Statistica Sinica **8**(1998), 801-812

NONPARAMETRIC TESTS FOR THE MULTIVARIATE MULTI-SAMPLE LOCATION PROBLEM

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Abstract: Nonparametric tests for the multi-sample multivariate location problem are proposed which extend the two-sample multivariate rank tests by Randles and Peters (1990) to the multi-sample setting. The asymptotic distributions of the proposed statistics under the null hypothesis and under certain contiguous alternatives are obtained for a class of elliptically symmetric distributions. Comparisons are made between the proposed statistics and several competitors via Pitman asymptotic relative efficiencies and Monte Carlo results. The tests proposed perform better than the Lawley-Hotelling generalized T^2 for heavy-tailed distributions. For normal to light-tailed distributions, the proposed statistics also perform better than other nonparametric competitors and the proposed analog of the signed-rank test performs better than the Lawley-Hotelling generalized T^2 for light-tailed distributions.

Key words and phrases: Interdirections, location problem, multi-sample, multivariate, nonparametric.

1. Introduction

Consider tests for the multivariate multi-sample location problem. Assume that $\boldsymbol{X}_i^{(\alpha)} = (X_{i1}^{(\alpha)}, \ldots, X_{ip}^{(\alpha)})', i = 1, \ldots, n_\alpha$, denotes a random sample of size n_α from a p-variate continuous population with density function $f(\boldsymbol{X} - \boldsymbol{\theta}_\alpha)$ and p-dimensional location parameter $\boldsymbol{\theta}_\alpha$. Here $\alpha = 1, \ldots, c$ indexes samples from c different populations. These samples are assumed to be mutually independent. We are interested in testing $H_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_c = \boldsymbol{\theta}$ against the general alternative that $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_c$ are not all equal.

A normal theory test for this problem can be based on the likelihood ratio criterion. Assume that the underlying distributions are all p-variate normal with common unknown covariance matrix $\boldsymbol{\Sigma}$ and mean vectors possibly different from each other. The Lawley-Hotelling generalized T^2 (Lawley (1938) and Hotelling (1951)) is defined as

$$T^2 = \sum_{\alpha=1}^{c} n_\alpha (\bar{\boldsymbol{X}}^{(\alpha)} - \bar{\boldsymbol{X}})' \boldsymbol{S}^{-1} (\bar{\boldsymbol{X}}^{(\alpha)} - \bar{\boldsymbol{X}}),$$

where

$$\boldsymbol{S} = \frac{1}{N-c}\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}^{(\alpha)})(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}^{(\alpha)})',$$

with $\bar{\boldsymbol{X}}^{(\alpha)} = \frac{1}{n_\alpha}\sum_{i=1}^{n_\alpha}\boldsymbol{X}_i^{(\alpha)}, \bar{\boldsymbol{X}} = \frac{1}{N}\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}\boldsymbol{X}_i^{(\alpha)}$, and $N = n_1+n_2+\cdots+n_c$. The Lawley-Hotelling generalized T^2 statistic is affine invariant. Thus if the data are transformed to form $\boldsymbol{Y}_i^{(\alpha)} = \boldsymbol{D}\boldsymbol{X}_i^{(\alpha)} + \boldsymbol{d}^*$ for $i = 1,\ldots,n_\alpha$ and $\alpha = 1,\ldots,c$, where $\boldsymbol{D}$ is a $p\times p$ nonsingular matrix of constants and $\boldsymbol{d}^*$ is $p\times 1$ vector of constants, then T^2 computed on the $\boldsymbol{Y}_i^{(\alpha)}$'s would be identical to T^2 computed on the original $\boldsymbol{X}_i^{(\alpha)}$'s. This property insures, for example, that the performance of T^2 is consistent over changes in the matrix $\boldsymbol{\Sigma}$ and in the direction of shift. We also note that the asymptotic null distribution of Hotelling's T^2 is chi-square with $p(c-1)$ degrees of freedom.

Puri and Sen (1971) proposed a nonparametric approach to this problem based on a component-wise ranking. Let $R_{ij}^{(\alpha)}$ denote the rank of $X_{ij}^{(\alpha)}$ among the N observations $X_{1j}^{(1)},\ldots,X_{n_1j}^{(1)},\ldots,X_{1j}^{(c)},\ldots,X_{n_cj}^{(c)}$ for each component $j = 1,\ldots,p$. Let $J(\cdot)$ denote a score function defined on (0,1) that is nondecreasing and nonconstant, and satisfies $\int_0^1 J(u)du = 0$ plus $\int_0^1 J^2(u)du < \infty$. Define $E_{ij}^{(\alpha)} = J((N+1)^{-1}R_{ij}^{(\alpha)})$ for $j = 1,\ldots,p$ and $\boldsymbol{E}_i^{(\alpha)} = (E_{i1}^{(\alpha)},\ldots,E_{ip}^{(\alpha)})'$ for $i = 1,\ldots,n_\alpha$ and $\alpha = 1,\ldots,c$. In addition, form the $p\times N$ matrix of scores:

$$\boldsymbol{E} = \begin{pmatrix} E_{11}^{(1)}\cdots E_{n_11}^{(1)}\cdots E_{n_c1}^{(c)} \\ \cdots \\ E_{1p}^{(1)}\cdots E_{n_1p}^{(1)}\cdots E_{n_cp}^{(c)} \end{pmatrix}.$$

