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TIME-SEQUENTIAL TESTS BY POISSON APPROXIMATION

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Abstract: Time-sequential testing problems are studied in the counting process context. With a random time change of the counting process, it is shown that the associated optimal stopping time problem gets close to that for a Poisson process as sample size gets large. By making use of this observation, we propose a class of time-sequential tests, which is shown to be optimal in the sense that it has minimal asymptotic expected total time on-test with given bounds on error probabilities.

Key words and phrases: Asymptotic optimality, counting process, intensity process, likelihood ratio process, Poisson process, time-sequential test, time-sequential probability ratio test.

1. Introduction

1.1. Preliminaries

In a follow-up study, arising in a clinical trial or a life testing problem, typically, the response (viz. failure) occurs sequentially over time. Further, because of time, cost and other limitations, it is often desirable to curtail observations at some early stage, prior to the last response, when enough statistical evidence is accumulated. Taking this into consideration, time-sequential procedures have been studied by many authors.

In particular, time-sequential estimation problems were studied by Sen (1980), Gardiner, Susarla & Van Ryzin (1986) and Chang & Hsiung (1990); and time-sequential testing problems were studied by Epstein & Sobel (1955), Sen (1981, 1985) and Liu (1995), among others. However, as pointed out by Gardiner & Susarla (1991) in reviewing the progress in time-sequential inference of a survival curve parameter, there still seems a need to develop the theory of time-sequential tests on its own, taking into account the specific objectives within its framework. It is the purpose of this paper to study one such time-sequential testing problem by a counting process approach.

Let $Y_1, \ldots, Y_n$ denote the life times of n items drawn at random from a population and placed on a life test. Assume that Y_1 has intensity $\lambda(t) = \theta\alpha(t)$, where θ is a constant and α is an unknown continuous function with $\alpha(0) = 1$. In this paper, we are interested in testing the hypothesis $H_0 : \theta = \theta_0$ versus the hypothesis $H_1 : \theta = \theta_1$, where $\theta_0 < \theta_1$, and the hypothesis testing is carried out in such a way that one may curtail the experiment at some early stage, prior to the last failure time, and make a terminal decision so as to reduce the total time on test.

The preceding time-sequential hypothesis testing problem was studied by Epstein & Sobel (1955), and Liu (1995). Assuming λ is a constant, or equivalently Y_1 is exponentially distributed, Epstein & Sobel (1955) proposed a continuous analogue of the SPRT of Wald for the problem and obtained some asymptotic properties of it. Liu (1995) continued to argue that the SPRT of Epstein & Sobel (1955) (henceforth time-sequential probability ratio test (TSPRT)) enjoys optimum properties similar to those of the classical SPRT of Wald (cf. Wald and Wolfowitz (1948)).

In fact, Liu (1995) introduces a Bayes auxiliary problem and transforms it into a classical Wald-Wolfowitz problem with finite horizon. Although the finite horizon introduces non-stationarity into the problem, Liu (1995) still can apply the optimal stopping theory of Chow, Robbins & Siegmund (1971) to find the optimal generalized time-sequential probability ratio test. With this result, Liu (1995) is able to show that TSPRT with constant boundaries is asymptotically optimal in the sense that, asymptotically, it has minimal expected total time on test among tests having equal or smaller error probabilities.

We remark that Liu (1995) requires that $\lambda(t) = \theta\alpha(t)$ with $\alpha(\cdot)$ a *known* function and permits curtailing the experiment only at failure times. In order to eliminate these restrictions, we take a counting process approach to study the problem under the condition that $\alpha(\cdot)$ is an *unknown* continuous function, which means we are dealing with composite hypotheses. As usual, the counting process approach has the added advantages of allowing for early termination between successive failure times and of allowing for independent censoring variables, although we do not consider independent censoring variables in this paper in order to simplify the presentation.

1.2. Counting process formulation of the problem

Let $Y_{n1} \le \cdots \le Y_{nn}$ be the order statistics of $Y_1, \ldots, Y_n$. Let $\mathcal{F}_k^{(n)} = \sigma\{Y_{nj} \mid j \le k\}$, denote the σ-field generated by $Y_{n1}, \ldots, Y_{nk}$. A time-sequential test is an $\mathcal{F}_k^{(n)}-$ stopping time τ together with an $\mathcal{F}_\tau^{(n)}-$ measurable test statistic δ. Let $W^{(n)}(Y_{n\tau}) = \sum_{i=1}^n (Y_i \wedge Y_{n\tau})$ be the total time on test of the test statistic (τ, δ).

Let

$$\bar{N}^{(n)}(t) = \sum_{i=1}^{n} 1_{[Y_i,\ \infty)}(t) \tag{1.1}$$

be the counting process which records the number of failures at (calendar) time t in the study. Let $\bar{\mathcal{F}}_t^{(n)} = \sigma\{\bar{N}^{(n)}(s) \mid 0 \le s \le t\}$ be the self-excited filtration.

