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ON TIME-REVERSIBILITY OF MULTIVARIATE LINEAR PROCESSES

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Abstract: We study the time-reversibility of multivariate linear processes, introducing a necessary and sufficient condition related to linear transforms of the multivariate linear process. Conditions analogous to Cheng's for univariate non-Gaussian linear processes are also explored; these are in terms of the noise distribution and the model parameters. The exploration results in an easily verifiable set of necessary and sufficient conditions for a multivariate non-Gaussian linear process driven by a univariate noise, leaving the case of multivariate noise as a challenging open problem.

Key words and phrases: Moving average, multivariate linear process, time-reversibility.

1. Introduction

Time reversibility is an important concept in statistical mechanics as well as in stochastic processes. It has also direct relevance to statistical inference of time series (e.g., Whittle (1963)) and deconvolution (e.g., Rosenblatt (2000)). If the series is time reversible, then the projection forward in time is equivalent to the projection backward in time. This has impact on orthogonalization operations, such as Levinson-Durbin's recursion, as can be seen in Whittle (1963). It is well known that the equivalence holds for univariate Gaussian time series. However, this need not be the case for a multivariate time series, even if it is Gaussian (Whittle (1963)). Moreover, at a conceptual level, it is pertinent to have a complete characterization of time reversible time series. The situation for a univariate stationary linear time series is now completely solved (e.g., Cheng (1999)). However, it is a curious fact that time reversibility for *multivariate* non-Gaussian time series is, as far as we know, hardly studied in the literature.

A multivariate stationary discrete-time process $\{X_t; t = \ldots, -1, 0, 1, \ldots\}$ is said to be *time-reversible* if, for any $k = 1, 2, \ldots$ and any k-tuple $t_1 < \cdots < t_k$, the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ is the same as that of $(X_{-t_1}, \ldots, X_{-t_k})$. In the univariate case, Weiss (1975) shows that the only time-reversible non-Gaussian ARMA processes are sub-classes of pure moving average processes, and Findley (1986), Hallin, Lefevre and Puri (1988), Breidt and Davis (1992) and Cheng (1990, 1999) studied time-reversibility and related problems in the context of general linear processes.

The following theorem on time-reversibility of univariate linear processes is due to Cheng (1999).

Theorem 1.1. *Let*

$$X_t = \sum_{j=-\infty}^{\infty} \alpha_j Z_{t-j} \tag{1.1}$$

be a non-Gaussian linear process, where $\{Z_t\}$ *is a sequence of independent and identically distributed random variables with zero mean and finite variance, and* $\{\alpha_j\}$ *is a square-summable sequence of constants. Suppose the spectral density* $f_X(\lambda)$ *of* $\{X_t\}$ *satisfies*

$$f_X(\lambda) = \frac{1}{2\pi}|\alpha(e^{-i\lambda})|^2 \mathrm{E}(Z_t^2) \neq 0, \qquad \lambda \in [-\pi, \pi] \tag{1.2}$$

almost everywhere, where $\alpha(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \alpha_j e^{-ij\lambda}$. *Then* $\{X_t\}$ *is time-reversible if and only if the following conditions hold for some integer* t_0 *and some non-zero constant* a*:*

(i) $\alpha_t = a\alpha_{t_0-t}$ *for all* $t \in \mathbb{Z}$;

(ii) $\{Z_t\}$ *and* $\{aZ_t\}$ *have the same distribution.*

Actually, from condition (i), $a = \pm 1$, and if $\{Z_t\}$ is not symmetrically distributed, $a = 1$.

In this paper, we study the time-reversibility of multivariate linear processes. A necessary and sufficient condition is given in Section 2 in terms of the distributions of multivariate linear transforms of a multivariate linear process. Conditions analogous to those in Theorem 1.1 are studied in Section 3− we prove that they are sufficient but not necessary conditions for a multivariate non-Gaussian linear process to be time-reversible. A multivariate non-Gaussian linear process driven by a univariate noise is studied in Section 4, and an easily verifiable set of necessary and sufficient conditions for time-reversibility of such a process is given.

It gives us great pleasure to offer this paper to honour Professor George Tiao, because multivariate (also called multiple) time series is close to his heart.