The test statistic is analogous to the Lawley-Hotelling generalized T^2 computed on the scores of the componentwise ranks. It rejects H_0 for large values of

$$L_N = \sum_{\alpha=1}^{c} n_\alpha(\bar{\boldsymbol{E}}^{(\alpha)} - \bar{\boldsymbol{E}})'\boldsymbol{V}^{-1}(\bar{\boldsymbol{E}}^{(\alpha)} - \bar{\boldsymbol{E}}),$$

where the elements of $\boldsymbol{V} = \{V_{st}\}$ are given by

$$V_{st} = \frac{1}{N}(\boldsymbol{E}^{[s]\prime}\boldsymbol{E}^{[t]}) - \frac{1}{N^2}(\mathbf{1}'\boldsymbol{E}^{[s]})(\mathbf{1}'\boldsymbol{E}^{[t]}),$$

where $\boldsymbol{E}^{[s]}(\boldsymbol{E}^{[t]})$ is the sth (tth) row of $\boldsymbol{E}$ and $\mathbf{1}$ is a $N\times 1$ vector of ones. Under the null hypothesis, L_n has a limiting chi-square distribution with $p(c-1)$ degrees of freedom. However, unlike T^2, L_n is not affine invariant. Thus its performance will depend on the form of the covariance matrix and the direction of shift.

An affine-invariant nonparametric test procedure was recently proposed by Hettmansperger and Oja (1994). Their test rejects H_0 for large values of $H =$

$\sum_{\alpha=1}^{c}(1-n_\alpha/n)H_\alpha$, where H_α is a two-sample difference in location statistic used to measure the difference in location between the αth sample and all the other samples combined. To describe the component statistics, consider H_1 and let $\hat{\boldsymbol{\theta}}$ denote the Oja median (Oja (1983)) of all c samples combined. Align the data points to form $\boldsymbol{Z}_i^{(1)} = \boldsymbol{X}_i^{(1)} - \hat{\boldsymbol{\theta}}$ for $i = 1, \ldots, n_1$, and $\boldsymbol{Z}_i^{(2)} = \boldsymbol{X}_{i*}^{(\alpha)} - \hat{\boldsymbol{\theta}}$ for $i = 1, \ldots, N - n_1$, where i^* and α delineate all the data points except those in the first sample. The statistic H_1 is then the Hettmansperger, Nyblom, Oja (1994) sign test statistic based on $Z_i^{(1)}$ for $i = 1, \ldots, n_1$ and $-\boldsymbol{Z}_i^{(2)}$ for $i = 1, \ldots, N - n_1$ combined into a sample of size N. The test statistic H is affine invariant and has a limiting chi-square distribution with $p(c-1)$ degrees of freedom under H_0.

An alternative class of affine-invariant, nonparametric test statistics for this problem, based on interdirections, is described in Section 2. Section 3 shows their limiting null distribution and develops some Pitman efficiencies relative to Lawley-Hotelling generalized T^2. In Section 4 these tests are compared to those described in the current section via Monte Carlo results. A data set is also used to show comparative results. Key steps to proofs required in Section 3 are contained in the appendix.

2. The Proposed Class of Statistics

Parametric multivariate tests perform poorly when the underlying distribution is heavy-tailed. They are particularly vulnerable to outliers. Moreover, outliers are quite difficult to identify in multivariate settings. Thus nonparametric tests are particularly important. The family of statistics proposed in this paper is built on the class of multivariate affine-invariant statistics, denoted by $Z_{N,\phi}$, proposed by Randles and Peters (1990) for testing the difference in location between two elliptically symmetric populations. The extension to the c-sample setting is made by summing all possible pairs of two-sample statistics. Thus the class of statistics proposed is given by

$$\begin{aligned}
W_{N,\phi} &= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} Z_{N,\phi}^{\alpha,\beta} \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} \frac{n_\alpha n_\beta}{NE(\phi^2)}\Big\{\frac{1}{n_\alpha^2}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\alpha}\cos\left(\pi\hat{p}_\alpha(i,j;\hat{\boldsymbol{\theta}})\right)\phi(\frac{R_{\alpha,i}}{N})\phi(\frac{R_{\alpha,j}}{N}) \\
&\qquad +\frac{1}{n_\beta^2}\sum_{i=1}^{n_\beta}\sum_{j=1}^{n_\beta}\cos\left(\pi\hat{p}_\beta(i,j;\hat{\boldsymbol{\theta}})\right)\phi(\frac{R_{\beta,i}}{N})\phi(\frac{R_{\beta,j}}{N}) \\
&\qquad -\frac{2}{n_\alpha n_\beta}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta}\cos\left(\pi\hat{p}_{\alpha,\beta}(i,j;\hat{\boldsymbol{\theta}})\right)\phi(\frac{R_{\alpha,i}}{N})\phi(\frac{R_{\beta,j}}{N})\Big\},
\end{aligned}$$

where $Z_{N,\phi}^{\alpha,\beta}$ compares the αth and βth samples' locations. Here $\hat{\boldsymbol{\theta}}$ denotes an affine equivariant estimator of the common location parameter $\boldsymbol{\theta}$ when H_0 is true and is based on all N observations. Randles and Peters (1990) used the sample mean of the N observations for $\hat{\boldsymbol{\theta}}$. For our case we use another affine equivariant estimator, the Oja median (Oja (1983)). In the above expression, $R_{\alpha,i}$ denotes the rank of distance $D_{\alpha,i}$ among all N distances $D_{1,1},\ldots,D_{1,n_1},\ldots,D_{c,1},\ldots,D_{c,n_c}$, where

$$D_{\alpha,i} = (\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})$$

and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}(\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})(\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})'$$