Then one can show that $\bar{\mathcal{F}}^{(n)}_{Y_{nk}} = \mathcal{F}^{(n)}_k$. With this observation, the time-sequential testing problem mentioned in Subsection 1.1 can now be formulated more satisfactorily as follows. A time-sequential test is, by definition, an $\bar{\mathcal{F}}^{(n)}_t-$ stopping time τ together with an $\bar{\mathcal{F}}^{(n)}_\tau-$ measurable test statistic δ. We are interested in finding time-sequential tests such that, asymptotically, they have minimal expected total time on test $W^{(n)}(\tau) = \sum_{i=1}^n (Y_i \wedge \tau)$ among those having equal or smaller error probabilities.

Since the total time on test $W^{(n)}(\tau)$ is to be minimized and is not easy to deal with in the filtration $\bar{\mathcal{F}}^{(n)}_t$, we reparametrize the time dimension of the counting process so that, in the new time scale, it is some simple quantity to be minimized, not the complicated on-test time functional to be minimized. This motivates the following random time-change for $\bar{N}^{(n)}(\cdot)$. The idea is to record the number of failures in terms of the total time on test of the experiment.

Since the total time on test $W^{(n)}(t)$ at calendar time t is a strictly increasing function in t, it is legitimate to perform the following time-change for (1.1). We define

$$N^{(n)}(t) = \bar{N}^{(n)}(s),$$

if $t = W^{(n)}(s)$. In fact, $N^{(n)}(t) = \sum_{k=1}^n 1_{[W^{(n)}_k,\ \infty)}(t)$, where $W^{(n)}_k = \sum_{j=1}^n Y_{nj} \wedge Y_{nk}$. Let

$$\mathcal{G}^{(n)}_t = \sigma\{N^{(n)}(s) \mid 0 \le s \le t\}$$

be the internal history of $N^{(n)}(t)$. Then we have the following proposition about the relation between $\bar{N}^{(n)}$ and $N^{(n)}$. Although Proposition 1.1 is naturally expected, its complete justification is somewhat complicated and is given at the end of this paper.

Proposition 1.1. *S is an $\bar{\mathcal{F}}^{(n)}_t-$ stopping time if and only if $W^{(n)}(S)$ is a $\mathcal{G}^{(n)}_t-$ stopping time and, in this case, $\bar{\mathcal{F}}^{(n)}_S = \mathcal{G}^{(n)}_{W^{(n)}(S)}$.*

Proposition 1.1 indicates that our statistical problem can be equally well-expressed in terms of the on-test time counting process $N^{(n)}(t)$, which records the failures in the follow-up study according to the on-test time, *not the calendar time*. Thus, in the rest of this paper, a time-sequential test for $N^{(n)}(t)$ is meant a $\mathcal{G}^{(n)}_t-$ stopping time τ together with a $\mathcal{G}^{(n)}_\tau-$ measurable $\{0,1\}-$ valued statistic δ, and H_0 is rejected if and only if $\delta = 1$.

We stress that this random time-change simplifies our statistical problem in that the problem reduces to finding a time-sequential test (τ, δ) for $\mathcal{G}^{(n)}_t$ which has minimal expected *stopping time* τ with given bounds on the error probabilities asymptotically, where the quantity of on-test time disappears. Although this simplification is the original motivation for performing the random time-change, the main advantage of working with $N^{(n)}(\cdot)$ comes from the fact that $N^{(n)}(\cdot)$ converges weakly to a Poisson process $N^{(0)}(\cdot)$.

Here are the major ideas and steps in our approach. A well-known result of Dvoretzky, Kiefer & Wolfowitz (1953) says that, for a Poisson process, the time-sequential probability ratio test with constant boundary is the solution of a Bayes auxiliary decision problem and has minimal expected duration of observation with given upper bounds of probability errors (cf. B. K. Ghosh (1970), Chap. 4). We show that the solution of the Bayes auxiliary decision problem for the Poisson process $N^{(0)}(\cdot)$ is the limit of the solution of the corresponding Bayes decision problem for $N^{(n)}(\cdot)$ on finite horizon. These imply that a time-sequential test $(\tau^{(n)}, \delta^{(n)})$ for $N^{(n)}(\cdot)$ is asymptotically optimal if it converges to the time-sequential probability ratio test for $N^{(0)}(\cdot)$; this motivates the proposed test in Subsection 2.1.

The plan of this paper is as follows. Subsection 2.1 proposes a time-sequential test and states the main result of this paper concerning its optimum property. Subsection 2.2 follows the classical approach of Wald & Wolfowitz (1948) to introduce a Bayes auxiliary problem and indicates that the solution of the Bayes auxiliary problem exists and converges to that for a Poisson process. Since some of the proofs are quite complicated, all the proofs are postponed to Section 3 in order to increase the readability of this paper.

