Extremal linear quantile regression with Weibull-type tails

Fengyang He

Department of Statistics, Hunan University of Technology and Business, Changsha, China

Huixia Judy Wang

Department of Statistics, George Washington University, Washington DC, USA

Tiejun Tong

Department of Mathematics, Hong Kong Baptist University, Hong Kong

Supplementary Material

This document serves as a supplement to the main manuscript and is organized as follows. In Section S1, we provide seven lemmas that are needed to derive the asymptotic results of the proposed estimators. In Section S2, we provide two propositions that are used in Section 4.3 of the manuscript. Technical proofs of all four theorems are presented in Section S3. In Section S4, we present Figure 1, which plots the root mean integrated squared errors of different estimators versus k_0 for the simulation study.

S1. Seven lemmas

For all $0 < s < 1$ and $m > 1$, we define $a_n = K^\theta(\mu_{\mathbf{X}})\sqrt{n\tau_n}[\bar{F}_u^{-1}(\tau_n) - \bar{F}_u^{-1}(m\tau_n)]^{-1}$ and $a_n(s) = K^\theta(\mu_{\mathbf{X}})\sqrt{ns\tau_n}[\bar{F}_u^{-1}(s\tau_n) - \bar{F}_u^{-1}(sm\tau_n)]$. Besides, we also define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

Note that the Weibull-type distributions have a common extreme value index at zero. Thus in addition to (2.3), $\bar{F}_u(\cdot)$ also satisfies

$$\lim_{z\to\infty} \frac{\bar{F}_u\left(z + xa(z)\right)}{\bar{F}_u\left(z\right)} = e^{-x}, \tag{S1.1}$$

for all $x \in \mathbb{R}$ by (1.2.4) in de Haan and Ferreira (2006), where $a(\cdot)$ is a suitable positive function. By Theorem 1.2.1 in de Haan and Ferreira (2006), we can choose $a(z) = \int_z^\infty \bar{F}_u(s)ds/\bar{F}_u(z)$ such that (S1.1) holds, and then $\int_z^\infty \bar{F}_u(s)ds < \infty$ for $z < \infty$.

Lemma 1. *For the function $a(z) = \int_z^\infty \bar{F}_u(s)ds/\bar{F}_u(z)$ in* (S1.1) *with $\int_z^\infty \bar{F}_u(s)ds < \infty$ for all $z \in \mathbb{R}$, we have*

$$\lim_{z\to\infty} \frac{a(z)H_u(z)}{z} = \theta.$$

Proof. Note that

$$\frac{a(z)H_u(z)}{z} = \frac{\int_z^\infty \bar{F}_u\left(s\right)ds}{z\bar{F}_u\left(z\right)/H_u(z)},$$

where $\int_z^\infty \bar{F}_u(s)ds \to 0$ as $z \to \infty$, and $\lim_{z\to\infty} z\bar{F}_u(z)/H_u(z) = 0$ by $\lim_{z\to\infty} z\bar{F}_u(z) = 0$ in de Haan and Ferreira (2006) and $H_u(z) \to \infty$ as

$z \to \infty$. It follows by L'Hospital's rule that

$$\lim_{z\to\infty} \frac{\int_z^\infty \bar{F}_u(s)\,ds}{z\bar{F}_u(z)/H_u(z)} = \lim_{z\to\infty} -\frac{\bar{F}_u(z)}{D_u(z)},$$

where $D_u(z) := \bar{F}_u'(z)(1+H_u(z))z/H_u^2(z) + \bar{F}_u(z)/H_u(z)$. Hence,

$$-\frac{D_u(z)}{\bar{F}_u(z)} = \frac{zH_u'(z)}{H_u(z)}\frac{1+H_u(z)}{H_u(z)} - \frac{1}{H_u(z)}.$$

It is easy to find that $\lim_{z\to\infty}[1+H_u(z)]/H_u(z) = 1$ and $\lim_{z\to\infty} 1/H_u(z) = 0$.

The following task is to calculate $\lim_{z\to\infty} zH_u'(z)/H_u(z)$. Because $H_u(z)$ is differentiable and $H_u(z) \in \mathcal{RV}_\infty(1/\theta)$, we can obtain that $H_u'(z) \in \mathcal{RV}_\infty(1/\theta-1)$. Noting also that $H_u(z)$ is strictly increasing and $H_u(z) \to \infty$ as $z \to \infty$, we can get by Theorem B.1.5 (Karamata's theorem) in de Haan and Ferreira (2006) that for some $z_0 > 0$

$$\lim_{z\to\infty} \frac{zH_u'(z)}{\int_{z_0}^z H_u'(s)ds} = \lim_{z\to\infty} \frac{zH_u'(z)}{H_u(z) - H_u(z_0)} = \frac{1}{\theta}.$$

Hence, it is readily seen that $\lim_{z\to\infty} zH_u'(z)/H_u(z) = 1/\theta$. This leads to $\lim_{z\to\infty}[-\bar{F}_u(z)]/D_u(z) = \theta$, and hence proves the lemma. □

Lemma 2. *Suppose condition* (C5) *holds. Let* $0 < s < 1$, *and* $\tau_n \to 0$ *as* $n \to \infty$. *Then,*

$$\ln q_Y(s\tau_n|\mathbf{x}) - \ln q_Y(\tau_n|\mathbf{x}) = \frac{\ln(1/s)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln \tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] + \varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) + R_n,$$

where $R_n = O((\ln(1/\tau_n))^{-2} \vee \varpi^2(s, \tau_n^{1/K(\mathbf{x})}))$.

Proof. It is easy to deduce that

$$\begin{aligned}\ln q_Y(s\tau_n|\mathbf{x}) - \ln q_Y(\tau_n|\mathbf{x}) &= \ln\left[q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)\left(1+\varpi\left(s,\tau_n^{1/K(\mathbf{x})}\right)\right)/q_u\left(\tau_n^{1/K(\mathbf{x})}\right)\right]\\ &= \ln\frac{q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)}{q_u\left(\tau_n^{1/K(\mathbf{x})}\right)} + \ln\left(1+\varpi\left(s,\tau_n^{1/K(\mathbf{x})}\right)\right)\\ &= \ln\frac{q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)}{q_u\left(\tau_n^{1/K(\mathbf{x})}\right)} + \varpi\left(s,\tau_n^{1/K(\mathbf{x})}\right) + O\left(\varpi^2\left(s,\tau_n^{1/K(\mathbf{x})}\right)\right)\\ &= T_q + \varpi\left(s,\tau_n^{1/K(\mathbf{x})}\right) + O\left(\varpi^2\left(s,\tau_n^{1/K(\mathbf{x})}\right)\right),\end{aligned}$$

where $T_q = \ln[q_u((s\tau_n)^{1/K(\mathbf{x})})/q_u(\tau_n^{1/K(\mathbf{x})})]$. Noting that $\ln_{-2}(z) = \ln[\ln(1/z)]$, we have

$$\begin{aligned}T_q &= \ln\left[\frac{\ln^\theta(1/(s\tau_n))}{\ln^\theta(1/(\tau_n))}\frac{l(-\ln(s\tau_n)/K(\mathbf{x}))}{l(-\ln\tau_n/K(\mathbf{x}))}\right]\\ &= \theta\left[\ln_{-2}(s\tau_n) - \ln_{-2}(\tau_n)\right] + \ln\left[\frac{l(-\ln(s\tau_n)/K(\mathbf{x}))}{l(-\ln\tau_n/K(\mathbf{x}))}\right]\\ &= \theta\left[\ln_{-2}(s\tau_n) - \ln_{-2}(\tau_n)\right] + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\int_1^{\ln(s\tau_n)/\ln\tau_n} t^{\rho-1}dt\,(1+o(1)) \qquad \text{(S1.2)}\\ &= \theta\left[\frac{\ln(1/s)}{\ln(1/\tau_n)} + O\left((\ln(1/\tau_n))^{-2}\right)\right] + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)D_\rho\left(\frac{\ln(s\tau_n)}{\ln\tau_n}\right)(1+o(1))\\ &= \frac{\ln(1/s)}{\ln(1/\tau_n)}\left[\theta + b\left(-K^{-1}(\mathbf{x})\ln\tau_n\right)(1+o(1))\right] + O\left((\ln(1/\tau_n))^{-2}\right),\end{aligned}$$

where (S1.2) is by $D_\rho(\ln(s\tau_n)/\ln\tau_n) \sim \ln s/\ln\tau_n$ as $\tau_n \to 0$ with $\rho \le 0$ and the second order condition in (C5). This proves the lemma. □

Lemma 3. *Suppose conditions* (C1), (C2), *and* (C5) *hold, and let* $\tau_n \to 0$

as $n \to \infty$. Then for any $k \in (0,1) \cup (1,\infty)$, and each $\mathbf{x} \in \mathcal{X}$,

$$\bar{F}_U^{-1}(k\tau_n|\mathbf{x}) - \bar{F}_U^{-1}(\tau_n|\mathbf{x}) \sim \bar{F}_u^{-1}\left((k\tau_n)^{1/K(\mathbf{x})}\right) - \bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right).$$

Proof. Note that

$$\bar{F}_U^{-1}(k\tau_n|\mathbf{x}) - \bar{F}_U^{-1}(\tau_n|\mathbf{x}) = \bar{F}_U^{-1}(\tau_n|\mathbf{x})\left(\frac{\bar{F}_U^{-1}(k\tau_n|\mathbf{x})}{\bar{F}_U^{-1}(\tau_n|\mathbf{x})} - 1\right)$$

and

$$\bar{F}_u^{-1}\left((k\tau_n)^{1/K(\mathbf{x})}\right) - \bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right) = \bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right)\left(\frac{\bar{F}_u^{-1}\left((k\tau_n)^{1/K(\mathbf{x})}\right)}{\bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right)} - 1\right).$$

Let $T = \bar{F}_u^{-1}((k\tau_n)^{1/K(\mathbf{x})})/\bar{F}_u^{-1}(\tau_n^{1/K(\mathbf{x})}) - 1$, then

$$\begin{aligned}
T &= \frac{H_u^{-1}(-\ln(k\tau_n)/K(\mathbf{x}))}{H_u^{-1}(-\ln\tau_n/K(\mathbf{x}))} - 1 \\
&= \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta \frac{l(-\ln(k\tau_n)/K(\mathbf{x}))}{l(-\ln\tau_n/K(\mathbf{x}))} - 1 \\
&= \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta \left(\frac{l(-\ln(k\tau_n)/K(\mathbf{x}))}{l(-\ln\tau_n/K(\mathbf{x}))} - 1\right) + \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta - 1 \\
&= \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta I_A + I_B,
\end{aligned}$$

where $I_A = l(-\ln(k\tau_n)/K(\mathbf{x}))/l(-\ln\tau_n/K(\mathbf{x})) - 1$ and $I_B = (\ln(k\tau_n)/\ln\tau_n)^\theta - 1$.

Consider the case of $k > 1$ firstly. For I_A now, when $l(\cdot)$ is a non-constant slowly varying function, by (C5) it yields that

$$I := \ln\left[\frac{l(-\ln\tau_n/K(\mathbf{x}))}{l(-\ln(k\tau_n)/K(\mathbf{x}))}\right] = b\left(-\frac{\ln(k\tau_n)}{K(\mathbf{x})}\right) D_\rho\left(\frac{\ln\tau_n}{\ln(k\tau_n)}\right)(1 + o(1)),$$

with $D_\rho(\ln \tau_n/\ln(k\tau_n)) = \int_1^{\ln \tau_n/\ln(k\tau_n)} t^{\rho-1}dt \sim \ln k/\ln(1/\tau_n)$ as $\tau_n \to 0$. Hence, $I \to 0$ as $\tau_n \to 0$ such that $I_A = \exp(-I) - 1$ satisfies $I_A \sim b(-\ln(k\tau_n)/K(\mathbf{x}))\ln k/\ln \tau_n$. When $l(\cdot)$ is a constant function, $I_A \equiv 0$. Similarly the term I_B, we have $I_B \sim \theta \ln k/\ln \tau_n$ as $\tau_n \to 0$ after some simple calculations.