(A more robust estimator of the variance-covariance matrix could be used in place of $\widehat{\boldsymbol{\Sigma}}$, but the performance is virtually the same.) The term $\hat{p}_\alpha(i,j;\hat{\boldsymbol{\theta}})$ is the sample proportion of hyperplanes formed by $\hat{\boldsymbol{\theta}}$ and $p-1$ of the observations $\boldsymbol{X}_{i'}^{(\alpha)}(i'=1,\ldots,n_\alpha$, but $i \neq i' \neq j)$ such that $\boldsymbol{X}_i^{(\alpha)}$ and $\boldsymbol{X}_j^{(\alpha)}$ are on opposite sides of the hyperplane formed. The term $\hat{p}_\beta(i,j;\hat{\boldsymbol{\theta}})$ is similarly defined. Here $\boldsymbol{X}_i^{(\alpha)}(\boldsymbol{X}_i^{(\beta)})$ is the ith observation from the αth (βth) sample. The term $\hat{p}_{\alpha,\beta}(i,j;\hat{\theta})$ represents the sample proportion of hyperplanes formed by $\hat{\boldsymbol{\theta}}$ and $p-1$ of the $n_\alpha + n_\beta - 2$ observations $(\boldsymbol{X}_{i'}^{(\alpha)}, i' = 1,\ldots,n_\alpha$ with $i' \neq i$, and $\boldsymbol{X}_{j'}^{(\beta)}, j' = 1,\ldots,n_\beta$ with $j' \neq j)$ such that $\boldsymbol{X}_i^{(\alpha)}$ (coming from the αth sample) and $\boldsymbol{X}_j^{(\beta)}$ (coming from the βth sample) are on opposite sides of the hyperplane formed. The score function ϕ is nondecreasing on (0,1) and satisfies $E(\phi^2) = \int \phi^2(t)dt < \infty$. In particular, $\phi_1(u) = 1$ and $\phi_2(u) = u$ are emphasized.

3. Pitman Asymptotic Relative Efficiencies

This section establishes the asymptotic null distribution of $W_{N,\phi}$ under the class of elliptically symmetric distributions and displays some Pitman asymptotic relative efficiencies. Proofs of the results are sketched in the appendix. Let $\boldsymbol{X}_i^{(\alpha)}, i = 1,\ldots,n_\alpha$ and $\alpha = 1,\ldots,c$ be independent random samples from the same elliptically symmetric distribution with dispersion matrix $\boldsymbol{\Sigma}$ and a location parameter $\boldsymbol{\theta}$. We assume that both $\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma} = O_p(N^{-1/2})$ and $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} = O_p(N^{-1/2})$ and that $\lim_{N\to\infty}(n_\alpha/N) = \lambda_\alpha$ with $0 < \lambda_\alpha < 1$, and $\lambda_1+\lambda_2+\cdots\lambda_c = 1$. Because the statistics $Z_{N,\phi}^{\alpha,\beta}$ are all affine invariant, we assume without loss of generality that $\boldsymbol{\Sigma} = \boldsymbol{I}(p\times p)$ and that $\boldsymbol{\theta}$ is the origin. Under H_0, we let $\boldsymbol{X}_i^{(\alpha)} = (Q_{\alpha,i})^{1/2}\boldsymbol{U}_i^{(\alpha)}$ where the $\boldsymbol{U}_i^{(\alpha)}$ are all i.i.d. uniformly distributed on the p-dimensional unit sphere, the $Q_{\alpha,i}$'s are all i.i.d. positive R.V.'s and independent of the $\boldsymbol{U}_i^{(\alpha)}$'s. We use an approximating statistic which has the same asymptotic null distribution as $W_{N,\phi}$.

Randles and Peters (1990) showed that an approximating quantity for $Z_{N,\phi}^{\alpha,\beta}$ in the two sample case is:

$$\begin{aligned}Z_{N,\phi}^{*,\alpha,\beta} = \frac{n_\alpha n_\beta}{NE(\phi^2)}\Big\{&\frac{1}{n_\alpha^2}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\alpha}\cos(\pi p_\alpha(i,j;\boldsymbol{\theta}))\phi(H(Q_{\alpha,i}))\phi(H(Q_{\alpha,j}))\\ &+\frac{1}{n_\beta^2}\sum_{i=1}^{n_\beta}\sum_{j=1}^{n_\beta}\cos(\pi p_\beta(i,j;\boldsymbol{\theta}))\phi(H(Q_{\beta,i}))\phi(H(Q_{\beta,j}))\\ &-\frac{2}{n_\alpha n_\beta}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta}\cos(\pi p_{\alpha,\beta}(i,j;\boldsymbol{\theta}))\phi(H(Q_{\alpha,i}))\phi(H(Q_{\beta,j}))\Big\},\end{aligned}$$

where $Q_{\alpha,i} = \boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{X}_i^{(\alpha)}$, $Q_{\beta,i} = \boldsymbol{X}_i^{(\beta)\prime}\boldsymbol{X}_i^{(\beta)}$, H is the distribution function of $Q_{\alpha,i}$ (and $Q_{\beta,i}$) under H_0, and $p_\alpha(i,j;\boldsymbol{\theta})$ is the angle in radians between $\boldsymbol{X}_i^{(\alpha)}$ - $\boldsymbol{\theta}$ and $\boldsymbol{X}_j^{(\alpha)}$ - $\boldsymbol{\theta}$ (similarly for $p_\beta(i,j;\boldsymbol{\theta})$ and $p_{\alpha,\beta}(i,j;\boldsymbol{\theta})$). Therefore our approximating quantity for $W_{N,\phi}$ is given by:

$$W_{N,\phi}^* = Z_{N,\phi}^{*,1,2} + Z_{N,\phi}^{*,1,3} + \cdots + Z_{N,\phi}^{*,c-1,c}.$$

Since $Z_{N,\phi}^{\alpha,\beta} - Z_{N,\phi}^{*,\alpha,\beta} = o_p(1)$ under H_0 (Randles and Peters (1990)), W_N, ϕ and $W_{N,\phi}^*$ have the same asymptotic null distribution. The following theorem results from this asymptotic approximation. Its proof is sketched in the appendix.