The idea of the proofs is carried out by first studying the limiting behavior of a discretized version of the Bayes auxiliary problem and then using a continuity argument to piece together the solutions for the discretized problem. One of the most important observations used here says that the stochastic processes $X_t^{(n)}$ associated with the Bayes auxiliary problem enjoys certain uniform continuity and equi-continuity properties as described in Lemma 3.2, whose proof uses the quadratic variation formula of a counting process. Another crucial step is to create a new sequence of stochastic processes in Lemma 3.4 so that the method of backward induction becomes tractable in the study of the limiting behavior of the discretized auxiliary Bayes problem.

Finally, we note that this type of optimality for hypothesis testing problems was also studied by Dvoretzky, Kiefer & Wolfowitz (1953), Kiefer & Wolfowitz (1956) and Bhat (1988) in the context of stochastic processes. In particular, Bhat (1988) considered stochastic processes which can be transformed exactly, by a random time-change, into a Brownian motion or a Poisson process. In this connection, we remark that our counting processes are transformable, also by a random time-change, into counting processes that are only approximately Poissonian.

Remark. Since our methods work for a more general situation, we consider in the rest of this paper the hypotheses $H_i : \lambda(t) = \lambda_i(t)$, where $\lambda_i(0) = \theta_i$ and λ_i is an unknown nonnegative continuous function. When $\lambda_1(t)/\lambda_2(t)$ is constant in t, this reduces to the situation discussed in Subsection 1.1. We also note that, under H_i, the counting process $N^{(n)}(\cdot)$ has the intensity

$$\lambda_i^{(n)}(t) = \sum_{k=0}^{n-1} \lambda_i\Big(\frac{t - W_k^{(n)}}{n-k} + Y_{nk}\Big) 1_{(W_k^{(n)},\ W_{k+1}^{(n)}]}(t). \tag{1.2}$$

2. Main Results

2.1. An asymptotically optimum test

For $n = 0, 1, \ldots,$ we define

$$S^{(n)} = \inf\{t > 0 \mid \tilde{\mathcal{L}}^{(n)}(t) \notin (u_n(t), v_n(t))\}, \tag{2.1}$$

where $u_n(t) = u + (1-u)t/b_n,\ v_n(t) = v + (1-v)t/b_n$ for $0 \le t \le b_n,\ 0 < u < 1 < v,\ b_n > 0,\ b_0 = \infty$ and $\tilde{\mathcal{L}}^{(n)}(t)$ satisfies

$$\log \tilde{\mathcal{L}}^{(n)}(t) = \Big(\log \frac{\theta_1}{\theta_0}\Big) N^{(n)}(t) + (\theta_0 - \theta_1)t. \tag{2.2}$$

For the hypothesis testing problem introduced in Subsection 1.2, we propose to make a decision at $S^{(n)}$ with the decision rule $\beta^{(n)}$ which rejects H_0 if and only if $\tilde{\mathcal{L}}^{(n)}(S^{(n)}) \ge v_n(S^{(n)})$. We show that the time-sequential test $(S^{(n)}, \beta^{(n)})$ has the following asymptotically optimum property, if b_n tends to infinity slowly enough. We note that sequential tests with this kind of boundary were introduced by Anderson (1960) in order to avoid the maximum expected sample size of SPRT.

Let $\alpha_0(S^{(n)})$ and $\alpha_1(S^{(n)})$ denote respectively the type 1 and type 2 error probabilities of $(S^{(n)}, \beta^{(n)})$. Let $(\tau^{(n)}, \delta^{(n)})$ be a time-sequential test of H_0 versus H_1, based on $N^{(n)}(t)$, whose type 1 and type 2 error probabilities are denoted by $\alpha_0^{(n)}$ and $\alpha_1^{(n)}$ respectively.

Theorem 2.1. *Assume $\tau^{(n)} \le M$ for some constant $M > 0$. Then there exists $b_1, b_2, \ldots$ tending to infinity such that if*

$$\liminf_{n\to\infty} [\alpha_i(S^{(n)}) - \alpha_i^{(n)}] \ge 0 \tag{2.3}$$

for $i = 0$ and 1, then

$$\limsup_{n\to\infty} [E_i(S^{(n)}) - E_i(\tau^{(n)})] \le 0 \tag{2.4}$$

for $i = 0$ and 1, where E_i is the expectation taken when H_i holds.

Note that, in Theorem 2.1, E_i is the expectation specified by intensity function λ_i with $\lambda_i(0) = \theta_i$, which is the only assumption we make about λ_i, and we do not need to know λ_i in performing the test (2.1) and Theorem 2.1 holds true so long as $\lambda_i(0) = \theta_i$.