2. A Necessary and Sufficient Condition on Time-reversibility for Multivariate Linear Processes

Let $\{\mathbf{X}_t\}$ be a stationary p-variate ($p \geq 1$) linear process defined, in the mean-squares sense, by

$$\mathbf{X}_t = \sum_{j=-\infty}^{\infty} A_j \mathbf{Z}_{t-j}, \qquad t \in \mathbb{Z}, \tag{2.1}$$

where the noise $\{\mathbf{Z}_t\}$ is a sequence of i.i.d. q-variate ($q \geq 1$) random vectors, and $\{A_j\}$ is a square-summable sequence of $p \times q$ matrices in the sense that $\sum_{j=-\infty}^{\infty} A_j A_j' < \infty$. Moreover, we assume that

$$\mathrm{E}(\mathbf{Z}_t) = \mathbf{0}, \qquad \mathrm{E}(\mathbf{Z}_t \mathbf{Z}_t') = \Sigma > 0, \tag{2.2}$$

where $\mathbf{0}$ is the zero-vector in $\mathbb{R}^q$, and $\Sigma > 0$ stands for positive-definiteness. When $p = q = 1$, $\{\mathbf{X}_t\}$ reduces to a univariate linear process defined by (1.1). In this paper, whenever $p + q > 2$, we call $\{\mathbf{X}_t\}$ a multivariate linear process. Note that we have not assumed $p = q$ (Cf. Hannan (1970, p.14).) Further, although our discussion will be conducted for the most part without any restriction as to whether $p \geq q$ or $p \leq q$, it is the former case that is the more relevant in many practical situations.

Now, even if a multivariate linear process is marginally time-reversible, it may not be jointly time-reversible. The following is a simple counter-example.

Example 2.1. Suppose that $\{\mathbf{X}_t = (U_t, V_t)'\}$ is a bivariate moving average time series, where $U_t = \varepsilon_t + \alpha\varepsilon_{t-1} + \varepsilon_{t-2}, V_t = \eta_t + \beta\eta_{t-1} + \beta\eta_{t-2} + \eta_{t-3}$ are two univariate moving average processes, $\{\varepsilon_t\}$ and $\{\eta_t\}$ are two sequences of i.i.d. random variables, with $\mathrm{Cov}\,(\varepsilon_t, \eta_t) = \gamma \neq 0$, and ε_t is independent of η_s whenever $t \neq s$. Noting that $\mathrm{Cov}\,(U_1, V_2) = (\beta + \alpha\beta + 1)\gamma$ and $\mathrm{Cov}\,(U_2, V_1) = (\alpha + \beta)\gamma$, we may conclude that $\{\mathbf{X}_t\}$ is not jointly time-reversible although it is marginally so.

As far as time-reversibility of linear processes is concerned, there is another fundamental difference between the univariate case and the p-variate ($p > 1$) case. In the former instance any stationary Gaussian linear process is time-reversible, but in the latter this is generally *not* true.

In fact, the following theorem on time-reversibility of multivariate Gaussian linear processes is obvious.

Theorem 2.2. *Let $\{\mathbf{X}_t\}$ be a p-variate ($p \geq 1$) Gaussian linear process, and $\Gamma(j) = \mathrm{E}(\mathbf{X}_t \mathbf{X}_{t-j}')$, $j \in \mathbb{Z}$, be the auto-covariance matrices of $\{\mathbf{X}_t\}$. Then $\{\mathbf{X}_t\}$ is time-reversible if and only if $\Gamma(j)$ is symmetric for all $j \in \mathbb{Z}$.*

Remark 2.3. For any integer $p \geq 1$, Whittle (1963) has developed the 'forward' and 'backward' equations in the Levinson-Durbin recursion, which are identical for the time reversible case.

One way to exploit existing results on the time-reversibility of univariate linear processes is to consider linear combinations of components of a multivariate linear process.