Theorem 1. *Under* $H_0, W_{N,\phi} \xrightarrow{d} \chi^2_{p(c-1)}$.

Establishing Pitman efficiencies requires the asymptotic distribution of the test statistics under a sequence of alternatives approaching the null hypothesis. Assume $\boldsymbol{X}_1^{(\alpha)}, \ldots, \boldsymbol{X}_{n_\alpha}^{(\alpha)}$ are iid from $f(\boldsymbol{x} - \boldsymbol{d}_\alpha/\sqrt{N})$ where $\boldsymbol{d}_\alpha = (d_{\alpha 1}, \ldots, d_{\alpha p})'$ satisfies $\sum_{\alpha=1}^c \lambda_\alpha \boldsymbol{d}_\alpha = \boldsymbol{0}$. The class of multivariate distributions used for this comparison is the exponential power family, with density function $f(\boldsymbol{x}) = k_0 \exp\{-[(\boldsymbol{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})/c_0]^\nu\}$, where $\nu > 0, c_0 = [p\Gamma(p/2\nu)/\Gamma((p+2)/2\nu)]$ and $k_0 = \nu\Gamma(p/2)/[\Gamma(p/2\nu)(\pi c_0)^{p/2}]$. Because both tests are affine invariant we again assume, without loss of generality, that $\boldsymbol{\Sigma} = \boldsymbol{I}$ and $\boldsymbol{\theta} = \boldsymbol{0}$. The rationale of Hajek and Sidak ((1967), pp. 208-213) shows that the alternatives are contiguous to the null hypothesis. The proof of the following theorem is sketched in the Appendix.

Theorem 2. *Under the sequences of alternatives,*

$$W_{N,\phi} \xrightarrow{d} \chi^2_{p(c-1)}(\delta),$$

where

$$\delta = \frac{4\nu^2}{pc_0^{2\nu}E(\phi^2)}E^2\Big(\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big)\sum_{\alpha=1}^c \lambda_\alpha \boldsymbol{d}_\alpha{}'\boldsymbol{d}_\alpha.$$

Pitman asymptotic relative efficiency will be used to make a comparison between $W_{N,\phi}$ and the Lawley-Hotelling generalized T^2. The asymptotic distribution of the generalized T^2 under the same sequence of contiguous alternatives described above is chi-square with noncentrality parameter $\sum_{\alpha=1}^{c} \lambda_\alpha \boldsymbol{d}'_\alpha \boldsymbol{d}_\alpha$ (Puri and Sen (1971), p. 212). Thus

$$\text{ARE}(W_{N,\phi}, T^2) = \frac{4\nu^2}{pc_0^{2\nu} E(\phi^2)} E^2\Big(\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big).$$

Using the score functions $\phi_1(u) = 1$ and $\phi_2(\text{u}) = \text{u}$, the asymptotic relative efficiencies are identical to the one-sample asymptotic relative efficiencies of the sign and signed-rank tests, respectively reported in Randles (1989) and in Peters and Randles(1990). So that, after evaluating the expectation, we have

$$\text{ARE}(W_{N,\phi_1}, T^2) = \frac{4\nu^2 \Gamma^2(\frac{2\nu+p-1}{2\nu})\Gamma(\frac{p+2}{2\nu})}{p^2\Gamma^3(\frac{p}{2\nu})},$$

and

$$\text{ARE}(W_{N,\phi_2}, T^2) = \frac{12\nu^2 \Gamma(\frac{p+2}{2\nu})\Gamma^2(\frac{2p+2\nu-1}{2\nu})\beta^2(\frac{1}{2};\frac{p}{2\nu};\frac{2p+2\nu-1}{2\nu})}{p^2\Gamma^5(\frac{p}{2\nu})},$$

where $\beta(\frac{1}{2};a;b) = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx, a, b > 0$. It is of interest to note that $\lim_{\nu\to\infty}$ ARE $(W_{N,\phi_1}, T^2) = \frac{p}{p+2}$ (≤ 1) and $\lim_{\nu\to\infty}$ ARE $(W_{N,\phi_2}, T^2) = \frac{3p}{p+2}(\geq 1)$ for fixed p. Thus we expect that for light-tailed distributions W_{N,ϕ_2} will perform better than both W_{N,ϕ_1} and the Lawley-Hotelling generalized T^2 when p > 1. Table 1 displays the Pitman asymptotic relative efficiencies of $W_{N,\phi}$ relative to the Lawley-Hotelling generalized T^2 for these two score functions and for selected values of ν and p.

Table 1. ARE(W_{N,ϕ_1}, T^2) and ARE (W_{N,ϕ_2}, T^2)

	ν									
	0.1		0.5		1.0		2.0		5.0	
p	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2
1	252252	739	2.00	1.50	.637	.954	.411	.873	.347	.907
2	367	25	1.50	1.13	.785	.985	.590	1.051	.519	1.221
3	54.8	7.40	1.33	1.00	.849	.975	.688	1.099	.620	1.346
4	21.0	3.93	1.25	0.94	.884	.961	.749	1.109	.687	1.395
5	11.7	2.67	1.20	0.90	.905	.949	.790	1.106	.734	1.412

As can be seen, W_{N,ϕ_2} is more efficient than W_{N,ϕ_1} and T^2 for light-tailed distributions ($\nu = 2$ and 5) when p> 1. Even under the multivariate normal distribution and other distributions close to the normal, W_{N,ϕ_2} continues to be efficient relative to T^2. For heavy-tailed distributions ($\nu = 0.1$ and 0.5), W_{N,ϕ_1} is superior to both W_{N,ϕ_2} and T^2.