Consequently, $T \sim \theta \ln k/\ln \tau_n$ as $\tau_n \to 0$. Combining this relation, (C2) and $\bar{F}_U^{-1}(\tau_n|\mathbf{x}) \sim \bar{F}_u^{-1}(\tau_n^{1/K(\mathbf{x})})$ as $\tau_n \to 0$, we can get

$$\bar{F}_U^{-1}(k\tau_n|\mathbf{x}) - \bar{F}_U^{-1}(\tau_n|\mathbf{x}) \sim \bar{F}_u^{-1}\left((k\tau_n)^{1/K(\mathbf{x})}\right) - \bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right).$$

For the case of $0 < k < 1$, the claim of this lemma can be proved in a similar way. □

Lemma 4. *If $K(\mathbf{x}) \in (0,\infty)$ for all $\mathbf{x} \in \mathcal{X}$, and $k \in (0,1) \cup (1,\infty)$, under condition* (C5)*, and $\tau_n \to 0$ as $n \to \infty$, we have*

$$\bar{F}_u^{-1}(k\tau_n) - \bar{F}_u^{-1}(\tau_n) \sim [K(\mathbf{x})]^\theta \left[\bar{F}_u^{-1}\left((k\tau_n)^{1/K(\mathbf{x})}\right) - \bar{F}_u^{-1}\left(\tau_n^{1/K(\mathbf{x})}\right)\right]. \tag{S1.3}$$

Proof. We first consider $k > 1$. Noting that

$$\bar{F}_u^{-1}(\tau_n) = H_u^{-1}(-\ln \tau_n) = (-\ln \tau_n)^\theta\, l(-\ln \tau_n),$$

the left-hand side of (S1.3) is

$$\begin{aligned}\bar{F}_u^{-1}(k\tau_n) - \bar{F}_u^{-1}(\tau_n) &= H_u^{-1}\left(-\ln(k\tau_n)\right) - H_u^{-1}\left(-\ln\tau_n\right)\\ &= [-\ln(k\tau_n)]^{\theta}\, l\left(-\ln(k\tau_n)\right) - [-\ln\tau_n]^{\theta}\, l\left(-\ln\tau_n\right)\\ &= [-\ln\tau_n]^{\theta}\, l\left(-\ln\tau_n\right)\left[\left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^{\theta}\frac{l\left(-\ln(k\tau_n)\right)}{l\left(-\ln\tau_n\right)} - 1\right]\\ &=: \Delta_1.\end{aligned}$$

If we further let $\Delta_2 = [K(\mathbf{x})]^{\theta}[\bar{F}_u^{-1}((k\tau_n)^{1/K(\mathbf{x})}) - \bar{F}_u^{-1}(\tau_n^{1/K(\mathbf{x})})]$ be the right-hand side of (S1.3), then

$$\begin{aligned}\Delta_2 &= [K(\mathbf{x})]^{\theta}\left[H_u^{-1}\left(-\frac{\ln(k\tau_n)}{K(\mathbf{x})}\right) - H_u^{-1}\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\right]\\ &= [-\ln(k\tau_n)]^{\theta}\, l\left(-\frac{\ln(k\tau_n)}{K(\mathbf{x})}\right) - [-\ln\tau_n]^{\theta}\, l\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\\ &= (-\ln\tau_n)^{\theta}\, l\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\left[\left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^{\theta}\frac{l\left(-\ln(k\tau_n)/K(\mathbf{x})\right)}{l\left(-\ln\tau_n/K(\mathbf{x})\right)} - 1\right].\end{aligned}$$

When $l(\cdot)$ is a non-constant slowly varying function, by (C5) it yields that

$$\begin{aligned}\delta_1 := \ln\left(\frac{l\left(-\ln\tau_n\right)}{l\left(-\ln(k\tau_n)\right)}\right) &\sim b\left(-\ln\left(k\tau_n\right)\right)\int_1^{\ln\tau_n/\ln(k\tau_n)} t^{\rho-1}dt\\ &\sim b\left(-\ln\left(k\tau_n\right)\right)\ln k/\ln(1/\tau_n)\\ &\to 0 \quad \text{as} \quad \tau_n \to 0\end{aligned}$$

and

$$\begin{aligned}
\delta_2 := \ln\left(\frac{l\left(-\ln\tau_n/K(\mathbf{x})\right)}{l\left(-\ln(k\tau_n)/K(\mathbf{x})\right)}\right) &\sim b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\int_1^{\ln\tau_n/\ln(k\tau_n)} t^{\rho-1}dt \\
&\sim b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\ln k/\ln(1/\tau_n) \\
&\to 0 \quad \text{as} \quad \tau_n \to 0.
\end{aligned}$$

When $l(\cdot)$ is a constant function, $\delta_1 = \delta_2 \equiv 0$. Also, we have $[\ln(k\tau_n)/\ln\tau_n]^\theta$ $-1 \sim \theta\ln k/\ln\tau_n$ as $\tau_n \to 0$ by straightforward calculations. In addition, $\exp(x) - 1 \sim x$ as $x \to 0$ and $b(z) \to 0$ as $z \to \infty$. Combining the above results, we have

$$\begin{aligned}
\Delta_1 &= (-\ln\tau_n)^\theta\, l\left(-\ln\tau_n\right)\left[\left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta(\exp(-\delta_1)-1) + \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta - 1\right] \\
&\sim (-\ln\tau_n)^\theta\, l\left(-\ln\tau_n\right)\theta\ln k/\ln\tau_n
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2 &= (-\ln\tau_n)^\theta\, l\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\left[\left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta(\exp(-\delta_2)-1) + \left(\frac{\ln(k\tau_n)}{\ln\tau_n}\right)^\theta - 1\right] \\
&\sim (-\ln\tau_n)^\theta\, l\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\theta\ln k/\ln\tau_n.
\end{aligned}$$

Therefore, under $\lim_{z\to\infty} l(\zeta z)/l(z) = 1$ locally uniformly in ζ on $(0,\infty)$, we have $\Delta_1/\Delta_2 \to 1$ as $\tau_n \to 0$.

For $0 < k < 1$, we can prove the result accordingly so that Lemma 4 holds. □

Lemma 5. *Let* $\bar{F}_u^{-1}(\tau) = \inf\{z : \bar{F}_u(z) \leq \tau\}$ *for any* $\tau \in (0,1)$*. By* $(S1.1)$*, for any* $x > 0$*, we have*

$$\lim_{\tau \to 0} \frac{\bar{F}_u^{-1}(x\tau) - \bar{F}_u^{-1}(\tau)}{a\left(\bar{F}_u^{-1}(\tau)\right)} = -\ln x.$$

Proof. Let $V(z) = \inf\{y : 1/\bar{F}_u(y) \geq z\}$. By Theorem 1.1.6 in de Haan and Ferreira (2006), (S1.1) is equivalent to

$$\lim_{t \to \infty} \frac{V(tx) - V(t)}{b(t)} = \ln x \tag{S1.4}$$

for a positive function $b(t)$ and $x > 0$, where $a(t) = b(1/\bar{F}_u(t))$. Noting also that $\bar{F}_u(z)$ is differentiable and strictly decreasing so that $b(1/\tau) = a(\bar{F}_u^{-1}(\tau))$, (S1.4) can be rewritten as

$$\lim_{\tau \to 0} \frac{V(x/\tau) - V(1/\tau)}{b(1/\tau)} = \lim_{\tau \to 0} \frac{\bar{F}_u^{-1}(\tau/x) - \bar{F}_u^{-1}(\tau)}{a(\bar{F}_u^{-1}(\tau))} = \ln x$$

for $x > 0$. Therefore, for any $x > 0$,

$$\lim_{\tau \to 0} \frac{\bar{F}_u^{-1}(x\tau) - \bar{F}_u^{-1}(\tau)}{a\left(\bar{F}_u^{-1}(\tau)\right)} = -\ln x.$$

□

For a given positive integer J and $l > 0$, let $s_j > 0$, $j = 1, \ldots, J$, be a sequence of real numbers, $\hat{Z}_n = a_n(\hat{\beta}(\tau_n) - \beta(\tau_n))$ and $\hat{Z}_n(s) = a_n(s)(\hat{\beta}(s\tau_n) - \beta(s\tau_n))$.

Lemma 6. *Suppose conditions* (C1)-(C5) *hold,* $\tau_n \to 0$*, and* $n\tau_n \to \infty$*. Then,*

$$\hat{Z}_n \xrightarrow{d} Z_\infty \stackrel{d}{=} \mathrm{N}(\mathbf{0}, \Omega_0),$$
$$\left(\hat{Z}_n(s_1)', \ldots, \hat{Z}_n(s_J)'\right)' \xrightarrow{d} \left(Z_\infty(s_1)', \ldots, Z_\infty(s_J)'\right)' \stackrel{d}{=} \mathrm{N}(\mathbf{0}, \Omega),$$
$$E Z_\infty(s_j) Z_\infty(s_{j'})' = \frac{s_j \wedge s_{j'}}{\sqrt{s_j s_{j'}}}\Omega_0,$$

for $j,\ j' = 1, \ldots, J$*, where* $\Omega_0 = (\ln m)^{-2}\mathcal{Q}_H^{-1}\mathcal{Q}_{\mathbf{X}}\mathcal{Q}_H^{-1}$*,* $\mathcal{Q}_{\mathbf{X}} = \mathrm{E}(\mathbf{X}\mathbf{X}')$*,* $\mathcal{Q}_H = \mathrm{E}[(H(\mathbf{X}))^{-1}\mathbf{X}\mathbf{X}']$ *and* $H(\mathbf{X}) = [K(\mu_{\mathbf{X}})/K(\mathbf{X})]^\theta$ *for* $\theta > 0$*.*

Proof. Step 1 outlines the overall proof by convexity lemma in Knight (1989), while the principal Step 2 proves some preliminary results in Step 1. Step 3 shows joint convergence of several regression quantile statistics. Hereinafter, we note τ for τ_n for simplicity.

Step 1. With reference to (2.2), and note that $\hat{Z}_n = a_n(\hat{\beta}(\tau) - \beta(\tau))$ minimizes

$$R_n(z, \tau) = \frac{a_n}{\sqrt{\tau n}} \sum_{i=1}^n \left(\rho_\tau\left(Y_i - \mathbf{X}_i'\beta(\tau) - \mathbf{X}_i' z/a_n\right) - \rho_\tau\left(Y_i - \mathbf{X}_i'\beta(\tau)\right)\right)$$

Making use of Knight's identity,

$$\rho_\tau(u - v) - \rho_\tau(u) = -v\left(\mathrm{I}(u > 0) - \tau\right) + \int_0^v (\mathrm{I}(u > 0) - \mathrm{I}(u > t))dt$$

then,

$$R_n(z,\tau) = W_n(\tau)'z + G_n(z,\tau),$$
$$W_n(\tau) = \frac{-1}{\sqrt{\tau n}} \sum_{i=1}^{n} \left(\mathrm{I}[Y_i \geq \mathbf{X}_i'\beta(\tau)] - \tau\right) \mathbf{X}_i, \tag{S1.5}$$
$$G_n(z,\tau) = \frac{a_n}{\sqrt{\tau n}} \sum_{i=1}^{n} \left(\int_0^{\mathbf{X}_i' z/a_n} [\mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > 0) - \mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > t)] dt \right).$$

By Lemma 7, $W_n(\tau) \xrightarrow{d} W \stackrel{\mathrm{d}}{=} \mathrm{N}(\mathbf{0}, \mathrm{E}\mathbf{X}\mathbf{X}')$, and by Step 2,

$$G_n(z,\tau) \xrightarrow{p} \frac{\ln m}{2} z'\mathcal{Q}_H z, m > 1,$$

where $\mathcal{Q}_H = \mathrm{E}[(H(\mathbf{X}))^{-1}\mathbf{X}\mathbf{X}']$, $H(\mathbf{X}) = [K(\mu_{\mathbf{X}})/K(\mathbf{X})]^{\theta}$ for Weibull-type tails. Thus, the weak marginal limit of $R_n(z)$ is given by

$$R_\infty(z) = W'z + \frac{\ln m}{2} z'\mathcal{Q}_H z.$$

We know that $\mathrm{E}(\mathbf{X}\mathbf{X}')$ is positive definite by (C3) and $0 < H(\mathbf{X}) < d < \infty$ for some constant d. Thus, $\mathcal{Q}_H$ is finite and $\mathcal{Q}_H$ is positive definite. In fact, $z'\mathcal{Q}_H z = \mathrm{E}[(\mathbf{X}'z)^2/H(\mathbf{X})] = 0$ for some $z \neq 0$ if and only if $\mathbf{X}'z = 0$ a.s.. Hence, the marginal limit $R_\infty(z)$ is uniquely minimized at $Z_\infty = -(\ln m)^{-1}\mathcal{Q}_H^{-1}W \stackrel{\mathrm{d}}{=} \mathrm{N}(\mathbf{0}, (\ln m)^{-2}\mathcal{Q}_H^{-1}\mathrm{E}(\mathbf{X}\mathbf{X}')\mathcal{Q}_H^{-1})$. According to the convexity lemma in Knight (1989), we get $\hat{Z}_n \xrightarrow{d} Z_\infty$.

