Testing and modelling for the structural change in covariance matrix time series with multiplicative form

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Supplementary Material

This supplement provides four appendices for the paper. Appendix A gives the proofs of Theorems 1–2, Appendix B offers the proofs of Theorems 3–5, Appendix C lists some basic derivatives results, and Appendix D provides some numerical evidences on spurious long memory phenomena caused by the structural change. In what follows, we define the pseudo data $\mathbf{u}_t = \mathbf{u}_{-t}$, $\Sigma_t = \Sigma_{-t}$ for $-[Th] \le t \le -1$, and $\mathbf{u}_t = \mathbf{u}_{2T-t}$, $\Sigma_t = \Sigma_{2T-t}$ for $T+1 \le t \le T+[Th]$ obtained by the reflection method.

A Proofs of Theorems 1–2

Define $v_t = \text{vech}(\mathbf{y}_t - \Sigma_0 - \Sigma_{1t}/c_T)$ and let

$$\Pi_1(x) = \frac{1}{T}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-s/T)v_s,$$

$$\Pi_2(x) = \Big[\frac{1}{T}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-s/T)-1\Big]\text{vech}(\Sigma_0)$$

$$+ c_T^{-1}\Big[\frac{1}{T}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-s/T)\mathrm{vech}(\Sigma_{1s}) - \frac{1}{T}\sum_{s=1}^{T}\mathrm{vech}(\Sigma_{1s})\Big],$$

and $\Pi_3 = \frac{1}{T}\sum_{s=1}^{T} v_s$. Then,

$$\begin{aligned} Th^{1/2}\widehat{\mathcal{S}} =& h^{1/2}\sum_{t=1}^{T}[\Pi_1(t/T)+\Pi_2(t/T)-\Pi_3]'[\Pi_1(t/T)+\Pi_2(t/T)-\Pi_3] \\ \triangleq& S_1 - 2S_2 + S_3 + S_4, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} S_1 &= h^{1/2}\sum_{t=1}^{T}\Pi_1(t/T)'\Pi_1(t/T), \quad S_2 = h^{1/2}\sum_{t=1}^{T}\Pi_1(t/T)'\Pi_3, \\ S_3 &= h^{1/2}\sum_{t=1}^{T}\Pi_3'\Pi_3, \quad S_4 = h^{1/2}\sum_{t=1}^{T}[2\Pi_1(t/T)+\Pi_2(t/T)-2\Pi_3]'\Pi_2(t/T). \end{aligned}$$

Next, let $\Sigma_{t*} = [(\Sigma_0 + \Sigma_{1t}/c_T)^{1/2}]^{\otimes 2} D_n$. Under H_{1T},

$$v_t = \Sigma_{t*} z_t, \tag{A.2}$$

where z_t is defined as in (2.6). Particularly, $v_t = v_{0t}$ under H_0 (i.e., $\Sigma_{1t} \equiv 0$), where $v_{0t} = \Sigma_{0*} z_t$ is a stationary process, and Σ_{0*} is defined as in (2.5). Since Σ_0 and Σ_{1t} are bounded deterministic matrices, Assumption 2 implies

$$\begin{aligned} &v_t \text{ is a strictly stationary } \beta\text{-mixing process with mixing coefficients} \quad \text{(A.3)} \\ &\beta(j) \text{ satisfying } \sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/(1+\delta)} < \infty \text{ for some } 0 < \delta < 1; \\ &\max_t E\left\|v_t\right\|^{4(1+\delta)} < \infty. \quad \text{(A.4)} \end{aligned}$$

Moreover, since $c_T \to \infty$, we have $\left\|\Sigma_0^{-1/4}\Sigma_{1t}\Sigma_0^{-1/4}/c_T\right\| \ll 1$. Hence,

$$\begin{aligned}(\Sigma_0 + \Sigma_{1t}/c_T)^{1/2} =& \Sigma_0^{1/4}(I_n + \Sigma_0^{-1/2}\Sigma_{1t}\Sigma_0^{-1/2}/c_T)^{1/2}\Sigma_0^{1/4} \\ =& \Sigma_0^{1/4}\Big[I_n + \frac{1}{2c_T}\Sigma_0^{-1/2}\Sigma_{1t}\Sigma_0^{-1/2} + O\Big(\frac{1}{c_T^2}\Big)\Big]\Sigma_0^{1/4} \\ =& \Sigma_0^{1/2} + \frac{1}{2c_T}\Sigma_0^{-1/4}\Sigma_{1t}\Sigma_0^{-1/4} + O\Big(\frac{1}{c_T^2}\Big)\end{aligned}$$

by Taylor's expansion, and it entails

$$\Sigma_{t*} - \Sigma_{0*} = \epsilon_t/c_T, \tag{A.5}$$

where $\epsilon_t = [\Sigma_0^{-1/4}\Sigma_{1t}\Sigma_0^{-1/4} \otimes \Sigma_0]/2 + [\Sigma_0 \otimes \Sigma_0^{-1/4}\Sigma_{1t}\Sigma_0^{-1/4}]/2 + O(1/c_T^2)$.

In order to prove Theorems 1 and Theorem 2 (i)–(ii), Propositions A.1–A.5 below are needed. These five propositions and their related lemmas are all proved under H_{1T} with $T^{1/2}h^{1/4} = O(c_T)$, and Assumptions 2–4.

Proposition A.1. *$S_1 - \mathcal{B} \to_{\mathcal{L}} N(0, \mathcal{V})$, where $\mathcal{B} = h^{-1/2}tr(M)\big[\int K^2(x)dx\big]$, M is defined in (2.5), and $\mathcal{V}$ is defined as in Theorem 2.*

Proposition A.2. *$S_2 = o_p(1)$.*

Proposition A.3. *$S_3 = o_p(1)$.*

Proposition A.4. *$S_4 - \frac{Th^{1/2}}{c_T^2}\mathcal{B}_l = o_p(1)$, where $\mathcal{B}_l$ is defined as in Theorem 2.*

Proposition A.5. *$\widehat{M} - M = o_p(h^{1/2})$, where M is defined in (2.5).*

Proof of Proposition A.1. Note that

$$
\begin{aligned}
S_1 =& \frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor}K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor}K\Big(\frac{t-s}{Th}\Big)v_s\Big]\\
=& \frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\sum_{s=1}^{T}K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\sum_{s=1}^{T}K\Big(\frac{t-s}{Th}\Big)v_s\Big]\\
&+\frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]\\
&+\frac{2}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\sum_{r=1}^{T}K\Big(\frac{t-r}{Th}\Big)v_r\Big]\\
=& \frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}K^2(0)v_t'v_t+\frac{1}{T^2h^{3/2}}\sum_{s\neq t}^{T}K^2\Big(\frac{t-s}{Th}\Big)v_s'v_s\\
&+\frac{2}{T^2h^{3/2}}\sum_{s\neq t}^{T}K\Big(\frac{t-s}{Th}\Big)K(0)v_s'v_t+\frac{1}{T^2h^{3/2}}\sum_{s\neq r,s\neq t,r\neq t}^{T}K\Big(\frac{t-s}{Th}\Big)K\Big(\frac{t-r}{Th}\Big)v_s'v_r\\
&+\frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]\\
&+\frac{2}{T^2h^{3/2}}\sum_{t=1}^{T}\Big[\Big(\sum_{s=1-\lfloor Th\rfloor}^{0}+\sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big)K\Big(\frac{t-s}{Th}\Big)v_s\Big]'\Big[\sum_{r=1}^{T}K\Big(\frac{t-r}{Th}\Big)v_r\Big]\\
\triangleq& \sum_{i=1}^{6}S_{1i}.
\end{aligned}
$$

By Lemmas A.1–A.4 below,

$$S_1-\mathcal{B}_1-\mathcal{B}_2=S_{142}+o_p(1). \tag{A.6}$$

Next, we have that $\sum_{j=-\infty}^{\infty}E(v_{0t}v_{0t+j}')=\mathrm{tr}(M)$, which entails that $\mathcal{B}_1+\mathcal{B}_2=\mathcal{B}$. Therefore, the conclusion holds by (A.6) and Lemma A.5 below.

□

Lemma A.1. *$S_{11} = o_p(1)$.*

Lemma A.2. *$S_{12} - \mathcal{B}_1 = o_p(1)$, where $\mathcal{B}_1 = h^{-1/2}[E(v_{0t}'v_{0t})]\int K^2(x)dx$.*

Lemma A.3. *$S_{13} = o_p(1)$.*

Lemma A.4. *$S_{14} - \mathcal{B}_2 = S_{142} + o_p(1)$, where S_{142} is defined as in (A.14) below, and $\mathcal{B}_2 = h^{-1/2}[\sum_{j=1}^{\infty} E(v_{0t}'v_{0t+j})]\int K^2(x)dx$.*

Lemma A.5. *$S_{15} = o_p(1)$.*

Lemma A.6. *$S_{16} = o_p(1)$.*

Lemma A.7. *$S_{142} \to_{\mathcal{L}} N(0, \mathcal{V})$.*

Proof of Lemma A.1. By (A.4), $E(S_{11}) = O\left(\frac{1}{Th^{3/2}}\right) = o(1)$ and

$$
\begin{aligned}
&\mathrm{Var}(S_{11})\\
=&\frac{K^4(0)}{T^4h^3}\sum_{t=1}^{T}\mathrm{Var}(v_t'v_t) + \frac{K^4(0)}{T^4h^3}\sum_{s\neq t}^{T}\mathrm{Cov}(v_t'v_t, v_s'v_s)\\
=&O\Big(\frac{1}{T^3h^3}\Big) + O\Big(\frac{1}{T^4h^3}\Big)\sum_{s\neq t}^{T}\mathrm{Cov}(v_t'v_t, v_s'v_s)\\
=&O\Big(\frac{1}{T^3h^3}\Big) + O\Big(\frac{1}{T^4h^3}\Big)\times\\
&\left[\sum_{t=1}^{T}\sum_{j=1}^{t-1}\mathrm{Cov}(v_t'v_t, v_{t-j}'v_{t-j}) + \sum_{t=1}^{T}\sum_{j=1}^{T-t}\mathrm{Cov}(v_t'v_t, v_{t+j}'v_{t+j})\right]. \qquad \text{(A.7)}
\end{aligned}
$$

By (A.4) and Davydov's inequality (Davydov, 1968),

$$
|\mathrm{Cov}(v_t'v_t, v_{t\pm j}'v_{t\pm j})| \leq C\beta(j)^{\delta/(1+\delta)}\left\|v_t'v_t\right\|_{2(1+\delta)}\left\|v_{t\pm j}'v_{t\pm j}\right\|_{2(1+\delta)}
$$

$$\leq C\beta(j)^{\delta/(1+\delta)} \tag{A.8}$$

for some $\delta \in (0,1)$ and all $t \geq 1$. Hence, $\sum_{t=1}^{T}\sum_{j=1}^{t-1}|\mathrm{Cov}(v_t'v_t, v_{t-j}'v_{t-j})| + \sum_{t=1}^{T}\sum_{j=1}^{T-t}|\mathrm{Cov}(v_t'v_t, v_{t+j}'v_{t+j})| \leq C\sum_{t=1}^{T}\sum_{j=1}^{\infty}\beta(j)^{\delta/(1+\delta)} = O(T)$ by (A.3). Together with (A.7), it follows that $\mathrm{Var}(S_{11}) = O\big(\frac{1}{T^3h^3}\big) = o(1)$. Now, we can conclude $S_{11} = o_p(1)$ by Chebyshev's inequality. □

Proof of Lemma A.2. By the symmetry of $K(\cdot)$, we can write $S_{12} = \frac{1}{T^2h^{3/2}}\sum_{s=1}^{T}(\Delta_{1s}+\Delta_{2s})$, where

$$\Delta_{1s} = \Big[\sum_{j=1}^{s-1}K^2\Big(\frac{j}{Th}\Big)\Big]v_s'v_s \ \text{ and } \ \Delta_{2s} = \Big[\sum_{j=1}^{T-s}K^2\Big(\frac{j}{Th}\Big)\Big]v_s'v_s.$$

By (A.2) and (A.5), we have $v_t = \Sigma_{*t}(\Sigma_{0*})^{-1}v_{0t} = (I_n + \epsilon_t\Sigma_{0*}^{-1}/c_T)v_{0t}$ and

$$\begin{aligned} v_s'v_t &= v_{0s}'v_{0t} + v_{0s}'\epsilon_t\Sigma_{0*}^{-1}v_{0t}/c_T \\ &\quad + v_{0s}'\Sigma_{0*}^{-1}\epsilon_s v_{0t}/c_T + v_{0s}'\Sigma_{0*}^{-1}\epsilon_s\epsilon_t\Sigma_{0*}^{-1}v_{0t}/c_T^2, \end{aligned} \tag{A.9}$$

where v_{0t} is stationary with mean zero by Assumption 2.