Theorem 2.4. *A p-variate ($p > 1$) linear process $\{\mathbf{X}_t, t \in \mathbb{Z}\}$ defined by* (2.1) *is time-reversible if and only if for any integer n, any p-dimensional vectors $\mathbf{l}_1, \ldots, \mathbf{l}_n$, and any $\{t_1, \ldots, t_n\} \subset \mathbb{Z}$, $t_1 < \cdots < t_n$, $\mathbf{U} = (\mathbf{l}_1'\mathbf{X}_{t_1}, \ldots, \mathbf{l}_n'\mathbf{X}_{t_n})'$ has the same distribution as $\mathbf{V} = (\mathbf{l}_1'\mathbf{X}_{-t_1}, \ldots, \mathbf{l}_n'\mathbf{X}_{-t_n})'$.*

Proof. Let

$$\varphi_{X+}(\mathbf{s}_1, \ldots, \mathbf{s}_n) = \mathrm{E}[\exp\{i(\mathbf{X}_{t_1}'\mathbf{s}_1 + \cdots + \mathbf{X}_{t_n}'\mathbf{s}_n)\}], \tag{2.3}$$

$$\varphi_{X-}(\mathbf{s}_1, \ldots, \mathbf{s}_n) = \mathrm{E}[\exp\{i(\mathbf{X}_{-t_1}'\mathbf{s}_1 + \cdots + \mathbf{X}_{-t_n}'\mathbf{s}_n)\}] \tag{2.4}$$

be the characteristic functions of (np)-dimensional random vectors $(\mathbf{X}_{t_1}', \ldots, \mathbf{X}_{t_n}')'$ and $(\mathbf{X}_{-t_1}', \ldots, \mathbf{X}_{-t_n}')'$, respectively, where $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are n p-dimensional vectors. Similarly, let $\varphi_U(s_1, \ldots, s_n) = \mathrm{E}[\exp\{i(\mathbf{X}_{t_1}'\mathbf{l}_1 s_1 + \cdots + \mathbf{X}_{t_n}'\mathbf{l}_n s_n)\}]$ and $\varphi_V(s_1, \ldots, s_n) = \mathrm{E}[\exp\{i(\mathbf{X}_{-t_1}'\mathbf{l}_1 s_1 + \cdots + \mathbf{X}_{-t_n}'\mathbf{l}_n s_n)\}]$ be the characteristic functions of n-dimensional random vectors $\mathbf{U}$ and $\mathbf{V}$ respectively, where $s_1, \ldots, s_n$ are n scalars.

Now we can easily see that while $\mathbf{l}_1, \ldots, \mathbf{l}_n$ take values over the p-dimensional real space, so do $\mathbf{s}_1, \ldots, \mathbf{s}_n$. This fact completes the proof of the theorem.

3. A Sufficient Condition for Time-reversibility of Multivariate Linear Processes

Henceforth, we consider only non-Gaussian time series unless specified otherwise.

Prompted by the conditions for time-reversibility of the univariate linear process in Theorem 1.1, obvious conditions on A_j's and $\{\mathbf{Z}_t\}$ in representation (2.1) can be given as follows.

(SC) There exist some integer t_0 and some non-zero $q \times q$ constant matrix B such that the following conditions hold:

(i) $A_t = A_{t_0 - t}B$ for all $t \in \mathbb{Z}$;

(ii) $\{\mathbf{Z}_t\}$ and $\{B\mathbf{Z}_t\}$ have the same distribution.

Condition (i) effectively imposes reflective symmetry of the coefficient matrices about the time origin t_0 and up to multiplication by a constant matrix. By (ii), it is necessary that $B\Sigma B' = \Sigma$, where $\Sigma = \mathrm{E}(\mathbf{Z}_t\mathbf{Z}_t') > 0$ is the covariance matrix of $\mathbf{Z}_t$. However, this does not imply that $BB' = I$, the $q \times q$ identity matrix. Moreover, we have $rank(B) = q$, i.e., B is non-singular.

Example 3.1. Let $\{X_t\}$ be a univariate linear process generated by a sequence of i.i.d. symmetric bivariate random variables, $\{\mathbf{Z}_t = (\varepsilon_t, \eta_t)'\}$,

$$X_t = (1, 0)\mathbf{Z}_t + (0, 1)\mathbf{Z}_{t-1} = \varepsilon_t + \eta_{t-1}, \tag{3.1}$$

with $\mathrm{Cov}\,(\varepsilon_t, \eta_t) = \gamma \neq 0$. (Note that if $\{\varepsilon_t\}$ and $\{\eta_t\}$ are independent, $\{X_t\}$ itself is time-reversible.) It is not difficult to check that $\{X_t\}$ in Example 3.1 satisfies conditions **(SC)**, with $t_0 = 1$ and $B = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$.