4. Monte Carlo Study and Example

In this section we display results from a Monte Carlo study in which the number of samples is $c = 3$, the dimension is $\mathrm{p} = 3$, and each sample size is $n_\alpha = 15$, $\alpha = 1, 2, 3$. The five statistics W_{N,ϕ_1}, W_{N,ϕ_2}, Lawley-Hotelling generalized T^2, and the statistics L_N and H were compared using samples from three distributions which span the spectrum of tailweights among elliptically symmetric distributions. Specifically, the distributions considered are a very light-tailed member of the exponential power family with $\nu = 25$, the multiviariate normal (exponential power with $\nu = 1$) and the multiviariate Cauchy distribution with density function $f(\boldsymbol{x}) = k_1(1 + \boldsymbol{x}'\boldsymbol{x})^{-(p+1)/2}$, where $k_1 = \pi^{-(p+1)/2}\Gamma((p+1)/2)$. Here we also use $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, except in the multiviariate normal case, where the individual components of $\boldsymbol{X}$ have mean 0 and variance 1, but each pair of components are highly correlated with $\rho = .90$. The latter is chosen to illustrate how its lack of affine invariance can cause L_N to perform poorly. The distributions were chosen to illustrate performance extremes from very light-tailed to very heavy-tailed cases.

The number of repetitions at each case was 1000. In each Monte Carlo simulation, the proportion of times out of 1000 in which each test statistic exceeded its critical value is reported. The asymptotic null distribution χ^2_6 is used to determine the critical value for tests W_{N,ϕ_1}, W_{N,ϕ_2}, L_N and H. For Lawley-Hotelling generalized T^2, the test rejects H_0 if T^2 exceeds $\frac{84}{13}F_{6,78}$ where F_{ν_1,ν_2} denotes the upper 5th percentile of an F distribution with ν_1 and ν_2 degrees of freedom.

In Table 2 we present the results of the Monte Carlo study for various values of the locations $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$. The first sample was selected in each case from the indicated population centered at the origin. The second and third samples were selected from the same population but centered at shifted values $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ respectively, as indicated in Table 2. The locations for the second and third populations were selected so that the results would represent a reasonable range of powers under the alternative hypotheses. The directions of shift are fairly irrelevant to the performances of the affine invariant procedures, but not so to L_N.

In general, the tests W_{N,ϕ_1} and H perform similarly, with W_{N,ϕ_1} performing slightly better in this study due to the fact that its levels are closer to the target $\alpha = .05$ using the asymptotic cutoff. All of the tests could be conducted using permutation cutoffs which would equate their null levels, but would greatly increase the computational difficulty in a simulation such as this. Both W_{N,ϕ_1} and H are superior to W_{N,ϕ_2}, T^2 and L_N when the distribution is the multivariate

Cauchy. When the underlying distribution is multivariate normal, T^2 is best, but W_{N,ϕ_1}, H and especially W_{N,ϕ_2} are very competitive in this case. Here the performance of L_N is poor due to its lack of affine-invariance. For the light-tailed ($\nu = 25$) case, W_{N,ϕ_2} is best with T^2 the only legitimate competitor.

We make another comparison among those five statistics using the example Hettmansperger and Oja (1994) took from Seber (1984). The data contains x_1, the amount of phospate (mg/ml) and x_2, the amount of calcium (mg/ml) in urine samples from 36 men in three different weight groups, with $n_1 = 10$, $n_2 = 12$ and $n_3 = 14$. The result of the analysis are given in Table 3. Here the approximate distribution of Lawley-Hotelling generalized T^2 is $F_{4,62}$ and the others have an approximate χ^2_4 distribution. All five statistics indicate that these three weight groups do differ in urine phosphate and calcium levels. The asymptotic p-values associated with W_{N,ϕ_1} and W_{N,ϕ_2} are the smallest among the nonparametric tests.

Table 2. Monte Carlo results with $n = 15$, reps = 1000

			Statistics				
$\boldsymbol{\theta}_1$	$\boldsymbol{\theta}_2$	$\boldsymbol{\theta}_3$	W_{N,ϕ_1}	W_{N,ϕ_2}	T^2	L_N	H
			Cauchy				
(0,0,0)	(0,0,0)	(0,0,0)	.058	.059	.012	.045	.038
(0,0,0)	(.4,.4,.4)	(0,−.4,0)	.264	.199	.035	.199	.213
(0,0,0)	(.6,.6,.6)	(0,−.6,0)	.463	.299	.075	.349	.417
(0,0,0)	(.8,.8,.8)	(0,−.8,0)	.688	.405	.097	.566	.660
			normal, correlations = 0.9				
(0,0,0)	(0,0,0)	(0,0,0)	.062	.063	.049	.030	.037
(0,0,0)	(−.1,−.1,0)	(.1,.1,0)	.139	.152	.141	.054	.105
(0,0,0)	(−.2,−.2,0)	(.2,.2,0)	.430	.459	.505	.100	.386
(0,0,0)	(−.3,−.3,0)	(.3,.3,0)	.808	.840	.897	.203	.780
			exp. power $\nu = 25$				
(0,0,0)	(0,0,0)	(0,0,0)	.065	.071	.048	.042	.038
(0,0,0)	(.2,.2,.2)	(0,−.2,0)	.162	.174	.150	.123	.126
(0,0,0)	(.4,.4,.4)	(0,−.4,0)	.207	.367	.278	.130	.172
(0,0,0)	(.6,.6,.6)	(0,−.6,0)	.396	.730	.620	.261	.352

Table 3. Statistical analysis of urine data

	Statistics				
	W_{N,ϕ_1}	W_{N,ϕ_2}	T^2	L_N	H
Value	13.8931	15.0518	6.262	12.2479	12.511
p-value	0.0076	0.0046	0.0003	0.0156	0.0139

