there exist constants A_n and $B_n > 0$ such that

$$F_3(A_n + B_n y|a_n + b_n x, a_n + b_n x') \longrightarrow H(y|x, x'), \tag{2}$$

as $n \to \infty$, for all x, x' and y, where $H(y|x, x')$ is a c.d.f. Then, as $n \to \infty$,

$$P(V_{r_n,s_n,n} \leq A_n + B_n y) \longrightarrow \iint_{u>v} [H(y|u, v)]^{c^*} g_{c+2,c-c^*+1}(u, v)dudv. \tag{3}$$

Theorem 5(a) in Section 5 is an illustration of Result 1 with $B_n = b_n$ in (2). Note that $g_{c+2,c-c^*+1}(\cdot,\cdot)$ is a p.d.f. We will demonstrate how a conditional probability problem in (3) can be reduced to an unconditional one by setting proper conditions on $H(y|x, x')$. Note that as $n \to \infty$, $a_n + b_n x \to \omega(F_X)$ and $a_n + b_n x' \to \omega(F_X)$, for all x, x'. Suppose the conditional c.d.f. $F_{Y|X}(y|x)$ converges to $H(y)$ as x tends to $\omega(F_X)$. Then (2) holds with $H(y|u, v) = H(y)$, $A_n = 0$, and $B_n = 1$. This fact is formalized in the following lemma.

Lemma 2. *Suppose that $\lim_{x\to\omega(F_X)} F_{Y|X}(y|x) = H(y)$ holds for some y. Then*

$$\lim_{x'\leq x'';x',x''\to\omega(F_X)} F_3(y|x', x'') = H(y).$$

If the assumption of Lemma 2 holds and the appropriate von Mises condition holds for F_X, then $P(V_{r_n,s_n,n} \leq y) \longrightarrow [H(y)]^{c^*}$ as $n \to \infty$. Thus if H is a c.d.f., $V_{r_n,s_n,n} \to_d V$, i.e., V behaves like the maximum of a random sample of size c^* from the c.d.f. H. Theorem 5(b) in Section 5 is an illustration of the above discussions.

3. Asymptotic Distribution of $V_{r_n,s_n,n}$ in the Quantile Case

In this section we investigate the limiting behavior of $V_{r_n,s_n,n}$ when $\lim_{n\to\infty}(n - r_n) = \infty$. In fact, we only consider $r_n/n = r_0 + o(n^{-1/2})$ and $s_n/n = s_0 + o(n^{-1/2})$, where $0 < r_0 \leq s_0 < 1$. Set $x_1 = \xi_{F_X}(r_0)$ and $x_2 = \xi_{F_X}(s_0)$. Hereafter, let $t_n = s_n - r_n + 1$ and assume $f_X(x_1) > 0$ and $f_X(x_2) > 0$.

For constants A_{t_n} and $B_{t_n} > 0$ free of x, x' and for fixed y, define $H_n(y|x,x') = [F_3(A_{t_n} + B_{t_n} y|x,x')]^{t_n}$. Note that $X_{r_n:n} = x_1 + O_P(n^{-1/2})$ and $X_{s_n:n} = x_2 + O_P(n^{-1/2})$. As long as the limit of $H_n(y|x_1 + u_1/\sqrt{n}, x_2 + u_2/\sqrt{n})$ remains unchanged as u_1 and u_2 vary, it is expected that $P(V_{r_n,s_n,n} \leq A_{t_n} + B_{t_n} y)$ will converge to the limit of $H_n(y|x_1,x_2)$. Again as in Section 2, we reduce a conditional probability problem in (3) to an unconditional one. The following result makes the above heuristic argument precise. Its proof can be obtained easily by employing the same argument used in establishing Result 2 of Nagaraja and David (1994).

Result 2. Assume that as $n \to \infty$, for u_1, $u_2 \in R$,

$$H_n\left(y \,\middle|\, x_1 + \frac{u_1}{\sqrt{n}}, x_2 + \frac{u_2}{\sqrt{n}}\right) \longrightarrow H(y|x_1,x_2), \tag{4}$$

where $H(y|x_1,x_2)$ is a c.d.f. Then

$$P(V_{r_n,s_n,n} \leq A_{t_n} + B_{t_n} y) \longrightarrow H(y|x_1,x_2). \tag{5}$$

Remark 1. When $n - s_n$ tends to a constant as $n \to \infty$ (hence $n - s_n$ equals that constant when n is large enough), $P(V_{r_n,s_n,n} \leq A_{t_n} + B_{t_n} y) \longrightarrow H(y|x_1)$. Hence, our Result 2 is indeed an extension of Result 2 of Nagaraja and David (1994).

Suppose that $t_n = c_1$ where c_1 is a positive integer. Then $x_1 = x_2$. For this case, we expect that the asymptotic distribution of $V_{r_n,s_n,n}$ can be viewed as the maximum of a sample of size c_1 with c.d.f. $F_{Y|X}(y|x_1)$ if $F_{Y|X}(y|x)$ is continuous at x_1.

Result 3. Assume $c_1 = \lim_{n\to\infty} t_n$ and $F_{Y|X}(y|x)$ is continuous at x_1. Then $P(V_{r_n,s_n,n} \leq y) \longrightarrow [F_{Y|X}(y|x_1)]^{c_1}$.

This can be established easily by the continuity of $F_{Y|X}(y|x)$ at x_1 and identifying $H(y|x_1,x_1) = [F_{Y|X}(y|x_1)]^{c_1}$ in Result 2.

When $t_n \to \infty$, we will give sufficient conditions along the lines of Nagaraja and David (1994) to guarantee that $V_{r_n,s_n,n}$(appropriately normalized) has the same limiting distribution as that of the maximum of a random sample of size t_n from the c.d.f. $F_3(y|x_1,x_2)$. Note that the derivation of the extreme value distributions lies in finding a good approximation to the product of tail probabilities. Hence, the above claim can be established easily if $1-F_Y(y)$ is of the same magnitude as $1-F_3(y|x,x')$ as $y\to\omega(F_Y)$, i.e., $F_3(y|x,x')$ and $F_Y(y)$ are tail equivalent. Note that $F_3(y|x,x')$ and $F_Y(y)$ are said to be tail equivalent if there exists a finite positive function $c(\cdot,\cdot)$ such that for any (x,x') in S,

$$\lim_{y\to\omega(F_Y)} \frac{1-F_Y(y)}{1-F_3(y|x,x')} = c(x,x'). \tag{6}$$

If (6) holds, $F_Y \in D(G)$ with normalizing constants A_{t_n} and B_{t_n}, and $t_n\to\infty$, then by Resnick (1987, p.67, Proposition 1.19), $H_n(y|x,x') \to H(y|x,x')$, with $H(y|x,x') = G(A(x,x') + B(x,x')y)$. Furthermore,

$$\begin{aligned} &A(x,x') = 0 \text{ and } B^{\alpha}(x,x') = c(x,x'), && \text{if } G=\Phi_\alpha;\\ &A(x,x') = 0 \text{ and } B^{-\alpha}(x,x') = c(x,x'), && \text{if } G=\Psi_\alpha;\\ &A(x,x') = \log c(x,x') \text{ and } B(x,x') = 1, && \text{if } G=\Lambda. \end{aligned} \tag{7}$$

Since G is a continuous c.d.f., if $c(x,x')$ is continuous in a set S, then from (7) it is clear that $H(y|x,x')$ is also continuous in S. We then conclude that (4) holds. Combining the above discussion, we have the following result.

