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LE CAM MAXIMIN TESTS FOR SYMMETRY OF CIRCULAR DATA BASED ON THE CHARACTERISTIC FUNCTION

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Abstract: We consider asymptotic inferences for circular data based on empirical characteristic functions. More precisely, we provide tests for reflective symmetry of circular data based on the imaginary part of the empirical characteristic function. We show that the proposed tests have many attractive features including the property of being locally and asymptotically maximin in the Le Cam sense under sine-skewed alternatives in the specified mean direction case. To the best of our knowledge, this result provides the first instance of such an optimality property for empirical characteristic functions. For the unspecified mean direction case, we provide corrected versions of the original tests that retain nice asymptotic power properties. The results are illustrated using a well-known data set and are checked using Monte-Carlo simulations.

Key words and phrases: Characteristic function, directional statistics, reflective symmetry.

1. Introduction

Statistical modeling and the corresponding analyses of circular data have attracted much attention in the recent years. For instance Jones, Pewsey and Kato (2015) proposed copulas for circular distributions and Kato and Jones (2010), Kato and Jones (2013), Kato and Jones (2015) introduced families of distributions on the circle obtained using various techniques. Density estimation on the circle is considered in García–Portugués, Crujeiras and González–Manteiga (2013) while Oliveira, Crujeiras and Rodríguez-Casal (2014a) and Oliveira, Crujeiras and Rodríguez-Casal (2014b) provided practical tools to deal with circular data. The popularity of circular statistics stems from various disciplines including the study of wind direction or animal orientation. Traditional methods dealing with circular data are well summarized in the monographs Mardia and Jupp (2000) and Jammalamadaka and SenGupta (2001).

Symmetry is one of the most important structural assumptions made on underlying distributions. Circular distributions are not an exception to this rule because (i) most circular distributions are *reflectively symmetric* around a fixed direction, and (ii) most inferential procedures require the reflective symmetry structure in order to be valid or asymptotically valid. Nevertheless, non-symmetric models have grown in popularity owing to their their flexibility and practical usefulness; for example see Umbach and Jammalamadaka (2009), Kato and Jones (2010), Abe and Pewsey (2011) and Jones and Pewsey (2012). Thus, testing for symmetry in the circular data context has become increasingly important. Pewsey (2002) and Pewsey (2004) proposed procedures for testing symmetry around an unspecified and a specified mean direction respectively. The proposed tests are based on the second-order trigonometric moments. In the specified mean direction case, the Pewsey (2004) test has been shown to be locally and asymptotically optimal under natural skewed alternatives by Ley and Verdebout (2014), who provided a family of testing procedures.

In general, many types of multivariate symmetric distributions exist including spherically symmetric distributions, elliptically symmetric distributions, and so on. A particular symmetry structure often yields a certain shape of the corresponding characteristic function (CF). For instance, if a random vector $\mathbf{X}$, taking values in $\mathbb{R}^p$, is symmetric around some location parameter $\boldsymbol{\mu}$ (in the sense that $\mathbf{X} - \boldsymbol{\mu} =_d \boldsymbol{\mu} - \mathbf{X}$), then the imaginary part of the CF of $\mathbf{X} - \boldsymbol{\mu}$ vanishes. This is the central idea behind the tests for symmetry proposed in many works including Heathcote, Rachev and Cheng (1995), Neuhaus and Zhu (1998), Henze, Klar and Meintanis (2003), and Ngatchou–Wandji and Harel (2013). In this study, we also use the imaginary part of the empirical CF(ECF) process to provide tests for reflective symmetry of circular data. The resulting procedures enjoy many attractive features. First, they are asymptotically distribution-free, which is obviously an important property in the given context. More importantly, we show that in the specified mean direction case, the procedures based on the ECF are locally and asymptotically *maximin* in the Le Cam sense, under very general local alternatives. To the best of our knowledge, this is the first time that such a property has been shown to hold for procedures based on an ECF. Furthermore we also provide asymptotic procedures in the unspecified location case that are extremely competitive.

The rest of the paper is organized as follows. In Section 2, we discuss the properties of the CF of circular random variables. In Section 3, we present our test procedures in the specified location case, and show their optimality properties. In Section 4, we examine the unspecified location case. Section 5 is devoted to Monte Carlo simulations, and in Section 6 we illustrate the procedures using a real data set. In Section 7, we conclude the paper. The proofs of the main results are collected in the Appendix.

2. Properties of the CF

Let θ denote an arbitrary circular random variable with an absolutely continuous circular distribution function $F(t) = \mathbb{P}(\theta \leq t)$. The specificity of such a circular random variable or random angle θ is its periodicity, in the sense that letting f denote the density associated with F, we have

$$f(t) = f(t + 2k\pi)$$

for any integer k. As on the real line, the distribution of θ is in one-to-one correspondence with the CF defined as

$$\varphi_\theta(r) := \mathbb{E}\left[e^{ir\theta}\right] = \int_{-\pi}^{\pi} e^{irt} dF(t), r \in \mathbb{R}. \tag{2.1}$$

The CF can also be written in terms of Cartesian coordinates as

$$\varphi_\theta(r) = \mathbb{E}\left[\cos(r\theta)\right] + i\mathbb{E}\left[\sin(r\theta)\right] := \alpha_r + i\beta_r, \tag{2.2}$$

where α_r (resp. β_r) is the real (resp. the imaginary) part of φ_θ. Owing to its periodicity, and unlike real-line distributions, the CF of a circular random variable needs to be defined only at *integer values* $r = 0, \pm 1, \pm 2, \ldots$; see Jammalamadaka and SenGupta (2001).

Now, letting μ denote the mean direction of θ, defined as

$$(\cos(\mu), \sin(\mu))' := \frac{(\mathrm{E}[\cos(\theta)], \mathrm{E}[\sin(\theta)])'}{(\alpha_1^2 + \beta_1^2)^{1/2}},$$

θ is said to be reflectively symmetric around μ if its density f is such that $f(\mu + t) = f(\mu - t)$ for all $t \in [-\pi, \pi)$. Throughout this paper, the class $\mathcal{F}_\mu$ of reflectively symmetric densities around μ is denoted as

$$\mathcal{F}_\mu := \Big\{ f : f(t) > 0 \, \text{a.e.}, f(t + 2j\pi) = f(t) \; \forall j \in \mathbb{Z}, f(\mu + t) = f(\mu - t), \\ f \, \text{unimodal at} \, \mu, \int_{-\pi}^{\pi} f(t) dt = 1 \Big\}.$$

Following Jammalamadaka and SenGupta (2001), if θ is reflectively symmetric around μ, then the central trigonometric moments

$$\mathbb{E}\left[e^{ir(\theta - \mu)}\right] = \mathbb{E}\left[\cos(r(\theta - \mu))\right] + i\mathbb{E}\left[\sin(r(\theta - \mu))\right] := \bar{\alpha}_r + i\bar{\beta}_r, \tag{2.3}$$

are such that $\bar{\beta}_r = 0$ for all $r \in \mathbb{N}$. This is the basis of our test statistic for the null hypothesis of symmetry around μ.