Remark. When phrased in calendar time, (2.1) becomes

$$\begin{aligned}\bar{S}^{(n)} = \inf \Big\{ s > 0 \mid & \exp[(\log \frac{\theta_1}{\theta_0})\bar{N}^{(n)}(s) + (\theta_0 - \theta_1)W^{(n)}(s)] \\ & \notin (u_n(W^{(n)}(s)), v_n(W^{(n)}(s))) \Big\},\end{aligned}$$

and our proposal is to make a decision at calendar time $\bar{S}^{(n)}$ with the rule $\bar{\beta}^{(n)}$ of rejecting H_0 if and only if $\exp[(\log \frac{\theta_1}{\theta_0})\bar{N}^{(n)}(\bar{S}^{(n)}) + (\theta_0 - \theta_1)W^{(n)}(\bar{S}^{(n)})] \geq v_n(W^{(n)}(\bar{S}^{(n)}))$. The exact optimum property for $(\bar{S}^{(n)}, \bar{\beta}^{(n)})$ can be read off from (2.3) and (2.4) immediately, which concerns the amount of total time on-test.

2.2. A Bayes auxiliary decision problem

In this subsection, we assume further that the intensity function $\lambda(t)$ of Y_i is either $\lambda_0(t)$ or $\lambda_1(t)$, both of which are completely *known.*

Let $a > 0,\ b > 0,\ c > 0$ and $0 < \pi < 1$ be given. Define

$$\begin{aligned}&\gamma_n(\pi, \tau^{(n)}, \delta^{(n)}, a, b, c) \\ = \ & \pi[a\alpha_0(\tau^{(n)}, \delta^{(n)}) + cE_0(\tau^{(n)})] + (1-\pi)[b\alpha_1(\tau^{(n)}, \delta^{(n)}) + cE_1(\tau^{(n)})], \quad (2.5)\end{aligned}$$

where $\alpha_0(\tau^{(n)}, \delta^{(n)})$ and $\alpha_1(\tau^{(n)}, \delta^{(n)})$ are respectively the type 1 and type 2 error probabilities of the time-sequential test $(\tau^{(n)}, \delta^{(n)})$ based on $N^{(n)}(t)$. In this subsection, our main interest is in the asymptotic properties of solution of the Bayes decision problem (2.5), which is a time-sequential test $(\tau_*^{(n)}, \delta_*^{(n)})$ that minimizes (2.5). We show, in Lemma 2.1, that the Bayes decision problem (2.5) is, as usual, an optimal stopping time problem.

Let $\mathcal{P}_i^{(n),t}$ be the probability measure on $\mathcal{G}_t^{(n)}$ specified by the hypothesis H_i, for $i = 0, 1$. We note that both H_i are simple in this subsection. Then the likelihood ratio process is

$$\mathcal{L}^{(n)}(t) \equiv \frac{d\mathcal{P}_1^{(n),t}}{d\mathcal{P}_0^{(n),t}},$$

which satisfies

$$\log \mathcal{L}^{(n)}(t) = \int_0^t \log \mu^{(n)}(s) dN^{(n)}(s) + \int_0^t (1 - \mu^{(n)}(s))\lambda_0^{(n)}(s) ds, \quad (2.6)$$

where

$$\mu^{(n)}(t) = \frac{\lambda_1^{(n)}(t)}{\lambda_0^{(n)}(t)}$$

and $\lambda_i^{(n)}(t)$ is given in (1.2). We note that $\lambda_i^{(0)}(t) = \theta_i$ and $\lambda_i^{(n)}(t)$ is the intensity of $N^{(n)}(t)$ relative to $\mathcal{G}_t^{(n)}$, under $\mathcal{P}_i^{(n),t}$; and (2.6) is derived from the theory of point processes. (see, for example, p.59-61, p.187, Brémaud (1981)). Let

$$\begin{aligned} \pi_t^{(n)} &= \frac{\pi}{\pi + (1-\pi)\mathcal{L}^{(n)}(t)}, \\ \delta_*^{(n)}(t) &= 1_{[\pi_t^{(n)} a \,\leq\, (1-\pi_t^{(n)})b]}. \end{aligned} \tag{2.7}$$

Then we have

Lemma 2.1. *Assume that both λ_0 and λ_1 are bounded and bounded away from 0, then*

$$\gamma_n(\pi, \tau^{(n)}, \delta^{(n)}, a, b, c) \geq \gamma_n(\pi, \tau^{(n)}, \delta_*^{(n)}(\tau^{(n)}), a, b, c)$$

for every time-sequential test $(\tau^{(n)}, \delta^{(n)})$.