Now consider the finite-dimensional characteristic functions $\varphi_+(u_1, \ldots, u_n) = \mathrm{E}[\,\exp\{i(X_{t_1}u_1 + \cdots + X_{t_n}u_n)\}$ and $\varphi_-(u_1, \ldots, u_n) = \mathrm{E}[\,\exp\{i(X_{-t_1}u_1 + \cdots + X_{-t_n}u_n)\}]$, where $t_1 < \cdots < t_n$ is any n-tuple of times, $n = 1, 2, \ldots$. Actually if there is any couple (t_i, t_{i+1}) satisfying $t_{i+1} - t_i > 1$, the n-dimensional random vector $(X_{t_1}, \ldots, X_{t_n})$ can be divided into two independent sub-vectors $(X_{t_1}, \ldots, X_{t_i})$ and $(X_{t_{i+1}}, \ldots, X_{t_n})$. A similar remark applies to the time-reversed vector. Therefore, without loss of generality, we need only check the characteristic function of the simple n-tuple $(t_1, \ldots, t_n) = (1, \ldots, n)$.

By (3.1), we have $\varphi_+(u_1, \ldots, u_n) = \varphi_\eta(u_1)\varphi(u_1, u_2)\cdots\varphi(u_{n-1}, u_n)\varphi_\varepsilon(u_n)$, and $\varphi_-(u_1, \ldots, u_n) = \varphi_\varepsilon(u_1)\varphi(u_2, u_1)\cdots\varphi(u_n, u_{n-1})\varphi_\eta(u_n)$, where $\varphi_\varepsilon(\cdot)$, $\varphi_\eta(\cdot)$ and $\varphi(\cdot,\cdot)$ denote the characteristic functions of ε_t, η_t and (ε_t, η_t) respectively. If $\varphi(\cdot,\cdot)$ is symmetric, then $\varphi_\varepsilon = \varphi_\eta$, and therefore $\varphi_+ = \varphi_-$. So $\{X_t\}$ is time-reversible.

Example 3.2.(Elliptical symmetric distributions) As pointed out in Xia, Tong, Li and Zhu (2002), for a second-order stationary time series $\{Y_t\}$, if the random vector $\mathbf{X} = (Y_{t-1}, \ldots, Y_{t-p})'$ has an elliptical symmetric distribution for all p, then $\{Y_t\}$ is time-reversible. In fact, for a multivariate linear process defined by (2.1), if the white noise $\mathbf{Z}_t$ has an elliptical symmetric distribution, then it is easy to find some non-zero matrix B such that condition (ii) of **(SC)** holds. For such a matrix B, any multivariate linear processes with model parameters $\{A_j\}$ satisfying condition (i) of **(SC)** are time-reversible.

The following theorem shows that conditions **(SC)** are sufficient conditions under which the multivariate linear process $\{\mathbf{X}_t\}$ in (2.1) is time-reversible.

Theorem 3.3. *Let $\{\mathbf{X}_t\}$ be a multivariate linear process defined by* (2.1) *and suppose that* **(SC)** *holds. Then $\{\mathbf{X}_t\}$ is time-reversible.*

Proof. Let $\varphi_{X+}(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ and $\varphi_{X-}(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ be the characteristic functions defined by (2.3) and (2.4) respectively, where $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are n p-dimensional vectors. Then, by **(SC)**, we have

$$\begin{aligned}\varphi_{X-}(\mathbf{s}_1,\ldots,\mathbf{s}_n) &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\mathbf{X}'_{-t_k}\mathbf{s}_k\Big\}\Big)\\ &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{j=-\infty}^{\infty}\mathbf{Z}'_{-t_k-j}A'_j\mathbf{s}_k\Big\}\Big)\\ &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{j=-\infty}^{\infty}\mathbf{Z}'_{-t_k-j}B'A'_{t_0-j}\mathbf{s}_k\Big\}\Big)\\ &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{l=-\infty}^{\infty}\mathbf{Z}'_{-t_k-t_0+l}B'A'_l\mathbf{s}_k\Big\}\Big).\end{aligned}$$