Step 2. This step shows that as $\tau \to 0$, $\mathrm{E}\{G_n(z,\tau)\} \to 2^{-1} \ln m z'\mathcal{Q}_H z$, whereas Lemma 7 proves that $\mathrm{Var}(G_n(z,\tau)) \to 0$. Define $\bar{f}_U(z|\mathbf{X}_i) = \partial \bar{F}_U(z|\mathbf{X}_i)/\partial z$. In what follows, $\bar{F}_i$ and $\bar{f}_i$ denote $\bar{F}_U(\cdot|\mathbf{X}_i)$ and $\bar{f}_U(\cdot|\mathbf{X}_i)$, respectively, where U is the auxiliary quantity constructed in (C1). Since

$$\begin{aligned} G_n(z,\tau) &= \sum_{i=1}^{n} a_n \left(\int_0^{\mathbf{X}_i' z/a_n} \left[\frac{\mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > 0) - \mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > t)}{\sqrt{\tau n}} \right] dt \right) \\ &= \sum_{i=1}^{n} \left(\int_0^{\mathbf{X}_i' z} \left[\frac{\mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > 0\right) - \mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > t/a_n\right)}{\sqrt{\tau n}} \right] dt \right), \end{aligned} \tag{S1.6}$$

we get

$$\begin{aligned} \mathrm{E}\left\{G_n(z,\tau)\right\} &= n \cdot \mathrm{E}\left(\int_0^{\mathbf{X}_i' z} \frac{\bar{F}_i\left(\bar{F}_i^{-1}(\tau)\right) - \bar{F}_i\left(\bar{F}_i^{-1}(\tau) + t/a_n\right)}{\sqrt{\tau n}} dt \right) \\ &= n \cdot \mathrm{E}\left(\int_0^{\mathbf{X}_i' z} \frac{\bar{f}_i\left\{\bar{F}_i^{-1}(\tau) + o\left(\bar{F}_u^{-1}(\tau) - \bar{F}_u^{-1}(m\tau)\right)\right\}(-t)}{a_n\sqrt{\tau n}} dt \right) \end{aligned} \tag{S1.7}$$

$$\sim -n \cdot \mathrm{E}\left(\int_0^{\mathbf{X}_i' z} \frac{\bar{f}_i\left\{\bar{F}_i^{-1}(\tau)\right\}}{a_n\sqrt{\tau n}} t dt \right) \tag{S1.8}$$

$$\begin{aligned} &= -n \cdot \mathrm{E}\left(\frac{(\mathbf{X}_i' z)^2}{2} \frac{\bar{f}_i\left\{\bar{F}_i^{-1}(\tau)\right\}}{a_n\sqrt{\tau n}} \right) \\ &= \mathrm{E}\left(\frac{(\mathbf{X}_i' z)^2}{2} \frac{K^{-\theta}\left(\mu_{\mathbf{X}}\right)\left[\bar{F}_u^{-1}(m\tau) - \bar{F}_u^{-1}(\tau)\right]}{\tau\left(\bar{f}_i\left\{\bar{F}_i^{-1}(\tau)\right\}\right)^{-1}} \right) \end{aligned}$$

$$\sim \mathrm{E}\left(\frac{(\mathbf{X}_i' z)^2}{2} \frac{1}{H(\mathbf{X}_i)} \ln m \right) \tag{S1.9}$$

$$= \frac{\ln m}{2} z' \mathcal{Q}_H z.$$

(S1.7) is according to the definition of a_n and a first order Taylor expansion.

In fact, owing to $\tau n \to \infty$ uniformly over t in any compact subset of $\mathbb{R}$,

$$\frac{t}{a_n} = \frac{tK^{-\theta}(\mu_{\mathbf{X}})}{\sqrt{\tau n}} \left[\bar{F}_u^{-1}(\tau) - \bar{F}_u^{-1}(m\tau)\right]$$
$$= o\left(\bar{F}_u^{-1}(\tau) - \bar{F}_u^{-1}(m\tau)\right).$$

To prove (S1.8), it suffices to show that, for any sequence $v_\tau = o(\bar{F}_u^{-1}(\tau) - \bar{F}_u^{-1}(m\tau))$ with $m > 1$ as $\tau \to 0$,

$$\bar{f}_i\left\{\bar{F}_i^{-1}(\tau) + v_\tau\right\} \sim \bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right) \text{ uniformly in } i. \tag{S1.10}$$

This will be shown by using the assumption made in (C4), which is that uniformly in i, $1/\bar{f}_i(\bar{F}_i^{-1}(\tau)) \sim \partial\bar{F}_u^{-1}(\tau^{K(\mathbf{X}_i)})/\partial\tau$, where $\partial\bar{F}_u^{-1}(\exp(-z))/\partial z$ $= \partial H_u^{-1}(z)/\partial z \in \mathcal{RV}_\infty(\theta - 1)$.

To be clear, we first show (S1.10) for the special case of $\bar{f}_i = \bar{f}_u$ and $\bar{F}_i^{-1}(\tau) = \bar{F}_u^{-1}(\tau)$:

$$\bar{f}_u\left(\bar{F}_u^{-1}(\tau) + v_\tau\right) \sim \bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right), \tag{S1.11}$$

as $\tau \to 0$. By the regular variation property of $\partial\bar{F}_u^{-1}(\tau)/\partial\tau = 1/\bar{f}_u(\bar{F}_u^{-1}(\tau))$, locally uniformly in l in any compact subset of $(0, \infty)$,

$$\bar{f}_u\left(\bar{F}_u^{-1}(l\tau)\right) \sim l\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right). \tag{S1.12}$$

Namely, locally uniformly in l,

$$\bar{f}_u\left(\bar{F}_u^{-1}(\tau) + \left[\bar{F}_u^{-1}(l\tau) - \bar{F}_u^{-1}(\tau)\right]\right) \sim l\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right).$$

Hence, for any $l_\tau \to 1$,

$$\bar{f}_u\left(\bar{F}_u^{-1}(\tau) + \left[\bar{F}_u^{-1}(l_\tau\tau) - \bar{F}_u^{-1}(\tau)\right]\right) \sim \bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right).$$

Therefore, for any sequence $v_\tau = o(\bar{F}_u^{-1}(m\tau) - \bar{F}_u^{-1}(\tau))$ with $m > 1$ as $\tau \to 0$,

$$\bar{f}_u\left(\bar{F}_u^{-1}(\tau) + v_\tau\right) \sim \bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right),$$

because for any $\{v_\tau\}$, in view of

$$\frac{\bar{F}_u^{-1}(l_\tau\tau) - \bar{F}_u^{-1}(\tau)}{\bar{F}_u^{-1}(m\tau) - \bar{F}_u^{-1}(\tau)} \sim \frac{\ln l_\tau}{\ln m} \quad \text{as} \quad \tau \to 0$$

by Lemma 5, and we can choose a sequence $\{l_\tau\}$ such that $\{v_\tau\} = \{[\bar{F}_u^{-1}(l_\tau\tau) - \bar{F}_u^{-1}(\tau)]\}$ with $l_\tau \to 1$ as $\tau \to 0$.

Now, to prove (S1.8), we generalize the (S1.11) to (S1.10). It is easy to prove the following (a)-(d).

(a) $1/\bar{f}_i(\bar{F}_i^{-1}(\tau)) \sim \partial \bar{F}_u^{-1}(\tau^{K(\mathbf{X}_i)})/\partial\tau = 1/\{K(\mathbf{X}_i)\tau^{1-1/K(\mathbf{X}_i)}\bar{f}_u(\bar{F}_u^{-1}(\tau^{1/K(\mathbf{X}_i)}))\}$ uniformly in i by assumption (C4), and

(b) $\bar{f}_u(\bar{F}_u^{-1}(l\tau)^{1/K}) \sim l^{1/K}\bar{f}_u(\bar{F}_u^{-1}(\tau^{1/K}))$, locally uniformly in l and uniformly in $K \in \{K(\mathbf{x}) : \mathbf{x} \in \mathcal{X}\}$. Thus, combining (a) and (b), we have that locally uniformly in l and uniformly in i,

$$\bar{f}_i\left(\bar{F}_i^{-1}(l\tau)\right) \sim l\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right).$$

(c) $\bar{F}_i^{-1}(\tau) - \bar{F}_i^{-1}(k\tau) \sim \bar{F}_u^{-1}(\tau^{1/K(\mathbf{X}_i)}) - \bar{F}_u^{-1}((k\tau)^{1/K(\mathbf{X}_i)})$ for any $k \in$

$(0,1)\cup(1,\infty)$ and $\mathbf{X}_i \in \mathcal{X}$ by Lemma 3 and $\bar{F}_u^{-1}(\tau)-\bar{F}_u^{-1}(k\tau)\sim[K(\mathbf{X}_i)]^\theta$ $\times[\bar{F}_u^{-1}(\tau^{1/K(\mathbf{X}_i)})-\bar{F}_u^{-1}((k\tau)^{1/K(\mathbf{X}_i)})]$ by Lemma 4.

(d) $\bar{f}_u(\bar{F}_u^{-1}(\tau^l))\sim l^{1-\theta}\tau^{l-1}\bar{f}_u(\bar{F}_u^{-1}(\tau))$ locally uniformly in l.

As we have proved in (b) that locally uniformly in l

$$\bar{f}_i\left(\bar{F}_i^{-1}(\tau)+\left[\bar{F}_i^{-1}(l\tau)-\bar{F}_i^{-1}(\tau)\right]\right)\sim l\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right),$$

therefore, for any $l_\tau\to 1$ as $\tau\to 0$,

$$\bar{f}_i\left(\bar{F}_i^{-1}(\tau)+\left[\bar{F}_i^{-1}(l_\tau\tau)-\bar{F}_i^{-1}(\tau)\right]\right)\sim \bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right).$$

For any sequence $v_\tau=o(\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau))$ with $m>1$ as $\tau\to 0$, and noting that $\bar{F}_u^{-1}(\tau)-\bar{F}_u^{-1}(m\tau)\sim[K(\mathbf{X}_i)]^\theta[\bar{F}_i^{-1}(\tau)-\bar{F}_i^{-1}(m\tau)]$ by (c), then we have $v_\tau=o(\bar{F}_i^{-1}(m\tau)-\bar{F}_i^{-1}(\tau))$.