Proof. Making use of the fact that $\mathcal{L}^{(n)}(t)$ is a martingale relative to $\mathcal{P}_0^{(n),\infty}$, we obtain

$$\begin{aligned} &a\pi\alpha_0(\tau^{(n)}, \delta^{(n)}) + b(1-\pi)\alpha_1(\tau^{(n)}, \delta^{(n)}) \\ &= a\pi E_0(1_{[\delta^{(n)}=1]}) + b(1-\pi)E_1(1_{[\delta^{(n)}=0]}) \\ &= E_0(a\pi 1_{[\delta^{(n)}=1]} + b(1-\pi)1_{[\delta^{(n)}=0]}\mathcal{L}^{(n)}(\tau^{(n)})) \\ &= E_0(a\pi_{\tau^{(n)}}^{(n)} 1_{[\delta^{(n)}=1]} + b(1-\pi_{\tau^{(n)}}^{(n)})1_{[\delta^{(n)}=0]})(\pi + (1-\pi)\mathcal{L}^{(n)}(\tau^{(n)})) \\ &\geq a\pi\alpha_0(\tau^{(n)}, \delta_*^{(n)}(\tau^{(n)})) + b(1-\pi)\alpha_1(\tau^{(n)}, \delta_*^{(n)}(\tau^{(n)})). \end{aligned} \tag{2.8}$$

This completes the proof.

It follows from Lemma 2.1, or more precisely, from (2.7) and (2.8), that the Bayes decision problem (2.5) reduces to the following optimal stopping problem. Let

$$\begin{aligned} &h(z) = \min\ [az,\ b(1-z)], \quad 0 \leq z \leq 1, \\ &X_t^{(n)} = -h(\pi_t^{(n)}) - ct, \quad t \geq 0. \end{aligned} \tag{2.9}$$

We are interested in a $\mathcal{G}_t^{(n)}-$ stopping time $\tau_*^{(n)}$ such that $E(X_{\tau_*^{(n)}}^{(n)}) \geq E(X_{\tau^{(n)}}^{(n)})$ for every $\mathcal{G}_t^{(n)}-$ stopping time $\tau^{(n)}$, where $E(\cdot) = E_0(\pi + (1-\pi)\mathcal{L}^{(n)}(\infty))(\cdot)$. $\tau_*^{(n)}$

is called an optimal stopping time. The main theorem of this subsection concerns this optimal stopping time problem on a finite horizon $[0, M]$.

Theorem 2.2. *Let M be a positive number. Then*
(i) *for every $n = 0, 1, \ldots,$ there exists an optimal stopping time $\tau_*^{(n)}$ for the optimal stopping time problem* (2.9) *on* $[0, M]$,
(ii) $\lim_{n\to\infty} E(X^{(n)}_{\tau_*^{(n)}}) = E(X^{(0)}_{\tau_*^{(0)}})$. (2.10)

The proof for Theorem 2.2 is given in Section 3, which includes some preparatory lemmas.

3. Proofs

We need Theorem 2.2 to prove Theorem 2.1 and Theorem 2.2 itself is established by the following lemmas.

Lemma 3.1. *$N^{(n)}(\cdot)$ converges weakly to $N^{(0)}(\cdot)$.*

Lemma 3.2. *Let M be a positive number. There exists a constant $B^* > 0$ such that for any two stopping times $\tau_1^{(n)}$, $\tau_2^{(n)} \le M$ satisfying $|\tau_1^{(n)} - \tau_2^{(n)}| \le \eta$, the inequality*

$$\left| E(X^{(n)}_{\tau_1^{(n)}}) - E(X^{(n)}_{\tau_2^{(n)}}) \right| \le B^* \eta \tag{3.1}$$

holds for every $n = 1, 2, \ldots$.

Lemma 3.3. *For every $n = 0, 1, \ldots,$ there exists an optimal stopping time $\tau_*^{(n)}$ for $(X_t^{(n)}, \mathcal{G}_t^{(n)})$ on $[0, M]$.*

Lemma 3.4. *Assume M is an integer. For every $n = 0, 1, \ldots,$ and $l = 1, 2, \ldots,$ there exists an optimal stopping time $\tau_*^{(n,l)}$ for the stochastic sequence $\{(X^{(n)}_{k/2^l}, \mathcal{G}^{(n)}_{k/2^l}) \mid k = 0, 1, \ldots, 2^l M\}$ such that $\lim_{n\to\infty} E(X^{(n)}_{\tau_*^{(n,l)}}) = E(X^{(0)}_{\tau_*^{(0,l)}})$.*

Lemma 3.5. *If $\lim_{n\to\infty} b_n = \infty$, then $S^{(n)}$ converges in distribution to $S^{(0)}$ as n goes to infinity.*

If the sequence b_n goes to infinity slowly enough, it follows from Lemma 3.5, Fatou's lemma and a little reasoning that

Lemma 3.6. $\lim_{n\to\infty} E(S^{(n)}) = E(S^{(0)})$.

Proof of Lemma 3.1. Let $Z_k^{(n)} = W_k^{(n)} - W_{k-1}^{(n)}$, for $k = 1, \ldots, n$. Then, one can show that for every fixed k, $(Z_1^{(n)}, \ldots, Z_k^{(n)})$ converges weakly to $(Z_1, \ldots, Z_k)$ as n goes to infinity, where $Z_1, Z_2, \ldots$ is a sequence of i.i.d. exponential random variables with mean $1/\lambda(0)$. This together with Theorem 1.21 on p.15 of Karr (1991), concerning weak convergence of point processes, yields the lemma.