If we write $\{\mathbf{Y}_t\} = \{\mathbf{Z}_{-t}\}$, by **(SC)** we have $\{B\mathbf{Y}_t\} \overset{\mathrm{d}}{=} \{\mathbf{Y}_t\} \overset{\mathrm{d}}{=} \{\mathbf{Z}_t\}$, where "$\overset{\mathrm{d}}{=}$" stands for equality in distribution. Therefore,

$$\begin{aligned}\varphi_{X-}(\mathbf{s}_1,\ldots,\mathbf{s}_n) &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{l=-\infty}^{\infty}\mathbf{Y}'_{t_k+t_0-l}B'A'_l\mathbf{s}_k\Big\}\Big)\\ &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{l=-\infty}^{\infty}\mathbf{Z}'_{t_k+t_0-l}A'_l\mathbf{s}_k\Big\}\Big)\\ &= \mathrm{E}\Big(\exp\Big\{i\sum_{k=1}^{n}\sum_{l=-\infty}^{\infty}\mathbf{Z}'_{t_k-l}A'_l\mathbf{s}_k\Big\}\Big) = \varphi_{X+}(\mathbf{s}_1,\ldots,\mathbf{s}_n)\end{aligned}$$

This completes the proof.

The condition **(SC)** is not necessary. Here is a counter-example.

Example 3.4. Suppose that $\{\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)})'\}$ is a bivariate moving average time series defined by

$$\mathbf{X}_t = \mathbf{Z}_t + A_1\mathbf{Z}_{t-1} + A_2\mathbf{Z}_{t-2}, \tag{3.2}$$

where $\{\mathbf{Z}_t = (Z_t^{(1)}, Z_t^{(2)})'\}$ is a sequence of i.i.d. random variables with $\{Z_t^{(1)}\}$ independent of $\{Z_t^{(2)}\}$, both having mean zero and finite variance. Let $A_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $A_2 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$. For convenience, we take A_0 to be the identity matrix, and all other A_j's to be zero.

Now, for any non-singular 2×2 matrix B, denote the rank of matrix A_j, or equivalently A_jB, by r_j, $j\in\mathbb{Z}$. Obviously, we have $r_0 = 2, r_1 = r_2 = 1$, and $r_j = 0$ for all $j < 0$ or $j > 2$. If **(SC)** holds, by condition (i), it is necessary that there exists some integer t_0 such that $r_t = r_{t_0-t}$ for all $t\in\mathbb{Z}$. Obviously, this is impossible. Therefore, **(SC)** does not hold. However, $\{\mathbf{X}_t\}$ is still time-reversible.

Actually we can rewrite (3.2) as $X_t^{(1)} = Z_t^{(1)} + Z_{t-1}^{(1)}$, $X_t^{(2)} = Z_t^{(2)} + Z_{t-2}^{(2)}$. By Theorem 1.1, it is obvious that $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ are both time-reversible. By independence of $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$, $\{\mathbf{X}_t\}$ is jointly time-reversible.

4. Time-reversibility of a Multivariate Linear Process Driven by a Univariate Noise of Multivariate Linear Processes

Throughout this section, we assume that

$$\mathbf{X}_t = \sum_{j=-\infty}^{\infty} \mathbf{m}_j Z_{t-j}, \qquad t \in \mathbb{Z}, \tag{4.1}$$

where $\{\mathbf{m}_j\}$ is a sequence of square-summable $p \times 1$ column vectors, and $\{Z_t\}$ is a sequence of i.i.d. (scalar) random variables. The above process may be used to model a panel of time series, which are interconnected through a common noise source (e.g., Hjellvik and Tjøstheim (1999)). If $p = 1$, then $\{\mathbf{X}_t\}$ is a univariate linear process as defined by (1.1).

We begin with the sufficient conditions **(SC)**, which now simplify to the following form.

(SC-UN) There exist some integer t_0 and some non-zero constant a such that the following conditions hold:

(i) $\mathbf{m}_t = a\mathbf{m}_{t_0-t}$;

(ii) $\{Z_t\}$ and $\{aZ_t\}$ have the same distribution.