3. Optimal Tests Based on the Empirical Characteristic Function

Let $\theta_1, \ldots, \theta_n$, be an identically and independently distributed (i.i.d.) random sample with mean direction μ. First, we consider testing for symmetry around a known center. Thus throughout the section and without loss of generality, we assume that $\mu = 0$. The aforementioned properties of the CF of symmetric circular random variables discussed lead directly to consider the imaginary part of the ECF,

$$b_n(r) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sin(r\theta_i), \tag{3.1}$$

for any $r \in \mathbb{N}$. The simplest tests for reflective symmetry are based on $b_n(1)$ or $b_n(2)$ such as the tests proposed by Pewsey (2004) and Ley and Verdebout (2014). In particular, Ley and Verdebout (2014) showed that a natural test based on $b_n(r)$ is locally and asymptotically most powerful under *r-skewed alternatives* (see below). In this study we use the empirical process $b_n(r)$ to construct new tests for reflective symmetry. Specifically, we use the following result, which is a direct consequence of the central limit theorem.

Proposition 1. *Assume that $\theta_1, \ldots, \theta_n$ is an i.i.d. sequence of circular random variables with density f_0 in $\mathcal{F}_0$. Then, as $n \to \infty$, the process $\{b_n(r)\}_{r\in\mathbb{N}}$ converges in (finite-dimensional) distribution to a Gaussian process $B(.)$ with mean zero and covariance kernel (the expectation is taken under f_0)*

$$K(s,t) = \mathbb{E}[\sin(s\theta_1)\sin(t\theta_1)]. \tag{3.2}$$

Proposition 1 states that any vector of the form

$$\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)} := (b_n(k_1), \ldots, b_n(k_m))$$

converges weakly to a centered multinormal distribution with a covariance matrix $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$ determined by the kernel in (3.2). Letting $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ denote a consistent estimator of $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$, this suggests to consider test statistics of the form

$$W^{(n)}_{(k_1,\ldots,k_m)} := (\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)})' \hat{\mathbf{\Sigma}}^{-1}_{(k_1,\ldots,k_m);f_0} \mathbf{B}^{(n)}_{(k_1,\ldots,k_m)}. \tag{3.3}$$

Note that a consistent estimator $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ can be obtained by substituting $K(s,t)$ in (3.2) by its empirical counterpart

$$\hat{K}(s,t) := \frac{1}{n} \sum_{i=1}^{n} \sin(s\theta_i)\sin(t\theta_i).$$

Indeed the law of large numbers directly implies that $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0} - \mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0} =$

$o_{\mathrm{P}}(1)$ as $n \to \infty$ under the null (and, therefore, under contiguous alternatives). Test statistics similar to (3.3) have been suggested by Koutrouvelis (1985) for conventional (non-circular) distributions; see also Csörgő and Heathcote (1987). In addition note that if $m = 1$ and $k_1 = r$ say, the test coincides with that proposed by Ley and Verdebout (2014) which is locally and asymptotically most powerful in the Le Cam sense under r sine-skewed alternatives. If more that one component is selected in $\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)}$, then the local most powerfulness against a particular alternative, as in Abe and Pewsey (2011) and Ley and Verdebout (2014) is lost. However as we explain now, the test remains locally and asymptotically *maximin* under more general local alternatives.

The alternatives to reflective symmetry considered in Abe and Pewsey (2011) and Ley and Verdebout (2014) are characterized by densities of the form

$$f_\lambda(t) := f_0(t)(1 + \lambda \sin(kt)), \quad t \in [-\pi, \pi), k \in \mathbb{N}, \lambda \in [-1, 1], \tag{3.4}$$

where f_0 belongs to the class $\mathcal{F}_0$, defined in Section 2. Within this family of distributions, the null hypothesis of reflective symmetry coincides with the subset of distributions with $\lambda = 0$.

Now, as explained in the previous section, we have that $\mathrm{E}[\sin(r\theta_1)] = 0$ for any $r \in \mathbb{N}$ under reflective symmetry. As a result, more general alternatives are absolutely continuous distributions, with densities of the form

$$f_{\boldsymbol{\lambda}}(t) := f_0(t)\left(1 + \sum_{i=1}^{m} \lambda_{k_i} \sin(k_i t)\right) \quad t \in [-\pi, \pi), \tag{3.5}$$

where $(k_1, \ldots, k_m) \in \mathbb{N}^m$ is an m-tuple of distinct integers and $\boldsymbol{\lambda} := (\lambda_{k_1}, \ldots, \lambda_{k_m}) \in \mathcal{C}^m := [-1/m, 1/m]^m$ is a multivariate skewness parameter. Note that $\boldsymbol{\lambda} \in \mathcal{C}^m$ guarantees that $f_{\boldsymbol{\lambda}}$ in (3.5) is a (positive) density function. In the remainder of the paper we write $\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}$ for the joint distribution of an n-tuple $\theta_1, \ldots, \theta_n$ of circular random variables with common density (3.5). Testing for symmetry against such alternatives is more difficult because it becomes a multivariate problem owing to the parameter space attached to the underlying probability space being multidimensional. More precisely, we can test for symmetry against these alternatives by considering the problem $\mathcal{H}_0 : \boldsymbol{\lambda} = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\lambda} \neq \mathbf{0}$. Here we show that when based on the m-tuple $(k_1, \ldots, k_m)$, $W^{(n)}_{(k_1,\ldots,k_m)}$ is locally and asymptotically *maximin* under local alternatives of the form $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$ for some bounded sequence $\boldsymbol{\ell}^{(n)} = (\ell^{(n)}_{k_1}, \ldots, \ell^{(n)}_{k_m}) \in \mathbb{R}^m$ such that $n^{-1/2}\boldsymbol{\ell}^{(n)} \in \mathcal{C}^m$. A test ϕ^* is called maximin in the class $\mathcal{C}_\alpha$ of level-α tests for $\mathcal{H}_0$ against $\mathcal{H}_1$ if (i) ϕ^*

has level α and (ii) the power of ϕ^* is such that

$$\inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi^*] \geq \sup_{\phi\in\mathcal{C}_\alpha} \inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi].$$

We have the following result.