Since $\frac{1}{Th}\sum_{j=1}^{T-1}K^2\big(\frac{j}{Th}\big) = \int_0^1 K^2(x)dx + O\big(\frac{1}{Th}\big)$ and $\frac{1}{Th}\sum_{j=1}^{T-1}\frac{j}{T}K^2\big(\frac{j}{Th}\big) = h[\int_0^1 xK^2(x)dx] + O\big(\frac{1}{T}\big)$, by the boundedness of ϵ_t and Σ_{0*}, (A.9) and the stationarity of v_{0t}, it is not hard to see

$$\begin{aligned} E(S_{12}) &= \sum_{s=1}^{T}\frac{2E(v_s'v_s)}{T^2h^{3/2}}\left[\sum_{j=1}^{s-1}K^2\Big(\frac{j}{Th}\Big) + \sum_{j=1}^{T-s}K^2\Big(\frac{j}{Th}\Big)\right] \\ &= \sum_{s=1}^{T}\frac{E(v_s'v_s)}{Th^{1/2}}\Big[\frac{2}{Th}\sum_{j=1}^{T-1}\Big(1-\frac{j}{T}\Big)K^2\Big(\frac{j}{Th}\Big)\Big] \end{aligned}$$

$$=\frac{E(v_{0s}'v_{0s})}{h^{1/2}}\Big[\frac{2}{Th}\sum_{j=1}^{T-1}\Big(1-\frac{j}{T}\Big)K^2\Big(\frac{j}{Th}\Big)\Big]+O\Big(\frac{1}{c_Th^{1/2}}\Big)+O\Big(\frac{1}{c_T^2h^{1/2}}\Big)$$

$$=\mathcal{B}_1+O\Big(\frac{1}{Th^{3/2}}\Big)+O(h^{1/2})+O\Big(\frac{1}{c_Th^{1/2}}\Big)+O\Big(\frac{1}{c_T^2h^{1/2}}\Big)\to\mathcal{B}_1.$$

Moreover, since $\frac{1}{Th}\sum_{j=1}^{T-1}K^2\big(\frac{j}{Th}\big)=O(1)$, we can show

$$\begin{aligned}
\mathrm{Var}\Big(\frac{1}{T^2h^{3/2}}\sum_{s=1}^{T}\Delta_{1s}\Big)=&\frac{1}{T^4h^3}\sum_{s=1}^{T}\mathrm{Var}(\Delta_{1s})+\frac{2}{T^4h^3}\sum_{s=1}^{T}\sum_{s<r}\mathrm{Cov}(\Delta_{1s},\Delta_{1r})\\
\le&\frac{1}{T^4h^3}\sum_{s=1}^{T}\Big[\sum_{j=1}^{T-1}K^2\Big(\frac{j}{Th}\Big)\Big]^2\mathrm{Var}(v_s'v_s)\\
&+\frac{2}{T^4h^3}\sum_{s=1}^{T-1}\sum_{r=s+1}^{T-1}\Big[\sum_{i=1}^{T-1}K^2\Big(\frac{i}{Th}\Big)\Big]^2|\mathrm{Cov}(v_s'v_s,v_r'v_r)|\\
\le&O\Big(\frac{1}{Th}\Big)+O\Big(\frac{1}{T^2h}\Big)\sum_{s=1}^{T-1}\sum_{j=1}^{T-s-1}|\mathrm{Cov}(v_s'v_s,v_{s+j}'v_{s+j})|\\
\le&O\Big(\frac{1}{Th}\Big)+O\Big(\frac{1}{T^2h}\Big)\sum_{s=1}^{T-1}\sum_{j=1}^{T-s-1}\beta(j)^{\delta/(1+\delta)}\\
=&O\Big(\frac{1}{Th}\Big)=o(1),
\end{aligned}$$

where the last inequality holds by (A.8). Similarly, $\mathrm{Var}\big(\frac{1}{T^2h^{3/2}}\sum_{s=1}^{T}\Delta_{2s}\big)=o(1)$, which implies that $\mathrm{Var}(S_{12})=o(1)$ by the Cauchy-Schwarz inequality.

□

Proof of Lemma A.3. Define $\Psi_1(\psi_s,\psi_t)=K\big(\frac{s-t}{Th}\big)v_t'v_s$, where $\psi_s=(v_s,\frac{s}{Th})$, and $\Psi_1(\cdot,\cdot):\mathbb{R}^{(n+1)(n+2)/2}\times\mathbb{R}^{(n+1)(n+2)/2}\to\mathbb{R}$ is a symmetric function. Then, $S_{13}=\frac{4K(0)}{T^2h^{3/2}}\sum_{s<t}^{T}\Psi_1(\psi_s,\psi_t)$.

By the symmetry and boundedness of $K(\cdot)$, we have

$$\begin{aligned}
|E(S_{13})| =& \left|\frac{4K(0)}{T^2h^{3/2}}\sum_{t=1}^{T}\sum_{j=1}^{t-1}K\Big(\frac{j}{Th}\Big)E(v_t'v_{t-j})\right| \\
\leq& \frac{C}{T^2h^{3/2}}\sum_{t=1}^{T}\sum_{j=1}^{t-1}|E(v_t'v_{t-j})| \\
\leq& \frac{C}{Th^{3/2}}\sum_{j=1}^{\infty}\beta(j)^{\delta/(1+\delta)} = O\left(\frac{1}{Th^{3/2}}\right) = o(1), \qquad \text{(A.10)}
\end{aligned}$$

where the last inequality holds by a similar argument as for (A.8).

Moreover, since $E[\Psi_1(\psi_s, x)] = 0$ for any fixed $x \in \mathbb{R}^{(n+1)(n+2)/2}$, by (A.3)–(A.4) we have

$$\begin{aligned}
\mathrm{Var}(S_{13}) =& \frac{16K^2(0)}{T^4h^3}\mathrm{Var}\Big(\sum_{s<t}^{T}\Psi_1(\psi_s,\psi_t)\Big) \\
\leq& \frac{C}{T^2h^3}\max_{s<t}\Big[E|\Psi_1(\psi_s,\psi_t)|^{2(1+\delta)}\Big]^{1/(1+\delta)}\sum_{j=1}^{\infty}j\beta(j)^{\delta/(1+\delta)} \\
=& O\Big(\frac{1}{T^2h^3}\Big) = o(1), \qquad \text{(A.11)}
\end{aligned}$$

where the inequality holds by Lemma A(ii) of Hjellvik et al. (1998), which relies on some minor modifications for the proof of Lemma 1 in Yoshihara (1976). Hence, by (A.10)–(A.11) it follows that $S_{13} = o_p(1)$. □

Proof of Lemma A.4. Write

$$\begin{aligned}
S_{14} =& \frac{1}{T^2h^{3/2}}\sum_{s\neq r, s\neq t, r\neq t}^{T}K\Big(\frac{t-s}{Th}\Big)K\Big(\frac{t-s}{Th}+\frac{s-r}{Th}\Big)v_s'v_r \\
=& \frac{1}{Th^{1/2}}\sum_{s\neq r}^{T}\Big[\int K(x)K\Big(x+\frac{s-r}{Th}\Big)dx\Big]v_s'v_r + O\Big(\frac{1}{T^2h^{3/2}}\Big)\sum_{s\neq r}^{T}v_s'v_r
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{Th^{1/2}}\sum_{s\neq r}^{T}\Big[\int K(x)K\Big(x+\frac{s-r}{Th}\Big)dx\Big]v_s'v_r+O_p\Big(\frac{1}{Th^{3/2}}\Big)\\
\triangleq&S_{14}^*+O_p\Big(\frac{1}{Th^{3/2}}\Big),
\end{aligned}
\tag{A.12}
$$

where the second equality holds since $\frac{1}{Th}\sum_{t\neq s,t\neq r}^{T}K\big(\frac{t-s}{Th}\big)K\big(\frac{t-s}{Th}+\frac{s-r}{Th}\big)=\int K(x)K\big(x+\frac{s-r}{Th}\big)dx+O\big(\frac{1}{Th}\big)$ for any s and r, and the third equality holds since $\sum_{s\neq r}^{T}v_s'v_r=O_p(T)$ by a similar argument as for (A.10).

Next, we introduce a truncation lag p_T such that

$$
p_T\to\infty,\ p_T=o(Th),\ p_Th^{3/2}\to\infty \text{ and } \sum_{j=p_T}^{\infty}j^2\beta(j)<Cp_T^{-1}. \tag{A.13}
$$

Denote $\mathbb{S}_1=\{(s,r):1\leq|s-r|\leq p_T,1\leq r\neq s\leq T\}$ and $\mathbb{S}_2=\{(s,r):p_T<|s-r|<T,1\leq r\neq s\leq T\}$. Then,

$$
S_{14}^*=\frac{1}{Th^{1/2}}\sum_{\mathbb{S}_1}\Delta_{3rs}v_s'v_r+\frac{1}{Th^{1/2}}\sum_{\mathbb{S}_2}\Delta_{3rs}v_s'v_r\triangleq S_{141}+S_{142}, \tag{A.14}
$$

where $\Delta_{3rs}=\int K(x)K\big(x+\frac{s-r}{Th}\big)dx$.

Furthermore, we re-write S_{141} as

$$
\begin{aligned}
S_{141}=&\Big[\int K^2(x)dx\Big]\frac{1}{Th^{1/2}}\sum_{\mathbb{S}_1}v_s'v_r+\frac{1}{Th^{1/2}}\sum_{\mathbb{S}_1}\Delta_{4rs}v_s'v_r\\
\triangleq&S_{1411}+S_{1412},
\end{aligned}
\tag{A.15}
$$

where $\Delta_{4rs}=\int K(x)\big[K\big(x+\frac{s-r}{Th}\big)-K(x)\big]dx$.

For S_{1411}, $E(S_{1411})=\big[\int K^2(x)dx\big]\big[\frac{1}{Th^{1/2}}\sum_{r=1}^{T}E\sum_{j=1}^{p_T}(v_r'v_{r+j}+v_r'v_{r-j})\big]$ via some simple calculations. Hence, by (A.2), (A.5), the stationarity of z_t, and a similar argument as for (A.10), we can show

$$E(S_{1411}) = \mathcal{B}_2 + O\Big(\frac{1}{c_T h^{1/2}}\Big) = \mathcal{B}_2 + o(1). \tag{A.16}$$

Moreover, by defining $\varpi_1(\psi_s, \psi_r) = v_s' v_r - E(v_s' v_r)$, we have

$$\begin{aligned}
\mathrm{Var}(S_{1411}) &= \Big[\int K^2(x)dx\Big]^2 \frac{1}{T^2 h} \sum_{s,r\in\mathbb{S}_1} \sum_{k,l\in\mathbb{S}_1} E(v_s' v_r v_k' v_l) - E(v_s' v_r) E(v_k' v_l) \\
&= \Big[\int K^2(x)dx\Big]^2 \frac{1}{T^2 h} \sum_{s,r\in\mathbb{S}_1} \sum_{k,l\in\mathbb{S}_1} E[\varpi_1(\psi_s, \psi_r)\varpi_1(\psi_k, \psi_l)] \\
&= O\Big(\frac{p_T}{Th}\Big) = o(1),
\end{aligned} \tag{A.17}$$

where the third equality holds by Proposition A.4 in Hong et al. (2017), and the fourth equality holds by (A.13). Here, Proposition A.4 in Hong et al. (2017) is valid due to some minor modifications for the proof of Lemma 1 in Yoshihara (1976). By (A.16)–(A.17), it follows that $S_{1411} = \mathcal{B}_2 + o_p(1)$.