Result 4. Suppose $F_Y\in D(G)$, that is, there exist constants A_{t_n} and $B_{t_n}>0$ such that $P(Y_{t_n:t_n} \le A_{t_n} + B_{t_n}y) \to G(y)$, for all real y. Also assume there exist neighborhoods I_1 and I_2 of x_1 and x_2, respectively, such that (6) holds with $c(x,x')$ being continuous at $(x,x') = (x_1,x_2)$. Then $P(V_{r_n,s_n,n} \le A_{t_n} + B_{t_n}y) \longrightarrow G(A(x_1,x_2) + B(x_1,x_2)y)$ as $t_n\to\infty$. The functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ depend on $c(x_1,x_2)$ and the form of G as at (7).

Convenient choices for A_{t_n} and B_{t_n} can be found in Galambos (1987, pp.53-54).

4. Asymptotic Distributions of Range and Midrange

As mentioned in Section 2, it is not difficult to modify the conditions and give the corresponding results for $W_{r_n,s_n,n}$. In this section we will study the limiting distributions of the range $R_{r_n,s_n,n}$ and midrange $M_{r_n,s_n,n}$ of concomitants. We will discuss the extreme and quantile cases separately in Sections 4.1 and 4.2. Also we denote by $F\in D(L)(\min)$ if $\lim_{n\to\infty}[1-F(c_n+d_nx)]^n = 1-L(x)$ holds, where $d_n>0$.

4.1. Extreme case

We first state the following theorem.

Theorem 1. *Suppose the conditions of Result* 1 *for both maximum and minimum versions hold and assume that there exist* A_n, $B_n > 0$, C_n *and* $D_n > 0$, *such that*

$$F_3(A_n+B_nv|a_n+b_nx, a_n+b_nx')-F_3(C_n+D_nw|a_n+b_nx, a_n+b_nx') \longrightarrow H(v, w|x, x'),$$

as $n \to \infty$, *for all* (x, x') *and* (v, w). *Then, as* $n \to \infty$,

$$\begin{aligned}&P(W_{r_n,s_n,n} \leq C_n + D_nw, V_{r_n,s_n,n} \leq A_n + B_nv)\\ &\longrightarrow \iint_{x'>x} \left[H^{c^*}(v|x, x') - H^{c^*}(v, w|x, x')\right] g_{c+2,c-c^*+1}(x, x')dxdx'.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}&P(W_{r_n,s_n,n} \geq C_n + D_nw, V_{r_n,s_n,n} \leq A_n + B_nv)\\ =&\int\int_{x<x'} [F_3(A_n + B_nv|x, x') - F_3(C_n + D_nw|x, x')]^{c^*} f_{r_n-1,s_n+1:n}(x, x')dxdx'\\ =&\int\int_{x<x'} [F_3(A_n+B_nv|a_n+b_nx, a_n+b_nx')-F_3(C_n+D_nw|a_n+b_nx, a_n+b_nx')]^{c^*}\\ &\qquad\cdot\left\{s_n^2 f_{r_n-1,s_n+1:n}(a_n + b_nx, a_n + b_nx')\right\} dxdx'.\end{aligned}$$

By a similar argument as in Result 1, we conclude that

$$\begin{aligned}&P(W_{r_n,s_n,n} \geq C_n + D_nw, V_{r_n,s_n,n} \leq A_n + B_nv)\\ =&\int\int_{x<x'} \{H(v, w|x, x')\}^{c^*} g_{c+2,c-c^*+1}(x, x')dxdx'.\end{aligned}$$

This completes the proof of Theorem 1.

As an extension of Lemma 2, it can be shown similarly that if for some y and y',

$$\lim_{x\to\omega(F_X)} F_{Y|X}(y|x) = H(y) \quad \text{and} \quad \lim_{x\to\omega(F_X)} F_{Y|X}(y'|x) = H(y'), \tag{8}$$

then $\lim_{x'\leq x'';x',x''\to\omega(F_X)} F_3(y|x', x'') - F_3(y'|x', x'') = H(y) - H(y')$. So we can conclude that if (8) holds and the appropriate von Mises condition holds for F_X, then as $n \to \infty$, $P(W_{r_n,s_n,n} \leq w, V_{r_n,s_n,n} \leq v) \longrightarrow H^{c^*}(v) - [H(v) - H(w)]^{c^*}$. From this we can obtain the asymptotic distributions of $R_{r_n,s_n,n}$ and $M_{r_n,s_n,n}$.

4.2. Quantile case

First we determine the asymptotic joint distribution of $(W_{r_n,s_n,n}, V_{r_n,s_n,n})$ for $\lim_{n\to\infty} t_n = \infty$. The method developed here is inspired by Section 2.9 of Galambos (1987).

Theorem 2. *Assume $F_Y \in D(H)(max)$, $F_Y \in D(L)(min)$, and that $F_Y(y)$ and $F_3(y|x,x')$ are tail equivalent. In particular,*

$$\lim_{y\to\omega(F_Y)} \frac{1-F_Y(y)}{1-F_3(y|x,x')} = c(x,x') \quad \text{and} \quad \lim_{y\to\alpha(F_Y)} \frac{F_Y(y)}{F_3(y|x,x')} = d(x,x'),$$

where $c(x,x')$ and $d(x,x')$ are continuous at $(x,x') = (x_1,x_2)$, and there exist constants A_{t_n}, $B_{t_n} > 0$, C_{t_n} and $D_{t_n} > 0$ such that

$$\lim_{t_n\to\infty} [F_3(A_{t_n} + B_{t_n}y|x_1,x_2)]^{t_n} = H(A(x_1,x_2) + B(x_1,x_2)y), \tag{9}$$

$$\lim_{t_n\to\infty} [1 - F_3(C_{t_n} + D_{t_n}y|x_1,x_2)]^{t_n} = 1 - L(C(x_1,x_2) + D(x_1,x_2)y). \tag{10}$$

Then

$$\begin{aligned}&\lim_{t_n\to\infty} P(W_{r_n,s_n,n} \le C_{t_n} + D_{t_n}w, V_{r_n,s_n,n} \le A_{t_n} + B_{t_n}v)\\ &= L(C(x_1,x_2) + D(x_1,x_2)w)\cdot H(A(x_1,x_2) + B(x_1,x_2)v).\end{aligned}$$