Proposition 2. *Assume that $\theta_1, \ldots, \theta_n$ is an i.i.d. sequence such that under $\mathrm{P}^{(n)}_{\mathbf{0};f_0}$ with f_0 in $\mathcal{F}_0$, $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$ has full rank and $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0} - \mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$. Furthermore, let $\boldsymbol{\ell}^{(n)} = (\ell^{(n)}_{k_1}, \ldots, \ell^{(n)}_{k_m})$ be a bounded sequence of $\mathbb{R}^m$ such that (i) $n^{-1/2}\boldsymbol{\ell}^{(n)} \in \mathcal{C}^m$,and (ii) $\boldsymbol{\ell}^{(n)}$ converges to $\boldsymbol{\ell} := \lim_{n\to\infty} \boldsymbol{\ell}^{(n)}$. Then, we have that*

(i) $W^{(n)}_{(k_1,\ldots,k_m)}$ *is asymptotically chi-square with m degrees of freedom under $\cup_{f_0\in\mathcal{F}}\mathrm{P}_{\mathbf{0};f_0}$;*

(ii) $W^{(n)}_{(k_1,\ldots,k_m)}$ *is asymptotically chi-square with m degrees of freedom and with non-centrality parameter $\boldsymbol{\ell}'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}$ under $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$,*

(iii) the test $\phi^{(n)}_{\mathrm{MV};(k_1,\ldots,k_m)}$ that rejects the null hypothesis when $W^{(n)}_{(k_1,\ldots,k_m)} > \chi^2_{m;1-\alpha}$ is locally and asymptotically maximin when testing $\cup_{f_0\in\mathcal{F}}\mathrm{P}_{\mathbf{0};f_0}$ against

$$\cup_{f_0\in\mathcal{F}}\mathrm{P}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}.$$

The results obtained in Proposition 2 extend those of Ley and Verdebout (2014) which were obtained under simpler local alternatives of the form (3.4); point (iii) of the proposition states that $\phi^{(n)}_{\mathrm{MV};(k_1,\ldots,k_m)}$ is locally and asymptotically maximin under any reference density $f_0 \in \mathcal{F}$. Because k in (3.4) can not realistically be selected a priori, following Proposition 2 above it is more appropriate to perform the test $\phi^{(n)}_{\mathrm{MV};(k_1,\ldots,k_m)}$ which rejects the symmetry hypothesis $\mathcal{H}_0$ when $W^{(n)}_{(k_1,\ldots,k_m)} > \chi^2_{m;1-\alpha}$ for some m-tuple $(k_1, \ldots, k_m)$.

A reasonable criterion for the selection of the number m and the specific location $(k_1, \ldots, k_m)$ of the ECF arguments clearly remains an issue. In fact, for the conventional (non-circular) ECF the problem dates back to Feigin and Heathcote (1976) and Csörgő and Heathcote (1987) in the case of a single argument ($m = 1$). Feurverger and McDunnough (1981) provide a fundamental contribution connecting the efficiency of the ECF estimation procedures with the efficiency of the maximum likelihood. However this refers to point estimation rather than testing and, moreover, assumes a fixed parametric model. On the other hand, and in the context of hypothesis testing, the finite-sample results of Koutrouvelis (1980) and Epps and Singleton (1986) imply that although a larger m may be asymptotically preferable, it is advisable to use a value of $m := m_n$ that depends on the sample size in an increasing fashion. Here we note that selecting $m > n$ typically yields an estimator $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ that is singular making it be impossible to perform the corresponding test in practice. Numerical problems may occur even when $m \leq n$ if m and n are of the same order of magnitude.

The only theoretically viable solution to a related problem has been provided by Teinreiro (2009). This corresponds to the optimal choice of the weight parameter a in our test in (4.2) below when the weight function $w(r)$ is already fixed. However, Teinreiro's solution falls within the strict parametric context of testing normality and depends heavily on the direction of departure from the null hypothesis of normality. Such a choice is quite different to our context, which is nonparametric, even under the null hypothesis. Furthermore it is too specific to provide guidance in the present situation, but clearly shows the complexity of the problem of choosing m and $(k_1, \ldots, k_m)$. Now, in terms of asymptotics, a reasonable approach is to select both m and the m-tuple $(k_1, \ldots, k_m)$ to maximize the local power under $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$, that is, selecting $\tilde{m}$ and the $\tilde{m}$-tuple $(k_1, \ldots, k_{\tilde{m}})$ such that $\boldsymbol{\ell}'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}$ is maximal. As mentioned above, the problem is complicated because the local power depends on the perturbation $\boldsymbol{\ell}$ (which is also m-dimensional) and on f_0. One "ad-hoc" way of selecting $(k_1, \ldots, k_{\tilde{m}})$ is to take $(k_1, \ldots, k_{\tilde{m}}) = \mathrm{argmax}_{\cup_{m\in\mathbb{N}}\mathcal{K}_m}\mathrm{tr}(\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0})/m$, where $\mathcal{K}_m$ is the set of all possible m-tuples of distinct natural numbers. Rather than providing a solution to this difficult "non-parametric" problem, we instead offer some advice. Assume that the density f_0 is indexed by some positive concentration parameter κ, such that when $\kappa = 0$, the distribution is uniform on $\mathcal{S}^1$ and when $\kappa \to \infty$, the distribution tends to a point mass on the location parameter μ. Many well-known densities, such as the von Mises densities, are of this type. For such a $f_0^{(\kappa)}$, when $\kappa = 0$, then $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0^{(\kappa)}} = 2\mathbf{I}_m$. Thus $\beta_{(k_1,\ldots,k_m);f_0^{(\kappa)}} := \mathrm{tr}(\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0^{(\kappa)}})/m$ is constant for any choice of m and $(k_1, \ldots, k_m)$. Therefore, for distributions with small concentration, $\beta_{(k_1,\ldots,k_m);f_0^{(\kappa)}}$ does vary much in which case it may be worth staying with a small m. Now, as κ grows, the variability of the trigonometric moments diminishes, especially for small order moments. Therefore it is more natural to select more higher order moments than lower order moments when the data are more concentrated.

In summary we note that the test $\phi^{(n)}_{\mathrm{MV};(k_1,\ldots,k_m)}$ still suffers from several drawbacks:

(i) Although the new test takes into account the ECF process over a fixed grid of values $(k_1, \ldots, k_m)$, there remains the problem of consistency against general alternatives. Such consistency can be obtained by considering arbitrarily large grids with $m = m_n$ that diverges to ∞. However, this is a complex issue, as explained above.

(ii) The assumption of a known symmetry center is sometimes unrealistic.

In the next section, we consider testing procedures for reflective symmetry whose objective is to resolve these drawbacks.

4. Tests in the Unknown Mean Direction Case

Let $\theta_1, \ldots, \theta_n$, be independent circular random variables with mean direction μ, and consider the centered observations $\vartheta_i = \theta_i - \mu$, for $i = 1, \ldots, n$. A well-known nonparametric estimate of the mean direction μ is given by

$$\widehat{\mu} := \arctan\left(\frac{\sum_{i=1}^n \sin\theta_j}{\sum_{i=1}^n \cos\theta_i}\right);$$

see Jammalamadaka and SenGupta (2001). The estimator $\hat{\mu}$ naturally yields estimated versions of $b_n(r)$ in (3.1), given by

$$\widehat{b}_n(r) = n^{-1/2}\sum_{i=1}^n \sin(r\widehat{\vartheta}_i),$$

where $\widehat{\vartheta}_i = \theta_i - \widehat{\mu},\ i = 1, \ldots, n$. In the following result, we study the asymptotic properties of $\widehat{b}_n(r)$ under the null hypothesis of symmetry. The proof of the result requires the use of a discretized version of $\hat{\mu}$. More precisely it requires $\hat{\mu}$ to be *locally and asymptotically discrete*: $\hat{\mu}$ only takes a bounded number of distinct values in μ-centered intervals with $O(n^{-1/2})$ radius. Note that this discretization condition is a purely technical requirement (e.g. see (Ley et al. (2013); Hallin et al. (2013), and Hallin, Paindaveine and Verdebout (2014))), with few practical implications (in fixed-n practice, such discretizations are irrelevant because the discretization radius can be taken arbitrarily large). Therefore, for the sake of simplicity, we tacitly assume in the sequel that $\widehat{\mu}$ is locally and asymptotically discrete. Defining $\gamma := \sqrt{\alpha_1^2 + \beta_1^2}$ (see (2.2)), we have the following result.