As before, condition (i) effectively imposes on the model parameters a reflective symmetry about the time origin t_0 up to a constant multiplier. Similar to the univariate case in Theorem 1.1, we still have $a = \pm 1$.

Moreover, we can prove that conditions **(SC-UN)** are not only sufficient but also necessary conditions for the time-reversibility of the multivariate linear process defined by (4.1). We have the following theorem.

Theorem 4.1. *Let $\{\mathbf{X}_t\}$ be a p-variate ($p \geq 1$) non-Gaussian linear process defined by* (4.1)*, where the noise $\{Z_t\}$ is a sequence of i.i.d. random variables with zero mean and finite variance. Suppose that the spectral density matrix or spectrum of $\{\mathbf{X}_t\}$,*

$$\mathbf{f}(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \Gamma(j) = (f_{uv}(\lambda))_{1 \leq u,v \leq p}, \qquad \lambda \in [-\pi, \pi], \tag{4.2}$$

is positive-definite almost everywhere, where $\{\Gamma(j), j \in \mathbb{Z}\}$ is the auto-covariance matrix function of $\{\mathbf{X}_t\}$. Then $\{\mathbf{X}_t\}$ is time-reversible if and only if **(SC-UN)** *holds.*

Proof. We need only consider $p > 1$ and, after Theorem 3.3, we need only prove necessity.

Suppose that $\{\mathbf{X}_t\}$ is time-reversible. Then, for any p-dimensional column vector $\mathbf{l} \in \mathbb{R}^p$, the linear combination of $\{\mathbf{X}_t\}$,

$$\mathbf{l}'\mathbf{X}_t = \sum_{j=-\infty}^{\infty} (\mathbf{l}'\mathbf{m}_j) Z_{t-j} = \sum_{j=-\infty}^{\infty} \alpha_j(\mathbf{l}) Z_{t-j}, \tag{4.3}$$

is time-reversible too, where $\{\alpha_j(\mathbf{l}) = \mathbf{l}'\mathbf{m}_j, j \in \mathbb{Z}\}$ is a sequence of scalar functions of $\mathbf{l} \in \mathbb{R}^p$. Moreover, denote the auto-covariance function for the univariate linear process $\{\mathbf{l}'\mathbf{X}_t\}$ by $\{\gamma_j(\mathbf{l}), j \in \mathbb{Z}\}$. Then the spectral density of $\{\mathbf{l}'\mathbf{X}_t\}$ is

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \gamma_j(\mathbf{l}) \\ &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \mathrm{E}(\mathbf{l}'\mathbf{X}_t \mathbf{X}'_{t-j}\mathbf{l}) \\ &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \mathbf{l}'\Gamma(j)\mathbf{l} \\ &= \mathbf{l}'\mathbf{f}(\lambda)\mathbf{l}, \qquad\qquad \lambda \in [-\pi, \pi]. \end{aligned}$$

By (4.2), $f(\lambda)$ is positive almost everywhere for any non-zero $\mathbf{l} \in \mathbb{R}^p$.

In order to apply Theorem 1.1 to the univariate linear processes $\{\mathbf{l}'\mathbf{X}_t\}$, we have to show that $\mathbf{l}'\mathbf{X}$ is non-Gaussian. However, this may not be true for all $\mathbf{l} \in \mathbb{R}^p$. Consider the subset $\mathbf{l}(\mathbf{X}) = \{\mathbf{l} \in \mathbb{R}^p : \ \{\mathbf{l}'\mathbf{X}_t\} \text{ is Gaussian}\}$. It is not difficult to see that $\mathbf{l}(\mathbf{X})$ is a subspace of $\mathbb{R}^p$. By the assumption that $\{\mathbf{X}_t\}$ is non-Gaussian, $\mathbf{l}(\mathbf{X}) \neq \mathbb{R}^p$ and $\dim(\mathbf{l}(\mathbf{X})) < p$, where $\dim(\mathbf{l}(\mathbf{X}))$ denotes the dimension of the subspace $\mathbf{l}(\mathbf{X})$.