Theorem 2 points out that if, after suitable normalization, the asymptotic distribution of $(W_{r_n,s_n,n}, V_{r_n,s_n,n})$ exists, then the normalized $W_{r_n,s_n,n}$ and $V_{r_n,s_n,n}$ are asymptotically independent. The normalizing constants in Theorem 2, A_{t_n}, B_{t_n}, C_{t_n} and D_{t_n}, depend only on the c.d.f. F_Y (see Galambos (1987), pp.58-59) and can be obtained easily. Also the functions $C(\cdot,\cdot)$, $D(\cdot,\cdot)$ are given as follows (see Resnick (1987), p.67):

$$\begin{aligned}&C(x_1,x_2) = 0 \text{ and } D^{\alpha}(x_1,x_2) = d(x_1,x_2), \quad \text{if } L(x) = 1-\Phi_\alpha(-x);\\ &C(x_1,x_2) = 0 \text{ and } D^{-\alpha}(x_1,x_2) = d(x_1,x_2), \quad \text{if } L(x) = 1-\Psi_\alpha(-x);\\ &C(x_1,x_2) = \log d(x_1,x_2) \text{ and } D(x_1,x_2) = 1, \quad \text{if } L(x) = 1-\Lambda(-x).\end{aligned}$$

Proof of Theorem 2. First we give the joint distribution of $W_{r_n,s_n,n}$ and $V_{r_n,s_n,n}$.

$$\begin{aligned}&P(W_{r_n,s_n,n} \ge w, V_{r_n,s_n,n} \le v) = P(w \le Y_{[r_n:n]} \le v, \ldots, w \le Y_{[s_n:n]} \le v)\\ &= \int\int_{x<x'} [F_3(v|x,x') - F_3(w|x,x')]^{t_n} f_{r_n-1,s_n+1:n}(x,x')dxdx'.\end{aligned}$$

We now study the limiting behavior of $[F_3(v|x,x') - F_3(w|x,x')]^{t_n}$, as $n\to\infty$. As in the proof of Result 2 of Nagaraja and David (1994), we only need to show that, with proper normalization, $[F_3(v|x,x') - F_3(w|x,x')]^{t_n}$ converges to a proper limit.

Substitute v and w with $A_{t_n} + B_{t_n}v$ and $C_{t_n} + D_{t_n}w$. For $v > \alpha(H)$ and $w < \omega(L)$, we have

$$\lim_{t_n\to\infty} F_3(A_{t_n} + B_{t_n}v|x,x') = 1, \quad \lim_{t_n\to\infty} F_3(C_{t_n} + D_{t_n}w|x,x') = 0. \tag{11}$$

Note that $\log s = -(1-s) + o(1)$, as $s \to 1$. We have

$$\begin{aligned}&[F_3(A_{t_n} + B_{t_n} v|x,x') - F_3(C_{t_n} + D_{t_n} w|x,x')]^{t_n}\\ &= \exp\{-t_n[1 - F_3(A_{t_n} + B_{t_n} v|x,x')] - t_n F_3(C_{t_n} + D_{t_n} w|x,x') + o(1)\}, \quad (12)\end{aligned}$$

as $t_n \to \infty$, where $\alpha(L) < w < \omega(L)$ and $\alpha(H) < v < \omega(H)$. Then by (9), (10) and (11), as $t_n \to \infty$,

$$\begin{aligned}\lim_{t_n\to\infty} t_n[1 - F_3(A_{t_n} + B_{t_n} v|x,x')] &= -\log[H(A(x,x') + B(x,x')v)],\\ \lim_{t_n\to\infty} t_n F_3(C_{t_n} + D_{t_n} w|x,x') &= -\log[1 - L(C(x,x') + D(x,x')w].\end{aligned}$$

We thus conclude that the right side of (12) tends to $[H(A(x,x')+B(x,x')v)][1-L(C(x,x')+D(x,x')w)]$. On the other hand, if $w > \omega(L)$ or $v < \alpha(H)$, the limit is evidently zero. Finally, for $w < \alpha(L)$ or $v > \omega(H)$, the limit is $H(A(x,x') + B(x,x')v)$ or $1 - L(C(x,x') + D(x,x')w)$, respectively. In all of these cases, the limit can be written in the form of $[H(A(x,x') + B(x,x')v)][1 - L(C(x,x') + D(x,x')w)]$. This completes the proof.

For finding the limiting behavior of $R_{r_n,s_n,n}$ and $M_{r_n,s_n,n}$, we need the following lemma (see Section 2.9 of Galambos (1987)).

Lemma 3. *Assume that the distribution function of the vector (Y_n, U_n) converges weakly to $T(y)S(u)$, where $T(y)$ and $S(u)$ are continuous distribution functions. Then, as $n \to \infty$,*

$$\lim_{n\to\infty} P(Y_n + U_n < x) = \int_{-\infty}^{\infty} S(x-y)dT(y).$$

Now we can state and prove the main result.

Theorem 3. *If $D_{t_n} = B_{t_n}$ and the conditions in Theorem 2 hold, then, as $t_n \to \infty$,*

$$\begin{aligned}&\lim_{t_n\to\infty} P(R_{r_n,s_n,n} < A_{t_n} - C_{t_n} + B_{t_n} x)\\ &= \int_{\infty}^{\infty} [1 - L(C(x_1,x_2) + D(x_1,x_2)(y-x))]dH(A(x_1,x_2) + B(x_1,x_2)y),\\ &\lim_{t_n\to\infty} P(2M_{r_n,s_n,n} < A_{t_n} + C_{t_n} + B_{t_n} x)\\ &= \int_{\infty}^{\infty} L(C(x_1,x_2) + D(x_1,x_2)(x-y))dH(A(x_1,x_2) + B(x_1,x_2)y).\end{aligned}$$

Proof. By Theorem 2, the random variables $Y_{t_n} = (V_{r_n,s_n,n} - A_{t_n})/B_{t_n}$ and $U_{t_n} = (W_{r_n,s_n,n} - C_{t_n})/B_{t_n}$ are asymptotically independent. Hence Lemma 3 is applicable to both $(Y_{t_n}, -U_{t_n})$ and (Y_{t_n}, U_{t_n}), from which Theorem 3 follows.

By a similar argument as in the proof of Result 2 and Lemma 3, we obtain the following result for $\lim_{n\to\infty} t_n = c_1$.

Theorem 4. *If the conditions of Result* 3 *for both maximum and minimum versions hold, then as* $n \to \infty$,

$$P(W_{r_n,s_n,n} \le w, V_{r_n,s_n,n} \le v) \longrightarrow F_{Y|X}^{c_1}(v|x_1) - \Big[F_{Y|X}(v|x_1) - F_{Y|X}(w|x_1)\Big]^{c_1},$$

where $c_1 = \lim_{n\to\infty} t_n$.

5. Locating the Maximizer of a Regression Function

Now we apply the results of Section 2 to derive the limiting distribution of $V^*_{k,k_1,n} - \omega(F_Y)$ when k and k_1 are fixed. The basic setup that we will consider is as follows. Suppose that $\{(X_i, Y_i),\ i = 1, \ldots, n\}$, is a random sample from (X, Y) where $X = Y + \epsilon$, Y and ϵ are independent, $E\epsilon = 0$, and $\omega(F_Y) < \infty$. We further assume that only the X_i's are observed.