Proposition 3. *Assume that $\theta_1, \ldots, \theta_n$ is an i.i.d. sequence of circular random variables with density $f \in \mathcal{F}_\mu$. Then, as $n \to \infty$, the process $\{\widehat{b}_n(r)\}_{r\in\mathbb{N}}$ converges in (finite-dimensional) distribution to a Gaussian process $\tilde{B}(.)$ with mean zero* *and covariance kernel (see* (2.3) *for a definition of* $\bar{\alpha}_r$*)*

$$\begin{aligned}\tilde{K}(s,t) &= \mathrm{E}[\sin(s\vartheta_1)(\sin(t\vartheta_1)] - \gamma^{-1}s\bar{\alpha}_s\mathrm{E}[\sin(\vartheta_1)(\sin(t\vartheta_1)] \\ &\quad - \gamma^{-1}t\bar{\alpha}_t\mathrm{E}[\sin(\vartheta_1)\sin(s\vartheta_1)] + \gamma^{-2}st\bar{\alpha}_s\bar{\alpha}_t\mathrm{E}[\sin^2(\vartheta_1)],\end{aligned} \tag{4.1}$$

where the expectations are taken under $f \in \mathcal{F}_\mu$*.*

As in the previous section, it follows directly from Proposition 3 that the vector

$$\widehat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)} := (\widehat{b}_n(k_1),\ldots,\widehat{b}_n(k_m))$$

converges weakly to a centered multinormal distribution, with a covariance matrix $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ determined by the kernel in (4.1). Note that in (4.1),

$$\tilde{K}(1,1) = \left(1 - \frac{\bar{\alpha}_1}{\gamma}\right)^2 \mathrm{E}[\sin^2(\vartheta_1)].$$

From standard trigonometry, we have

$$\begin{aligned}\bar{\alpha}_1 &= \mathbb{E}\left[\cos(\theta_1 - \mu)\right] = \cos(\mu)\mathbb{E}[\cos(\theta_1)] + \sin(\mu)\mathbb{E}[\sin(\theta_1)] \\ &= \gamma\cos^2(\mu) + \gamma\sin^2(\mu) = \gamma.\end{aligned}$$

Thus we readily obtain that $\tilde{K}(1,1) = 0$. This is in line with the well-known identity (see, e.g., (Pewsey (2002)))

$$\sum_{i=1}^{n} \sin(\theta_i - \widehat{\mu}) = 0,$$

which states that $\widehat{b}_n(1)$ is equal to zero for any n and, therefore, has no (asymptotic) variance. As a result, we recommend choosing a m-tuple of indices $(k_1,\ldots, k_m)$, such that $1 \notin (k_1,\ldots,k_m)$. Taking such an m-tuple and letting $\widehat{\tilde{\mathbf{\Sigma}}}_{(k_1,\ldots,k_m)}$ denote a consistent estimator of $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ (as in the case of known μ, such an estimator can be obtained by replacing expectations by empirical means and μ with $\hat{\mu}$ in (4.1)), the test statistic

$$\hat{W}^{(n)}_{(k_1,\ldots,k_m)} := (\hat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)})'\widehat{\tilde{\mathbf{\Sigma}}}^{-1}_{(k_1,\ldots,k_m)}\hat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)}$$

is asymptotically chi-square with m degrees of freedom if $1 \notin (k_1,\ldots,k_m)$, and with $m-1$ degrees of freedom if $1 \in (k_1,\ldots,k_m)$.

Now, even if the test that rejects the null for large values of $\hat{W}^{(n)}_{(k_1,\ldots,k_m)}$ is asymptotically valid in the case of an unspecified μ, the consistent estimation of $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ remains an issue when m is large, especially when $m > n$. Furthermore, the asymptotic properties of $\hat{W}^{(n)}_{(k_1,\ldots,k_m)}$ under local alternatives require a careful study of the Fisher information for both the symmetry and the location parameters which is beyond the scope of this paper.

Therefore, we propose a test that takes into account the full empirical characteristic process $\{\widehat{b}_n(r)\}_{r\in\mathbb{N}}$. This test rejects the null hypothesis of reflective symmetry for large values of

$$T_{n,w} = \sum_{r=1}^{\infty} w(r)\widehat{b}_n^2(r), \tag{4.2}$$

where $w(r),\ r \geq 1$, is a sequence of positive weights. It follows easily from the definition of $\widehat{b}_n(r)$ that

$$\widehat{b}_n^2(r) = \frac{1}{2n}\left(\sum_{i,j=1}^{n}\cos(r\widehat{\vartheta}^-_{i,j,n}) - \sum_{i,j=1}^{n}\cos(r\widehat{\vartheta}^+_{i,j,n})\right),$$

where $\widehat{\vartheta}^{\pm}_{i,j,n} = \widehat{\vartheta}_i \pm \widehat{\vartheta}_j$. Thus letting $C_w(\vartheta) := \sum_{r=1}^{\infty} w(r)\cos(r\vartheta)$, the test statistic in (4.2) may be rewritten as

$$T_{n,w} = \frac{1}{2n}\sum_{i,j=1}^{n}\left(C_w(\widehat{\vartheta}^-_{i,j,n}) - C_w(\widehat{\vartheta}^+_{i,j,n})\right).$$

Although the test statistic $T_{n,w}$ is defined using infinite sums, some choices of the weights $w(r)$ make $T_{n,w}$ easy to compute. More precisely, following the results in Gradshteyn and Ryzhik (1994), the sequences $w(r)$ that provide closed forms for $T_{n,w}$ include $w(r) = a^r,\ |a| < 1$, which yields

$$C_w(\vartheta) = \frac{1}{2}\left(\frac{1-a^2}{1-2a\cos\vartheta + a^2} - 1\right)$$

and $w(r) = e^{-ar},\ a > 0$, which yields

$$C_w(\vartheta) = \frac{1}{2}\left(\frac{e^a - e^{-a}}{e^a + e^{-a} - 2\cos\vartheta} - 1\right).$$

The following proposition implies the strong (almost sure) consistency of the test statistic $T_{n,w}$ against fixed alternatives.