For S_{1412}, since $|K(x + \frac{j}{Th}) - K(x)| \le C\frac{j}{Th}$, a similar argument as for (A.16) entails

$$\begin{aligned}
|E(S_{1412})| \le& \frac{C\int K(x)dx}{Th^{1/2}} \sum_{r=1}^{T} \sum_{j=1}^{p_T} \frac{j}{Th} [|E(v_r' v_{r+j})| + |E(v_r' v_{r-j})|] \\
=& O\Big(\frac{1}{Th^{3/2}} \sum_{j=1}^{p_T} j\beta(j)^{\delta/(1+\delta)}\Big) = O\Big(\frac{1}{Th^{3/2}}\Big) = o(1).
\end{aligned} \tag{A.18}$$

Moreover, since $\Delta_{4rs} \le C\frac{|s-r|}{Th}$, we have

$$\Big|\mathrm{Var}(S_{1412})\Big| = \Big|\frac{1}{T^2 h} \sum_{s,r\in\mathbb{S}_1} \sum_{k,l\in\mathbb{S}_1} \Delta_{4rs}\Delta_{4lk}[E(v_s' v_r v_k' v_l) - E(v_s' v_r)E(v_k' v_l)]\Big|$$

$$\leq \frac{C}{T^2 h} \sum_{s,r \in \mathbb{S}_1} \sum_{k,l \in \mathbb{S}_1} \frac{|s-r|}{Th} \frac{|k-l|}{Th} E|\varpi_1(\psi_s, \psi_r)\varpi_1(\psi_k, \psi_l)|$$
$$\leq \frac{Cp_T^2}{T^4 h^3} \sum_{s,r \in \mathbb{S}_1} \sum_{k,l \in \mathbb{S}_1} E|\varpi_1(\psi_s, \psi_r)\varpi_1(\psi_k, \psi_l)|$$
$$= O\Big(\frac{p_T^3}{T^3 h^3}\Big) = o(1), \tag{A.19}$$

where (A.19) holds by a similar argument as for (A.17). Hence, by (A.18)–(A.19), it follows that $S_{1412} = o_p(1)$. Now, the conclusion holds by (A.12) and (A.14)–(A.15). □

Proof of Lemma A.5. By the Cauchy-Schwarz inequality, we only need to prove

$$S_{151} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \Big[\sum_{s=1-\lfloor Th \rfloor}^{0} K\Big(\frac{t-s}{Th}\Big) v_s \Big]' \Big[\sum_{s=1-\lfloor Th \rfloor}^{0} K\Big(\frac{t-s}{Th}\Big) v_s \Big] = o_p(1),$$
$$S_{152} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \Big[\sum_{s=T+1}^{T+\lfloor Th \rfloor} K\Big(\frac{t-s}{Th}\Big) v_s \Big]' \Big[\sum_{s=T+1}^{T+\lfloor Th \rfloor} K\Big(\frac{t-s}{Th}\Big) v_s \Big] = o_p(1).$$

Since S_{151} deals with the left boundary while S_{152} deals with the right boundary, by symmetry, we only have to prove the result for S_{151}.

By the symmetry of $K(\cdot)$ and the fact that $\mathbf{y}_s = \mathbf{y}_{-s}$ for $\lfloor Th \rfloor \leq s \leq 0$, we have

$$S_{151} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \Big[\sum_{s=1}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big) v_s \Big]' \Big[\sum_{s=1}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big) v_s \Big]$$
$$= \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{s=1}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big)^2 v_s' v_s$$

$$
\begin{aligned}
&+\frac{1}{T^2h^{3/2}}\sum_{t=1}^{T}\sum_{s\neq r}^{\lfloor Th\rfloor}K\Big(\frac{t+s-1}{Th}\Big)K\Big(\frac{t+r-1}{Th}\Big)v_s'v_r\\
\triangleq & S_{1511}+S_{1512}.
\end{aligned}
$$

Note that $K(x)=0$ for $x>1$. Hence,

$$
E|S_{1511}|\leq\frac{1}{T^2h^{3/2}}\sum_{t=1}^{\lfloor Th\rfloor}\sum_{s=1}^{\lfloor Th\rfloor}K\Big(\frac{t+s-1}{Th}\Big)^2E|v_s'v_s|=O(h^{1/2})=o(1),
$$

which implies that $S_{1511}=o_p(1)$.

By the fact that

$$
\frac{1}{Th}\sum_{t=1}^{\lfloor Th\rfloor}\sum_{s\neq r}^{\lfloor Th\rfloor}K\Big(\frac{t+s-1}{Th}\Big)K\Big(\frac{t+r-1}{Th}\Big)=\int K\Big(\frac{s}{Th}+x\Big)K\Big(\frac{r}{Th}+x\Big)dx+o(1),
$$

we can obtain

$$
\begin{aligned}
S_{1512}=&\frac{1}{Th^{1/2}}\sum_{s\neq r}^{\lfloor Th\rfloor}\Big[\int K\Big(\frac{s}{Th}+x\Big)K\Big(\frac{r}{Th}+x\Big)dx+o(1)\Big]v_s'v_r\\
=&\frac{1}{Th^{1/2}}\sum_{s\neq r}^{\lfloor Th\rfloor}[\Delta_{5rs}+o(1)]v_s'v_r,
\end{aligned}
$$

where $\Delta_{5rs}=\int K(\frac{s}{Th}+x)K(\frac{r}{Th}+x)dx$.

By the similar arguments used in the proof of (A.11), we can prove

$$
\mathrm{Var}(S_{1512})\leq\frac{CT^2h^2}{T^2h}=O(h)=o(1),
$$

which implies that $S_{1512}=o_p(1)$ by Chebyshev's inequality. Hence, it follows that $S_{151}=o_p(1)$. $\square$

Proof of Lemma A.6. Note that

$$\begin{aligned}
S_{16} =& \frac{2}{T^2h^{3/2}} \sum_{t=1}^{T} \sum_{s=1-\lfloor Th \rfloor}^{0} \sum_{r=1}^{T} K\Big(\frac{t-s}{Th}\Big) K\Big(\frac{t-r}{Th}\Big) v_s' v_r \\
&+ \frac{2}{T^2h^{3/2}} \sum_{t=1}^{T} \sum_{s=T+1}^{T+\lfloor Th \rfloor} \sum_{r=1}^{T} K\Big(\frac{t-s}{Th}\Big) K\Big(\frac{t-r}{Th}\Big) v_s' v_r \\
\triangleq& S_{161} + S_{162}.
\end{aligned}$$

By symmetry, we only need to prove that $S_{161} = o_p(1)$.

By the symmetry of $K(\cdot)$ and the fact that $\mathbf{y}_s = \mathbf{y}_{-s}$ for $\lfloor Th \rfloor \leq s \leq 0$, we can further decompose S_{161} as

$$\begin{aligned}
S_{161} =& \frac{2}{T^2h^{3/2}} \sum_{t=1}^{\lfloor Th \rfloor} \sum_{s=1}^{\lfloor Th \rfloor} \sum_{r=1}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big) K\Big(\frac{t-r}{Th}\Big) v_s' v_r \\
=& \frac{2}{T^2h^{3/2}} \sum_{t=1}^{\lfloor Th \rfloor} \sum_{s=1}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big) K\Big(\frac{t-s}{Th}\Big) v_s' v_s \\
&+ \frac{2}{T^2h^{3/2}} \sum_{t=1}^{\lfloor Th \rfloor} \sum_{s\neq r}^{\lfloor Th \rfloor} K\Big(\frac{t+s-1}{Th}\Big) K\Big(\frac{t-r}{Th}\Big) v_s' v_r \\
\triangleq& S_{1611} + S_{1612}.
\end{aligned}$$

Then, we have

$$E|S_{1611}| \leq C\frac{T^2h^2}{T^2h^{3/2}} = O(h^{1/2}) = o(1),$$

which implies that $S_{1611} = o_p(1)$ by Markov's inequality. Using the similar arguments used in the proof of (A.11), we can obtain that $\mathrm{Var}(S_{1612}) = O(h) = o(1)$, and so $S_{1612} = o_p(1)$. Hence, it follows that $S_{161} = o_p(1)$. $\square$

Proof of Lemma A.7. First, by the symmetry of $K(\cdot)$, we have

$$S_{142} = \frac{1}{Th^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Big\{ \int K(x)\Big[K\Big(x+\frac{s-r}{Th}\Big)+K\Big(x+\frac{r-s}{Th}\Big)\Big]dx\Big\} v_s' v_r$$
$$= \frac{2}{Th^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs} v_s' v_r.$$

By the boundedness of $K(\cdot)$, (A.13), and a similar argument as for (A.10), it follows that $|E(S_{142})| \le \frac{C}{Th^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} |E(v_s' v_r)| \le \frac{C}{h^{1/2}} \sum_{j=p_T+1}^{T}$ $|E(v_r' v_{r+j})| \le \frac{C}{h^{1/2}} \sum_{j=p_T+1}^{T} \beta(j) = O\big(\frac{1}{p_T h^{1/2}}\big) = o(1)$.

Next, define $S_{0,142}$ in the same way as S_{142} with v_t replaced by v_{0t}. Then, since $E(S_{142}) = o(1)$, we have that $E(S_{0,142}) = o(1)$ and

$$\mathrm{Var}(S_{0,142}) = E(S_{0,142}^2) + o(1), \tag{A.20}$$

where

$$E(S_{0,142}^2) = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{l=1}^{T-p_T-1} \sum_{k=l+p_T+1}^{T} \Delta_{3rs}\Delta_{3lk} E(v_{0s}' v_{0r} v_{0k}' v_{0l})$$
$$= V_1 + V_2 + V_3 + V_4 \tag{A.21}$$

with

$$V_1 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs}^2 E(v_{0s}' v_{0r} v_{0s}' v_{0r}),$$
$$V_2 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=r+p_T+1, k\ne s}^{T} \Delta_{3rs}\Delta_{3rk} E(v_{0s}' v_{0r} v_{0k}' v_{0r}),$$
$$V_3 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{l=1, l\ne r}^{T-p_T-1} \sum_{s=\max\{r,l\}+p_T+1}^{T} \Delta_{3rs}\Delta_{3ls} E(v_{0s}' v_{0r} v_{0s}' v_{0l}),$$

$$V_4 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{l=1, l\neq r}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1, k\neq s}^{T} \Delta_{3rs}\Delta_{3lk} E(v_{0s}' v_{0r} v_{0k}' v_{0l}).$$

Define $\delta_j = \int K(x)K\big(x+\frac{j}{Th}\big)dx$. By (A.21) and Lemmas A.8–A.11 below, we can obtain

$$\begin{aligned} E(S_{0,142}^2) &= V_1^* + V_2^* + V_3^* + V_4^* \\ &= \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{j=p_T+1}^{T-r} \delta_j^2 \left[\sum_{m=-\min\{T-p_T-1-r,p_T\}}^{\min\{T-p_T-1-r,p_T\}} E\mathrm{vec}(v_{0r} v_{0r+m}') \right]' \\ &\quad \times \left[\sum_{m'=-\min\{T-j-r,p_T\}}^{\min\{T-j-r,p_T\}} E\mathrm{vec}(v_{0r+j} v_{0r+j+m'}') \right]. \end{aligned} \tag{A.22}$$

Note that $\frac{2}{Th}\sum_{j=p_T+1}^{T-r} \delta_j^2 = \int \big[\int K(x)K(x+\lambda)dx\big]^2 d\lambda + o(1)$ for all r, and $v_{0t} = \Sigma_{0*} z_t$. By (A.22), it follows that $E(S_{0,142}^2) = V + o(1)$. Hence, by (A.20) and Lemma A.4 in Kim et al. (2011), we have that $S_{0,142} \to_{\mathcal{L}} N(0, \mathcal{V})$.