Besides, using (c) and Lemma 5, we obtain that

$$\begin{aligned}\frac{\bar{F}_i^{-1}(l_\tau\tau)-\bar{F}_i^{-1}(\tau)}{\bar{F}_i^{-1}(m\tau)-\bar{F}_i^{-1}(\tau)} &\sim -\frac{\bar{F}_u^{-1}(l_\tau\tau)-\bar{F}_u^{-1}(\tau)}{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}\\ &\sim -\frac{\ln l_\tau}{\ln m},\end{aligned}$$

where $\ln l_\tau/\ln m\to 0$ for any $l_\tau\to 1$ as $\tau\to 0$. Hence, for the above sequence $\{v_\tau\}$, it satisfies that $\{v_\tau\}=\{[\bar{F}_u^{-1}(l_\tau\tau)-\bar{F}_u^{-1}(\tau)]\}$ by selecting a sequence $\{l_\tau\}$, where $l_\tau\to 1$ as $\tau\to 0$. Consequently, the required conclusion (S1.10) holds.

(S1.9) can be shown as follows. By (a), uniformly in i,

$$\frac{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}{\tau\left(\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right)\right)^{-1}} \sim \frac{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}{\tau\left[K(\mathbf{X}_i)\tau^{1-1/K(\mathbf{X}_i)}\bar{f}_u\left(\bar{F}_u^{-1}\left(\tau^{1/K(\mathbf{X}_i)}\right)\right)\right]^{-1}}. \quad \text{(S1.13)}$$

By (d), we have uniformly in i,

$$K(\mathbf{X}_i)\tau^{1-1/K(\mathbf{X}_i)}\bar{f}_u\left(\bar{F}_u^{-1}\left(\tau^{1/K(\mathbf{X}_i)}\right)\right) \sim K(\mathbf{X}_i)^{\theta}\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right). \quad \text{(S1.14)}$$

Joining (S1.13) together with (S1.14), we have uniformly in i,

$$\begin{aligned}\frac{K^{-\theta}\left(\mu_{\mathbf{X}}\right)\left[\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)\right]}{\tau\left(\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right)\right)^{-1}} &\sim \frac{K^{-\theta}\left(\mu_{\mathbf{X}}\right)}{K(\mathbf{X}_i)^{-\theta}}\frac{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}{\tau\left(\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)\right)^{-1}} \\ &= \frac{1}{H(\mathbf{X}_i)}\frac{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}{\tau\left(\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)\right)^{-1}}, \quad \text{(S1.15)}\end{aligned}$$

where $H(\mathbf{X}_i)=[K(\mu_{\mathbf{X}})/K(\mathbf{X}_i)]^{\theta}$ for $\theta>0$.

In addition, by $\partial\bar{F}_u^{-1}(\eta)/\partial\eta = 1/\bar{f}_u\left(\bar{F}_u^{-1}(\eta)\right)$ and (S1.12),

$$\begin{aligned}\frac{\bar{F}_u^{-1}(m\tau)-\bar{F}_u^{-1}(\tau)}{\tau\left(\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)\right)^{-1}} &= \frac{\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)}{\tau}\int_1^m \frac{\partial\bar{F}_u^{-1}(t\tau)}{\partial t}dt \\ &= \frac{\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)}{\tau}\int_1^m \frac{\tau}{\bar{f}_u\left(\bar{F}_u^{-1}(t\tau)\right)}dt \\ &= \int_1^m \frac{\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)}{\bar{f}_u\left(\bar{F}_u^{-1}(t\tau)\right)}dt \\ &\sim \int_1^m t^{-1}dt \\ &= \ln m. \quad \text{(S1.16)}\end{aligned}$$

Combining (S1.15) and (S1.16) proves (S1.9).

STEP 3. By the definition of $\hat{Z}_n(s)$, we note that

$$(\hat{Z}_n(s_j), j = 1, \ldots, J) = \underset{\mathbf{z} \in \mathbb{R}^{d \times J}}{\arg\min} \left[R_n(z_1, s_1\tau) + \cdots + R_n(z_J, s_J\tau)\right]$$
$$= \underset{\mathbf{z} \in \mathbb{R}^{d \times J}}{\arg\min} \sum_{j=1}^{J} \left[W_n(s_j\tau)' z_j + G_n(z_j, s_j\tau)\right],$$

where $\mathbf{z} = (z_1', \ldots, z_J')'$ and the functions $R_n(\cdot, \cdot)$, $W_n(\cdot)$ and $G_n(\cdot, \cdot)$ are defined in (S1.5), respectively. As this objective function is a sum of the objective functions in the preceding steps, it retains the properties of the elements summed. Accordingly, the previous arguments can be used to conclude that the marginal limit of this objective function is given by $\sum_{j=1}^{J}[W(s_j)' z_j + G(z_j, s_j)]$, where $(W(s_j), 1 \leq j \leq J) \overset{\mathrm{d}}{=} N(0, \Sigma)$ with $G(z_j, s_j) = (\ln m/2) z' \mathcal{Q}_H z$ and $\mathrm{E}[W(s_j) W(s_{j'})'] = \mathrm{E}(\mathbf{X}\mathbf{X}')((s_j \wedge s_{j'})/\sqrt{s_j s_{j'}})$ for $j, j' = 1, \ldots, n$. This limit objective function is minimized at $(Z_\infty(s_j), 1 \leq j \leq J) = (-(\ln m)^{-1} \mathcal{Q}_H^{-1} W(s_j), 1 \leq j \leq J)$. Indeed,

$$\Phi(\mathbf{z}) := \sum_{j=1}^{J} \left[W(s_j)' z_j + G(z_j, s_j)\right] = \mathbf{W}'\mathbf{z} + \frac{\ln m}{2} \mathbf{z}' \mathcal{Q} \mathbf{z},$$

where $\mathbf{W} = (W(s_1)', \ldots, W(s_J)')'$ and $\mathcal{Q} = \mathrm{diag}(\mathcal{Q}_H, \ldots, \mathcal{Q}_H)$. Hence, $\Phi(\mathbf{z})$ is uniquely minimized at $\mathbf{z} = -(\ln m)^{-1} \mathcal{Q}^{-1} \mathbf{W} = (-(\ln m)^{-1} \mathcal{Q}_H^{-1} W(s_j)$ $, 1 \leq j \leq J)$. Thus, $(\hat{Z}_n(s_j), 1 \leq j \leq J) \xrightarrow{d} (Z_\infty(s_j), 1 \leq j \leq J)$ by the convexity lemma of Knight (1989). □

Lemma 7. *Let $\{Y_i, \mathbf{X}_i, i = 1, \ldots, n\}$ be an i.i.d. sequence. We have the following statements for $W_n(\cdot)$ and $G_n(\cdot, \cdot)$ in* (S1.5)*, as $\tau \to 0$ and $\tau n \to$*

∞.

(a) *For any constants* $s_j > 0$, $j = 1, \ldots, J$,

$$(W_n(s_1\tau)', \ldots, W_n(s_J\tau)')' \xrightarrow{d} (W(s_1)', \ldots, W(s_J)')' \stackrel{d}{=} \mathrm{N}(0, \Sigma),$$

with $\mathrm{E}[W(s_j)W(s_{j'})'] = ((s_j \wedge s_{j'})/\sqrt{s_j s_{j'}})\mathrm{E}(\mathbf{X}\mathbf{X}')$, *for* $j, j' = 1, \ldots, J$.

(b) $\mathrm{Var}(G_n(z, \tau)) \to 0$.

Proof. The proof of this lemma follows arguments similar to those used in the proof of Lemma A.5 in He et al. (2016).

(a) Let

$$\mathbf{W} = (W_n(s_1\tau)', \ldots, W_n(s_J\tau)')', \quad \mathbf{B} = (\beta_1', \ldots, \beta_J')',$$

where each $\beta_j \in \mathbb{R}^d$, $j = 1, \ldots, J$. By the Cramér-Wold theorem, part (1) can be proved by finding the limit distributions of the sequence of real variables

$$\mathbf{B}'\mathbf{W} = \sum_{j=1}^{J} \beta_j' W_n(s_j\tau) =: \sum_{i=1}^{n} Z_{i,n},$$

where, for all $i = 1, \ldots, n$, the random variables $Z_{i,n}$ are defined by

$$Z_{i,n} = -\sum_{j=1}^{J} \frac{1}{\sqrt{n s_j \tau}} \left(\mathrm{I}[Y_i \geq \mathbf{X}_i'\beta(s_j\tau)] - s_j\tau\right) \beta_j' \mathbf{X}_i.$$

Note that $\{Z_{i,n}\}_{i=1}^{n}$ is a set of *i.i.d.* random variables, and their expectation and variance are

$$\mathrm{E}(Z_{i,n}) = \mathrm{E}[\mathrm{E}(Z_{i,n}|\mathbf{X}_i)] = 0, \quad \mathrm{Var}(Z_{i,n}) = \frac{1}{n\tau}\mathbf{B}'C(\mathbf{X})\mathbf{B},$$

where $C(\mathbf{X})$ is the covariance matrix with sub-matrix defined for $(j,j') \in \{1,\ldots,J\}^2$ by

$$C_{j,j'}(\mathbf{X}) = \frac{A_{j,j'}(\mathbf{X})}{\sqrt{s_j s_{j'}}},$$

$$\begin{aligned}A_{j,j'}(\mathbf{X}) &= \mathrm{Cov}\left(\left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_j\tau)] - s_j\tau\right)\mathbf{X}, \left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_{j'}\tau)] - s_{j'}\tau\right)\mathbf{X}\right) \\ &= \mathrm{E}\left\{\left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_j\tau)] - s_j\tau\right)\left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_{j'}\tau)] - s_{j'}\tau\right)\mathbf{X}\mathbf{X}'\right\} \\ &= \mathrm{E}\left\{\mathrm{E}\left(\left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_j\tau)] - s_j\tau\right)\left(\mathrm{I}[Y \geq \mathbf{X}'\beta(s_{j'}\tau)] - s_{j'}\tau\right)\mathbf{X}\mathbf{X}'|\mathbf{X}\right)\right\} \\ &= \left[(s_j \wedge s_{j'})\tau - s_j s_{j'}\tau^2\right]\mathrm{E}\left(\mathbf{X}\mathbf{X}'\right).\end{aligned}$$

Hence, $\mathrm{Var}(\sum_{i=1}^n Z_{i,n}) = \tau^{-1}\mathbf{B}'C(\mathbf{X})\mathbf{B} \to \mathbf{B}'\Sigma\mathbf{B}$ as $\tau \to 0$, where the sub-matrix for $(j,j') \in \{1,\ldots,J\}^2$ is $\Sigma_{j,j'} = ((s_j \wedge s_{j'})/\sqrt{s_j s_{j'}})\mathrm{E}(\mathbf{X}\mathbf{X}')$. In addition, by the central limit theorem,

$$\frac{\mathbf{B}'\mathbf{W}}{\sqrt{\mathrm{Var}\left(\sum\limits_{i=1}^n Z_{i,n}\right)}} = \frac{\sum\limits_{i=1}^n Z_{i,n}}{\sqrt{\mathrm{Var}\left(\sum\limits_{i=1}^n Z_{i,n}\right)}} \xrightarrow{d} \mathrm{N}(0,1).$$

Further, by Slutsky's theorem, we have

$$\frac{\mathbf{B}'\mathbf{W}}{\sqrt{\mathbf{B}'\Sigma\mathbf{B}}} \xrightarrow{d} \mathrm{N}(0,1),$$

or equivalently, $\mathbf{B}'\mathbf{W} \xrightarrow{d} \mathrm{N}(0, \mathbf{B}'\Sigma\mathbf{B})$. Thereby, by the Cramér-Wold theorem, we get $\mathbf{W} \xrightarrow{d} \mathrm{N}(\mathbf{0}, \Sigma)$ with $\mathrm{E}[W(s_j)W(s_{j'})'] = ((s_j \wedge s_{j'})/\sqrt{s_j s_{j'}}) \times \mathrm{E}(\mathbf{X}\mathbf{X}')$.