The above setting is partly motivated by the problem of locating the maximizer of a nonlinear regression functions θ, i.e. $X = \theta(Z) + \epsilon$, as considered in Chen, Huang and Huang (1996). When Z falls in a compact set and $\theta(\cdot)$ is continuous, $\theta(Z)$ is bounded (i.e., $\omega(F_Y) < \infty$). Note that $V^*_{k,1,n} = Y_{[n-k+1:n]}$ and $W^*_{k,k,n} = \min\{Y_{[n-k+1:n]}, \ldots, Y_{[n:n]}\}$. If $W^*_{k,k,n} - \omega(F_Y) = o_P(1)$, then any of $\{Z_{[n-k+1:n]}, \ldots, Z_{[n:n]}\}$ is a consistent estimate of the maximizer of $\theta(\cdot)$. This result suggests the following algorithm to locate the maximizer of $\theta(\cdot)$. We first apply a clustering algorithm to $\{Z_{[n-k+1:n]}, \ldots, Z_{[n:n]}\}$ to identify a region of Z which contains z_0, the maximizer of $\theta(\cdot)$, and then do a local modeling on that region to get an estimate of z_0. This algorithm is attractive when a global modeling for $\theta(\cdot)$ is not feasible (i.e., the dimensionality of Z is high). A theoretical study of this algorithm will be reported elsewhere.

The algorithm just described is a modification of the best-r-points-average method for locating z_0, based on n samples $(Z_1, X_1), \ldots, (Z_n, X_n)$ under the assumption that $Z_1, \ldots, Z_n$ are over a bounded interval C in R according to a certain distribution. The best-r-points-average method estimates z_0 by the average of those Z_i's corresponding to the r largest order statistics X_i's. Denote this estimate by $\hat{z}_{rn}$. The utility of the best-r-points-average method depends on whether the unobservable regression function value $\theta(\cdot)$ corresponding to the extreme order statistics $X_{n-r+1:n}, \ldots, X_{n:n}$ is close to the maximum of $\{\theta(Z_1), \ldots, \theta(Z_n)\}$. Chen, Huang and Huang (1996) derived the convergence rates of $\hat{z}_{rn} - z_0$.

5.1. Main results and discussions

First we describe conditions on the distributions of Y and ϵ. For simplicity, for a random variable U we write $\omega(F_U)$ and $\alpha(F_U)$ as ω_U and α_U, respectively.

Condition 1. $\omega_U < \infty$ and f_U is positive on (u, ω_U) where $u < \omega_U$. Also, $(\omega_U - u)f_U(u)/[1 - F_U(u)]$ tends to a positive limit as $u \to \omega_U$.

Remark 2. Suppose that $X = \theta(Z) + \epsilon$ where Z takes values in a bounded interval C. Let z_0 denote the unique global maximizer of $\theta(Z)$ over Z. When $\theta^{'}(z_0) = 0$, $\theta^{''}(z_0) < 0$, and the p.d.f. of Z is bounded away from zero and infinity on C, $Y = \theta(Z)$ satisfies Condition 1 with limit 2. Refer to Chen, Huang and Huang (1996) for the motivation of Condition 1.

Suppose Y satisfies Condition 1 with limit β, which guarantees that $F_Y \in D(\Psi_\beta)(max)$. We assume that ϵ satisfies either Condition 1 or the following condition.

Condition 2. $\omega_\epsilon = \infty$ and f_ϵ satisfies $f_\epsilon \sim ABvx^{-u+v-1}\exp(-Bx^v)$, as $x \to \infty$, where $v > 1$, $u \geq 0$, and A, B are positive constants. (Here "$g(x) \sim h(x)$ as $x \to \infty$" means $\lim_{x\to\infty} g(x)/h(x) = 1$.)

If ϵ satisfies Condition 2, then $F_\epsilon \in D(\Lambda)(max)$.

Now we state the main theorem on $V^*_{k,k_1,n}$. Similar results can be obtained for $W^*_{k,k_1,n}$.

Theorem 5.

(a) *Suppose Y and ϵ satisfy Condition 1 with limits β and α, respectively. Then $X \in D(\Psi_{\alpha+\beta})(max)$, $\omega_Y - V^*_{k,k_1,n} = O_p(n^{-1/(\alpha+\beta)})$ and*

$$\begin{aligned} &P\Big(c_X^{-1/(\alpha+\beta)} n^{1/(\alpha+\beta)}(V^*_{k,k_1,n} - \omega_Y) \leq y\Big) \\ &= \int\int_{x'>x} \left[H(y|x,x')\right]^{k_1} g_{k+1,k-k_1}(x,x')dxdx' \end{aligned}$$

as $n \to \infty$, where

$$g_{k+1,k-k_1}(x,x') = \frac{g(x')}{G(x')} \cdot \frac{[-\log G(x')]^{k-k_1-1}}{(k-k_1-1)!} \cdot g(x) \cdot \frac{[\log G(x') - \log G(x)]^{k_1}}{k_1!},$$

with $x, x' \in R$, $G(x) = \Psi_{\alpha+\beta}(-x)$, $g = G^{'}$, and

$$H(y|x,x') = \begin{cases} \frac{B(y,x)-B(y,x')}{x^\gamma-(x')^\gamma}, & \textit{if } x > x' \geq y \geq 0, \\ \frac{B(y,x)}{x^\gamma-(x')^\gamma}, & \textit{if } x > y \geq x' \geq 0. \end{cases}$$

Here $\gamma = \alpha + \beta$, $B(y,x) = x^\gamma \int_{y/x}^1 t^{\beta-1}(1-t)^\alpha dt/Beta(\beta, \alpha+1)$, and $c_X = \lim_{t\to\omega_X}[1 - F_X(\omega_X - \eta)]/\eta^{\alpha+\beta}$. Note that $\omega_X = \omega_Y + \omega_\epsilon$.

(b) *When Y satisfies Condition 1 with limit β and ϵ satisfies Condition 2, $\omega_Y - V^*_{k,k_1,n} = O_p((\log n)^{-(v-1)/v})$ and the limiting c.d.f. of $(B^{1/v}v\log n)^{(v-1)/v}(\omega_Y - V^*_{k,k_1,n})$ is the distribution of maximum of a random sample of size k_1 from the gamma distribution with parameter β.*

Remark 3. Chen, Huang and Huang (1996) discusses the convergence rate of $\omega_Y - Y_{[n:n]}$ (i.e., $\omega_Y = \theta(z_0)$), which can be translated to the rate of convergence of $z_0 - \hat{z}_{1n}$ if the shape of $\theta(\cdot)$ near z_0 is known. Theorem 5 confirms the validity of the best-r-points-average method for locating the maximizer for a certain type of error distributions and also provides results on the asymptotic distribution of $\omega_Y - V^*_{r,r,n}$. It even improves the convergence rates derived in Chen, Huang and Huang (1996). It is possible to use Theorem 5 to find the asymptotic distribution of $z_0 - \hat{z}_{rn}$. However, such a result depends critically on the shape of $\theta(\cdot)$ near z_0 such as whether $\theta(\cdot)$ is symmetric at z_0. We will report this result elsewhere.