Third, it only suffices to show that $S_{142} - S_{0,142} = o_p(1)$. By (A.9), we can get

$$\begin{aligned} S_{142} - S_{0,142} &= \frac{2}{Th^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs} \\ &\quad \times [v_{0s}'\epsilon_t \Sigma_{0*}^{-1} v_{0t}/c_T + v_{0s}' \Sigma_{0*}^{-1}\epsilon_s v_{0t}/c_T + v_{0s}'\Sigma_{0*}^{-1}\epsilon_s\epsilon_t\Sigma_{0*}^{-1} v_{0t}/c_T^2]. \end{aligned}$$

Because ϵ_t and Σ_{0*} are bounded, by using the similar argument as for (A.22), we can prove that $E(S_{142} - S_{0,142})^2 = O\big(\frac{1}{c_T^2}\big) = o(1)$, which implies $S_{142} - S_{0,142} = o_p(1)$ by Chebyshev's inequality. □

Lemma A.8. $V_1 = V_1^* + o(1)$, *where*

$$V_1^* = \frac{4}{Th} \sum_{r=1}^{T-p_T-1} \sum_{j=p_T+1}^{T-r} \delta_j^2 E\mathrm{vec}(v_{0r+j}v_{0r+j}')' E\mathrm{vec}(v_{0r}v_{0r}').$$

Lemma A.9. $V_2 = V_2^* + o(1)$, *where*

$$V_2^* = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-2} \sum_{j=p_T+1}^{T-r-1} \delta_j^2 \sum_{m'=1}^{\min\{T-r-j,p_T\}} \Big[E\mathrm{vec}(v_{0r+j+m'}v_{0r+j}')' E\mathrm{vec}(v_{0r}v_{0r}') \\ + E\mathrm{vec}(v_{0r+j}v_{0r+j+m'}')' E\mathrm{vec}(v_{0r}v_{0r}') \Big].$$

Lemma A.10. $V_3 = V_3^* + o(1)$, *where*

$$V_3^* = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-2} \sum_{j=p_T+1}^{T-r-1} \delta_j^2 \sum_{m=1}^{\min\{T-p_T-1-r,p_T\}} \Big[E\mathrm{vec}(v_{0r+j}v_{0r+j}')' E\mathrm{vec}(v_{0r}v_{0r+m}') \\ + E\mathrm{vec}(v_{0r+j}v_{0r+j}')' E\mathrm{vec}(v_{0r+m}v_{0r}')].$$

Lemma A.11. $V_4 = V_4^* + o(1)$, *where*

$$V_4^* = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-2} \sum_{j=p_T+1}^{T-r-1} \delta_j^2 \sum_{m=1}^{\min\{T-p_T-1-r,p_T\}} E\mathrm{vec}(v_{0r}v_{0r+m}' + v_{0r+m}v_{0r}')' \\ \times \sum_{m'=1}^{\min\{T-r-j,p_T\}} E\mathrm{vec}(v_{0r+j+m'}v_{0r+j}' + v_{0r+j}v_{0r+j+m'}').$$

In the sequel, we only give the proof of Lemma A.11, since the proofs of Lemmas A.8–A.10 are similar and much easier.

Proof of Lemma A.11. By noting that

$$v_{0s}'v_{0r}v_{0k}'v_{0l} = v_{0s}'v_{0r}v_{0l}'v_{0k} = \mathrm{tr}(v_{0s}'v_{0r}v_{0l}'v_{0k})$$

$$= \mathrm{tr}(v_{0k}v_{0s}'v_{0r}v_{0l}') = \mathrm{vec}(v_{0k}v_{0s}')'\mathrm{vec}(v_{0r}v_{0l}'), \tag{A.23}$$

we can re-write

$$\begin{aligned}
V_4 =& \frac{4}{T^2h}\sum_{r\neq l}^{T-p_T-1}\sum_{s=r+p_T+1}^{T}\sum_{k=l+p_T+1,k\neq s}^{T}\delta_{3rs}\delta_{3lk}E\mathrm{vec}(v_{0r}v_{0l}')'\mathrm{vec}(v_{0k}v_{0s}') \\
=& \frac{4}{T^2h}\sum_{r\neq l}^{T-p_T-1}\sum_{s=r+p_T+1}^{T}\sum_{k=l+p_T+1,k\neq s}^{T}\delta_{3rs}\delta_{3lk}E\mathrm{vec}(v_{0r}v_{0l}')'E\mathrm{vec}(v_{0k}v_{0s}') \\
& + \frac{4}{T^2h}\sum_{r\neq l}^{T-p_T-1}\sum_{s=r+p_T+1}^{T}\sum_{k=l+p_T+1,k\neq s}^{T}\delta_{3rs}\delta_{3lk} \\
& \quad \times [E\mathrm{vec}(v_{0r}v_{0l}')'\mathrm{vec}(v_{0k}v_{0s}') - E\mathrm{vec}(v_{0r}v_{0l}')'E\mathrm{vec}(v_{0k}v_{0s}')] \\
\triangleq& V_{41} + V_{42}.
\end{aligned}$$

First, we consider V_{41} by splitting it into four parts:

$$V_{41} = V_{411} + V_{412} + V_{413} + V_{414}, \tag{A.24}$$

where V_{41i} are defined according to the following constraints on the indexes:

$V_{411} : 0 < |r-l| \leq p_T, 0 < |s-k| \leq p_T; V_{412} : |r-l| > p_T, 0 < |s-k| \leq p_T;$

$V_{412} : 0 < |r-l| \leq p_T, |s-k| > p_T; \qquad V_{414} : |r-l| > p_T, |s-k| > p_T.$

For V_{411}, some calculations lead to

$$V_{411} = \frac{4}{T^2h}\sum_{\substack{r,l=1 \\ 0<|r-l|\leq p_T}}^{T-p_T-1} E\mathrm{vec}(v_{0r}v_{0l}')'\sum_{s=r+p_T+1}^{T}\sum_{\substack{k=l+p_T+1 \\ 0<|k-s|\leq p_T}}^{T}\delta_{3rs}\delta_{3lk}E\mathrm{vec}(v_{0k}v_{0s}')$$

$$
\begin{aligned}
=&\frac{4}{T^2h}\sum_{r=1}^{T-p_T-2}\sum_{m=1}^{\min\{T-p_T-1-r,p_T\}}E\text{vec}(v_{0r}v_{0r+m}'+v_{0r+m}v_{0r}')'\\
&\times\sum_{s=r+p_T+1}^{T-1}\sum_{m'=1}^{\min\{T-s,p_T\}}E\text{vec}(v_{0s+m'}v_{0s}'+v_{0s}v_{0s+m'}')\\
&\times\int K(x)K\Big(x+\frac{r-s}{Th}\Big)dx\int K(x)K\Big(x+\frac{s-r+c_{mm'}}{Th}\Big)dx\\
=&\frac{4}{T^2h}\sum_{r=1}^{T-p_T-2}\sum_{m=1}^{\min\{T-p_T-1-r,p_T\}}E\text{vec}(v_{0r}v_{0r+m}'+v_{0r+m}v_{0r}')'\\
&\times\sum_{j=p_T+1}^{T-r-1}\sum_{m'=1}^{\min\{T-r-j,p_T\}}E\text{vec}(v_{0r+j+m'}v_{0r+j}'+v_{0r+j}v_{0r+j+m'}')\\
&\times\int K(x)K\Big(x+\frac{j}{Th}\Big)dx\int K(x)K\Big(x+\frac{j+c_{mm'}}{Th}\Big)dx,
\end{aligned}
$$

where $c_{mm'}$ is $m-m'$, $m+m'$, $-m-m'$ or $-m+m'$.

Since $|c_{mm'}|<2p_T$, it follows that $\int K(x)\big|K\big(x+\frac{j+c_{mm'}}{Th}\big)-K\big(x+\frac{j}{Th}\big)\big|<\frac{Cp_T}{Th}$. Then, by the fact that $\frac{1}{Th}\int K(x)K\big(x+\frac{j}{Th}\big)dx<\infty$ and a similar argument as for (A.8), we can show

$$
\begin{aligned}
V_{411}=&\frac{4}{T^2h}\sum_{r=1}^{T-p_T-2}\sum_{m=1}^{\min\{T-p_T-1-r,p_T\}}E\text{vec}(v_{0r}v_{0r+m}'+v_{0r+m}v_{0r}')'\\
&\times\sum_{j=p_T+1}^{T-r-1}\sum_{m'=1}^{\min\{T-r-j,p_T\}}E\text{vec}(v_{0r+j+m'}v_{0r+j}'+v_{0r+j}v_{0r+j+m'}')\\
&\times\Big[\int K(x)K\Big(x+\frac{j}{Th}\Big)dx\Big]^2+O\Big(\frac{p_T}{Th}\Big)\\
=&V_4^*+o(1),
\end{aligned}
\tag{A.25}
$$

where the last equality holds by (A.13).

For V_{412}, we can show

$$
\begin{aligned}
|V_{412}| =& \Big|\frac{4}{T^2h}\sum_{\substack{r,l=1\\|r-l|>p_T}}^{T-p_T-1} E\mathrm{vec}(v_{0r}v_{0l}')'\sum_{s=r+p_T+1}^{T}\sum_{\substack{k=l+p_T+1\\0<|k-s|\le p_T}}^{T}\delta_{3rs}\delta_{3lk}E\mathrm{vec}(v_{0k}v_{0s}')\Big| \\
=& \frac{4}{T^2h}\Big|\sum_{r=1}^{T-2p_T-2}\sum_{m=p_T+1}^{T-p_T-1-r} E\mathrm{vec}(v_{0r}v_{0r+m}'+v_{0r+m}v_{0r}')' \\
&\times \sum_{s=r+p_T+1}^{T-1}\sum_{m'=1}^{\min\{T-s,p_T\}} E\mathrm{vec}(v_{0s+m'}v_{0s}'+v_{0s}v_{0s+m'}') \\
&\times \int K(x)K\Big(x+\frac{r-s}{Th}\Big)dx\int K(x)K\Big(x+\frac{s-r+c_{mm'}}{Th}\Big)dx\Big| \\
\le& \frac{C}{T^2h}\sum_{r=1}^{T-2p_T-2}\sum_{m=p_T+1}^{T-p_T-1-r}\Big|E\mathrm{vec}(v_{0r}v_{0r+m}'+v_{0r+m}v_{0r}')'\Big| \\
&\times \sum_{s=r+p_T+1}^{T-1}\sum_{m'=1}^{\min\{T-s,p_T\}}\Big|E\mathrm{vec}(v_{0s+m'}v_{0s}'+v_{0s}v_{0s+m'}')\Big| \\
\le& \frac{C}{T^2h}\sum_{r=1}^{T-2p_T-2}\sum_{m=p_T+1}^{T-p_T-1-r}\beta(m)^{\delta/(1+\delta)}\sum_{s=r+p_T+1}^{T-1}\sum_{m'=1}^{\min\{T-s,p_T\}}\beta(m')^{\delta/(1+\delta)} \\
\le& \frac{C}{T^2h}\frac{T^2}{p_T}=O\Big(\frac{h^{1/2}}{p_Th^{3/2}}\Big)=o(1),
\end{aligned}
$$

where the first inequality holds by the integrability of $K(\cdot)$, the second inequality holds by a similar argument as for (A.8), and the third inequality holds by Assumption 2(i) and (A.13). Hence, $V_{412}=o(1)$. Similarly, we can prove that $V_{413}=o(1)$ and $V_{414}=o(1)$. By (A.24)–(A.25), it follows that $V_{41}=V_4^*+o(1)$. By (A.13) and the similar argument as for (A.17), we can show that $V_{42}=o(1)$, and hence the conclusion holds. □