Proof of Lemma 3.2. For $s < t$, we have

$$
\begin{aligned}
& \mid X_t^{(n)} - X_s^{(n)} \mid \\
\leq\ & \mid h(\pi_t^{(n)}) - h(\pi_s^{(n)}) \mid + c \mid t - s \mid \\
\leq\ & B_1 \mid \pi_t^{(n)} - \pi_s^{(n)} \mid + c \mid t - s \mid \\
\leq\ & B_1 B_2 \mid \mathcal{L}^{(n)}(t) - \mathcal{L}^{(n)}(s) \mid + c \mid t - s \mid \\
\leq\ & B_1 B_2 e^{t_*} \mid \log \mathcal{L}^{(n)}(t) - \log \mathcal{L}^{(n)}(s) \mid + c \mid t - s \mid \\
\leq\ & B_1 B_2 e^{B_3 N^{(n)}(t) + B_4 t} [B_3(N^{(n)}(t) - N^{(n)}(s)) + B_4(t-s)] + \ c \mid t - s \mid . \quad (3.2)
\end{aligned}
$$

Here $B_1 = \max\{a, b\}$, $B_2 = (1-\pi)/\pi$, t_* lies between $\log \mathcal{L}^{(n)}(t)$ and $\log \mathcal{L}^{(n)}(s)$, B_3 is a bound for $\mid \log \mu^{(n)}(\cdot) \mid$, and B_4 is a bound for $\mid (1 - \mu^{(n)}(\cdot))\lambda_0^{(n)}(\cdot) \mid$.

Since $(N^{(n)}(t) - \int_0^t \lambda^{(n)}(u)du)^2 - \int_0^t \lambda^{(n)}(u)du$ is a martingale (cf. Jacobsen (1982, p.39)), we know

$$
\begin{aligned}
& E(N^{(n)}(t) - N^{(n)}(s))^2 \\
=& E\Big(\int_s^t \lambda^{(n)}(u)du\Big) - E\Big(\int_s^t \lambda^{(n)}(u)du\Big)^2 + 2E\Big[(N^{(n)}(t) - N^{(n)}(s))\int_s^t \lambda^{(n)}(u)du\Big]. \quad (3.3)
\end{aligned}
$$

It follows from (3.3) that, there is a B_5 such that

$$
E(N^{(n)}(t) - N^{(n)}(s))^2 \leq B_5 \mid t - s \mid . \tag{3.4}
$$

Putting (3.2) and (3.4) together, we get (3.1) by making use of Lemma 3.1. This completes the proof.

Proof of Lemma 3.3. Let l be a positive integer. Let T^l be an optimal stopping time problem for the stochastic sequence $\{(X_{k/2^l}^{(n)}, \mathcal{G}_{k/2^l}^{(n)}) \mid k = 0, 1, \ldots, 2^l M\}$. It follows from Lemma 4.2 on p.64 of Chow, Robbins & Siegmund (1971) that we can assume T^l admissible. Let

$$
T_*^l = \max\ \{T^1, \ldots, T^l\}, \qquad \tau_*^{(n)} = \lim_{l\to\infty} T_*^l.
$$

Using the arguments in the proof of Lemma 3.2, we obtain

$$
\lim_{l\to\infty} E(X_{T_*^l}^{(n)}) = E(X_{\tau_*^{(n)}}^{(n)}), \tag{3.5}
$$

by the Lebesgue dominated convergence theorem applied to $\tau_*^{(n)} - T_*^l$. Note that $E(X_{T_*^l}^{(n)}) \leq E(X_{T_*^{l+1}}^{(n)})$ for every $l = 1, 2, \ldots$.

Let $\tau^{(n)}$ be an arbitrary $\mathcal{G}_t^{(n)}-$ stopping time with $\tau^{(n)} \leq M$. Let

$$
\tau^{(n),l} = \inf\Big\{\frac{k}{2^l} \in [0, M] \mid k \text{ is an integer, } \frac{k}{2^l} > \tau^{(n)}\Big\}.
$$

Then, using (3.1) and (3.5), we get

$$E(X^{(n)}_{\tau^{(n)}}) = \lim_{l\to\infty} E(X^{(n)}_{\tau^{(n)},l}) \le \lim_{l\to\infty} E(X^{(n)}_{T^l_*}) = E(X^{(n)}_{\tau^{(n)}_*}).$$

This completes the proof.

Proof of Lemma 3.4. The idea of the proof is to create a new sequence $\{\tilde{X}^{(n)}_{1/2^l},\ \tilde{X}^{(n)}_{2/2^l},\ldots\}$ which is close to the sequence $\{X^{(n)}_{1/2^l},\ X^{(n)}_{2/2^l},\ldots\}$ and has a discrete distribution as $\{X^{(0)}_{1/2^l},\ X^{(0)}_{2/2^l},\ \ldots\}$ does. Applying backward induction to this new sequence, we conclude the proof.