When ϵ is normally distributed, Theorem 5(b) states that $V^*_{k,k_1,n}$ converges to ω_Y at a logarithmic rate. This is much slower than that of the curve fitting approach considered in Müller (1989). But the proposed estimate is much simpler and easily understood so that it can be implemented in practical applications easily. Refer to Remarks 1 and 2, and Section 4 of Chen, Huang and Huang (1996) for related discussions.

5.2. Examples

In this subsection, we will present a key lemma used in proving Theorem 5 and give two examples to illustrate Theorem 5. Note that when k and k_1 are fixed, an expression for the limiting c.d.f. of appropriately normalized $V^*_{k,k_1,n}$ is given in Result 1. The following lemma, based on Propositions 1.15-1.17 and the representation (1.3) in Resnick (1987), gives a prescription for the choice of normalizing constants $\{a_n\}$ and $\{b_n\}$ that appear in Result 1.

Lemma 4. *Assume that f_ϵ is positive on $(t_\epsilon, \omega_\epsilon)$ where $t_\epsilon < \omega_\epsilon$.*

(a) *If $\omega_\epsilon = \infty$, and $\lim_{t\to\infty} tf_\epsilon(t)/[1 - F_\epsilon(t)] = \alpha$ for some $\alpha > 0$, then there are constants a_n and $b_n > 0$ such that the distribution of $(\epsilon_{n:n} - a_n)/b_n$ converges to Φ_α. Moreover, the constants can be chosen as $a_n = 0$ and $b_n = F_\epsilon^{-1}(1 - 1/n)$.*

(b) *If $\omega_\epsilon < \infty$, and $\lim_{t\to\omega_\epsilon}(\omega_\epsilon - t)f_\epsilon(t)/[1 - F_\epsilon(t)] = \alpha$, then there are constants a_n and $b_n > 0$ such that the distribution of $(\epsilon_{n:n} - a_n)/b_n$ converges to Ψ_α. Moreover, the constants can be chosen as $a_n = \omega_\epsilon$ and $b_n = \omega_\epsilon - F_\epsilon^{-1}(1-1/n)$.*

(c) *If $\int_{-\infty}^{\omega_\epsilon}(1 - F_\epsilon(t))dt < \infty$ and $\lim_{t\to\omega_\epsilon} f_\epsilon(t)[1 - F_\epsilon(t)]^{-2}\int_t^{\omega_\epsilon}[1 - F_\epsilon(u)]du = 1$, then there are constants a_n and $b_n > 0$ such that the distribution of $(\epsilon_{n:n} - a_n)/b_n$ converges to Λ. Moreover, the constants can be chosen as $a_n = F_\epsilon^{-1}(1 - 1/n)$ and $b_n = [1 - F_\epsilon(a_n)]/f_\epsilon(a_n)$.*

Example 1. Suppose that $Y \sim U[0, 1]$ and $\epsilon \sim U[-1, 1]$. We now apply Lemma 4 and Theorem 5 to derive the asymptotic distribution of $B_n^{-1}(Y_{[n-k+1:n]} - A_n)$ with proper normalizing constants A_n and B_n. Note that X is the sum of two uniformly distributed variables. Hence, X has a triangular density concentrated on $[-1, 2]$. By Lemma 4, $F_X \in D(\Psi_2)$ and we can choose $a_n = 2$ and $b_n = 2 - F_X^{-1}(1 - n^{-1}) = 2n^{-1/2}$. Also, $F_Y, F_\epsilon \in D(\Psi_1)$. Hence Y satisfies Condition 1 with $\omega_Y = 1$ and limit 1, ϵ satisfies Condition 1 with limit 1, and X satisfies Condition 1 with $\omega_X = 2$ and limit 2.

It can be shown that $P(Y \le A_n + B_n y | a_n + b_n x < X < a_n + b_n x')$ converges to a conditional distribution function $H(y|x, x')$ with $A_n = 1$ and $B_n = 2n^{-1/2}$, where

$$H(y|x,x') = \begin{cases} 1, & \text{if } 0 \ge y, \\ (x + x' - 2y)/(x + x'), & \text{if } x > x' \ge y \ge 0, \\ (x - y)^2/(x^2 - x'^2), & \text{if } x > y \ge x' \ge 0, \\ 0, & \text{if } y > x > x' \ge 0. \end{cases}$$

Obviously the above conditional probability is continuous everywhere.

Recall that X satisfies Condition 1 with limit 2. By Lemma 4, the limiting p.d.f. of $(X_{[n:n]} - \omega_X)/2n^{-1/2}$ is $-2xe^{-x^2}$ with $x \le 0$. Then by Theorem 5 for $y \ge 0$, as $n \to \infty$,

$$P\Big(\frac{1 - Y_{[n-k+1:n]}}{2n^{-1/2}} \le y\Big) \longrightarrow \int\int_{x'>x} H(y|x, x') g_{k+1,k-1}(x, x') dx dx',$$

where $g_{k+1,k-1}(x, x') = 4x'^{2k-3} x \cdot e^{-x^2}(x^2 - x'^2)/(k-2)!$.

Example 2. Suppose that Y satisfies Condition 1 with limit 1 and $\epsilon \sim N(0, 1)$. Then $F_Y \in D(\Psi_1)(max)$ with $a_n = 1$ and $b_n = n^{-1}$, and ϵ satisfies Condition 2 with $u = 1$ and $v = 2$.

Note that $P(Y \le 1 - b_n y | a_n - b_n x < X < a_n - b_n x') = \exp(-y)$ if $b_n^{-1} > y > 0$, $= 0$ if $y > b_n^{-1}$, $= 1$ if $0 > y$. By Lemma 4(c), the limiting c.d.f. of $(X_{[n:n]} - a_n)/b_n$ is $\exp(-e^{-x})$. It follows from Theorem 5(b) that, as $n \to \infty$, for $y \ge 0$, $P\Big(b_n^{-1}(\omega_Y - Y_{[n-k+1:n]}) \le y\Big) \longrightarrow \int\int_{x'>x} H(y|x, x') g_{k+1,k-1}(x, x') dx dx'$, where $g_{k+1,k-1}(x, x')$ is the p.d.f. of the form $\exp(-x') \frac{\exp[-(k-2)x']}{(k-2)!} \exp(-x) \exp(-e^{-x})$ $[\exp(-x') - \exp(-x)]$. Hence, $P(b_n^{-1}(\omega_Y - Y_{[n-k+1:n]}) \le y) \longrightarrow \exp(-y)$. We conclude that $(2 \log n)^{1/2}(\omega_Y - Y_{[n-k+1:n]})$ converges in distribution to a standard exponential distribution.

5.3. Proof of Theorem 5

Proof of Theorem 5(a)

Recall that $X = Y + \epsilon$, $F_Y \in D(\Psi_\beta)(max)$, and $F_\epsilon \in D(\Psi_\alpha)(max)$. The proof will proceed in two steps. First, we show that $F_X \in D(\Psi_\gamma)(max)$ with $\gamma = \alpha + \beta$. Second, we derive the limiting distributions of $Y_{[n-k+1:n]}$ and $V^\star_{k,k_1,n}$.