Proof of Proposition A.2. Write

$$
\begin{aligned}
S_2 =& \frac{1}{T^2h^{1/2}}\sum_{t=1}^{T}\sum_{s,r=1}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_r + \frac{1}{T^2h^{1/2}}\sum_{t=1}^{T}\Big[\sum_{s=1-\lfloor Th\rfloor}^{0} + \sum_{s=T+1}^{T+\lfloor Th\rfloor}\Big]\sum_{r=1}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_r \\
=& \frac{1}{T^2h^{1/2}}\sum_{t=1}^{T}K(0)v_t'v_t + \frac{1}{T^2h^{1/2}}\sum_{t\neq r}^{T}K(0)v_t'v_r + \frac{1}{T^2h^{1/2}}\sum_{t\neq s}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_t \\
&+ \frac{1}{T^2h^{1/2}}\sum_{t\neq s}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_s + \frac{1}{T^2h^{1/2}}\sum_{t\neq s, s\neq r, r\neq t}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_r \\
&+ \frac{1}{T^2h^{1/2}}\sum_{t=1}^{T}\sum_{s=1-\lfloor Th\rfloor}^{0}\sum_{r=1}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_r + \frac{1}{T^2h^{1/2}}\sum_{t=1}^{T}\sum_{s=T+1}^{T+\lfloor Th\rfloor}\sum_{r=1}^{T}K\Big(\frac{s-t}{Th}\Big)v_s'v_r \\
\triangleq& \sum_{i=1}^{7}S_{2i}.
\end{aligned}
$$

By Lemmas A.1–A.3, $S_{21} = \frac{h}{K(0)}S_{11} = o_p(1)$ and $S_{23} = \frac{h}{2K(0)}S_{13} = o_p(1)$. For S_{22}, we can show that $E(S_{22}) = O\big(\frac{1}{Th^{1/2}}\big) = o(1)$ by using a similar proof as for Lemma A.3. Moreover, by Lemma A(ii) of Hjellvik et al. (1998), it entails that $\mathrm{Var}(S_{22}) = O\big(\frac{1}{T^2h}\big) = o(1)$, which implies $S_{22} = o_p(1)$ by Chebyshev's inequality. For S_{24}, by using a similar proof as for Lemma A.2, we can show that $|E(S_{24})| = O(h^{1/2}) + O\big(\frac{h^{1/2}}{c_T}\big) + O\big(\frac{h^{1/2}}{c_T^2}\big) = o(1)$ and $\mathrm{Var}(S_{24}) = O\big(\frac{h}{T}\big) = o(1)$, leading to $S_{24} = o_p(1)$. For S_{25}, we write it as $S_{25} = h \cdot S_{25}^{*}$. Then, by a similar argument as for S_{14} in (A.12), we have $S_{25}^{*} = O_p(1)$, and so $S_{25} = o_p(1)$.

Note that

$$
S_{26} = \frac{1}{T^2h^{1/2}}\sum_{t=1}^{\lfloor Th\rfloor}\sum_{s=1}^{\lfloor Th\rfloor}\sum_{r=1}^{T}K\Big(\frac{t+s-1}{Th}\Big)v_s'v_r
$$

$$=\frac{1}{T^2h^{1/2}}\sum_{t=1}^{\lfloor Th\rfloor}\sum_{s=1}^{\lfloor Th\rfloor}\sum_{r=1}^{\lfloor Th\rfloor}K\Big(\frac{t+s-1}{Th}\Big)v_s'v_r+\frac{1}{T^2h^{1/2}}\sum_{t=1}^{\lfloor Th\rfloor}\sum_{s=1}^{\lfloor Th\rfloor}\sum_{r=\lfloor Th\rfloor+1}^{T}K\Big(\frac{t+s-1}{Th}\Big)v_s'v_r$$

$$\triangleq S_{261}+S_{262}.$$

Here, the similar arguments used in Lemma A.6 indicate that $S_{261}=o_p(1)$. Next, since

$$\frac{1}{Th}\sum_{t=1}^{\lfloor Th\rfloor}K\Big(\frac{t+s-1}{Th}\Big)=\int_0^1K\Big(x+\frac{s}{Th}\Big)ds+o(1),$$

we have

$$S_{262}=\frac{h^{1/2}}{T}\sum_{s=1}^{\lfloor Th\rfloor}\sum_{r=\lfloor Th\rfloor+1}^{T}\Big[\int_0^1K\Big(x+\frac{s}{Th}\Big)ds+o(1)\Big]v_s'v_r.$$

Hence, Davydov's inequality implies $|ES_{262}|\le Ch^{3/2}=o(1)$. Furthermore, by Lemma A(ii) of Hjellvik et al. (1998), $\mathrm{Var}(S_{262})=O(\frac{h}{T^2}\times T^2h)=O(h^2)$, this implies that $S_{262}=o_p(1)$ by Chebyshev's inequality. Hence $S_{26}=o_p(1)$, and similarly, $S_{27}=o_p(1)$.

Now, we can conclude that $S_2=o_p(1)$. □

Proof of Proposition A.3. Write $S_3=S_{31}+S_{32}$, where $S_{31}=\frac{h^{1/2}}{T^2}\sum_{s=1}^{T}v_s'v_s$ and $S_{32}=\frac{h^{1/2}}{T^2}\sum_{s\neq t}^{T}v_s'v_t$. Since $S_{31}=\frac{h^2}{K^2(0)}S_{11}$ and $S_{32}=\frac{h}{K(0)}S_{22}$, we have $S_{31}=o(1)$ and $S_{32}=o(1)$, which imply $S_3=o_p(1)$. □

Proof of Proposition A.4. Write $S_4=2S_{41}+S_{42}-2S_{43}$, where $S_{41}=h^{1/2}\sum_{t=1}^{T}\Pi_1(t/T)'\ \Pi_2(t/T)$, $S_{42}=h^{1/2}\sum_{t=1}^{T}\Pi_2(t/T)'\Pi_2(t/T)$, and

$S_{43} = h^{1/2} \sum_{t=1}^{T} \Pi_3' \Pi_2(t/T)$.

Since $\sup_{x\in[0,1]} \left|\frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x-s/T)-1\right| = O(\frac{1}{Th})$, and $\sup_{x\in[0,1]} \frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor}$ $K_h(x-s/T) = O(1)$, we have

$$\begin{aligned}
\sup_t \|\Pi_2(t/T)\| \leq & \sup_t \Big|\frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h\Big(\frac{t-s}{T}\Big) - 1\Big| \Big\|\mathrm{vech}(\Sigma_0)\Big\| \\
& + c_T^{-1} \sup_t \Big\|\frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h\Big(\frac{t-s}{T}\Big) \mathrm{vech}(\Sigma_{1s})\Big\| + O(c_T^{-1}) \\
\leq & O\Big((Th)^{-1} + c_T^{-1}\Big)
\end{aligned}$$

by the boundedness of Σ_0 and $\Sigma_1(\cdot)$. Hence, $\sup_s \frac{1}{Th} \sum_{t=1}^{T} K\left(\frac{t-s}{Th}\right) \|\Pi_2(t/T)\| =$ $O((Th)^{-1} + c_T^{-1})$. Furthermore, since

$$\begin{aligned}
S_{41} =& h^{1/2} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} v_s' \frac{1}{Th} \sum_{t=1}^{T} K\Big(\frac{t-s}{Th}\Big) \Pi_2(t/T) \\
=& h^{1/2} \Big[\sum_{s=1}^{T} + \sum_{s=1-\lfloor Th \rfloor}^{0} + \sum_{s=T+1}^{T+\lfloor Th \rfloor}\Big] v_s' \frac{1}{Th} \sum_{t=1}^{T} K\Big(\frac{t-s}{Th}\Big) \Pi_2(t/T) \\
\triangleq & S_{411} + S_{412} + S_{413}.
\end{aligned}$$

By (A.23), Assumption 2, and Davydov's inequality, it follows that

$$\begin{aligned}
ES_{411}^2 =& h \sum_{s=1}^{T} E\Big[v_s' \frac{1}{Th} \sum_{t=1}^{T} K\Big(\frac{t-s}{Th}\Big) \Pi_2(t/T)\Big]^2 \\
& + 2h \sum_{s=1}^{T} \sum_{j=1}^{T-s} E\Big\{v_s' \Big[\frac{1}{Th} \sum_{t=1}^{T} K\Big(\frac{t-s}{Th}\Big) \Pi_2(t/T)\Big] \\
& \times v_{s+j}' \Big[\frac{1}{Th} \sum_{t=1}^{T} K\Big(\frac{t-s-j}{Th}\Big) \Pi_2(t/T)\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
\leq & O\Big(\frac{Th}{c_T^2}+\frac{1}{Th}\Big)+O\Big(\frac{h}{c_T^2}+\frac{1}{T^2h}\Big)\sum_{s=1}^{T}\sum_{j=1}^{T-s}\beta(j)^{\delta/(1+\delta)}\\
= & O\Big(\frac{Th}{c_T^2}\Big)+o(1)=o(1).
\end{aligned}
$$

By Chebyshev's inequality, we have that $S_{411}=o_p(1)$. Similarly, $S_{412}=o_p(1)$ and $S_{413}=o_p(1)$, implying that $S_{41}=o_p(1)$. Using the similar arguments, we also have that $S_{43}=o_p(1)$.

Next, it suffices to show that $S_{42}=\frac{Th^{1/2}}{c_T^2}\mathcal{B}_l+o(1)$. Let

$$
\begin{aligned}
\Pi_{21}(x)= & \Big[\frac{1}{T}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor}K_h\Big(x-\frac{s}{T}\Big)-1\Big]\mathrm{vech}(\Sigma_0),\\
\Pi_{22}(x)= & \Big[\frac{1}{T}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor}K_h\Big(x-\frac{s}{T}\Big)\mathrm{vech}(\Sigma_{1s})-\frac{1}{T}\sum_{s=1}^{T}\mathrm{vech}[\Sigma_{1s}]\Big].
\end{aligned}
$$

Then, $S_{42}=S_{421}+S_{422}+S_{423}$, where

$$
\begin{aligned}
S_{421}= & \frac{h^{1/2}}{c_T^2}\sum_{t=1}^{T}\Pi_{22}(t/T)'\Pi_{22}(t/T),\\
S_{422}= & h^{1/2}\sum_{t=1}^{T}\Pi_{21}(t/T)'\Pi_{21}(t/T),\\
S_{423}= & \frac{2h^{1/2}}{c_T}\sum_{t=1}^{T}\Pi_{21}(t/T)'\Pi_{22}(t/T).
\end{aligned}
$$

Note that $\sup_x\|\Pi_{21}(x)\|=O\big(\frac{1}{Th}\big)$ and $\sup_x\|\Pi_{22}(x)\|=O(1)$. Hence, $|S_{422}|\leq O\big(\frac{1}{Th^{3/2}}\big)=o(1)$ and $|S_{423}|\leq O\big(\frac{1}{h^{1/2}c_T}\big)=o(1)$.

Moreover, by letting $\overline{\Sigma}_1=\frac{1}{T}\sum_{t=1}^{T}\Sigma_{1t}$ and $\overline{\Sigma}_{1t}=\frac{1}{Th}\sum_{s=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor}K\big(\frac{t-s}{Th}\big)\Sigma_{1s}$, we have

$$
\begin{aligned}
S_{421} =& \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\overline{\Sigma}_{1t} - \overline{\Sigma}_1)' \text{vech}(\overline{\Sigma}_{1t} - \overline{\Sigma}_1) \\
=& \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \overline{\Sigma}_1 + \overline{\Sigma}_{1t} - \Sigma_{1t})' \text{vech}(\Sigma_{1t} - \overline{\Sigma}_1 + \overline{\Sigma}_{1t} - \Sigma_{1t}) \\
\triangleq& S_{4211} + S_{4212} + S_{4213},
\end{aligned}
$$

where

$$
\begin{aligned}
S_{4211} =& \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \overline{\Sigma}_1)' \text{vech}(\Sigma_{1t} - \overline{\Sigma}_1), \\
S_{4212} =& \frac{2h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \overline{\Sigma}_1)' \text{vech}(\overline{\Sigma}_{1t} - \Sigma_{1t}), \\
S_{4213} =& \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\overline{\Sigma}_{1t} - \Sigma_{1t})' \text{vech}(\overline{\Sigma}_{1t} - \Sigma_{1t}).
\end{aligned}
$$

It follows easily that $S_{4211} = \frac{Th^{1/2}}{c_T^2}\mathcal{B}_l + o(1)$. Since $\overline{\Sigma}_1$, Σ_{1t}, and $\overline{\Sigma}_1$ all are bounded and $\|\overline{\Sigma}_{1t} - \Sigma_{1t}\| = o(1)$ except for at most $[2Th]$ points, we can show that $S_{4212} = O(\frac{Th^{3/2}}{c_T^2}) = o(1)$ and $S_{4213} = O(\frac{Th^{3/2}}{c_T^2}) = o(1)$. Therefore, it follows that $S_{421} = \frac{Th^{1/2}}{c_T^2}\mathcal{B}_l + o(1)$. This completes the proof. □

Proof of Proposition A.5. Since $\widehat{M} - M = (\widetilde{M} - M) + (\widehat{M} - \widetilde{M})$, it suffices to show that $\widetilde{M} - M = o_p(h^{1/2})$ and $\widehat{M} - \widetilde{M} = o_p(h^{1/2})$, where $\widetilde{M}$ is defined in the same way as $\widehat{M}$ in (2.4) with $\widehat{v}_t$ replaced by v_t. By Lemmas A.12–A.13 below, the conclusion follows. □

Lemma A.12. *$\widetilde{M} - M = o_p(h^{1/2})$.*

Lemma A.13. $\widehat{M} - \widetilde{M} = o_p(h^{1/2})$.