For $k = 0, 1, \ldots,$ we let

$$\tilde{X}^{(n)}_{\frac{k}{2^l}} = -h(\tilde{\pi}^{(n)}_{k/2^l}) - c\frac{k}{2^l},$$

where $\tilde{\pi}^{(n)}_t = \pi/[\pi + (1-\pi)\tilde{\mathcal{L}}^{(n)}(t)]$ and $\tilde{\mathcal{L}}^{(n)}(t)$ is given in (2.2).

Let $\tau^{(n),l}$ be any $\mathcal{G}^{(n)}_{k/2^l}-$ stopping time satisfying $\tau^{(n),l} \le M$. Then

$$\Big|E(X^{(n)}_{\tau^{(n)},l}) - E(\tilde{X}^{(n)}_{\tau^{(n)},l})\Big| \le \sum_{k=0}^{2^l M} E\Big|X^{(n)}_{\frac{k}{2^l}} - \tilde{X}^{(n)}_{\frac{k}{2^l}}\Big|. \tag{3.6}$$

It follows from (2.2), (2.6), and Lemma 3.1 that both $X^{(n)}_{k/2^l}$ and $\tilde{X}^{(n)}_{k/2^l}$ converge weakly to $X^{(0)}_{k/2^l}$. Since these are bounded random variables, we know (3.6) converges to 0.

Let $\tau_1^{(n,l)}$ be an optimal stopping time for $\{\tilde{X}^{(n)}_{k/2^l}\}$. Then (3.6) implies that

$$\lim_{n\to\infty}\Big|E(X^{(n)}_{\tau_*^{(n,l)}}) - E(\tilde{X}^{(n)}_{\tau_1^{(n,l)}})\Big| = 0. \tag{3.7}$$

Consider now $U^{(n)} \equiv (\tilde{X}^{(n)}_{0/2^l},\ \tilde{X}^{(n)}_{1/2^l},\ldots,\tilde{X}^{(n)}_{2^l M/2^l})$, which converges weakly to $U^{(0)} \equiv (X^{(0)}_{0/2^l},\ X^{(0)}_{1/2^l},\ \ldots,\ X^{(0)}_{2^l M/2^l})$ as n goes to infinity. With a change of probability space, we may assume without loss of generality that $U^{(n)}$ converges almost surely to $U^{(0)}$ as n goes to infinity (See, for example, Pollard (1984, p.71)). Since both $U^{(n)}$ and $U^{(0)}$ are discrete random vectors, it follows from backward induction (cf. Chow, Robbins & Siegmund (1971, p.50)) that there exists an optimal stopping time $\tilde{\tau}_*^{(n,l)}$ for the sequence $\{\tilde{X}^{(n)}_{k/2^l}\}$ which converges almost surely to $\tau_*^{(0,l)}$, an optimal stopping time for $U^{(0)}$. This together with (3.7) completes the proof.

Proof of Lemma 3.5. With Lemma 3.1, it follows from the strong representation theorem (cf. Pollard (1984, p.71)) that there exist counting processes $N_*^{(n)}(\cdot)$

and $N_*^{(0)}(\cdot)$ identically distributed as $N^{(n)}(\cdot)$ and $N^{(0)}(\cdot)$ respectively such that $N_*^{(n)}(\cdot)$ converges to $N_*^{(0)}(\cdot)$ almost surely. The conclusion of this lemma follows easily from this observation. This completes the proof.

Proof of Theorem 2.2. Part (i) is just Lemma 3.3. We need only prove (2.10). It follows from (3.1) that, for every l,

$$E(X^{(n)}_{\tau_*^{(n,l)}}) \le E(X^{(n)}_{\tau_*^{(n)}}) \le E(X^{(n)}_{\tau_*^{(n),l}}) + B^* \frac{1}{2^l} \le E(X^{(n)}_{\tau_*^{(n,l)}}) + B^* \frac{1}{2^l}. \tag{3.8}$$

Letting n go to infinity in (3.8), we get

$$E(X^{(0)}_{\tau_*^{(0,l)}}) \le \liminf_{n\to\infty} E(X^{(n)}_{\tau_*^{(n)}}) \le \limsup_{n\to\infty} E(X^{(n)}_{\tau_*^{(n)}}) \le E(X^{(0)}_{\tau_*^{(0,l)}}) + B^* \frac{1}{2^l}. \tag{3.9}$$

Since (3.9) is valid for every l, and $\lim_{l\to\infty} E(X^{(0)}_{\tau_*^{(0,l)}}) = E(X^{(0)}_{\tau_*})$, we get (2.10). This completes the proof.