(b) By (S1.6), we have

$$G_n(z,\tau) = \sum_{i=1}^{n} \left(\int_0^{\mathbf{X}_i' z} \left[\frac{\mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > 0\right) - \mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > t/a_n\right)}{\sqrt{\tau n}} \right] dt \right).$$

Consequently, $\mathrm{Var}(G_n(z,\tau)) = \mathrm{Var}(\lambda_1)/\tau$ for

$$\lambda_i = \int_0^{\mathbf{X}_i' z} \left[\mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > 0\right) - \mathrm{I}\left(Y_i - \mathbf{X}_i'\beta(\tau) > t/a_n\right) \right] dt.$$

By (C3), $|\lambda_i| \le K_1(z)|\mu_i|$ for $\mu_i = \mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > 0) - \mathrm{I}(Y_i - \mathbf{X}_i'\beta(\tau) > \mathbf{X}_i' z/a_n)$ and some $K_1(z) \in (0,\infty)$. Therefore,

$$\begin{aligned} \mathrm{Var}(\lambda_1) &= O\left(\mathrm{E}\left({\lambda_1}^2\right)\right) \\ &= O\left(\mathrm{E}\left(\mu_1^2\right)\right) \qquad \text{(S1.17)} \\ &= O\left(\mathrm{E}\left(|\mu_1|\right)\right), \qquad \text{(S1.18)} \end{aligned}$$

where (S1.17) is by $|\lambda_1| \le K_1(z)|\mu_1|$ and (S1.18) is by $|\mu_1| \in \{0,1\}$. For $\mathrm{E}(|\mu_1|)$, we have $\mathrm{E}(|\mu_1|) = \mathrm{E}[\mathrm{E}(|\mu_1||\mathbf{X}_1)]$, where

$$\begin{aligned} \mathrm{E}\left(|\mu_1||\mathbf{X}_1\right) &= \mathrm{P}\left(|\mu_1| = 1|\mathbf{X}_1\right) \\ &\le \mathrm{P}\left(\mu_1 = -1|\mathbf{X}_1\right) + \mathrm{P}\left(\mu_1 = 1|\mathbf{X}_1\right) \\ &\le |I_1| + |I_2|, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \mathrm{P}\left(\mathbf{X}_i'\beta(\tau) + \mathbf{X}_i' z/a_n \le Y_i \le \mathbf{X}_i'\beta(\tau)|\mathbf{X}_i\right) \\ &\le \left|\bar{F}_i\left(\bar{F}_i^{-1}(\tau) + \mathbf{X}_i' z/a_n\right) - \bar{F}_i\left(\bar{F}_i^{-1}(\tau)\right)\right| \end{aligned}$$

and

$$I_2 = \mathrm{P}\left(\mathbf{X}_i'\beta(\tau) \le Y_i \le \mathbf{X}_i'\beta(\tau) + \mathbf{X}_i'z/a_n|\mathbf{X}_i\right)$$
$$\le \left|\bar{F}_i\left(\bar{F}_i^{-1}(\tau)\right) - \bar{F}_i\left(\bar{F}_i^{-1}(\tau) + \mathbf{X}_i'z/a_n\right)\right|.$$

Accordingly, $\mathrm{E}(|\mu_1||\mathbf{X}_1) \le 2|\bar{F}_i(\bar{F}_i^{-1}(\tau) + \mathbf{X}_i'z/a_n) - \bar{F}_i(\bar{F}_i^{-1}(\tau))|$. Additionally, $\bar{F}_i(\bar{F}_i^{-1}(\tau) + \mathbf{X}_i'z/a_n) - \bar{F}_i(\bar{F}_i^{-1}(\tau)) \sim [\bar{f}_i(\bar{F}_i^{-1}(\tau))(\mathbf{X}_i'z)]/a_n$ by (S1.7) and (S1.8). Then,

$$\mathrm{E}\left(|\mu_1||\mathbf{X}_1\right) = O\left(\left|\bar{F}_i\left(\bar{F}_i^{-1}(\tau) + \mathbf{X}_i'z/a_n\right) - \bar{F}_i\left(\bar{F}_i^{-1}(\tau)\right)\right|\right)$$
$$= O\left(\left|\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right)\left(\mathbf{X}_i'z\right)\right|/a_n\right).$$

By (S1.13) and (S1.14), as $\tau \to 0$,

$$\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right) \sim K^{\theta}\left(\mathbf{X}_i\right)\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right). \tag{S1.19}$$

This leads to $\mathrm{E}(|\mu_1|) = \mathrm{E}[\mathrm{E}(|\mu_1||\mathbf{X}_1)] = O(|\bar{f}_u(\bar{F}_u^{-1}(\tau))|/a_n)$. Moreover, by (S1.15) and (S1.16) and the fact that $-\sqrt{n\tau}/a_n = K^{-\theta}(\mu_{\mathbf{X}})[\bar{F}_u^{-1}(m\tau) - \bar{F}_u^{-1}(\tau)]$,

$$-\sqrt{\frac{n}{\tau}}\frac{\bar{f}_i\left(\bar{F}_i^{-1}(\tau)\right)}{a_n} \sim \frac{1}{H(\mathbf{X}_i)}\ln m. \tag{S1.20}$$

Combining (S1.19) with (S1.20), we have

$$-K^{\theta}\left(\mu_{\mathbf{X}}\right)\sqrt{\frac{n}{\tau}}\frac{\bar{f}_u\left(\bar{F}_u^{-1}(\tau)\right)}{a_n} \sim \ln m,$$

which implies that $|\bar{f}_u(\bar{F}_u^{-1}(\tau))|/a_n = O(\sqrt{\tau/n})$. Hence, $\mathrm{Var}(G_n(z,\tau)) =$

$O(1/\sqrt{n\tau})$ by noting that $\mathrm{E}(|\mu_1|) = O(\sqrt{\tau/n})$, and then $\mathrm{Var}(G_n(z,\tau)) = o(1)$ as $n\tau \to \infty$. $\square$

S2. Two propositions

Proposition 1. *Suppose condition* (C5) *holds. Let $\tau_n \to 0$ as $n \to \infty$ such that $\kappa_n = \ln \psi_n / \ln \tau_n \to \kappa \in (1,\infty)$. Under the location shift model in* (M1)*, we have the following results.*

Case (i): If $0 < s < 1$, $\varepsilon(z) = c \ln z / z$ with $c \in (-\infty, 0) \cup (0, \infty)$, and $l(z) \to c_0 > 0$ as $z \to \infty$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-1-\theta}\right),$$
$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-\theta}\right).$$

Case (ii): If $0 < s < 1$, $\varepsilon(z) = 0$ for any $z > 0$, and $l(z) = \lambda > 0$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-1-\theta}\right),$$
$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-\theta}\right).$$

Case (iii): If $0 < s < 1$, $\varepsilon(z) = 1/\ln z$, and $l(z) = \alpha \ln z$ with $\alpha > 0$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(x)}\right) = O\left((\ln(1/\tau_n))^{-1-\theta}\,(\ln\ln(1/\tau_n))^{-1}\right),$$
$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(x)}\right) = O\left((\ln(1/\tau_n))^{-\theta}\,(\ln\ln(1/\tau_n))^{-1}\right).$$

Proof. The proof of this proposition is a simplified version of proof of proposition 2, and is therefore omitted here. □

Proposition 2. *Suppose condition* (C5) *holds. Let* $\tau_n \to 0$ *as* $n \to \infty$ *such that* $\kappa_n = \ln\psi_n/\ln\tau_n \to \kappa \in (1,\infty)$. *Under the heteroscedastic model in* (M2)*, we have the following results.*

Case (i): If $0 < s < 1$, $\varepsilon(z) = c\ln z/z$ with $c \in (-\infty, 0) \cup (0, \infty)$, and $l(z) \to c_0 > 0$ as $z \to \infty$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln\ln(1/\tau_n))(\ln(1/\tau_n))^{-2} \vee (\ln(1/\tau_n))^{-1-\theta}\right),$$
$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln\ln(1/\tau_n))(\ln(1/\tau_n))^{-1} \vee (\ln(1/\tau_n))^{-\theta}\right).$$

Case (ii): If $0 < s < 1$, $\varepsilon(z) = 0$ for any $z > 0$, and $l(z) = \lambda > 0$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-1-\theta}\right),$$
$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-\theta}\right).$$

Case (iii): If $0 < s < 1$, $\varepsilon(z) = 1/\ln z$, and $l(z) = \alpha\ln z$ with $\alpha > 0$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln(1/\tau_n))^{-1}\left(\ln\ln(1/\tau_n)\right)^{-2}\right),$$

$$\varpi\left(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}\right) = O\left((\ln\ln(1/\tau_n))^{-2}\right).$$

Proof. Because $q_U(\tau) = q_u(\tau^{1/K(\mathbf{x})})(1+\alpha(\tau))$ for any $\tau \in (0,1)$, we have

$$\frac{q_Y(s\tau|\mathbf{x})}{q_Y(\tau|\mathbf{x})} = \frac{q_u\left((s\tau)^{1/K(\mathbf{x})}\right)}{q_u\left(\tau^{1/K(\mathbf{x})}\right)} \frac{1+\alpha(s\tau)+\mathbf{x}'\beta_r/q_u\left((s\tau)^{1/K(\mathbf{x})}\right)}{1+\alpha(\tau)+\mathbf{x}'\beta_r/q_u\left(\tau^{1/K(\mathbf{x})}\right)}.$$

It is easy to obtain that

$$\begin{aligned}\varpi\left(s, \tau_n^{1/K(\mathbf{x})}\right) &= \frac{\alpha(s\tau_n)-\alpha(\tau_n)+\mathbf{x}'\beta_r\left[1/q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)-1/q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)\right]}{1+\alpha(\tau_n)+\mathbf{x}'\beta_r/q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)} \\ &= \frac{I_1+\mathbf{x}'\beta_r I_2}{1+\alpha(\tau_n)+\mathbf{x}'\beta_r/q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)},\end{aligned}$$

where $I_1 = \alpha(s\tau_n) - \alpha(\tau_n)$ and $I_2 = 1/q_u((s\tau_n)^{1/K(\mathbf{x})}) - 1/q_u({\tau_n}^{1/K(\mathbf{x})})$. Similarly, we can get

$$\varpi\left(\psi_n/\tau_n, {\tau_n}^{1/K(\mathbf{x})}\right) = \frac{\tilde{I}_1+\mathbf{x}'\beta_r\tilde{I}_2}{1+\alpha(\tau_n)+\mathbf{x}'\beta_r/q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)},$$

where $\tilde{I}_1 = \alpha(\psi_n) - \alpha(\tau_n)$ and $\tilde{I}_2 = 1/q_u({\psi_n}^{1/K(\mathbf{x})}) - 1/q_u({\tau_n}^{1/K(\mathbf{x})})$. Also, for the heteroscedastic model in (M2), we can find that

$$\alpha(\tau) = (\mathbf{x}'\xi)\,\frac{q_u(\tau)}{q_u(\tau^{1/K(\mathbf{x})})} - 1.$$

Furthermore, we have

$$\alpha(s\tau_n) - \alpha(\tau_n) = (\mathbf{x}'\xi)\left[\frac{q_u(s\tau_n)}{q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)} - \frac{q_u(\tau_n)}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\right]$$

and

$$\alpha(\psi_n) - \alpha(\tau_n) = (\mathbf{x}'\xi)\left[\frac{q_u(\psi_n)}{q_u\left({\psi_n}^{1/K(\mathbf{x})}\right)} - \frac{q_u(\tau_n)}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\right].$$

First we study the convergence rate of $\varpi(s, {\tau_n}^{1/K(\mathbf{x})}) \to 0$ as $n \to \infty$ for $0 < s < 1$. For I_1, by using $q_u(\tau) = H_u^{-1}(-\ln\tau) = (-\ln\tau)^\theta l(-\ln\tau)$ for all $\tau \in (0,1)$ and $\mathbf{x}'\xi = K^{-\theta}(\mathbf{x})$, we obtain

$$I_1 = \frac{l(-\ln(s\tau_n))}{l(-\ln(s\tau_n)/K(\mathbf{x}))} - \frac{l(-\ln\tau_n)}{l(-\ln\tau_n/K(\mathbf{x}))}.$$