Proposition 4. *Let $T_{n,w}$ denote the the test statistic given in* (4.2)*. Then*

$$\frac{T_{n,w}}{n} \longrightarrow \sum_{r=1}^{\infty} w(r)\, b_r^2 := T_w,\ \text{a.s. as } n \to \infty. \tag{4.3}$$

Because $T_w = 0$ only under the null hypothesis $\mathcal{H}_0$, (4.3) *implies the strong consistency of the test that rejects $\mathcal{H}_0$ for large values of $T_{n,w}$.*

The main difficulty with $T_{n,w}$ is that it is not distribution-free under the null. In fact, it may be argued that $T_{n,w}$ is asymptotically distributed as $\sum_{r=1}^{\infty} w(r)V(r)$, where $V(r)$ is the Gaussian process defined in Proposition 3. Following Neuhaus and Zhu (1998), critical values of a test based on $T_{n,w}$ can be obtained using permutational arguments. First, note that it follows from the proof of Proposition 3 that, for all r,

$$\hat{b}_n(r) = n^{-1/2}\sum_{i=1}^{n}\sin(r\vartheta_i) - \frac{r\mathrm{E}[\cos(r\vartheta_1)]}{\sqrt{\mathbb{E}^2[\cos(\theta_1)] + \mathbb{E}^2[\sin(\theta_1)]}} n^{-1/2}\sum_{i=1}^{n}\sin(\vartheta_i) + o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathcal{H}_0$. Let $\mathbf{e} = (e_1, \ldots, e_n)$ be an i.i.d. sequence of random variables such that $e_i = 1$ with probability 0.5 and $e_i = -1$ with probability 0.5. Then define

$$\tilde{b}_n^{(\mathbf{e})}(r) := n^{-1/2}\sum_{i=1}^{n}\left(\sin(re_i\hat{\vartheta}_i) - \frac{r(\sum_{j=1}^{n}\cos(re_j\hat{\vartheta}_j))}{((\sum_{j=1}^{n}\cos(e_j\hat{\vartheta}_j))^2 + (\sum_{j=1}^{n}\sin(e_j\hat{\vartheta}_j))^2)^{1/2}}\sin(e_i\hat{\vartheta}_i)\right). \tag{4.4}$$

Using the data $\vartheta_1, \ldots, \vartheta_n$, the critical values of the test based on $T_{n,w}$ at the level α are approximated as follows:

(i) Generate M i.i.d. random sequences of n-dimensional vectors $\mathbf{e}_1, \ldots, \mathbf{e}_M$ distributed as $\mathbf{e}$ above;

(ii) For all $j = 1, \ldots, M$, compute $Q_j := \sum_{r=1}^{m} w(r)(\tilde{b}_n^{(\mathbf{e}_j)}(r))^2$, for large m;

(iii) As the critical value, select the empirical $1-\alpha$ quantile of $Q_1, \ldots, Q_M$.

5. Simulations

In this section, our objective is to compare the properties of the proposed procedures with those of other well-known tests for the same problem. We perform two sets of simulations: one in the specified-μ situation and one in the unspecified-μ situation.

For the first problem, which consists in testing for reflective symmetry around a known center, we generate $N = 2{,}500$ mutually independent samples of i.i.d. circular random variables

$$\theta_{\ell;j}^{(\rho)}, \quad \rho = 1, \ldots, 6,\ \ell = 0, \ldots, 3,\ j = 1, \ldots, n = 100,$$

following various skewed distributions with concentration one and mean direction zero as in Abe and Pewsey (2011) and Ley and Verdebout (2014). The $\theta_{\ell;j}^{(1)}$'s are von Mises 1-sine-skewed (obtained by taking $k = 1$ in (3.4)), with skewness parameter $\lambda = \ell/25$; the $\theta_{\ell;j}^{(2)}$'s are von Mises 2-sine-skewed (obtained by taking $k = 2$ in (3.4)), with skewness parameter $\lambda = \ell/25$; the $\theta_{\ell;j}^{(3)}$'s are von Mises 4-sine-skewed (obtained by taking $k = 4$ in (3.4)), with skewness parameter $\lambda = \ell/25$ and the $\theta_{\ell;j}^{(4)}$'s are von Mises 6-sine-skewed (obtained by taking $k =$

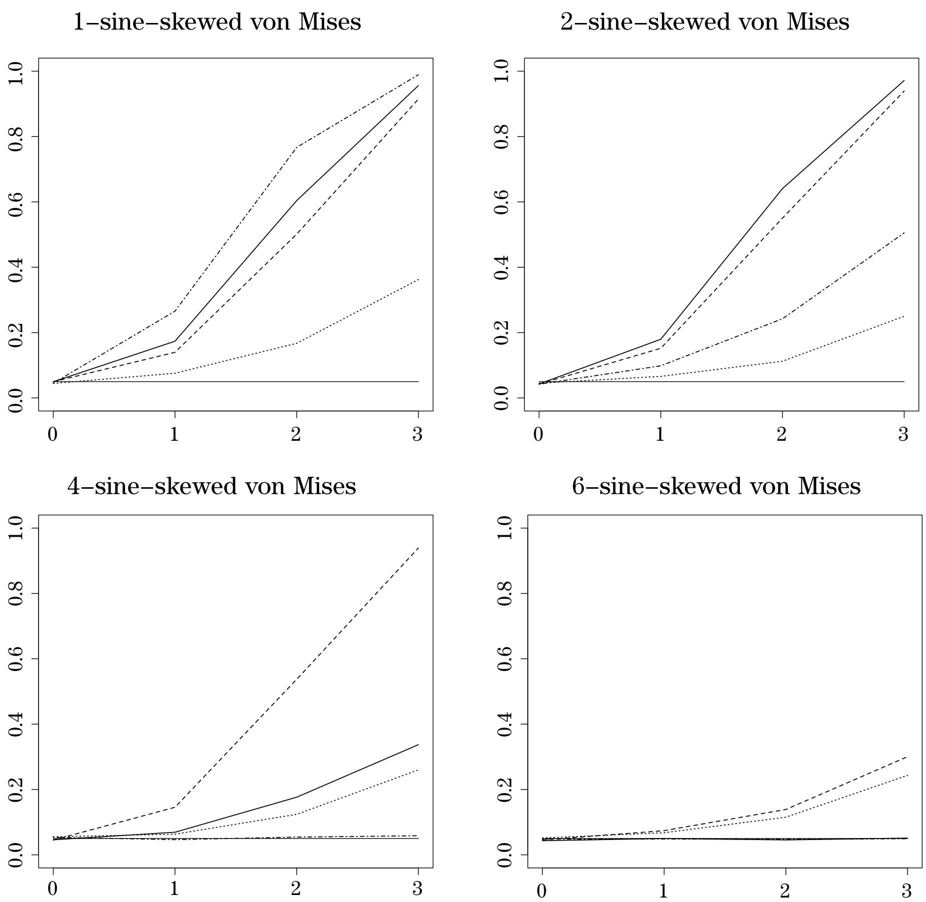

Figure 1. Power curves of (i) the Ley and Verdebout (2014) locally and asymptotically optimal test against 1-sine-skewed-von Mises alternatives (dot-dashed line), (ii) our test $\phi^{(n)}_{\mathrm{MV};(1,2,3)}$ (solid line), (iii) our test $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ (dashed line), and (iv) the modified runs test of Pewsey (2004) (dotted line). The sample size is $n = 100$.