Step 1: Observe that

$$
\begin{aligned}
1-F_X(\omega_X-\eta)=&\int_{\omega_Y-\eta}^{\omega_Y} f_Y(y)[1-F_\epsilon(\omega_X-\eta-y)]dy\\
=&\int_{\omega_Y-\eta}^{\omega_Y}(\eta+y)f_Y(y)\left[\frac{1-F_\epsilon(\omega_X-\eta-y)}{(\eta+y)f_\epsilon(\omega_X-\eta-y)}-\frac{1}{\alpha}\right]f_\epsilon(\omega_X-\eta-y)dy\\
&+\int_{\omega_Y-\eta}^{\omega_Y}\frac{\eta+y}{\alpha}f_Y(y)f_\epsilon(\omega_X-\eta-y)dy,
\end{aligned}
$$

and $f_X(\omega_X-\eta)=\int_{\omega_Y-\eta}^{\omega_Y} f_Y(y)f_\epsilon(\omega_X-\eta-y)dy$. Note that both ϵ and Y satisfy the von Mises conditions, i.e., $\lim_{t\to\omega_\epsilon}(\omega_\epsilon-t)f_\epsilon(t)/[1-F_\epsilon(t)]=\alpha$ and $\lim_{t\to\omega_Y}(\omega_Y-t)f_Y(t)/[1-F_Y(t)]=\beta$. Roughly speaking, as $\eta\to 0$, we have

$$
\begin{aligned}
1-F_\epsilon(\omega_\epsilon-\eta)\sim c_\epsilon\eta^\alpha,\ f_\epsilon(\omega_\epsilon-\eta)\sim\alpha c_\epsilon\eta^{\alpha-1},\\
1-F_Y(\omega_Y-\eta)\sim c_Y\eta^\beta,\ f_Y(\omega_Y-\eta)\sim\beta c_Y\eta^{\beta-1}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
1-F_X(\omega_X-\eta)&\sim c_\epsilon c_Y\beta\int_{\omega_Y-\eta}^{\omega_Y}(\omega_Y-y)^{\beta-1}(\eta+y-\omega_Y)^\alpha dy\\
&=c_\epsilon c_Y\beta\eta^{\alpha+\beta}\int_0^1 s^{\beta-1}(1-s)^\alpha ds=c_\epsilon c_Y\beta\eta^{\alpha+\beta}Beta(\beta,\alpha+1),\\
f_X(\omega_X-\eta)&\sim c_\epsilon c_Y\alpha\beta\int_{\omega_Y-\eta}^{\omega_Y}(\omega_Y-y)^{\beta-1}(\eta+y-\omega_Y)^{\alpha-1}dz\\
&=c_\epsilon c_Y\alpha\beta\eta^{\alpha+\beta-1}Beta(\beta,\alpha).
\end{aligned}
$$

Hence $\lim_{t\to\omega_X}(\omega_X-t)f_X(t)/[1-F_X(t)]=\alpha+\beta=\gamma$. By Lemma 1 of Nagaraja and David (1994) and Lemma 4, we have $F_X\in D(\Psi_\gamma)(max)$, $a_n=\omega_X=\omega_Y+\omega_\epsilon$, and $b_n=c_X^{-1/\gamma}n^{-1/\gamma}$ where $c_X=c_\epsilon c_Y\beta Beta(\beta,\alpha+1)$.

Step 2: Now we derive the limiting distribution of $Y_{[n-k+1:n]}$. By Result 1, we will show that $P(Y\le A_n-B_ny|a_n-b_nx<X<a_n-b_nx')$ converges to a nondegenerate conditional distribution function $H(y|x,x')$ with $A_n=\omega_Y$, $B_n=b_n$, $a_n=\omega_X$ and $b_n=c_X^{-1/\gamma}n^{-1/\gamma}$. Since $X=Y+\epsilon$ and $\omega_X=\omega_Y+\omega_\epsilon$, we have $P(Y\le\omega_Y-b_ny|\omega_X-b_nx<X<\omega_X-b_nx')=1$ for $y<0$, and $P(Y\le\omega_Y-b_ny|\omega_X-b_nx<X<\omega_X-b_nx')=0$ for $y>x>x'>0$. Note that $X\in D(\Psi_\gamma)(max)$ and for $\delta>0$,

$$
P(X_{n:n}\le\omega_X-c_X^{-1/\gamma}n^{-1/(\gamma+\delta)})\sim(1-n^{-\gamma/(\gamma+\delta)})^n\to 0.
$$

Then we only need to consider the case that $x>x'\ge 0$ and $\omega_X-b_nx\ge\omega_X-b_nn^{\delta_0}$ for some $\delta_0>0$. For $x\ge y\ge 0$, we have

$$
P(Y\le\omega_Y-b_ny,X>\omega_X-b_nx)=\int_{\omega_Y-b_nx}^{\omega_Y-b_ny}f_Y(t)[1-F_\epsilon(\omega_X-b_nx-t)]dt
$$

$$\sim \beta c_Y c_\epsilon b_n^{\alpha+\beta} \int_y^x s^{\beta-1}(x-s)^\alpha ds = n^{-1} \frac{x^\gamma}{Beta(\beta, \alpha+1)} \int_{y/x}^1 t^{\beta-1}(1-t)^\alpha dt.$$

Observe that

$$\begin{aligned}
&P(Y \leq \omega_Y - b_n y, \omega_X - b_n x < X < \omega_X - b_n x') \\
=&P(Y \leq \omega_Y - b_n y, X > \omega_X - b_n x) - P(Y \leq \omega_Y - b_n y, X > \omega_X - b_n x') \\
\sim&\frac{n^{-1}}{Beta(\beta, \alpha+1)} \left[x^\gamma \int_{y/x}^1 t^{\beta-1}(1-t)^\alpha dt - (x')^\gamma \int_{y/x'}^1 t^{\beta-1}(1-t)^\alpha dt \right], \text{ if } x > x' \geq y \geq 0. \\
&P(Y \leq \omega_Y - b_n y, \omega_X - b_n x < X < \omega_X - b_n x') = P(Y \leq \omega_Y - b_n y, X > \omega_X - b_n x) \\
\sim& n^{-1} \frac{x^\gamma}{Beta(\beta, \alpha+1)} \int_{y/x}^1 t^{\beta-1}(1-t)^\alpha dt, \text{ if } x > y \geq x' \geq 0.
\end{aligned}$$

Write $B(y,x) = x^\gamma \int_{y/x}^1 t^{\beta-1}(1-t)^\alpha dt / Beta(\beta, \alpha+1)$. Note that

$$P(\omega_X - b_n x < X < \omega_X - b_n x') \sim c_X(b_n x)^\gamma - c_X(b_n x')^\gamma = n^{-1}[x^\gamma - (x')^\gamma].$$