Appendix

Proof of Theorem 1. Noting that $\cos(\pi p_\alpha(i,j,\boldsymbol{\theta})) = \boldsymbol{U}_i^{(\alpha)'}\boldsymbol{U}_i^{(\alpha)}$, etc. we can write $Z_{N,\phi}^{*,\alpha,\beta} = \boldsymbol{S}_{N,\phi}^{\alpha,\beta\prime}\boldsymbol{S}_{N,\phi}^{\alpha,\beta}$, where

$$\boldsymbol{S}_{N,\phi}^{\alpha,\beta} = \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\Big\{\frac{1}{n_\alpha}\sum_{i=1}^{n_\alpha}\boldsymbol{U}_i^{(\alpha)}\phi(H(Q_{\alpha,i})) - \frac{1}{n_\beta}\sum_{j=1}^{n_\beta}\boldsymbol{U}_j^{(\beta)}\phi(H(Q_{\beta,j}))\Big\}.$$

Let $\boldsymbol{B} = (\boldsymbol{B}'_{1,2}, \boldsymbol{B}'_{2,3}, \ldots, \boldsymbol{B}'_{c-1,c})'$ where $\boldsymbol{B}_{\alpha,\beta}$ are arbitrary nonzero $p \times 1$ fixed vectors. Then

$$\begin{aligned}
&\boldsymbol{B}'\boldsymbol{S}_{N,\phi} = \boldsymbol{B}'_{1,2}\boldsymbol{S}_{N,\phi}^{1,2} + \boldsymbol{B}'_{1,3}\boldsymbol{S}_{N,\phi}^{1,3} + \cdots + \boldsymbol{B}'_{c-1,c}\boldsymbol{S}_{N,\phi}^{c-1,c}\\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\frac{1}{n_\alpha n_\beta}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta}\boldsymbol{B}'_{\alpha,\beta}\Big(\boldsymbol{U}_i^{(\alpha)}\phi(H(Q_{\alpha,i})) - \boldsymbol{U}_j^{(\beta)}\phi(H(Q_{\beta,j}))\Big)\\
&= \sum_{\alpha=1}^{c}\frac{1}{n_\alpha}\sum_{i=1}^{n_\alpha}\Big\{\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{\alpha,\beta}\boldsymbol{U}_i^{(\alpha)}\phi(H(Q_{\alpha,i}))\Big\},
\end{aligned}$$

where we set $\boldsymbol{B}_{\alpha,\beta} = -\boldsymbol{B}_{\beta,\alpha}$ when $\beta < \alpha$. Here we see that the summands in each term are i.i.d. random variables so that we can establish an asymptotic normal distribution for each term by the Central Limit Theorem. The αth term converges to $N(\boldsymbol{0}, (\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\sqrt{\lambda_\beta}B'_{\alpha,\beta})(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\sqrt{\lambda_\beta}B'_{\alpha,B})'))$, where $\lim_{N\to\infty}\frac{n_\alpha}{N} = \lambda_\alpha$, $\alpha = 1, \ldots, c$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_c = 1$. Also, since the terms are independent of one another, we obtain the asymptotic normality of $\boldsymbol{B}'\boldsymbol{S}_{N,\phi}$ via a multi-sample version of the Central Limit Theorem. Then $\boldsymbol{B}'\boldsymbol{S}_{N,\phi} \xrightarrow{d} N(\boldsymbol{0}, \Psi)$, where $\Psi = \boldsymbol{B}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{B}$ and

$$\boldsymbol{A} = \begin{pmatrix}
\sqrt{\lambda_2}\boldsymbol{I}_p & \sqrt{\lambda_3}\boldsymbol{I}_p & \cdots & \boldsymbol{0}_p\\
-\sqrt{\lambda_1}\boldsymbol{I}_p & \boldsymbol{0}_p & \cdot\cdot & \boldsymbol{0}_p\\
\boldsymbol{0}_p & -\sqrt{\lambda_1}\boldsymbol{I}_p & \cdot\cdot & \boldsymbol{0}_p\\
\cdot & \cdot & & \vdots\\
 & & & \sqrt{\lambda_c}\boldsymbol{I}_p\\
\cdot & \cdot & & -\sqrt{\lambda_{c-1}}\boldsymbol{I}_p
\end{pmatrix}_{cp\times\frac{(c-1)cp}{2}},$$

where $\boldsymbol{0}_p$ denotes a $p \times p$ matrix of zeros. Therefore, $\boldsymbol{S}_{N,\phi}\xrightarrow{\boldsymbol{d}} N(\boldsymbol{0}, \boldsymbol{V})$ where $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$ is a $\frac{(c-1)cp}{2} \times \frac{(c-1)cp}{2}$ variance-covaiance matrix.

In general, $\boldsymbol{A} = \boldsymbol{A}_1 \otimes \boldsymbol{I}_p$ where $\boldsymbol{A}_1$ is a $c \times \binom{c}{2}$ matrix of constants and $\boldsymbol{I}_p$ is a $p \times p$ identity matrix. Let $\boldsymbol{a}'_i$ denote the ith row of $\boldsymbol{A}_1$, $1 \leq i \leq c$, and $\boldsymbol{b}_j$ denote the jth column of $\boldsymbol{A}_1$, $1 \leq j \leq c(c-1)/2$. We see that $\boldsymbol{A}_1$ satisfies: (1) $\boldsymbol{a}'_\alpha\boldsymbol{a}_\alpha = (1 - \lambda_\alpha)$ for $\alpha = 1, \ldots, c$, (2) $\boldsymbol{a}'_\alpha\boldsymbol{a}_\beta = -\sqrt{\lambda_\alpha\lambda_\beta}$ for all $1 \leq \alpha < \beta \leq c$, (3)