6 in (3.4)), with skewness parameter $\lambda = \ell/25$. The $\theta^{(5)}_{\ell;j}$'s and the $\theta^{(6)}_{\ell;j}$'s are skewed Möbius distributions, as in Kato and Jones (2010) based on von Mises with concentration one and skewness parameter $\lambda = \ell/50$ respectively with $r = 0.25$ and $r = 0.5$ respectively (see (Kato and Jones (2010)) for details). The value $\ell = 0$ always yields a reflectively symmetric distribution belonging to the null hypothesis whereas the values $\ell = 1, 2$ and 3 provide distributions that are increasingly skew-symmetric.

The resulting rejection frequencies of the following tests for reflective symmetry, all at the nominal level 5% are plotted in Figures 1 and 2: the optimal 1-sine-skewed test of Ley and Verdebout (2014); the modified runs test of Pewsey

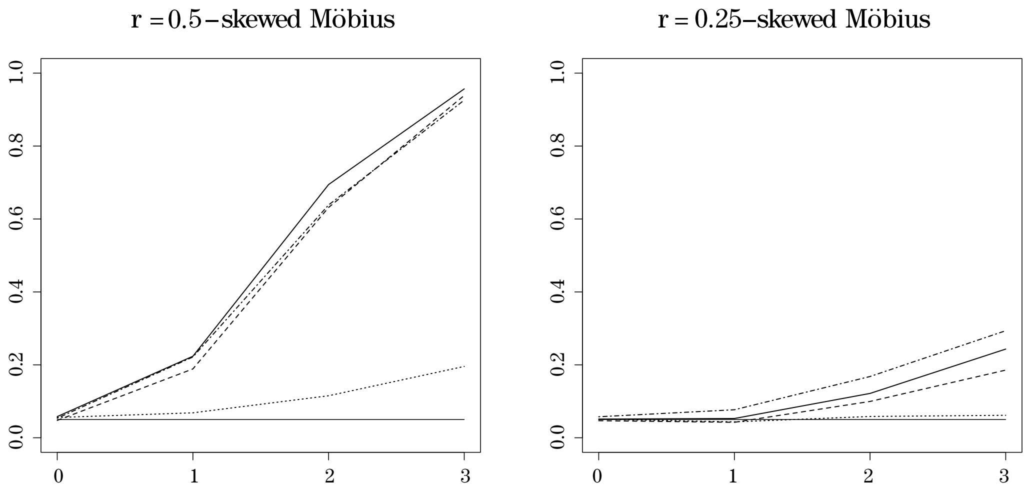

Figure 2. Power curves of (i) the Ley and Verdebout (2014) locally and asymptotically optimal test against 1-sine-skewed-von Mises alternatives (dot-dashed line), (ii) our test $\phi^{(n)}_{\mathrm{MV};(1,2,3)}$ (solid line), (iii) our test $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ (dashed line), and (iv) the modified runs test of Pewsey (2004) (dotted line). The sample size is $n = 100$.

(2004); the test $\phi^{(n)}_{\mathrm{MV};(1,2,3)}$ based on $W^{(n)}_{(1,2,3)}$; and the test $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ based on $W^{(n)}_{(1,\ldots,5)}$. Figures 1 and 2 reveal the following features: (i) all of the tests reach the correct asymptotic level; (ii) as expected, the optimal 1-sine-skewed test of Ley and Verdebout (2014) is optimal under 1-sine-skewed alternatives, but is dominated by the tests $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$ and $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ under all other alternatives; and (iii) $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ still behaves correctly under 4-sine-skewed alternatives, while the same is not true for the test based on $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$. The modified runs test of Pewsey (2004) behaves similarly under the sine-skewed alternatives considered. Under skewed Möbius distributions, both the optimal 1-sine-skewed test and $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$ behave well.

In a second simulation, we compare the following tests for reflective symmetry around an unspecified symmetry center: the Pewsey (2002) test; our tests $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ and $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$, based on the asymptotic critical values of $\widehat{W}_{(2,3)}$ and $\widehat{W}_{(2,\ldots,5)}$, respectively; and the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$, based on $T_{n,w}$ with weights $w_1(r) = 0.5^r$ and $w_2(r) = e^{-r/2}$, respectively. The critical values of $T_{n,w}$ are computed using the bootstrap procedure described below Proposition 4, with $M = 10{,}000$ and $m = 40$.

As for the first simulation scheme, we generate $N = 1{,}500$ mutually inde-

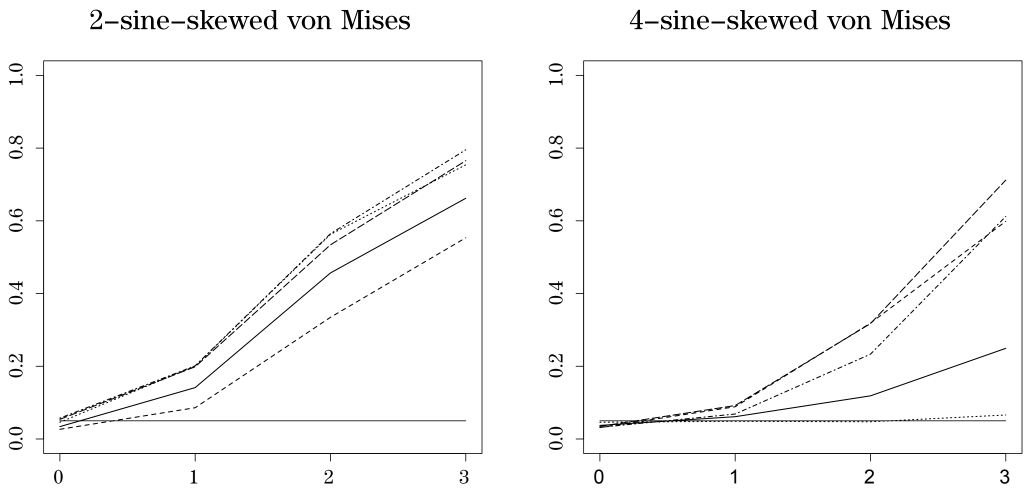

Figure 3. Power curves of the Pewsey (2002) test (dotted line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ (solid line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ (dashed line), $\phi_{T_{n,1}}$ (dot-dashed line) and $\phi_{T_{n,2}}$ (long-dashed line). The sample size is $n = 100$.