Noting that $l(z) = c\exp\{\int_1^z \varepsilon(t)/tdt\}$ in (C5)(ii), we have

$$\begin{aligned}\ln\left[\frac{l(-\ln(s\tau_n))}{l(-\ln(s\tau_n)/K(\mathbf{x}))}\right] &= \int_1^{-\ln(s\tau_n)} \frac{\varepsilon(u)}{u}du - \int_1^{-\ln(s\tau_n)/K(\mathbf{x})} \frac{\varepsilon(u)}{u}du \\ &=: I_{1,1} \to 0 \quad \text{as} \quad \tau_\mathrm{n} \to 0\end{aligned}$$

and

$$\begin{aligned}\ln\left[\frac{l(-\ln\tau_n)}{l(-\ln\tau_n/K(\mathbf{x}))}\right] &= \int_1^{-\ln(\tau_n)} \frac{\varepsilon(u)}{u}du - \int_1^{-\ln(\tau_n)/K(\mathbf{x})} \frac{\varepsilon(u)}{u}du \\ &=: I_{1,2} \to 0 \quad \text{as} \quad \tau_\mathrm{n} \to 0.\end{aligned}$$

Hence, if $I_{1,1} - I_{1,2} \neq 0$, $I_1 = e^{I_{1,2}}(e^{I_{1,1}-I_{1,2}} - 1) \sim I_{1,1} - I_{1,2}$ as $\tau_n \to 0$. Note that

$$\begin{aligned}I_{1,1} - I_{1,2} &= \int_{-\ln(\tau_n)}^{-\ln(s\tau_n)} \frac{\varepsilon(u)}{u}du - \int_{-\ln(\tau_n)/K(\mathbf{x})}^{-\ln(s\tau_n)/K(\mathbf{x})} \frac{\varepsilon(u)}{u}du \\ &=: J_{1,1} - J_{1,2}. \qquad \text{(S2.1)}\end{aligned}$$

Case (i): $\varepsilon(z) = c \ln z/z$ with $c \in (-\infty, 0) \cup (0, \infty)$ and $l(z) \to c_0 > 0$ as $z \to \infty$.

By some calculations, we can derive that

$$J_{1,1} = c\left[\frac{1+\ln[\ln(1/\tau_n)]}{\ln(1/\tau_n)} - \frac{1+\ln[\ln(1/(s\tau_n))]}{\ln(1/(s\tau_n))}\right]$$

and

$$J_{1,2} = cK(\mathbf{x})\left[\frac{1+\ln\ln(1/\tau_n) - \ln K(\mathbf{x})}{\ln(1/\tau_n)} - \frac{1+\ln\ln(1/(s\tau_n)) - \ln K(\mathbf{x})}{\ln(1/(s\tau_n))}\right].$$

Letting $\tilde{\tau}_n = \ln(1/\tau_n)$, $A = 1 - K(\mathbf{x})$ and $B = K(\mathbf{x})\ln K(\mathbf{x})$, we have

$$\begin{aligned}
I_1 \sim I_{1,1} - I_{1,2} = J_{1,1} - J_{1,2}& \\
&= c\frac{(A+B)\ln(1/s) + A\ln(1/s)\ln\tilde{\tau}_n - A\tilde{\tau}_n\ln(1+\ln(1/s)/\tilde{\tau}_n)}{\tilde{\tau}_n(\ln(1/s)+\tilde{\tau}_n)} \\
&\sim \frac{cA\ln(1/s)\ln\tilde{\tau}_n}{\tilde{\tau}_n^2} \\
&= \frac{c\ln(1/s)(1-K(\mathbf{x}))\ln\ln(1/\tau_n)}{(\ln(1/\tau_n))^2}.
\end{aligned}$$

Case (ii): $\varepsilon(z) = 0$ for any $z > 0$ and $l(z) = \lambda > 0$.

In this case, it is clear that $I_1 = 0$.

Case (iii): $\varepsilon(z) = 1/\ln z$ and $l(z) = \alpha \ln z$ with $\alpha > 0$.

For convenience, we let $\ddot{\tau}_n = \ln\ln(1/\tau_n)$ and $\dot{\tau}_{s,n} = \ln(1 + \ln s/\ln\tau_n)$. By some calculations, we get $J_{1,2} = \ln\{1 + [\dot{\tau}_{s,n}/(\ddot{\tau}_n - \ln K(\mathbf{x}))]\}$ and $J_{1,1} = \ln\{1 + (\dot{\tau}_{s,n}/\ddot{\tau}_n)\}$.

Hence, by $I_1 \sim I_{1,1} - I_{1,2}$ and (S2.1), we have $I_1 \sim \ln s \ln K(\mathbf{x})/[\ln(1/\tau_n)$

$\ln\ln(1/\tau_n)]^2$, where

$$\begin{aligned}J_{1,1}-J_{1,2}&=\ln\left(1-\frac{\dot{\tau}_{s,n}\ln K(\mathbf{x})}{\ddot{\tau}_n\left(\ddot{\tau}_n-\ln K(\mathbf{x})+\dot{\tau}_{s,n}\right)}\right)\\&\sim-\frac{\dot{\tau}_{s,n}\ln K(\mathbf{x})}{\left(\ddot{\tau}_n\right)^2}\\&\sim\frac{\ln s\ln K(\mathbf{x})}{\ln(1/\tau_n)\left[\ln\ln(1/\tau_n)\right]^2}.\end{aligned}$$

For I_2, we have

$$\begin{aligned}I_2&=\frac{1}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\left[\frac{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}{q_u\left((s\tau_n)^{1/K(\mathbf{x})}\right)}-1\right]\\&=\frac{1}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\left[\frac{H_u^{-1}\left(-\ln\tau_n/K\left(\mathbf{x}\right)\right)}{H_u^{-1}\left(-\ln(s\tau_n)/K\left(\mathbf{x}\right)\right)}-1\right]\\&=\frac{1}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\left[\left(\frac{\ln\tau_n}{\ln(s\tau_n)}\right)^\theta\frac{l\left(-\ln\tau_n/K\left(\mathbf{x}\right)\right)}{l\left(-\ln(s\tau_n)/K\left(\mathbf{x}\right)\right)}-1\right]\\&=\frac{1}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}\left[\left(\frac{\ln\tau_n}{\ln(s\tau_n)}\right)^\theta I_{2,1}+I_{2,2}\right],\end{aligned}$$

with $I_{2,1}=l(-\ln\tau_n/K(\mathbf{x}))/l(-\ln(s\tau_n)/K(\mathbf{x}))-1$ and $I_{2,2}=(\ln\tau_n/\ln(s\tau_n))^\theta$ $-1\sim\theta\ln s/\ln(1/\tau_n)$ as $\tau_n\to0$. For $I_{2,1}$, under Case (i) and Case (iii), from the second order condition in (C5), we obtain

$$\begin{aligned}\Delta:=\ln\frac{l\left(-\ln(s\tau_n)/K\left(\mathbf{x}\right)\right)}{l\left(-\ln\tau_n/K\left(\mathbf{x}\right)\right)}&=b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\int_1^{\ln(s\tau_n)/\ln\tau_n}t^{\rho-1}dt\left(1+o(1)\right)\\&\sim b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\frac{\ln s}{\ln\tau_n}\\&\sim\left[K(\mathbf{x})\right]^{-\rho}b\left(-\ln\tau_n\right)\frac{\ln s}{\ln\tau_n}.\end{aligned}$$

Hence, $I_{2,1}=e^{-\Delta}-1\sim\ln s[K(\mathbf{x})]^{-\rho}b(-\ln\tau_n)/\ln(1/\tau_n)$. Under Case (ii),

$I_{2,1} = 0$. Also, by $q_u({\tau_n}^{1/K(\mathbf{x})}) = [-\ln \tau_n/K(\mathbf{x})]^\theta \; l(-\ln \tau_n/K(\mathbf{x}))$, we get

$$\begin{aligned} I_2 &\sim \frac{1}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)} \frac{\theta \ln s}{\ln\left(1/\tau_n\right)} \\ &= \frac{\left[K\left(\mathbf{x}\right)\right]^\theta \theta \ln s}{\left[-\ln \tau_n\right]^{1+\theta} l\left(-\ln \tau_n/K\left(\mathbf{x}\right)\right)}. \end{aligned}$$

Combining the results for I_1 and I_2, we can obtain the results of Proposition 2 for $\varpi(s, {\tau_n}^{1/K(\mathbf{x})})$.

For the convergence rate of $\varpi(\psi_n/\tau_n, {\tau_n}^{1/K(\mathbf{x})})$, with a similar argument as those for I_1 and I_2, we have the following results.

Case (i): $\varepsilon(z) = c \ln z/z$ with $c \in (-\infty, 0) \cup (0, \infty)$, and $l(z) \to c_0 > 0$ as $z \to \infty$.

$$\tilde{I}_1 \sim c\left(1 - K(\mathbf{x})\right)\left(1 - \kappa^{-1}\right) \frac{\ln \ln\left(1/\tau_n\right)}{\ln\left(1/\tau_n\right)},$$

$$\begin{aligned} \tilde{I}_2 &= \frac{\left[K\left(\mathbf{x}\right)\right]^\theta \left[\kappa^{-\theta}\left(1 + o(1)\right) - 1\right]}{\left(\ln(1/\tau_n)\right)^\theta l\left(\ln\left(1/\tau_n\right)/K\left(\mathbf{x}\right)\right)} \\ &= O\left(\left(\ln(1/\tau_n)\right)^{-\theta}\right). \end{aligned}$$

Case (ii): $\varepsilon(z) = 0$ for any $z > 0$, and $l(z) = \lambda > 0$.

$$\tilde{I}_1 = 0 \qquad \text{and} \qquad \tilde{\mathrm{I}}_2 = \mathrm{O}\left(\left(\ln(1/\tau_\mathrm{n})\right)^{-\theta}\right).$$

Case (iii): $\varepsilon(z) = 1/\ln z$, and $l(z) = \alpha \ln z$ with $\alpha > 0$.

$$\tilde{I}_1 \sim \frac{\ln K(\mathbf{x}) \ln(1/\kappa)}{\left(\ln \ln\left(1/\tau_n\right)\right)^2},$$

$$\tilde{I}_2 = \frac{[K(\mathbf{x})]^{\theta}\left[\kappa^{-\theta}(1+o(1))-1\right]}{(\ln(1/\tau_n))^{\theta}\, l\left(\ln(1/\tau_n)/K(\mathbf{x})\right)}$$
$$= O\left((\ln(1/\tau_n))^{-\theta}\left(\ln\ln(1/\tau_n)\right)^{-1}\right).$$

Combining the results for $\tilde{I}_1$ and $\tilde{I}_2$, we hence establish the results of $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})})$ in Proposition 2. □

For the five important examples of Weibull-type distributions in (E1)-(E5), the Gaussian, Gamma and extended Weibull distributions belong to Case (i) of the above two propositions, and the Weibull and modified Weibull distributions belong to Cases (ii) and (iii), respectively. By Propositions B.1 and B.2, we are able to give the desired rates of τ_n, respectively.

(E1) Let u follow $\mathrm{N}(\mu, \sigma^2)$ with $\sigma > 0$, $\theta = 1/2$ and $b(\ln(1/\tau_n)) = \ln\ln(1/\tau_n)/(4\ln(1/\tau_n))$.

Under both (M1) and (M2), $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-3/2})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1/2})$. Hence, if $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

(E2) Let u follow $\Gamma(\beta, \alpha)$ with $\theta = 1$ and $b(\ln(1/\tau_n)) = (1-\alpha)\ln\ln(1/\tau_n)/\ln(1/\tau_n)$.

Under (M1), $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-2})$ with $0 < s < 1$ and

$\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1})$. Hence, if $\tau_n = k_0(\ln\ln n)/n$ or $k_0 \ln n/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

Under (M2),$\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln\ln(1/\tau_n))(\ln(1/\tau_n))^{-2})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln\ln(1/\tau_n))(\ln(1/\tau_n))^{-1})$. Hence, if $\tau_n = k_0(\ln\ln n)/n$ or $k_0 \ln n/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

(E3) Let u follow $\mathrm{W}(\alpha, \lambda)$ with $\alpha, \lambda > 0$, $\theta = 1/\alpha$ and $b(\ln(1/\tau_n)) \equiv 0$.