PROOF OF LEMMA A.12. Denote $\Gamma_{v,j} = Ev'_{t+j}v_t$. Then, we have

$$\begin{aligned}|E(\widetilde{M} - M)| =& \Big|E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j} + \sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M\Big| \\ \leq& \Big|E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j}\Big| + \Big|\sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M\Big|.\end{aligned}$$

First, we can show

$$\begin{aligned}\Big|E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j}\Big| \leq& \sum_{j=-b_T}^{b_T} \Big|k\Big(\frac{j}{b_T}\Big)\frac{T-j}{T} - 1\Big||\Gamma_{v,j}| + \sum_{|j|>b_T} |\Gamma_{v,j}| \\ \leq& \frac{C}{b_T}\sum_{j=-b_T}^{b_T} |j\Gamma_{v,j}| + O\Big(\frac{1}{b_T}\Big) \leq O\Big(\frac{1}{b_T}\Big),\end{aligned}$$

where the second inequality holds by Lipschitz condition and the fact that $\sum_{|j|>b_T} |\Gamma_{v,j}| \leq C/b_T$ for large b_T, and the last inequality holds by Davydov's inequality and Assumption 2.

Second, we have

$$\begin{aligned}\Big|\sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M\Big| =& \Big|\sum_{j=-\infty}^{\infty} \Sigma_{t+j*}Ez_{t+j}z'_t\Sigma'_{t*} - \Sigma_{0*}Ez_{t+j}z'_t\Sigma'_{0*}\Big| \\ \leq& \sum_{j=-\infty}^{\infty} \Big|\Sigma_{t*}Ez_{t+j}z'_t\Sigma'_{t*} - \Sigma_{0*}Ez_{t+j}z'_t\Sigma'_{0*}\Big| \\ \leq& \frac{C}{c_T}\sum_{j=-\infty}^{\infty} \Big|Ez'_{t+j}z_t\Sigma'_{0*}\Big| = O\Big(\frac{1}{c_T}\Big),\end{aligned}$$

where we have used the fact that $\sup_t |\Sigma_{t*} - \Sigma_{0*}| = O(1/c_T)$ by (A.5). Hence, it follows that $E(\widetilde{M} - M) = o(h^{1/2})$.

Third, by a similar argument as for (A.17), we can show that $\mathrm{Var}(\widetilde{M}) = O\big(\frac{b_T}{T}\big)$, and then the result follows by Chebyshev's inequality and Assumption 4. $\square$

Proof of Lemma A.13. Write $\widehat{M} - \widetilde{M} = M_1 + M_2$, where $M_1 = \sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big)\frac{1}{T}\sum_{t=j+1}^{T}(\widehat{v}_t\widehat{v}_{t-j}' - v_t v_{t-j}')$ and $M_2 = \sum_{j=-(T-1)}^{-1} k\big(\frac{j}{b_T}\big)\frac{1}{T}\sum_{t=1-j}^{T} (\widehat{v}_{t+j}\widehat{v}_t' - v_{t+j}v_t')$.

It suffices to prove that $M_1 = o_p(h^{1/2})$, since the proof of M_2 is similar. Write $M_1 = M_{11} + M_{12} + M_{13}$, where $M_{11} = \frac{1}{T}\sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big)\sum_{t=j+1}^{T}(\widehat{v}_t - v_t)(\widehat{v}_{t-j} - v_{t-j})'$, $M_{12} = \frac{1}{T}\sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big)\sum_{t=j+1}^{T} v_t(\widehat{v}_{t-j} - v_{t-j})'$, and $M_{13} = \frac{1}{T}\sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big)\sum_{t=j+1}^{T}(\widehat{v}_t - v_t)v_{t-j}'$. Here, we note that under H_{1T},

$$\widehat{v}_t - v_t = -\frac{1}{T}\sum_{s=1}^{T} v_s + \mathrm{vech}(\overline{\Sigma}_1/c_T + \Sigma_{1t}/c_T) \tag{A.26}$$

for any t, hence it follows that

$$\sup_t \|\widehat{v}_t - v_t\| = O_p\Big(\frac{1}{\sqrt{T}}\Big) + O\Big(\frac{1}{c_T}\Big). \tag{A.27}$$

For M_{11}, by (A.27), Assumptions 3(ii) and 4(ii), and the fact that $\frac{1}{b_T}\sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big) = O(1)$, we have

$$\begin{aligned}
\|M_{11}\| \leq & \Big|\frac{1}{T}\sum_{j=0}^{T-1} k\Big(\frac{j}{b_T}\Big)\Big| \sup_j \Big\|\sum_{t=j+1}^{T}(\widehat{v}_t - v_t)(\widehat{v}_{t-j} - v_{t-j})'\Big\| \\
\leq & O\Big(\frac{b_T}{T}\Big)\Big|\sup_j \Big\|\sum_{t=j+1}^{T}(\widehat{v}_t - v_t)(\widehat{v}_{t-j} - v_{t-j})'\Big\| \\
\leq & O(b_T) \times O_p\Big(\frac{1}{T} + \frac{1}{c_T^2}\Big)
\end{aligned}$$

$$=O_p\Big(\frac{b_T}{T}+\frac{b_T}{c_T^2}\Big)=O_p\Big(\frac{b_T}{c_T^2}\Big).$$

For M_{12}, by (A.26) and Assumptions 3(ii) and 4(ii), the boundedness of $\overline{\Sigma}_1$ and $\Sigma_1(\cdot)$, and the fact that $\frac{1}{b_T}\sum_{j=0}^{T-1} k\big(\frac{j}{b_T}\big)=O(1)$, we have

$$\begin{aligned}
\|M_{12}\| \leq & O\Big(\frac{b_T}{T}\Big)\sup_j\Big\|\sum_{t=j+1}^{T} v_t(\widehat{v}_{t-j}-v_{t-j})'\Big\| \\
\leq & O\Big(\frac{b_T}{T}\Big)\Big[\sup_j\Big\|\sum_{t=j+1}^{T} v_t\Big(\frac{1}{T}\sum_{s=1}^{T} v_s\Big)'\Big\|+\frac{1}{c_T}\sup_j\sum_{t=j+1}^{T}\|v_t\|\Big] \\
\leq & O\Big(\frac{b_T}{T}\Big)\Big[O_p(\sqrt{T})+O_p\Big(\frac{T}{c_T}\Big)\Big] \\
= & O_p\Big(\frac{b_T}{\sqrt{T}}+\frac{b_T}{c_T}\Big)=O_p\Big(\frac{b_T}{c_T}\Big)
\end{aligned}$$

Similarly, $\|M_{13}\|\leq o_p(h^{1/2})$, and hence $M_1=o_p(h^{1/2})$. $\square$

Proof of Theorem 1. Under H_0 (i.e., $\Sigma_1(x)\equiv 0$), $\mathcal{B}_l\equiv 0$. Hence, the conclusion holds by (2.4), (A.1), and Propositions A.1–A.5. $\square$

Proof of Theorem 2(i)–(ii). The conclusion holds directly by (2.4), (A.1), Propositions A.1–A.5, and the facts that when $c_T=cT^{1/2}h^{1/4}$, $S_4=c^2\mathcal{B}_l+o_p(1)$; and when $T^{1/2}h^{1/4}=o(c_T)$, $S_4=o_p(1)$. $\square$

Proof of Theorem 2(iii). When $c_T=o(T^{1/2}h^{1/4})$, we consider three cases: (1) $c_T^{-1}(Th)^{1/2}<\infty$; (2) $c_T(Th)^{-1/2}\to 0$ but $c_Th^{1/2}\to\infty$; (3) $c_T=O(h^{-1/2})$.

Case (1). Since $c_Th^{1/2}\to\infty$, a detailed investigation indicates that the proofs of Propositions A.1–A.3 still hold. Moreover, under the assumption that $h^{-1/2}c_T^{-1}b_T \to 0$, we can show that Proposition A.5 holds. Using the fact that $S_4 = \frac{Th^{1/2}}{c_T^2}\mathcal{B}_l + O_p(1)$ while $\frac{Th^{1/2}}{c_T^2} \to \infty$, it follows that $\widehat{D} \to \infty$ in probability.

Case (2). As for Case (1), Propositions A.1–A.3 hold. Next, it is easy to see that $Th^{1/2}\widehat{\mathcal{S}} = O_p(\frac{Th^{1/2}}{c_T^2})$, $\widehat{\mathcal{B}} = O_p(h^{-1/2})$, and $\widehat{\mathcal{V}} = O_p(1)$. Since $\frac{Th}{c_T^2} \to \infty$, we can see that $\widehat{\mathcal{D}} = O_p(\frac{Th^{1/2}}{c_T^2})$, implying that $\widehat{D} \to \infty$ in probability.

Case (3). It is easy to see that $S_1 - 2S_2 + S_3 = O_p(Th^{1/2})$ and $S_1 - 2S_2 + S_3 > 0$. Furthermore, by noting that

$$\begin{aligned} S_4 = {} & h^{1/2}\sum_{t=1}^{T} 2\Pi_1(t/T)'\Pi_2(t/T) + h^{1/2}\sum_{t=1}^{T}\Pi_2(t/T)'\Pi_2(t/T) \\ & - h^{1/2}\sum_{t=1}^{T} 2\Pi_3(t/T)'\Pi_2(t/T), \end{aligned}$$

we can easily show

$$\begin{aligned} h^{1/2}\sum_{t=1}^{T}\Pi_1(t/T)'\Pi_2(t/T) &= O_p(T^{1/2}h), \\ h^{1/2}\sum_{t=1}^{T}\Pi_2(t/T)'\Pi_2(t/T) &= O\Big(\frac{Th^{1/2}}{c_T^2}\Big), \\ h^{1/2}\sum_{t=1}^{T}\Pi_3(t/T)'\Pi_2(t/T) &= O_p(T^{1/2}h). \end{aligned}$$

Hence, $Th^{1/2}\widehat{\mathcal{S}}$ is at least of order $O_p(Th^{3/2})$ by the fact that $c_T = O(h^{-1/2})$. Since $\widehat{\mathcal{B}} = O_p(h^{-1/2}) = o_p(Th^{3/2})$ and $\widehat{\mathcal{V}} = O_p(1)$, it follows that $\widehat{D} \to \infty$ in probability. $\square$

B Proofs of Theorems 3–5

Let $B(x) = C_2\Sigma''(x)/2$, $V_t(x) = \Sigma(x)^{1/2}(\mathbf{u}_t - I_n)\Sigma(x)^{1/2}$, and $\widetilde{\Sigma}(x) = \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-t/T)\Sigma_t$. The technical lemma below plays a key role in our proof.