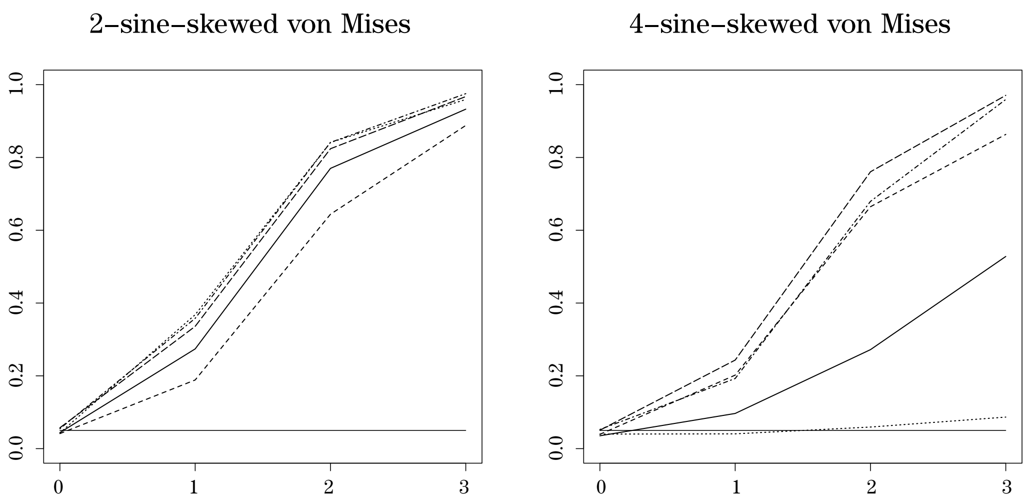

Figure 4. Power curves of the Pewsey (2002) test (dotted line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ (solid line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ (dashed line), $\phi_{T_{n,1}}$ (dot-dashed line) and $\phi_{T_{n,2}}$ (long-dashed line). The sample size is $n = 200$.

pendent samples of i.i.d. of circular random variables

$$\theta^{(\rho)}_{\ell;j}, \quad \rho = 1, 2,\ \ell = 0, \ldots, 3,\ j = 1, \ldots, n,$$

with sine-skewed von Mises distributions with concentration one and mean direction zero. The $\theta^{(1)}_{\ell;j}$'s are 2-sine-skewed (obtained by taking $k = 2$ in (3.4)),

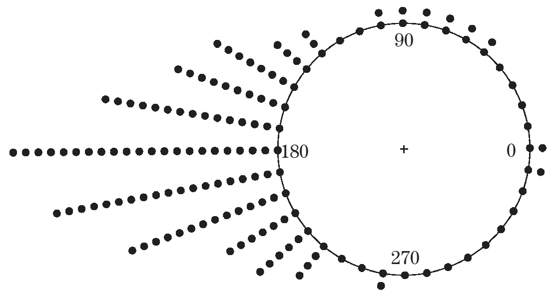

Figure 5. Raw circular plot of the Jander (1957) data set recorded during an orientation experiment with 730 red wood ants. Each dot represents the direction chosen by five ants.

with skewness parameter $\lambda = \ell/25$; and the $\theta^{(2)}_{\ell;j}$'s are 4-sine-skewed (obtained by taking $k = 4$ in (3.4)) with skewness parameter $\lambda = \ell/25$. The rejection frequencies of the five tests, all performed at the nominal level 5% are plotted in Figure 3 for $n = 100$ and in Figure 4 for $n = 200$. The comparisons between the Pewsey (2002) test and the tests $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$, and $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ are very similar to the case of a specified μ. However, note that the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ behave nicely in both cases, clearly detecting alternatives of the same magnitude as the other tests (deviations with rate $1/\sqrt{n}$ from the null). All of the tests are slightly conservative for $n = 100$, but improve with the higher sample size $n = 200$.

6. Real-data illustration

In this section, we illustrate the testing procedures described in the previous section using a well-known data set from an animal orientation experiment. This data set consists of the directions of 730 red wood ants originally placed in the center of an arena, with a black target positioned at an angle of 180° from the zero direction; see Figure 5. The question of interest is whether the directions chosen by the ants are symmetrically distributed around the median direction represented by the black target. The data set analyzed in Abe and Pewsey (2011) and Ley and Verdebout (2014) was originally created by Jander (1957).

Ley and Verdebout (2014) found that the locally and asymptotically most powerful (LAMP) test against 1-sine-skewed alternatives has a p-value of 0.778 and that the LAMP test against 2-sine-skewed alternatives has a p-value of 0.011. We compute our tests based on $W^{(n)}_{(1,2)}$, $W^{(n)}_{(1,2,3)}$, and $W^{(n)}_{(1,\ldots,5)}$, which have p-values 0.012, 0.013 and 0.022 respectively. Thus the tests yield rejection of the null hypothesis of reflective symmetry at the nominal level 0.05.

Because we focus on the symmetry around the black target, the problem can be viewed as a test for symmetry around a specified direction. Nevertheless we perform the tests based on the statistics $\hat{W}^{(n)}_{(1,2,3)}$ and $\hat{W}^{(n)}_{(1,\ldots,5)}$, and also perform the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ based on $T_{n,w}$ with weights $w_1(r) = 0.5^r$ and $w_2(r) = e^{-r/2}$, respectively; see Section 4 for details. The test based on $\hat{W}^{(n)}_{(1,2,3)}$ has a p-value of 0.054, and the test based on $\hat{W}^{(n)}_{(1,\ldots,5)}$ has a p-value of 0.046. Therefore at the nominal level 0.05, one test rejects the null but the other does not. The two tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ also reject the null (the critical values are computed using the bootstrap procedure described below Proposition 4, with $M = 10{,}000$ and $m = 40$).

7. Conclusion and Discussion

We suggest several tests for reflective symmetry based on the ECF. In the fixed location case, the new tests are locally and asymptotically optimal in the *maxmin* sense against certain alternatives. In the unknown location case, we suggested modifications of these optimal procedures as well as a new test that is consistent against each fixed alternative non-symmetric circular distribution. The finite-sample behavior is investigated via a simulation study, and the suggested tests are shown to perform well in comparison with other powerful symmetry procedures.

The following offer possible directions for future work. First the methods can be extended to dimension $p \geq 2$ using the *tangent-normal* decomposition

$$\mathbf{X} = (\mathbf{X}'\boldsymbol{\mu})\boldsymbol{\mu} + (\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X},$$

where $(\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X}$ follows a distribution that is spherically symmetric around zero under the assumption of rotational symmetry. Then, a test for rotational symmetry can be developed using a test for spherical symmetry of $(\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X}$ similarly to Henze, Hlávka and Meintanis (2013). Second goodness-of-fit tests can be developed for circular distributions based on quadratic forms analogous to those of Section 3 and Section 4. Then, the corresponding test statistics would include both the real part α_r and the imaginary part β_r of the CF.