Under both (M1) and (M2), $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-(\alpha+1)/\alpha})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1/\alpha})$. Hence, if $0 < \alpha < 2$, let $\tau_n = k_0(\ln\ln n)/n$ or $k_0 \ln n/n$ with $k_0 > 0$; if $\alpha \geq 2$, let $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

(E4) Let u follow $\mathrm{EW}(\alpha, \beta)$ with $\theta = 1/\alpha$ and $b(-\ln\tau_n) = -\beta\ln\ln(1/\tau_n) /(\alpha^2\ln\tau_n)^2$.

Under (M1), we can obtain $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-(\alpha+1)/\alpha})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1/\alpha})$. Hence, if $0 < \alpha < 2$, let $\tau_n = k_0(\ln\ln n)/n$ or $k_0 \ln n/n$ with $k_0 > 0$; if $\alpha \geq 2$, let $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

Under (M2), if $0 < \alpha \leq 1$, $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln\ln(1/\tau_n))(\ln(1/\tau_n))^{-2})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln\ln(1/\tau_n))(\ln(1/\tau_n)^{-1})$; if $\alpha > 1$, $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-(\alpha+1)/\alpha})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1/\alpha})$. Therefore, if $0 < \alpha < 2$, let $\tau_n = k_0(\ln\ln n)/n$ or $k_0 \ln n/n$ with $k_0 > 0$; if $\alpha \geq 2$, let $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

(E5) Let u follow MW(α) with $\theta = 1/\alpha$, $b(\ln(1/\tau_n)) = 1/\ln\ln(1/\tau_n)$ and $\alpha > 0$.

Under (M1), $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-(1+\alpha)/\alpha}(\ln\ln(1/\tau_n))^{-1})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1/\alpha}(\ln\ln(1/\tau_n))^{-1})$. Hence, if $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

Under (M2), $\varpi(s, \tau_n^{1/K(\mathbf{x})}) = O((\ln(1/\tau_n))^{-1}(\ln\ln(1/\tau_n))^{-2})$ with $0 < s < 1$ and $\varpi(\psi_n/\tau_n, \tau_n^{1/K(\mathbf{x})}) = O((\ln\ln(1/\tau_n))^{-2})$. Hence, if $\tau_n = k_0(\ln\ln n)/n$ with $k_0 > 0$, then all the conditions of theoretical results are fulfilled.

S3. Proofs of the theorems

Proof of Theorem 1

By Lemma 1, we have

$$\lim_{z\to\infty}\frac{a(z)H_u(z)}{z}=\theta.$$

Furthermore, if we let $z=\bar{F}_u^{-1}(\tau_n)\to\infty$ under $\tau_n\to 0$ as $n\to\infty$, then

$$\lim_{z\to\infty}\frac{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)\ln\left(1/\tau_n\right)}{\bar{F}_u^{-1}\left(\tau_n\right)}=\theta.$$

Under (C1), (2.5), and Remark 1, we have $\bar{F}_Y^{-1}(s\tau_n|\mathbf{x})\sim K^{-\theta}(\mathbf{x})\bar{F}_u^{-1}(\tau_n)$ as $\tau_n\to 0$ holds. For any $s>0$ and $\mathbf{x}\in\mathcal{X}$, we have

$$\begin{aligned}
\tilde{q}_n\left(s|\mathbf{x}\right)=&\sqrt{n\tau_n}\ln\left(1/\tau_n\right)\left(\frac{\hat{q}_n(s\tau_n|\mathbf{x})}{q_Y(s\tau_n|\mathbf{x})}-1\right)\\
=&\sqrt{n\tau_n}\ln\left(1/\tau_n\right)\frac{\mathbf{x}'\left(\hat{\beta}\left(s\tau_n\right)-\beta\left(s\tau_n\right)\right)}{\mathbf{x}'\beta\left(s\tau_n\right)}\\
=&\frac{\sqrt{n\tau_n}}{a_n(s)}\frac{\bar{F}_u^{-1}\left(\tau_n\right)}{\mathbf{x}'\beta\left(s\tau_n\right)a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)}\frac{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)\ln(1/\tau_n)}{\bar{F}_u^{-1}\left(\tau_n\right)}\mathbf{x}'a_n(s)\\
&\times\left(\hat{\beta}\left(s\tau_n\right)-\beta\left(s\tau_n\right)\right)\\
=&\frac{\sqrt{n\tau_n}}{a_n(s)}\frac{K^{\theta}(\mathbf{x})\left(1+o(1)\right)}{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)}\frac{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)\ln(1/\tau_n)}{\bar{F}_u^{-1}\left(\tau_n\right)}\mathbf{x}'\hat{Z}_n(s)\\
=&\frac{(1+o(1))}{\sqrt{s}}\frac{\bar{F}_u^{-1}\left(s\tau_n\right)-\bar{F}_u^{-1}\left(sm\tau_n\right)}{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)H(\mathbf{x})}\frac{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)\ln(1/\tau_n)}{\bar{F}_u^{-1}\left(\tau_n\right)}\mathbf{x}'\hat{Z}_n(s)\\
=&\frac{\ln m\left(1+o(1)\right)}{\sqrt{s}H(\mathbf{x})}\frac{a\left(\bar{F}_u^{-1}\left(\tau_n\right)\right)\ln(1/\tau_n)}{\bar{F}_u^{-1}\left(\tau_n\right)}\mathbf{x}'\hat{Z}_n(s)\\
=&\frac{\theta\ln m\left(1+o(1)\right)}{\sqrt{s}H(\mathbf{x})}\mathbf{x}'\hat{Z}_n(s),
\end{aligned}$$

according to $a_n(s) = \sqrt{ns\tau_n}/[K^{-\theta}(\mu_{\mathbf{X}})(\bar{F}_u^{-1}(s\tau_n) - \bar{F}_u^{-1}(sm\tau_n))]$, $H(\mathbf{x}) = [K(\mu_{\mathbf{X}})/K(\mathbf{x})]^\theta$ and Lemma 5. Hence,

$$\begin{pmatrix} \tilde{q}_n(s_1|\mathbf{x}) \\ \tilde{q}_n(s_2|\mathbf{x}) \\ \vdots \\ \tilde{q}_n(s_J|\mathbf{x}) \end{pmatrix} = \frac{\theta \ln m}{H(\mathbf{x})} \begin{pmatrix} s_1^{-\frac{1}{2}}\mathbf{x}' & 0 & \cdots & 0 \\ 0 & s_2^{-\frac{1}{2}}\mathbf{x}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_J^{-\frac{1}{2}}\mathbf{x}' \end{pmatrix} \begin{pmatrix} \hat{Z}_n(s_1) \\ \hat{Z}_n(s_2) \\ \vdots \\ \hat{Z}_n(s_J) \end{pmatrix} + \begin{pmatrix} o_P(1) \\ o_P(1) \\ \vdots \\ o_P(1) \end{pmatrix}.$$

Using Lemma 6, we get

$$\left(\hat{Z}_n(s_1)', \ldots, \hat{Z}_n(s_J)'\right)' \xrightarrow{d} \left(Z_\infty(s_1)', \ldots, Z_\infty(s_J)'\right)' \stackrel{\mathrm{d}}{=} \mathrm{N}(0, \Omega),$$

$$EZ_\infty(s_j)Z_\infty(s_{j'})' = \frac{\min(s_j, s_{j'})}{\sqrt{s_j s_{j'}}}\Omega_0,$$

where j, $j' = 1, \ldots, J$, and $\Omega_0 = \mathcal{Q}_H^{-1}\mathcal{Q}_{\mathbf{X}}\mathcal{Q}_H^{-1}\ln^{-2} m$. Consequently,

$$(\tilde{q}_n(s_1|\mathbf{x}), \ldots, \tilde{q}_n(s_J|\mathbf{x}))' \xrightarrow{d} (q_\infty(s_1), \ldots, q_\infty(s_J))' \stackrel{\mathrm{d}}{=} \mathrm{N}(\mathbf{0},\ \Sigma_{q(\mathbf{x})}),$$

where $(\Sigma_{q(\mathbf{x})})_{j,j'} = \theta^2(\mathbf{x}'\Omega_1 x)H^{-2}(\mathbf{x})(\max(s_j, s_{j'}))^{-1}$ for $j, j' = 1, \ldots, J$, $\Omega_1 = \mathcal{Q}_H^{-1}\mathcal{Q}_{\mathbf{X}}\mathcal{Q}_H^{-1}$, $\mathcal{Q}_H \equiv \mathrm{E}[(H(\mathbf{X}))^{-1}\mathbf{X}\mathbf{X}']$, $H(\mathbf{x}) = [K(\mu_{\mathbf{X}})/K(\mathbf{x})]^\theta$ and $\mathcal{Q}_{\mathbf{X}} = \mathrm{E}(\mathbf{X}\mathbf{X}')$.

Proof of Theorem 2

Define $I_{j+1} = \ln \hat{q}_n(s_{j+1}\tau_n|\mathbf{x}) - \ln q_Y(\tau_n|\mathbf{x})$ and $I_j = \ln \hat{q}_n(s_j\tau_n|\mathbf{x}) - \ln q_Y(\tau_n|\mathbf{x})$. By $I_j = \ln(q_Y(s_j\tau_n|\mathbf{x})/q_Y(\tau_n|\mathbf{x})) + \ln(\hat{q}_n(s_j\tau_n|\mathbf{x})/q_Y(s_j\tau_n|\mathbf{x}))$, we have by Lemma 2 that

$$\ln\left(\frac{q_Y(s_j\tau_n|\mathbf{x})}{q_Y(\tau_n|\mathbf{x})}\right) = \frac{\ln(1/s_j)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] + \varpi\left(s_j, {\tau_n}^{1/K(\mathbf{x})}\right) + R_{n,j},$$

where $R_{n,j} = O((\ln(1/\tau_n))^{-2} \vee \varpi^2(s_j, {\tau_n}^{1/K(\mathbf{x})}))$. In addition, let $\hat{q}_n(s_j\tau_n|\mathbf{x}) /q_Y(s_j\tau_n|\mathbf{x}) = 1 + \sigma_n\xi_{n,j}$, where $\sigma_n^{-1} = \sqrt{n\tau_n}\ln(1/\tau_n)$. It follows that $(\xi_{n,1}, \dots, \xi_{n,J})' \xrightarrow{d} \mathrm{N}(\mathbf{0}, \Sigma_{q(\mathbf{x})})$ by Theorem 1. We then yield

$$\begin{aligned} I_j =& \sigma_n\xi_{n,j} + O_P\left(\sigma_n^2\right) + \frac{\ln(1/s_j)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] \\ &+ \varpi\left(s_j, {\tau_n}^{1/K(\mathbf{x})}\right) + R_{n,j}. \end{aligned}$$

In a similar way, we can also get

$$\begin{aligned} I_{j+1} =& \sigma_n\xi_{n,j+1} + O_P\left(\sigma_n^2\right) + \frac{\ln(1/s_{j+1})}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] \\ &+ \varpi\left(s_{j+1}, {\tau_n}^{1/K(\mathbf{x})}\right) + R_{n,j+1}. \end{aligned}$$

Hence,

$$\begin{aligned} I_{j+1} - I_j =& \sigma_n\left(\xi_{n,j+1} - \xi_{n,j}\right) + \frac{\ln(1/r)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] \\ &+ O\left((\ln(1/\tau_n))^{-2} \vee \left(\vee_{j=1}^J\left|\varpi\left(s_j, \tau_n^{1/K(\mathbf{x})}\right)\right|\right)\right) + O_P\left(\sigma_n^2\right). \end{aligned}$$

Then,

$$\begin{aligned} \sqrt{n\tau_n}\left(\hat{\theta}_{n,\mathrm{P}}(\mathbf{x}) - \theta\right) =& \frac{1}{\ln(1/r)}\sum_{j=1}^J\left(w_{j-1} - w_j\right)\xi_{n,j} + \sqrt{n\tau_n}b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1)) \\ &+ O\left(\frac{\sqrt{n\tau_n}}{\ln(1/\tau_n)} \vee \left(\sqrt{n\tau_n}\ln(1/\tau_n) \vee_{j=1}^J\left|\varpi\left(s_j, {\tau_n}^{1/K(\mathbf{x})}\right)\right|\right)\right) \\ &+ O_P\left((\sqrt{n\tau_n}\ln(1/\tau_n))^{-1}\right) \\ =& \frac{1}{\ln(1/r)}\sum_{j=1}^J\left(w_{j-1} - w_j\right)\xi_{n,j} + o_P(1), \qquad \text{(S3.1)} \end{aligned}$$

where (S3.1) holds by conditions $\sqrt{n\tau_n}\max\left(1/\ln(1/\tau_n), |b(\ln(1/\tau_n))|\right) \to 0$ and $\sqrt{n\tau_n}\ln(1/\tau_n)\ \max_{j=1,\ldots,J}|\varpi(s_j, {\tau_n}^{1/K(\mathbf{x})})| \to 0$ of this theorem.