Lemma B.1. *Suppose Assumptions 1, 3(i) and 5–6 hold, and* $\Sigma(u)$ *is twice continuously differentiable on* $[0,1]$. *Then, almost surely,*

(i) $\sup_{x\in[0,1]} \big\|\widehat{\Sigma}(x)-\Sigma(x)-\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-t/T)V_t(t/T)-h^2B(x)\big\| = O\big(\frac{1}{Th}\big) + o(h^2)$;

(ii) if conditions in Theorem 4 hold, then almost surely,

$$\sup_{x\in[0,1]}\left\|\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-t/T)V_t(x)\right\| = O\Big(\sqrt{\frac{\log T}{Th}}\Big),$$
$$\sup_{x\in[0,1]}\left\|\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-t/T)[V_t(x)-V_t(t/T)]\right\| = O\Big(\sqrt{\frac{\log Th}{T}}\Big).$$

Proof of Lemma B.1. (i) Recall $\mathbf{u}_t = \mathbf{u}_{-t}$, $\Sigma_t = \Sigma_{-t}$ for $-[Th] \le t \le -1$, and $\mathbf{u}_t = \mathbf{u}_{2T-t}$, $\Sigma_t = \Sigma_{2T-t}$ for $T+1 \le t \le T+[Th]$. Note that

$$\begin{aligned}\widehat{\Sigma}(x)-\Sigma(x) =& \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h\Big(x-\frac{t}{T}\Big)V_t(t/T) + \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h\Big(x-\frac{t}{T}\Big)[\Sigma_t-\Sigma(x)]\\ &+\Sigma(x)\Big[\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h\Big(x-\frac{t}{T}\Big)-1\Big]\\ :=&\mathcal{T}_1(x)+\mathcal{T}_2(x)+\mathcal{T}_3(x).\end{aligned}$$

The proof of (i) is standard by using Taylor's expansion and the approximation of integrals.

(ii) Using the fact that $\|\Sigma(x) - \Sigma_t\| \leq C\mathbf{1}(|x - t/T| \leq h)$, it suffices to show

$$\sup_{x\in[0,1]} \|\mathcal{T}_1(x)\| = O\Big(\sqrt{\frac{\log T}{Th}}\Big).$$

Our proofs below follow the similar arguments as in Masry (1996), Hansen (2008) and Vogt (2012). Let $c_T = (\log T)^{1/2}$ and write

$$\begin{aligned}
\mathcal{T}_1(x) =& \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h\Big(x - \frac{t}{T}\Big)V_t(t/T)\mathbf{1}(\|V_t(t/T)\| > T^{1/s}c_T) \\
&+ \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h\Big(x - \frac{t}{T}\Big)V_t(t/T)\mathbf{1}(\|V_t(t/T)\| \leq T^{1/s}c_T) \\
:=& \mathcal{T}_{1,1}(x) + \mathcal{T}_{1,2}(x).
\end{aligned}$$

First, we consider $\mathcal{T}_{1,1}(x)$. Let $s > 2$. Then,

$$\sum_{t=1}^{\infty} P(\|V_t(t/T)\| > t^{1/s}c_t) \leq \sum_{t=1}^{\infty} t^{-1}c_t^{-s}E\|V_t(t/T)\|^{2s} \leq CE\|\mathbf{u}_t\|^{2s}\sum_{t=1}^{\infty} t^{-1}c_t^{-s} < \infty.$$

Hence, by Borel-Cantelli Lemma, for T sufficiently large, $\|V_t(t/T)\| \leq T^{1/s}c_T$ for $t \leq T$. That is, $\mathcal{T}_{1,1}(x) = 0$ almost surely.

Second, we consider $\mathcal{T}_{1,2}(x)$. Let $a_T = \sqrt{\frac{\log T}{Th}}$. Cover the interval $[0,1]$ with $\lfloor h^{-1}a_T^{-1}\rfloor + 1 := N$ balls $A_j = \{x : |x - x_j| \leq a_T h\}$. Then, for $x \in A_j$, $h^{-1}|x - x_j| \leq a_T$. Note that Assumption 3 ensures that for any $|x_1 - x_2| \leq \delta \leq 2$,

$$|K(x_2) - K(x_1)| \leq \delta K^*(x_1)$$

where $K^*(x) = C\mathbf{1}(|x| \leq 2)$. Hence,

$$\Big|K\Big(\frac{Tx-t}{Th}\Big) - K\Big(\frac{Tx_j-t}{Th}\Big)\Big| \leq a_T K^*\Big(\frac{Tx_j-t}{Th}\Big).$$

Denote

$$\widetilde{\mathcal{T}}_{1,2}(x) = \frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h^*\Big(x-\frac{t}{T}\Big)V_t(t/T)\mathbf{1}(\|V_t(t/T)\| \leq T^{1/s}c_T).$$

Note that there exists a constant $0 < M < \infty$ such that $\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h(x-\frac{t}{T})\|\Sigma_t\| \leq M$ and $\frac{1}{T}\sum_{t=1-\lfloor Th\rfloor}^{T+\lfloor Th\rfloor} K_h^*(x-\frac{t}{T})\|\Sigma_t\| \leq M$. By triangular inequality, we obtain

$$\sup_{x\in A_j}\|\mathcal{T}_{1,2}(x)\| \leq \|\mathcal{T}_{1,2}(x_j)\| + \|\widetilde{\mathcal{T}}_{1,2}(x_j)\| + 2Ma_T,$$

and hence,

$$\begin{aligned}&P\Big(\sup_{x\in[0,1]}\|\mathcal{T}_{1,2}(x)\| > 4Ma_T\Big)\\ \leq& N\max_{1\leq j\leq N} P\Big(\|\mathcal{T}_{1,2}(x_j)\| > Ma_T\Big) + N\max_{1\leq j\leq N} P\Big(\|\widetilde{\mathcal{T}}_{1,2}(x_j)\| > Ma_T\Big).\end{aligned}$$

By Theorem 2.1 in Liebscher (1996), the following statement holds: if the triangular array $\{Z_{t,T}\}_{t=1}^T$ satisfies $|Z_{t,T}| \leq b_T$ uniformly with triangular array α-mixing coefficient $\alpha_T(k)$, then for $T_0 \leq T$ and $\varepsilon > 4T_0 b_T$, we have

$$P\Big(\Big|\sum_{t=1}^T Z_{t,T}\Big| > \varepsilon\Big) \leq 4\exp\Big(-\frac{\varepsilon^2}{64T_0^{-1}TS_{T_0} + 3\varepsilon T_0 b_T}\Big) + 4\frac{T}{T_0}\alpha_T(T_0), \quad \text{(B.1)}$$

where $S_{T_0} = \sup_{0\leq j\leq T-1} E\big(\sum_{t=j+1}^{\min\{j+T_0,T\}} Z_{t,T}\big)^2$.

Recall that when $t \in [-\lfloor Th \rfloor, -1]$, $V_t(t/T) = V_{-t}(-t/T)$; and when $t \in [T+1, T+\lfloor Th \rfloor]$, $V_t(t/T) = V_{2T-t}(2-t/T)$. Hence,

$$\begin{aligned}\mathcal{T}_{1,2}(x) =& \frac{1}{Th}\sum_{t=1}^{\lfloor Th \rfloor}[K(x-t/T)+K(x+t/T)]V_t(t/T)\mathbf{1}(\|V_t(t/T)\| \le T^{1/s}c_T) \\ &+ \frac{1}{Th}\sum_{t=\lfloor Th \rfloor+1}^{T-\lfloor Th \rfloor}K(x-t/T)V_t(t/T)\mathbf{1}(\|V_t(t/T)\| \le T^{1/s}c_T) \\ &+ \frac{1}{Th}\sum_{t=T-\lfloor Th \rfloor+1}^{T}[K(x-t/T)+K(x+t/T-2)]V_t(t/T)\mathbf{1}(\|V_t(t/T)\| \le T^{1/s}c_T) \\ :=& \frac{1}{Th}\sum_{t=1}^{T}Z_{t,T}(x).\end{aligned}$$

Note that $P\Big(\|\mathcal{T}_{1,2}(x_j)\| > Ma_T\Big) = P\Big(\|\sum_{t=1}^{T}Z_{t,T}(x_j)\| > Ma_T Th\Big)$, and a straightforward extension of Theorem 1 in Hansen (2008) implies that $S_{T_0} \le C_0 T_0 h$ for some constant $0 < C_0 < \infty$. Hence, by letting $\varepsilon = Ma_T Th$, $b_T = T^{1/s}c_T$ and $T_0 = b_T^{-1}a_T^{-1}$ in (B.1), using the inequalities between mixing coefficients that $\alpha_T(k) \le \alpha(k) \le \beta(k)$, we have that for each x_j,

$$P\Big(\|\mathcal{T}_{1,2}(x_j)\| > Ma_T\Big) \le 4\exp\Big(-\frac{M^2 \log T}{64C_0 + 3M}\Big) + C\rho^{-T_0}T_0^{-1}T,$$

by the fact that $\beta(k) \le C\rho^k$ for some $\rho \in (0,1)$ and $C > 0$. Similarly, we have that for each x_j,

$$P\Big(\|\widetilde{\mathcal{T}}_{1,2}(x_j)\| > Ma_T\Big) \le 4\exp\Big(-\frac{M^2 \log T}{64C_0 + 3M}\Big) + C\rho^{-T_0}T_0^{-1}T.$$

Hence, recall $N = \lfloor h^{-1}a_T^{-1} \rfloor + 1$,

$$P\Big(\sup_{x\in[0,1]}\|\mathcal{T}_{1,2}(x)\| > 4M\sqrt{\frac{\log T}{Th}}\Big) \le C\Big[T^{-\frac{M^2}{64C_0+3M}}\sqrt{Th^{-1}} + \rho^{-T_0}T^2\Big] \triangleq \kappa^{(1)}(T) + \kappa^{(2)}(T).$$

Note that $\kappa^{(1)}(T) \leq T^{1-\frac{M^2}{64C_0+3M}}$ and $\frac{M^2}{64C_0+3CM} > 2$ with M sufficiently large. Since $\rho \in (0,1)$, it follows that

$$\sum_{t=1}^{\infty} \kappa^{(1)}(t) + \kappa^{(2)}(t) < \infty.$$

By Borel-Cantelli Lemma, the proof is completed. $\square$

Proof of Theorem 3. (i) Note that $h \to 0$ when T is sufficiently large. Hence, for any $x \in (0,1)$, we have that $h < x < 1-h$, and $\widehat{\Sigma}(x) = \frac{1}{T}\sum_{t=1}^{T} K_h\Big(x - \frac{t}{T}\Big)\mathbf{y}_t$. By the similar arguments as for Theorem 2 in Xu and Phillips (2008) and Assumption 3(i), we have that $\widetilde{\Sigma}(x) \to \Sigma(x-)\int_{-1}^{0} K(x)dx + \Sigma(x+)\int_{0}^{1} K(x)dx = \frac{1}{2}[\Sigma(x-)+\Sigma(x+)]$. Next, it suffices to show that for $x \in (0,1)$,

$$\left\|\widehat{\Sigma}(x) - \widetilde{\Sigma}(x)\right\| = o_p(1). \tag{B.2}$$

Since $\text{vech}(\mathbf{y}_t - \Sigma_t) = L_n(\Sigma_t^{1/2})^{\otimes 2} D_n z_t$, we have

$$\begin{aligned}
&E\Big[\frac{1}{T}\sum_{t=1}^{T} K_h\Big(x-\frac{t}{T}\Big)\text{vech}(\mathbf{y}_t - \Sigma_t)\Big]'\Big[\frac{1}{T}\sum_{t=1}^{T} K_h\Big(x-\frac{t}{T}\Big)\text{vech}(\mathbf{y}_t - \Sigma_t)\Big] \\
=&\frac{1}{T^2}\sum_{t=1}^{T} K_h^2\Big(x-\frac{t}{T}\Big) E\text{vech}(\mathbf{y}_t - \Sigma_t)'\text{vech}(\mathbf{y}_t - \Sigma_t) \\
&+ \frac{2}{T^2}\sum_{t=1}^{T}\sum_{j=1}^{T-t} K_h\Big(x-\frac{t}{T}\Big)K_h\Big(x-\frac{t+j}{T}\Big)E\text{vech}(\mathbf{y}_t - \Sigma_t)'\text{vech}(\mathbf{y}_{t+j} - \Sigma_{t+j}) \\
\leq&\frac{C}{Th}|Ez_t'z_t|\Big[\frac{1}{Th}\sum_{t=1}^{T} K\Big(x-\frac{t}{T}\Big)\Big]
\end{aligned}$$

$$+\frac{C}{T^2h^2}\sum_{t=1}^{T}K\Big(x-\frac{t}{T}\Big)\sum_{j=1}^{T-t}|Ez_t'z_{t+j}|=O\Big(\frac{1}{Th}\Big)$$

by Davydov's inequality, the stationarity of z_t, and the fact that $\sup_t\|L_n(\Sigma_t^{1/2})^{\otimes 2}D_n\|<\infty$ and $\frac{1}{Th}\sum_{t=1}^{T}K\big(x-\frac{t}{T}\big)<\infty$. Hence, it follows that (B.2) holds by Chebyshev's inequality.