$\sum_{\alpha=1}^{c}\sqrt{\lambda_\alpha}\boldsymbol{a}_\alpha' = \boldsymbol{0}'$, and (4) if the jth column of $\boldsymbol{A}_1$ corresponds to $\boldsymbol{B}_{\alpha,\beta}$, then $\boldsymbol{b}_j'\boldsymbol{b}_j = \lambda_\alpha + \lambda_\beta$ for all $j = 1, \ldots, c(c-1)/2$. Letting $\boldsymbol{V} = \boldsymbol{V}_1 \otimes \boldsymbol{I}_p$, we note that

$$\begin{aligned}\boldsymbol{V}_1 \cdot \boldsymbol{V}_1 &= \Big(\sum_{\alpha=1}^{c}\boldsymbol{a}_\alpha\boldsymbol{a}_\alpha'\Big)\Big(\sum_{\beta=1}^{c}\boldsymbol{a}_\beta\boldsymbol{a}_\beta'\Big)\\ &= \sum_{\alpha=1}^{c}\boldsymbol{a}_\alpha(1-\lambda_\alpha)\boldsymbol{a}_\alpha' + \sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{c}\sum_{\beta=1}^{c}\boldsymbol{a}_\alpha\left(-\sqrt{\lambda_\alpha\lambda_\beta}\right)\boldsymbol{a}_\beta'\\ &= \sum_{\alpha=1}^{c}\boldsymbol{a}_\alpha\boldsymbol{a}_\alpha' - \Big(\sum_{\alpha=1}^{c}\sqrt{\lambda_\alpha}\boldsymbol{a}_\alpha\Big)\Big(\sum_{\beta=1}^{c}\sqrt{\lambda_\beta}\boldsymbol{a}_\beta'\Big)\\ &= \sum_{\alpha=1}^{c}\boldsymbol{a}_\alpha\boldsymbol{a}_\alpha' = \boldsymbol{A}_1' \cdot \boldsymbol{A}_1 = \boldsymbol{V}_1.\end{aligned}$$

Since $\boldsymbol{V} = \boldsymbol{V}_1 \otimes \boldsymbol{I}_p$, and $\boldsymbol{V}\cdot\boldsymbol{V} = (\boldsymbol{V}_1 \otimes \boldsymbol{I}_p)(\boldsymbol{V}_1 \otimes \boldsymbol{I}_p) = (\boldsymbol{V}_1\cdot\boldsymbol{V}_1) \otimes (\boldsymbol{I}_p\cdot\boldsymbol{I}_p) = \boldsymbol{V}_1 \otimes \boldsymbol{I}_p = \boldsymbol{V}$, we see that $\boldsymbol{V}$ is idempotent. Therefore the asymptotic null distribution of $W_{N,\phi}^*$ is a chi-square distribution with degrees of freedom $\mathrm{tr}(\boldsymbol{V})$ (see Searle (1971)), where

$$\begin{aligned}\mathrm{tr}(\boldsymbol{V}) &= (\lambda_1+\lambda_2)p + (\lambda_1+\lambda_3)p + \cdots + (\lambda_{c-1}+\lambda_c)p\\ &= (c-1)p(\lambda_1+\lambda_2+\cdots+\lambda_c) = (c-1)p.\end{aligned}$$

This completes the proof.

Proof of Theorem 2. Under continguous alternatives, an asymptotic approximation to the log-likelihood function is:

$$\begin{aligned}T_N^* &= \frac{1}{\sqrt{N}}\Big\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}\boldsymbol{d}_\alpha{}'\frac{f'(\boldsymbol{X}_i^{(\alpha)})}{f(\boldsymbol{X}_i^{(\alpha)})}\Big\}\\ &= \frac{2\nu}{c_0^\nu\sqrt{N}}\Big\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}\boldsymbol{d}_\alpha{}'\boldsymbol{X}_i^{(\alpha)}(\boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{X}_i^{(\alpha)})^{\nu-1}\Big\}\\ &= \frac{2\nu}{c_0^\nu\sqrt{N}}\Big\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}\boldsymbol{d}_\alpha{}'(Q_{\alpha,i})^{\nu-1/2}\boldsymbol{U}_i^{(\alpha)}\Big\}.\end{aligned}$$

Let $\boldsymbol{a} = (a_1, a_2)'$ be an arbitrary fixed vector of constants in which both components are not zero. We first find the joint limiting distribution of T_N^* and $S_N^* = \boldsymbol{B}'\boldsymbol{S}_{N,\phi}$.

$$\begin{aligned}&\boldsymbol{a}'\begin{pmatrix}S_N^*\\ T_N^*\end{pmatrix} = a_1 S_N^* + a_2 T_n^*\\ &= \sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha}\Big[\frac{2a_2\nu}{c_0^\nu\sqrt{N}}(Q_{\alpha,i})^{\nu-1/2}\boldsymbol{d}_\alpha + a_1\phi(H(Q_{\alpha,i}))\big(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}\frac{\sqrt{n_\beta p}}{\sqrt{n_\alpha N E(\phi^2)}}\boldsymbol{B}_{\alpha,\beta}\big)\Big]'\boldsymbol{U}_i^{(\alpha)}.\end{aligned}$$

Since $E_{H_o}[S_N^*]$ and $E_{H_o}[T_N^*]$ are zero, $E_{H_o}[\boldsymbol{a}'(S_N^*, T_N^*)'] = 0$. Also,

$$V_{H_o}[\boldsymbol{a}'(S_N^*, T_N^*)'] = E_{H_o}[(\boldsymbol{a}'(S_N^*, T_N^*)')^2].$$