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Appendix: Proofs

Proof of Proposition 2. First, note that Point (i) follows from Proposition 1. For Points (ii) and (iii), we start the proof by showing that the sequence of models $\{\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}\}$ is locally and asymptotically normal in the vicinity of symmetry. First, note that

$$\log \frac{\mathrm{dP}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}}{\mathrm{dP}^{(n)}_{\mathbf{0};f_0}} = \sum_{i=1}^{n} \log(1 + n^{-1/2}(\boldsymbol{\ell}^{(n)})'\mathbf{S}_i^{(n)}), \tag{A.1}$$

where $\mathbf{S}_i^{(n)} := (\sin(k_1\theta_i), \ldots, \sin(k_m\theta_i))'$. From (A.1), the boundedness of $\mathbf{S}_i^{(n)}$ and $\log(1+v) = v - (1/2)v^2 + o(v^2)$, it follows that

$$\log \frac{\mathrm{dP}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}}{\mathrm{dP}^{(n)}_{\mathbf{0};f_0}} = (\boldsymbol{\ell}^{(n)})'\boldsymbol{\Delta}^{(n)} - \frac{1}{2}(\boldsymbol{\ell}^{(n)})'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}^{(n)} + o_{\mathrm{P}}(1) \tag{A.2}$$

as $n \to \infty$ under $\mathrm{P}_{\mathbf{0};f_0}$, where $\boldsymbol{\Delta}^{(n)} := n^{-1/2}\sum_{i=1}^{n}\mathbf{S}_i^{(n)}$ is asymptotically normal with mean zero and covariance $\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}$ still under $\mathrm{P}_{\mathbf{0};f_0}$. Therefore it follows from (A.2) that the sequence of models $\{\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}\}$ is locally and asymptotically normal. Now, from the local asymptotic normality, a locally and asymptotically maximin test for $\mathcal{H}_0 : \boldsymbol{\lambda} = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\lambda} \neq \mathbf{0}$ rejects the null when $(\boldsymbol{\Delta}^{(n)})'\boldsymbol{\Sigma}^{-1}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\Delta}^{(n)}$ exceeds the alpha upper quantile of the chi-square distribution with m degrees of freedom. Because

$$W_n = (\boldsymbol{\Delta}^{(n)})'\hat{\boldsymbol{\Sigma}}^{-1}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\Delta}^{(n)} = (\boldsymbol{\Delta}^{(n)})'\boldsymbol{\Sigma}^{-1}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\Delta}^{(n)} + o_{\mathrm{P}}(1),$$

Point (iii) follows. We now turn to Point (ii). It follows directly from (A.2) that $((\boldsymbol{\Delta}^{(n)})', \log(\mathrm{dP}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0})/\mathrm{dP}_{\mathbf{0};f_0})$ is asymptotically normal with mean $(\mathbf{0}', (-1/2)$ $\boldsymbol{\ell}'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);}\ f_0\boldsymbol{\ell})$ and variance

$$\begin{pmatrix} \boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0} & \boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell} \\ \boldsymbol{\ell}'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0} & \boldsymbol{\ell}'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell} \end{pmatrix}$$

under $\mathrm{P}_{\mathbf{0};f_0}$. Point (ii) then follows by applying Le Cam's third Lemma.

Proof of Proposition 3. First, note that

$$\begin{aligned} \hat{b}_n(r) &= n^{-1/2}\sum_{j=1}^{n}(\sin(r\widehat{\vartheta}_j) - \sin(r\vartheta_j) + \sin(r\vartheta_j)) \\ &= W_n(r) + V_n(r), \end{aligned}$$

where

$$V_n(r) := n^{-1/2}\sum_{j=1}^{n}\sin(r\vartheta_j)$$

and

$$W_n(r) := n^{-1/2} \sum_{j=1}^{n} (\sin(r\widehat{\vartheta}_j) - \sin(r\vartheta_j)).$$

Under $\mathcal{H}_0$, $\mathrm{E}[\sin(r\vartheta_j)] = 0$. Thus the central limit theorem directly implies that $V_n(r)$ converges weakly to a Gaussian random variable with mean zero and variance $\mathrm{E}[\sin^2(r\vartheta_j)]$. Now, for $W_n(r)$, assume for a moment that $\widehat{\mu}$ is discretized so that $n^{1/2}(\widehat{\mu} - \mu)$ can be replaced by a deterministic sequence in $W_n(r)$; see for instance Kreiss (1987) or Hallin, Paindaveine and Verdebout (2014). Then, a Taylor series development directly yields

$$W_n(r) = -r\mathrm{E}[\cos(r\vartheta_j)]n^{1/2}(\widehat{\mu} - \mu) + o_{\mathrm{P}}(1)$$

as $n \to \infty$. To obtain the asymptotic normality of $V_n(r) + W_n(r)$, we need to examine the asymptotic joint distribution of $n^{1/2}(\widehat{\mu} - \mu)$ and $V_n(r)$.
Now, applying the delta method, we easily obtain that

$$n^{1/2}(\widehat{\mu} - \mu) = \lambda^{-1} n^{-1/2} \sum_{i=1}^{n} \sin(\vartheta_i) + o_{\mathrm{P}}(1)$$

as $n \to \infty$, where $\lambda = \sqrt{\mathbb{E}^2[\cos(\theta_1)] + \mathbb{E}^2[\sin(\theta_1)]}$. In summary, we obtain that

$$\begin{pmatrix} V_n(r) \\ W_n(r) \end{pmatrix} = \begin{pmatrix} n^{-1/2} \sum_{j=1}^{n} \sin(r(\vartheta_j)) \\ -r(\mathrm{E}[\cos(r(\vartheta_j))])/(\lambda) n^{-1/2} \sum_{j=1}^{n} \sin(\vartheta_j) \end{pmatrix} + o_{\mathrm{P}}(1)$$

as $n \to \infty$. Thus the vector $\mathbf{B}_{m,n} := (\hat{b}_n(k_1), \ldots, \hat{b}_n(k_m))'$, for some fixed m, is such that

$$\mathbf{B}_{m,n} = \begin{pmatrix} n^{-1/2} \sum_{j=1}^{n} \sin(k_1\vartheta_j) \\ \vdots \\ n^{-1/2} \sum_{j=1}^{n} \sin(k_m\vartheta_j) \end{pmatrix} - \lambda^{-1} \begin{pmatrix} k_1\mathbb{E}[\cos(k_1\vartheta_j)] \\ \vdots \\ k_m\mathbb{E}[\cos(k_m\vartheta_j)] \end{pmatrix} n^{-1/2} \sum_{i=1}^{n} \sin(\vartheta_i) + o_{\mathrm{P}}(1).$$

Thus $\mathbf{B}_{m,n}$ is asymptotically normal with mean zero and covariance matrix $\mathbf{\Sigma} = (\Sigma_{st})$, where $\Sigma_{st} = \tilde{K}(s,t)$. The result follows.

Proof of Proposition 4. From the strong law of large numbers, we have, for $r \geq 1$,

$$n^{-1/2}\widehat{b}_n(r) \longrightarrow b(r), \text{ a.s. as } n \to \infty.$$

Therefore (4.3) follows. Moreover from Proposition 1, the almost sure limit T_w on the right–hand side of (4.3) is positive unless $\mathcal{H}_0$ holds. This in turn, implies that

$$T_{n,w} \longrightarrow \infty, \text{ a.s. as } n \to \infty,$$

under any fixed non-symmetric alternative distribution.

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