Therefore, $\sqrt{n\tau_n}(\hat{\theta}_{n,\mathrm{P}}(\mathbf{x}) - \theta) \xrightarrow{d} \mathrm{N}(0, (\ln r)^{-2}W'\Sigma_{q(\mathbf{x})}W)$, where $W = (w_0 - w_1, \ldots, w_{j-1} - w_j, \ldots, w_{J-1} - w_J)'$ with $w_0 = w_J = 0$.

Proof of Theorem 3

Let $T_j = \ln\hat{q}_n(s_j\tau_n|\mathbf{x}) - \ln q_Y(\tau_n|\mathbf{x})$ and $T_0 = \ln[\hat{q}_n(\tau_n|\mathbf{x})/q_Y(\tau_n|\mathbf{x})]$. Noting that $T_j = \ln(q_Y(s_j\tau_n|\mathbf{x})/q_Y(\tau_n|\mathbf{x})) + \ln(\hat{q}_n(s_j\tau_n|\mathbf{x})/q_Y(s_j\tau_n|\mathbf{x}))$, we have by Lemma 2 that

$$T_j = \frac{\ln(1/s_j)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] + \varpi\left(s_j, {\tau_n}^{1/K(\mathbf{x})}\right) + R_{n,j},$$

where $R_{n,j} = O((\ln(1/\tau_n))^{-2} \vee \varpi^2(s_j, {\tau_n}^{1/K(\mathbf{x})}))$ for $j = 1, 2, \ldots, J$.

In addition, it follows from Theorem 1 that

$$\frac{\hat{q}_n\left(s_j\tau_n|\mathbf{x}\right)}{q_Y\left(s_j\tau_n|\mathbf{x}\right)} = 1 + \sigma_n\xi_{n,j},$$

where $(\xi_{n,1}, \ldots, \xi_{n,J})' \xrightarrow{d} \mathrm{N}(\mathbf{0}, \Sigma_{q(\mathbf{x})})$ and $\sigma_n^{-1} = \sqrt{n\tau_n}\ln\left(1/\tau_n\right)$. Hence,

$$\begin{aligned} T_j =& \sigma_n\xi_{n,j} + O_P\left(\sigma_n^2\right) + \frac{\ln(1/s_j)}{\ln(1/\tau_n)}\left[\theta + b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\right] \\ &+ \varpi\left(s_j, {\tau_n}^{1/K(\mathbf{x})}\right) + R_{n,j}, \end{aligned}$$

for $j = 1, 2, \ldots, J$, and $T_0 = \sigma_n\xi_{n,1} + O_P(\sigma_n^2)$. Then, we can get the following expansion

$$\begin{aligned}\sqrt{n\tau_n}\left(\hat{\theta}_{n,\mathrm{H}}(\mathbf{x})-\theta\right) =& \sqrt{n\tau_n}\left(\ln(1/\tau_n)\left(\sum_{j=1}^{J}\ln(1/s_j)\right)^{-1}\sum_{j=1}^{J}(T_j-T_0)-\theta\right)\\ =& \frac{\sum\limits_{j=2}^{J}\xi_{n,j}-(J-1)\xi_{n,1}}{\sum\limits_{j=1}^{J}\ln(1/s_j)}+\sqrt{n\tau_n}b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)(1+o(1))\\ &+O\left(\frac{\sqrt{n\tau_n}}{\ln(1/\tau_n)}\vee\left(\sqrt{n\tau_n}\ln(1/\tau_n)\left(\vee_{j=1}^{J}\left|\varpi\left(s_j,{\tau_n}^{1/K(\mathbf{x})}\right)\right|\right)\right)\right)\\ &+O_P\left((\sqrt{n\tau_n}\ln(1/\tau_n))^{-1}\right)\\ =& \left(\sum_{j=1}^{J}\ln(1/s_j)\right)^{-1}\left[\sum_{j=2}^{J}\xi_{n,j}-(J-1)\xi_{n,1}\right]+o_P(1),\end{aligned} \tag{S3.2}$$

where (S3.2) holds by conditions $\sqrt{n\tau_n}\max\left(1/\ln(1/\tau_n),|b(\ln(1/\tau_n))|\right)\to 0$ and $\sqrt{n\tau_n}\ln(1/\tau_n)\max_{j=1,\ldots,J}|\varpi(s_j,{\tau_n}^{1/K(\mathbf{x})})|\to 0$ of this theorem. Let $\mathbf{J}_0=(1-J,1,\ldots,1)$, then $\mathbf{J}_0\Sigma_{q(\mathbf{x})}\mathbf{J}_0'=\theta^2(\mathbf{x}'\Omega_1\mathbf{x})H^{-2}(\mathbf{x})(\sum_{j=1}^{J}((2(J-j)+1)/s_j)-J^2)$. Hence,

$$\sqrt{n\tau_n}(\hat{\theta}_{n,\mathrm{H}}(\mathbf{x})-\theta)\xrightarrow{d}\mathrm{N}(0,\Lambda_J H^{-2}(\mathbf{x})\theta^2(\mathbf{x}'\Omega_1\mathbf{x})),$$

where

$$\Lambda_J=\left(\sum_{j=1}^{J}((2(J-j)+1)/s_j)-J^2\right)\left(\sum_{j=1}^{J}\log(1/s_j)\right)^{-2}.$$

The proof of this theorem is completed.

Proof of Theorem 4

The following expansion can be obtained easily,

$$\begin{aligned}\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\ln\frac{\hat{q}_{n,\mathrm{E}}\left(\psi_n|\mathbf{x}\right)}{q_Y(\psi_n|\mathbf{x})} =&\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left[\ln\hat{q}_n\left(\tau_n|\mathbf{x}\right)+\hat{\theta}_n\ln\kappa_n-\ln q_Y(\psi_n|\mathbf{x})\right]\\ =&\sqrt{n\tau_n}\left(\hat{\theta}_n-\theta\right)+\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\ln\frac{\hat{q}_n\left(\tau_n|\mathbf{x}\right)}{q_Y\left(\tau_n|\mathbf{x}\right)}\\ &+\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left(\ln q_Y\left(\tau_n|\mathbf{x}\right)-\ln q_Y(\psi_n|\mathbf{x})+\theta\ln\kappa_n\right)\\ =:&\, I_1+I_2+I_3.\end{aligned}$$

Under the assumptions of the theorem,

$$I_1=\sqrt{n\tau_n}\left(\hat{\theta}_n-\theta\right)\xrightarrow{d}\mathrm{N}(0,\sigma_\theta^2).$$

By Theorem 1, we have

$$I_2=\frac{\ln(1/\tau_n)\sqrt{n\tau_n}}{\ln(1/\tau_n)\ln\kappa_n}\ln\frac{\hat{q}_n\left(\tau_n|\mathbf{x}\right)}{q_Y\left(\tau_n|\mathbf{x}\right)}=O_P\left(\frac{1}{\ln(1/\tau_n)}\right)=o(1).$$

Finally, according to the second order condition in (C5) and $\lim_{z\to\infty}b(\zeta z)/b(z)=\zeta^\rho$ for any $\zeta>0$, the term I_3 can be written as

$$\begin{aligned}I_3=&-\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left(\ln\frac{q_u\left({\psi_n}^{1/K(\mathbf{x})}\right)}{q_u\left({\tau_n}^{1/K(\mathbf{x})}\right)}+\ln\left(1+\varpi\left(\psi_n/\tau_n,{\tau_n}^{1/K(\mathbf{x})}\right)\right)-\theta\ln\kappa_n\right)\\ =&-\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left(\ln\frac{l\left(-\ln\psi_n/K(\mathbf{x})\right)}{l\left(-\ln\tau_n/K(\mathbf{x})\right)}+\varpi\left(\psi_n/\tau_n,\tau_n^{1/K(\mathbf{x})}\right)(1+o(1))\right)\\ =&-\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left(b\left(-\frac{\ln\tau_n}{K(\mathbf{x})}\right)\int_1^{\kappa_n}t^{\rho-1}dt\,(1+o(1))+\varpi\left(\psi_n/\tau_n,{\tau_n}^{1/K(\mathbf{x})}\right)(1+o(1))\right)\\ =&-\frac{\sqrt{n\tau_n}}{\ln\kappa_n}\left(b\left(-\ln\tau_n\right)K^\rho(\mathbf{x})\int_1^{\kappa_n}t^{\rho-1}dt\,(1+o(1))+\varpi\left(\psi_n/\tau_n,{\tau_n}^{1/K(\mathbf{x})}\right)(1+o(1))\right)\\ =&\,o(1),\end{aligned}\tag{S3.3}$$

where we use the condition $\sqrt{n\tau_n}\max\{|b(\ln(1/\tau_n))|, |\varpi(\psi_n/\tau_n, {\tau_n}^{1/K(\mathbf{x})})|\}$ $\to 0$ to get (S3.3). The proof of this theorem is completed.

S4. Figures

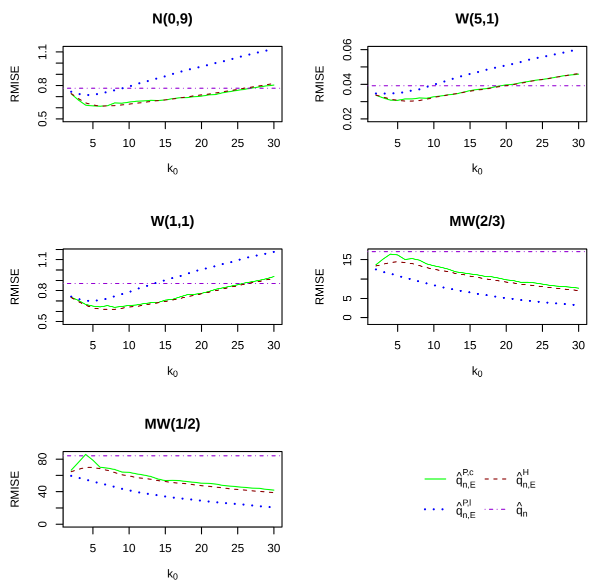

Figure 1: The root mean integrated squared error of different estimators versus k_0, where the dashed dotted horizontal line denotes the RMISE of the conventional quantile regression estimator $\hat{q}_n$ of $q_Y(\psi_n|\mathbf{x})$ with $\psi_n = 1/n^{1.01}$. Here, k_0 is the constant involved in $\tau_n = k_0(\ln\ln n)/n$, $\hat{q}^{\mathrm{P,c}}_{n,\mathrm{E}}$, $\hat{q}^{\mathrm{P,l}}_{n,\mathrm{E}}$, and $\hat{q}^{\mathrm{H}}_{n,\mathrm{E}}$ are the proposed extrapolation estimators based on the Pickand-type tail-coefficient estimators with constant and linear weights, and the Hill-type tail-coefficient estimator, respectively.

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