Using $E_{H_0}(\boldsymbol{U}_i^{(\alpha)}) = p^{-1}\boldsymbol{I}_p$, the variance of S_N^* is:

$$\sigma_{1,N}^2 = \frac{1}{N}\Big[\sum_{\alpha=1}^{c}(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\sqrt{n_\beta}\boldsymbol{B}'_{\alpha,\beta})'(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\sqrt{n_\beta}\boldsymbol{B}'_{\alpha,\beta})\Big]$$
$$\to \boldsymbol{B}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}'\boldsymbol{V}\boldsymbol{B} \text{ as } N \to \infty.$$

Also, as $N \to \infty$, the variance of T_N^*, goes to

$$\sigma_2^2 = \frac{4\nu^2}{c_0^{2\nu}p}E_{H_o}((Q_{1,i})^{2\nu-1})\Big(\sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}'_\alpha \boldsymbol{d}_\alpha\Big).$$

Finally, as $N \to \infty$, the covariance of S_N^* and T_N^* goes to

$$\sigma_{12} = \frac{2\nu}{c_0^\nu\sqrt{pE(\phi^2)}}E_{H_o}\Big[\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big]\Big[\sum_{\alpha=1}^{c}\lambda_\alpha(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{c}\frac{\sqrt{\lambda_\beta}}{\sqrt{\lambda_\alpha}}\boldsymbol{B}'_{\alpha,\beta})\boldsymbol{d}_\alpha\Big]$$
$$= \frac{2\nu}{c_0^\nu\sqrt{pE(\phi^2)}}E_{H_o}\Big[\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big]\boldsymbol{B}'\boldsymbol{G}\boldsymbol{d},$$

where $\boldsymbol{d}' = (\boldsymbol{d}'_1, \ldots, \boldsymbol{d}'_c)$ and

$$\boldsymbol{G} = \begin{pmatrix} \sqrt{\lambda_1\lambda_2}\boldsymbol{I}_p & -\sqrt{\lambda_1\lambda_2}\boldsymbol{I}_p & \boldsymbol{0}_p & \boldsymbol{0}_p \cdots & \boldsymbol{0}_p & \boldsymbol{0}_p \\ \sqrt{\lambda_1\lambda_3}\boldsymbol{I}_p & \boldsymbol{0}_p & -\sqrt{\lambda_1\lambda_3}\boldsymbol{I}_p & \boldsymbol{0}_p \cdots & \boldsymbol{0}_p & \boldsymbol{0}_p \\ \cdot & & & & & \\ \cdot & & & & & \\ \boldsymbol{0}_p & \boldsymbol{0}_p & \boldsymbol{0}_p & \boldsymbol{0}_p \cdots & \sqrt{\lambda_{c-1}\lambda_c}\boldsymbol{I}_p & -\sqrt{\lambda_{c-1}\lambda_c}\boldsymbol{I}_p \end{pmatrix}.$$

Then by LeCam's third Lemma, we have, under the sequence of alternatives,

$$S_N^* \xrightarrow{d} N(\mu_a^*, \boldsymbol{B}'\boldsymbol{V}\boldsymbol{B}),$$

where

$$\mu_a^* = \frac{2\nu}{c_0^\nu\sqrt{pE(\phi^2)}}E_{H_o}\Big[\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big]\boldsymbol{B}'\boldsymbol{G}\boldsymbol{d}.$$

Therefore, under the contiguous alternatives,

$$\boldsymbol{S}_N \xrightarrow{d} N\Big(\frac{2\nu}{c_0^\nu\sqrt{pE(\phi^2)}}E_{H_o}\Big[\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big]\boldsymbol{G}\boldsymbol{d}, \boldsymbol{V}\Big)$$

and

$$W_{N,\phi}^* \xrightarrow{d} \chi^2_{(c-1)p}\Big(\frac{4\nu^2}{c_0^{2\nu}pE(\phi^2)}E^2_{H_o}\Big[\phi(H(Q_{1,i}))(Q_{1,i})^{\nu-1/2}\Big]\boldsymbol{d}'\boldsymbol{G}'\boldsymbol{G}\boldsymbol{d}\Big).$$

To simplify the form of noncentrality parameter of the chi-square distribution, we use the fact that $\sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}_\alpha = \mathbf{0}$. Since

$$\begin{aligned}\boldsymbol{d}'\boldsymbol{G}'\boldsymbol{G}\boldsymbol{d} &= \boldsymbol{d}'\begin{pmatrix} \lambda_1(1-\lambda_1)\boldsymbol{I}_p & -\lambda_1\lambda_2\boldsymbol{I}_p & -\lambda_1\lambda_3\boldsymbol{I}_p \cdots & -\lambda_1\lambda_c\boldsymbol{I}_p \\ -\lambda_1\lambda_2\boldsymbol{I}_p & \lambda_2(1-\lambda_2)\boldsymbol{I}_p & -\lambda_2\lambda_3\boldsymbol{I}_p \cdots & -\lambda_2\lambda_c\boldsymbol{I}_p \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ -\lambda_c\lambda_1\boldsymbol{I}_p & -\lambda_c\lambda_1\boldsymbol{I}_p & -\lambda_c\lambda_3\boldsymbol{I}_p \cdots & \lambda_c(1-\lambda_c)\boldsymbol{I}_p \end{pmatrix}\boldsymbol{d} \\ &= \sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}'_\alpha \boldsymbol{d}_\alpha - \Big[\big(\sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}_\alpha\big)'\big(\sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}_\alpha\big)\Big] \\ &= \sum_{\alpha=1}^{c}\lambda_\alpha \boldsymbol{d}'_\alpha \boldsymbol{d}_\alpha, \end{aligned}$$

and we have the desired result.

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(Received September 1996; accepted February 1998)