By Theorem 1.1, for each non-zero $\mathbf{l} \in \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X})$, there exist an integer $t_0(\mathbf{l})$ and a non-zero constant $a(\mathbf{l})$ such that

$$\alpha_t(\mathbf{l}) = a(\mathbf{l}) \alpha_{t_0(\mathbf{l})-t}(\mathbf{l}), \qquad \{Z_t\} \stackrel{\mathrm{d}}{=} \{a(\mathbf{l}) Z_t\}. \tag{4.4}$$

Actually by the second equality in (4.4), $a(\mathbf{l})$ is a constant function of $\mathbf{l}$ which may take values ± 1. Without loss of generality, we assume that $a(\mathbf{l}) \equiv 1$. Then, we may re-write conditions (4.4) as follows. For each non-zero $\mathbf{l} \in \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X})$, there exists an integer $t_0(\mathbf{l})$, not necessarily unique, such that

$$\alpha_t(\mathbf{l}) - \alpha_{t_0(\mathbf{l})-t}(\mathbf{l}) = \mathbf{l}'(\mathbf{m}_t - \mathbf{m}_{t_0(\mathbf{l})-t}) = 0, \qquad \forall\ t \in \mathbb{Z}. \tag{4.5}$$

For each $\mathbf{l} \in \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X})$, define

$$T_0(\mathbf{l}) = \{t_0(\mathbf{l}) \in \mathbb{Z} : \ t_0(\mathbf{l}) \text{ satisfies equation (4.5)}\} \subset \mathbb{Z}.$$

Now, consider the map $\mathbf{l} \in \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X}) \longmapsto T_0(\mathbf{l}) \subset \mathbb{Z}$. Moreover, define $\mathbf{l}(j)$, $j \in \mathbb{Z}$, by $\mathbf{l}(j) = \{\mathbf{l} \in \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X}) : \ j \in T_0(\mathbf{l})\}$. For each $j \in \mathbb{Z}$, $\mathbf{l}(j)$ is a subset of $\mathbb{R}^p \backslash \mathbf{l}(\mathbf{X})$. Obviously, we have $\bigcup_{j=-\infty}^{\infty} \mathbf{l}(j) = \mathbb{R}^p \backslash \mathbf{l}(\mathbf{X})$. Therefore,

$$\mathbf{l}(\mathbf{X}) \cup \Big(\bigcup_{j=-\infty}^{\infty} \mathrm{sp}\{\mathbf{l}(j)\} \Big) = \mathbb{R}^p, \tag{4.6}$$

where $\mathrm{sp}\{\cdots\}$ denotes the space generated by $\{\cdots\}$. It is well known that any union of a countable number of subspaces of $\mathbb{R}^p$ each with dimension less than p is a subset of $\mathbb{R}^p$, but not equal to $\mathbb{R}^p$. Therefore, (4.6), together with the fact that $\dim(\mathbf{l}(\mathbf{X})) < p$, implies that there exists at least one integer j_0 such that $\mathrm{sp}\{\mathbf{l}(j_0)\} = \mathbb{R}^p$.

Finally, consider the subset $\mathbf{l}(j_0)$. By the definition of $T_0(\mathbf{l})$, we have $\mathbf{l}(j_0) = \{\mathbf{l} \in \mathbb{R}^p : \ \mathbf{l}'(\mathbf{m}_t - \mathbf{m}_{j_0-t}) = 0 \text{ for all } t \in \mathbb{Z}\}$. It is easy to check that $\mathbf{l}(j_0)$ is actually a linear space. Therefore, we have $\mathbf{l}(j_0) = \mathrm{sp}\{\mathbf{l}(j_0)\} = \mathbb{R}^p$. Or equivalently, $\mathbf{l}'(\mathbf{m}_t - \mathbf{m}_{j_0-t}) = 0$ for all $\mathbf{l} \in \mathbb{R}^p$ and $t \in \mathbb{Z}$. Hence, $\mathbf{m}_t - \mathbf{m}_{j_0-t}$ has to be the zero-vector, i.e., $\mathbf{m}_t = \mathbf{m}_{j_0-t}$ for all $t \in \mathbb{Z}$. This completes the proof.

5. Some Concluding Comments

For the general case of $q > 1$, the issue of necessary and sufficient conditions for time reversibility remains a challenging open problem. It is clear that the invariance of the distribution of Z under some group action has an important role to play.

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