Hence $P(Y \leq \omega_Y - b_n y | \omega_X - b_n x < X < \omega_X - b_n x') \to H(y|x,x')$, where $H(y|x,x')$ is defined as in Theorem 5(a). It can be seen that the above conditional probability is continuous everywhere. By Lemma 4(b), the limiting c.d.f. of $(\omega_X - X_{[n:n]})/b_n$ is $\exp(-x^\gamma)$ with $x \geq 0$. Then, from Result 1, for $y \geq 0$, as $n \to \infty$,

$$P\Big(\frac{\omega_Y - Y_{[n-k+1:n]}}{b_n} \leq y\Big) \to \iint_{x'>x} H(y|x,x') g_{k+1,k-1}(x,x') dx dx',$$

where $g_{k+1,k-1}(x,x') = \gamma^2 (x')^{\gamma k - \gamma - 1}[(k-2)!]^{-1} x^{\gamma-1}[x^\gamma - (x')^\gamma] \exp(-x^\gamma)$. Also

$$P\Big(\frac{\omega_Y - V^*_{k,k_1,n}}{b_n} \leq y\Big) \to \iint_{x'>x} \left[H(y|x,x')\right]^{k_1} g_{k+1,k-k_1}(x,x') dx dx'.$$

This completes the proof of Theorem 5(a).

Proof of Theorem 5(b)

Now we consider the case that $F_\epsilon \in D(\Lambda)(max)$ with $\omega_\epsilon = \infty$, and Y satisfies Condition 1 with limit β, $\beta \geq 1$. Then for this ϵ, we have $F_\epsilon \in D(\Lambda)(max)$ with $a_n = (B^{-1} \log n)^{1/v} - \frac{u \log(B^{-1} \log n)}{v^2 B^{1/v} (\log n)^{(v-1)/v}}$ and $b_n \sim (Bv)^{-1} a_n^{1-v}$. Without loss of generality, we further assume that $\omega_Y = 0$. Note that $F_Y \in D(\Psi_\beta)(max)$ with $\beta \geq 1$, $a_n = \omega_Y$, and $b_n = c_Y^{-1/\beta} n^{-1/\beta}$. Then by Corollary 1.14 and Proposition 1.16(c) in Resnick (1987), we have $1 - F_Y(y) \sim c_Y(-y)^\beta$ as $y \to 0^-$, where c_Y is a constant. The proof of Theorem 5(b) will proceed in two steps. We first show that $F_X \in D(\Lambda)(max)$ and give the normalizing constants for $X_{n:n}$. Then we

derive the limiting distribution.
Step 1: We show that the following lemma holds.

Lemma 5.
(a) $1 - F_X(x) \sim \frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)y^{-[u+(v-1)\beta]}\exp(-By^v)$ *as* $y \to \infty$.
(b) $F_X \in D(\Lambda)(max)$ *with*

$$a_n = (B^{-1}\log n)^{1/v} - \frac{\log v^\beta B^{-(u-\beta)/v} - \log[Ac_Y\Gamma(\beta+1)] + \frac{u+(v-1)\beta}{v}\log\log n}{vB^{1/v}(\log n)^{1-1/v}}$$

as $n \to \infty$ *and* $b_n = [nf_X(a_n)]^{-1} \sim (Bv)^{-1}a_n^{1-v}$.

Proof.
(a) We proceed by studying the behavior of $f_X(x)$, $1-F_X(x)$, and $\int_x^\infty[1-F_X(t)]dt$ as x tends to infinity. Since $E|Y|$ and $E|\epsilon|$ are finite and $X = Y + \epsilon$, we have $\int_{-\infty}^\infty[1-F_X(x)]dx < \infty$.

Note that $1 - F_\epsilon(x) \sim Ax^{-u}\exp(-Bx^v)$ as $x \to \infty$, by Lemma 5(a) in Chen, Huang and Huang (1996). For $\eta = -B^{-1}\beta x^{1-v}\log x$, we have

$$\begin{aligned}&\int_{\alpha_Y}^{\eta} f_Y(y)A(x-y)^{-u}\exp(-B(x-y)^v)dy\\ &\le A(x-\eta)^{-u}\exp[-B(x-\eta)^v] = o(x^{-[u+(v-1)\beta]}e^{-Bx^v}),\end{aligned}$$

and

$$\begin{aligned}&A\int_\eta^0 \frac{f_Y(y)}{(x-y)^u}e^{-B(x-y)^v}dy = A\int_0^{-\eta}\frac{f_Y(-t)}{(x+t)^u}e^{-B(x+t)^v}dy\\ &\sim A\beta c_Y x^{-u}e^{-Bx^v}\int_0^{B^{-1}\beta x^{1-v}\log x} t^{\beta-1}e^{-Bvx^{v-1}t}dt\\ &\sim A(Bv)^{-\beta}c_Y\Gamma(\beta+1)x^{-[u+(v-1)\beta]}e^{-Bx^v},\end{aligned}$$

by Condition 1, the Mean-Value Theorem, and $\lim_{x\to\infty}\int_0^{v\beta\log x} t^{\beta-1}e^{-t}dt = \Gamma(\beta+1)$. Hence

$$A\int_x^{x-\alpha_Y}\frac{f_Y(x-t)}{t^u}\exp(-Bt^v)dt \sim \frac{A}{(B-v)^\beta}c_Y\Gamma(\beta+1)x^{-[u+(v-1)\beta]}\exp\left(-Bx^v\right). \tag{13}$$

We conclude that, as $x \to \infty$,

$$1 - F_X(x) \sim \frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)x^{-[u+(v-1)\beta]}e^{-Bx^v}. \tag{14}$$

(b) Recall that $f_X(x) = ABv\int_{\alpha_Y}^0 f_Y(y)(x-y)^{-u+v-1}\exp(-Bx^v)dy$. Hence

$$f_X(x) \sim \frac{Ac_Y}{(Bv)^{\beta-1}}\Gamma(\beta+1)x^{-[u+(v-1)(\beta-1)]}\exp\left(-Bx^v\right) \tag{15}$$

as $x \to \infty$, by the same argument used in deriving (13). Then by (14), we have

$$\lim_{x\to\infty} \frac{\int_x^\infty [1-F_X(t)]dt}{\frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)x^{-[u+(v-1)(\beta+1)]}\exp(-Bx^v)}$$

$$= \lim_{x\to\infty} \frac{-[1-F_X(x)]}{-\frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)\{\frac{u+(v-1)(\beta+1)}{Bv}x^{-[u+(v-1)(\beta+1)+1]}+1\}x^{-[u+(v-1)\beta]}\exp(-Bx^v)} = 1.$$

It follows that

$$\int_x^\infty [1-F_X(t)]dt \sim \frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)x^{-[u+(v-1)(\beta+1)]}e^{-Bx^v}. \tag{16}$$

By (14), (15) and (16), we conclude that

$$\lim_{x\to\infty} \frac{f_X(x)}{[1-F_X(x)]^2}\int_x^\infty [1-F_X(t)]dt = 1.$$

This means that $F_Y \in D(\Lambda)(max)$ by Lemma 4(c).