Proof of Theorem 2.1. Let $\xi > 0$ be given. It follows from the Bayesian property of $S^{(0)}$ that there exists $c^* > 0,\ w^* \in (0,1)$ such that $S^{(0)}$ is the optimal stopping time for the stochastic process $\check{X}_t^{(0)}$. Here $\check{X}_t^{(0)}$ and, more generally, $\check{X}_t^{(n)}$ are defined by $\check{X}_t^{(n)} = -\check{h}(\check{\pi}_t^{(n)}) - c^* t$, $\check{\pi}_t^{(n)} = \xi/(\xi + (1-\xi)\mathcal{L}^{(n)}(t))$, $\check{h}(z) = \min\ [(1-w^*)z, w^*(1-z)]$.

Assume $\tau^{(n)} \le M$.

Let $\check{\tau}_*^{(n)}$ be the optimal stopping time for the stochastic process $\check{X}_t^{(n)}$ on $[0, M]$. Then it follows from Theorem 2.2 that

$$\lim_{n\to\infty} \check{E}(\check{X}^{(n)}_{\check{\tau}_*^{(n)}}) = \check{E}(\check{X}^{(0)}_{\check{\tau}_*^{(0)}}), \tag{3.10}$$

where $\check{E}(\cdot) = E_0(\xi + (1-\xi)\mathcal{L}^{(n)}(\infty))(\cdot)$.

Hence, using Lemma 3.5, Lemma 3.6 and (3.10), we have

$$\begin{aligned}
&\lim_{n\to\infty} \{\xi[(1-w^*)\alpha_0(S^{(n)}) + c^* E_0(S^{(n)})] + (1-\xi)[w^*\alpha_1(S^{(n)}) + c^* E_1(S^{(n)})]\} \\
=\ & -\check{E}(\check{X}^{(0)}_{S^{(0)}}) \le -\check{E}(\check{X}^{(0)}_{\check{\tau}_*^{(0)}}) \\
=\ & \lim_{n\to\infty} [-\check{E}(\check{X}^{(n)}_{\check{\tau}_*^{(n)}})] \le \liminf_{n\to\infty} [-\check{E}(\check{X}^{(n)}_{\tau^{(n)}})] \\
=\ & \liminf_{n\to\infty} \{\xi[(1-w^*)\alpha_0^{(n)} + c^* E_0(\tau^{(n)})] + (1-\xi)[w^*\alpha_1^{(n)} + c^* E_1(\tau^{(n)})]\}.
\end{aligned} \tag{3.11}$$

Since (3.11) is true for every $\xi \in (0,1)$, (2.4) is a consequence of (2.3) and (3.11). This completes the proof.

Proof of Proposition 1.1. Using the arguments in p.13-15, Jacobsen (1982), we can show that $\bar{\mathcal{F}}_S^{(n)} = \mathcal{G}^{(n)}_{W^{(n)}(S)}$ if S is a random time, i.e., S is a non-negative

$\bar{\mathcal{F}}^{(n)}_{\infty}-$ measurable random variable. We note that $\bar{\mathcal{F}}^{(n)}_{S}$ $(\mathcal{G}^{(n)}_{W^{(n)}(S)})$ is, in fact, the pre-algebra defined in 2.11, Jacobsen (1982), and is equal to the σ-field associated with a stopping time when S $(W^{(n)}(S))$ is a stopping time. With this, we first establish the proposition when S or $W^{(n)}(S)$ is constant, and then establish it for general stopping times.

Let $a > 0$ be a constant. Then, for $t \geq 0$,

$$[(W^{(n)})^{-1}(a) \leq t] = [a \leq W^{(n)}(t)] \in \mathcal{G}^{(n)}_{W^{(n)}(t)} = \bar{\mathcal{F}}^{(n)}_{t}.$$

Here we used the fact that a random time $W^{(n)}(t)$ is $\mathcal{G}^{(n)}_{W^{(n)}(t)}-$ measurable, which can be proved by making use of the definition of the pre-$W^{(n)}(t)$ algebra $\mathcal{G}^{(n)}_{W^{(n)}(t)}$. This shows that $(W^{(n)})^{-1}(a)$ is a $\bar{\mathcal{F}}^{(n)}_{t}-$ stopping time.

Next, let S be a $\bar{\mathcal{F}}^{(n)}_{t}-$ stopping time. Then

$$[W^{(n)}(S) \leq t] = [S \leq (W^{(n)})^{-1}(t)] \in \bar{\mathcal{F}}^{(n)}_{(W^{(n)})^{-1}(t)} = \mathcal{G}^{(n)}_{t},$$

which shows that $W^{(n)}(S)$ is a $\mathcal{G}^{(n)}_{t}-$ stopping time.

Assume $W^{(n)}(S)$ is a $\mathcal{G}^{(n)}_{t}-$ stopping time. Then

$$[S \leq t] = [W^{(n)}(S) \leq W^{(n)}(t)] \in \mathcal{G}^{(n)}_{W^{(n)}(t)} = \bar{\mathcal{F}}^{(n)}_{t},$$

which shows that S is a $\bar{\mathcal{F}}^{(n)}_{t}-$ stopping time. This completes the proof.

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