It remains to find acceptable choices of normalizing constants a_n and b_n. By (14), Lemma 4(c), and taking logarithm of both sides of $\frac{Ac_Y}{(Bv)^\beta}\Gamma(\beta+1)a_n^{-[u+(v-1)\beta]}$ $\exp(-Ba_n^v) = n^{-1}$, we get

$$-\log[Ac_Y\Gamma(\beta+1)] + \beta\log(Bv) + [u+(v-1)\beta]\log a_n + Ba_n^v = \log n. \tag{17}$$

Hence $a_n \to \infty$ and $a_n \sim (B^{-1}\log n)^{1/v}$ by dividing both sides of (17) by a_n^v. It follows from $b_n = [1-F_X(a_n)]/f_X(a_n)$, (14), and (15) that an acceptable choice for b_n is $(Bv)^{-1}a_n^{1-v}$. This tells us that in an expansion of a_n, we may neglect terms which are $o((\log n)^{1/v-1})$.

Next, write $a_n = (B^{-1}\log n)^{1/v} + r_n$ where r_n is a remainder which is $o((\log n)^{1/v})$. Substituting this into (17), we find

$$o(1) + \beta\log(Bv) - \log[Ac_Y\Gamma(\beta+1)] + \frac{u+(v-1)\beta}{v}(-\log B + \log\log n)$$
$$+ r_n\frac{v\log n}{(B^{-1}\log n)^{1/v}} = 0.$$

Hence we conclude

$$a_n = (B^{-1}\log n)^{1/v} - \frac{\log[v^\beta B^{-(u-\beta)/v}] - \log[Ac_Y\Gamma(\beta+1)] + \frac{u+(v-1)\beta}{v}\log\log n}{vB^{1/v}(\log n)^{1-1/v}}.$$

Step 2: Now we derive the limiting distribution of $b_n^{-1}(\omega_Y - Y_{[n-k+1:n]})$. The proof will proceed by first showing that the following hold:

$$P(X > a_n + b_n x) \sim n^{-1}e^{-x}e^{-[u+(v-1)\beta]x/\log n}, \tag{18}$$

$$P(b_n y \le Y \le 0, X > a_n + b_n x) \sim n^{-1}e^{-x}\frac{1}{\Gamma(\beta)}\int_0^{-y} s^{\beta-1}e^{-s}ds. \tag{19}$$

Proof of (18). Recall that $b_n \sim (Bv)^{-1} a_n^{1-v}$ by Lemma 5(b). We have

$$\log(a_n + b_n x) \sim \log a_n + \frac{x}{Bva_n^v} \text{ and } (a_n + b_n x)^v \sim a_n^v + B^{-1}x.$$

By (14) and (17), we have

$$\begin{aligned}
&\log P(X > a_n + b_n x)\\
&\sim -\beta \log Bv + \log[Ac_Y \Gamma(\beta+1)] - [u + (v-1)\beta] \log(a_n + b_n x) - B(a_n + b_n x)^v\\
&\sim -\log n - x - \frac{u + (v-1)\beta}{Ba_n^v} x.
\end{aligned}$$

Proof of (19). When $y \geq 0$, we have $P(Y \leq b_n y, X > a_n + b_n x) = P(X > a_n + b_n x)$, and when $\alpha_Y < b_n y < 0$, $P(b_n y \leq Y \leq 0, X > a_n + b_n x) = \int_{b_n y}^0 f_Y(t)[1 - F_\epsilon(a_n + b_n x - t)]dt$. By Lemma 5, $a_n^{u+(v-1)\beta} \sim B^{-[u+(v-1)\beta]/v} (\log n)^{-[u+(v-1)\beta]/v}$, and

$$\begin{aligned}
Ba_n^v &= B\{(B^{-1}\log n)^{1/v} - \frac{\log v^\beta B^{-(u-\beta)/v} - \log[Ac_Y\Gamma(\beta+1)] + \frac{u+(v-1)\beta}{v} \log\log n}{vB^{1/v}(\log n)^{1-1/v}}\\
&\sim \log n - \{\log v^\beta B^{-(u-\beta)/v} - \log[Ac_Y\Gamma(\beta+1)] + \frac{u+(v-1)\beta}{v} \log\log n\}.
\end{aligned}$$

Then (19) follows.

Now we show that $P(Y \leq A_n + B_n y | a_n + b_n x < X < a_n + b_n x')$ converges to a nondegenerate conditional distribution function $H(y|x, x')$ with $A_n = 0$ and $B_n = b_n$, where a_n and b_n are as described in Lemma 5(b). By (18) and (19), we have

$$\begin{aligned}
&P(a_n + b_n x < X < a_n + b_n x') \sim n^{-1}\exp(-x) - n^{-1}\exp(-x'),\\
&P(b_n y \leq Y, a_n + b_n x < X < a_n + b_n x') = P(a_n + b_n x < X < a_n + b_n x'), \text{ if } y \geq 0\\
&P(b_n y \leq Y, X > a_n + b_n x) - P(b_n y \leq Y, X > a_n + b_n x')\\
\sim &\left(n^{-1}e^{-x} - n^{-1}e^{-x'}\right) \frac{1}{\Gamma(\beta)} \int_0^{-y} s^{\beta-1} e^{-s} ds, \quad \text{if } 0 > y > b_n^{-1}\alpha_Y.
\end{aligned}$$

Hence $P(Y \leq b_n y | a_n + b_n x < X < a_n + b_n x') \to H(y|x, x') = [\Gamma(\beta)]^{-1} \int_{-y}^\infty s^{\beta-1} e^{-s} ds$ if $0 > y > b_n^{-1}\alpha_Y$, $= 0$ if $b_n^{-1}\alpha_Y \geq y$, $= 1$ if $y \geq 0$. Note that $H(y|x, x')$ does not depend on x and x'. By Lemma 4(c), the limiting c.d.f. of $(X_{[n:n]} - a_n)/b_n$ is $\exp(-e^{-x})$ with $x \in R$. Then, from Result 1, for $y \leq 0$, as $n \to \infty$,

$$P\Big(\frac{Y_{[n-k+1:n]} - \omega_Y}{b_n} \leq y\Big) \longrightarrow \iint_{x' > x} H(y|x, x') g_{k+1,k-1}(x, x') dx dx',$$

where $g_{k+1,k-1}(x,x') = e^{-x'}\frac{(e^{-x'})^{k-2}}{(k-2)!}e^{-x}e^{-e^{-x}}(e^{-x'} - e^{-x})$ by Lemma 1. Hence

$$P\Big(\frac{\omega_Y - Y_{[n-k+1:n]}}{b_n} \le y\Big) \to \frac{1}{\Gamma(\beta)}\int_0^y t^{\beta-1}e^{-t}dtI(y \ge 0),$$

and

$$P\Big(\frac{\omega_Y - V_{k,k_1,n}}{b_n} \le y\Big) \to \Big[\frac{1}{\Gamma(\beta)}\int_0^y t^{\beta-1}e^{-t}dt\Big]^{k_1} I(y \ge 0).$$

This completes the proof.

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