(ii) Recall $b(x)=\text{vech}(B(x))$. Let $v_t(x)\triangleq\text{vech}(V_t(x))=L_n(\Sigma(x)^{1/2})^{\otimes 2}D_nz_t$. By Lemma B.1(i), it follows that

$$\sqrt{Th}(\widehat{\sigma}(x)-\sigma(x)-h^2b(x))=\sqrt{Th}\frac{1}{T}\sum_{t=1}^{T}K_h(x-t/T)[v_t(x)+v_t(t/T)-v_t(x)].$$

By the CLT for mixing process (see Hall and Heyde (2014)),

$$\sqrt{Th}\frac{1}{T}\sum_{t=1}^{T}K_h(x-t/T)v_t(x)\to_{\mathcal{L}}N(0,V_\sigma(x)),$$

and by noting that $\|\Sigma(x)-\Sigma(t/T)\|\mathbf{1}(|x-t/T|\le h)=o(1)$, we have

$$\sqrt{Th}\frac{1}{T}\sum_{t=1}^{T}K_h(x-t/T)[v_t(t/T)-v_t(x)]=o_p(1).$$

Hence, Slutsky's Theorem implies the result once we give the expression of $V_\sigma(x)$. Take p_T as in (A.13). Then,

$$\begin{aligned}&\text{Var}\Big(\sqrt{Th}\frac{1}{T}\sum_{t=1}^{T}K_h\Big(x-\frac{t}{T}\Big)z_t\Big)\\=&\frac{1}{Th}\sum_{r=1}^{T}K^2\Big(\frac{Tx-r}{Th}\Big)Ez_rz_r'+\frac{1}{Th}\sum_{\mathbb{S}_1}K\Big(\frac{Tx-s}{Th}\Big)K\Big(\frac{Tx-r}{Th}\Big)Ez_rz_s'\\&+\frac{1}{Th}\sum_{\mathbb{S}_2}K\Big(\frac{Tx-s}{Th}\Big)K\Big(\frac{Tx-r}{Th}\Big)Ez_rz_s'\end{aligned}$$

$$\triangleq V_{z,1} + V_{z,2} + V_{z,3},$$

where $\mathbb{S}_1$ and $\mathbb{S}_2$ are defined as in (A.14). Note that $\left|K\left(\frac{Tx-r}{Th}\right) - K\left(\frac{Tx-s}{Th}\right)\right| \leq C\frac{|r-s|}{Th}$ by Assumption 3. Since $E(z_t z_{t-j}') = O(\rho^j)$ for some $0 < \rho < 1$ by Proposition 2(i), we can show that $V_{z,3} = o(1)$ and

$$\begin{aligned}
V_{z,2} =& \frac{1}{Th} \sum_{r=1}^{T-1} \sum_{j=1}^{\min\{r-1, p_T\}} K^2\Big(\frac{Tx-r}{Th}\Big)(Ez_r z_{r+j}' + Ez_{r+j} z_r') + o(1) \\
=& \frac{1}{Th} \sum_{r=1}^{T-1} \sum_{j=1}^{p_T} K^2\Big(\frac{Tx-r}{Th}\Big)(Ez_r z_{r+j}' + Ez_{r+j} z_r') \\
&- \frac{1}{Th} \sum_{r=2}^{p_T} \sum_{j=r-1}^{p_T} K^2\Big(\frac{Tx-r}{Th}\Big)(Ez_r z_{r+j}' + Ez_{r+j} z_r') \\
\to& \Big[\int K^2(x)dx\Big] \sum_{j=-\infty, j\neq 0}^{\infty} E(z_r z_{r-j}'),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\Big|\frac{1}{Th} \sum_{r=2}^{p_T} \sum_{j=r-1}^{p_T} K^2\Big(\frac{Tx-r}{Th}\Big)(Ez_r z_{r+j}' + Ez_{r+j} z_r')\Big| \\
\leq& C\frac{p_T}{Th} \sum_{j=1}^{p_T} |Ez_r z_{r+j}' + Ez_{r+j} z_r'| = o(1).
\end{aligned}$$

Since $V_{z,1} = \big[\int K^2(x)dx\big] E(z_r z_r') + o(1)$, it implies that $V_{z,1} + V_{z,2} + V_{z,3} = \big[\int K^2(x)dx\big] Z_\infty + o(1)$, and hence the expression of $V_\sigma(x)$ follows. □

To facilitate the proof of Theorem 4, we will introduce some notations. Denote

$$\ell_t(\phi) \triangleq \ell(\mathbf{y}_t, \Omega_t(\phi)) = \mathrm{tr}(\mathrm{G}_\mathrm{t}^{-1}(\phi)\mathbf{u}_\mathrm{t}) + \log\det(\Sigma_t^{1/2} G_t(\phi) \Sigma_t^{1/2}).$$

Recall the definitions of $G_t(\phi)$, $\widehat{G}_t(\phi)$, $\mathscr{L}_T(\phi)$ and $\widehat{\mathscr{L}}(\phi)$ in (3.2)–(3.5). We similarly define

$$\widetilde{\mathscr{L}}(\phi) = \sum_{t=1}^{T} \widetilde{\ell}_t(\phi) \quad \text{with} \quad \widetilde{\ell}_t(\phi) = \mathrm{tr}(\widetilde{G}_t(\phi)^{-1}\mathbf{u}_t) + \log\det(\Sigma_t^{1/2}\widetilde{G}_t(\phi)\Sigma_t^{1/2}), \tag{B.3}$$

where $\widetilde{G}_t(\phi)$ is defined in the same way as $\widehat{G}_t(\phi)$ in (3.5) with $\{\widehat{\mathbf{u}}_t\}_{t=1}^T$ replaced by $\{\mathbf{u}_t\}_{t=1}^T$.

In addition, we need Lemma B.2 herein, which is useful throughout the proof of Theorem 4. Specifically, Lemma B.2(i)-(ii) provide some useful results for $\Sigma(x)$ allowing for finite discontinuous points, and Lemma B.2(iii)-(iv) give some useful results for everywhere continuous $\Sigma(x)$. Let

$$\kappa_T \triangleq \sqrt{\frac{\log T}{Th}} + \sup_t(\widetilde{\Sigma}_t - \Sigma_t) \quad \text{and} \quad \Delta_t \triangleq \widehat{\mathbf{u}}_t - \mathbf{u}_t, \tag{B.4}$$

where $\widetilde{\Sigma}_t = \widetilde{\Sigma}(t/T)$.

Lemma B.2. *Suppose Assumptions 1, 3(i) and 5–8 hold. If $\Sigma(x)$ is twice continuously differentiable at continuous points on $(0,1)$, then almost surely,*

(i) $\widehat{\Sigma}(x)^{-1/2} = \widetilde{\Sigma}(x)^{-1/2} - \frac{1}{2}\widetilde{\Sigma}(x)^{-3/4}(\widehat{\Sigma}(x) - \widetilde{\Sigma}(x))\widetilde{\Sigma}(x)^{-3/4} + O(\kappa_T^2)$

holds uniformly for all x, and consequently,

(ii) $\Delta_t = \widetilde{\Sigma}_t^{-1/2}\mathbf{y}_t\widetilde{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2}\mathbf{y}_t\Sigma_t^{-1/2} + O(\kappa_T)\mathbf{y}_t$.

If $\Sigma(x)$ is twice continuously differentiable everywhere on $x \in (0,1)$,

(iii) $\widehat{\Sigma}(x)^{-1/2} = \Sigma(x)^{-1/2} - \frac{1}{2}\Sigma(x)^{-3/4}(\widehat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-3/4} + O(\kappa_T^2)$

holds uniformly for all x, and consequently,

(iv) $\Delta_t = -\frac{1}{2}\Sigma_t^{-3/4}(\widehat{\Sigma}_t-\Sigma_t)\Sigma_t^{-3/4}\mathbf{y}_t\Sigma_t^{-1/2}-\frac{1}{2}\Sigma_t^{-1/2}\mathbf{y}_t\Sigma_t^{-3/4}(\widehat{\Sigma}_t-\Sigma_t)\Sigma_t^{-3/4}+ O(\kappa_T^2)\mathbf{y}_t$.

Proof of Lemma B.2. We only prove (iii)-(vi), since the proofs of (i)-(ii) are similar.

(iii) Since $\Sigma(x)$ is continuous everywhere on $(0,1)$, then by Taylor's expansion, it follows easily that $\sup_t(\widetilde{\Sigma}_t - \Sigma_t) = O(1/Th)$, and hence

$$\kappa_T = O\Big(\sqrt{\frac{\log T}{Th}}\Big). \tag{B.5}$$

Moreover, since $\widehat{\Sigma}(x) = \Sigma(x)^{1/2}[I_n+\Sigma(x)^{-1/2}(\widehat{\Sigma}(x)-\Sigma(x))\Sigma(x)^{-1/2}]\Sigma(x)^{1/2}$, we have

$$\begin{aligned}
\widehat{\Sigma}(x)^{-1/2} =&\Sigma(x)^{-1/4}[I_n + \Sigma(x)^{-1/2}(\widehat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-1/2}]^{-1/2}\Sigma(x)^{-1/4}\\
=&\Sigma(x)^{-1/4}[I_n - \frac{1}{2}\Sigma(x)^{-1/2}(\widehat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-1/2}]\Sigma(x)^{-1/4} + O(\kappa_T^2)\\
=&\Sigma(x)^{-1/2} - \frac{1}{2}\Sigma(x)^{-3/4}(\widehat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-3/4} + O(\kappa_T^2),
\end{aligned}$$

where the second equality holds by Taylor's expansion of $(I_n + \epsilon)^{-1/2}$ for a $n \times n$ matrix ϵ, and the fact that

$$\sup_{x\in[0,1]} \Big\|\widehat{\Sigma}(x) - \Sigma(x)\Big\|^2 = O\Big(\frac{\log T}{Th}\Big) + O(h^4) = O(\kappa_T^2), \quad a.s.$$

by Lemma B.1.

(iv) By using the result in (iii), it is straightforward to see

$$\begin{aligned}\Delta_t =&(\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2})\mathbf{y}_t\Sigma_t^{-1/2} + \Sigma_t^{-1/2}\mathbf{y}_t(\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2})\\ &+ (\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2})\mathbf{y}_t(\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2})\\ =&-\frac{1}{2}\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-3/4}\mathbf{y}_t\Sigma_t^{-1/2}\\ &-\frac{1}{2}\Sigma_t^{-1/2}\mathbf{y}_t\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-3/4} + O(\kappa_T^2)\mathbf{y}_t.\end{aligned}$$

This completes all of the proofs. □

Proof of Theorem 4(i). The conclusion holds by Theorem 4.1.1 in Amemiya (1985) and the Propositions B.1–B.4 below. □

Proposition B.1. *ϕ_0 is the unique minimizer of $E\ell_t(\phi)$ for $\phi \in \Phi$.*

Proposition B.2. $\sup_{\phi\in\Phi} |T^{-1}\mathscr{L}_T(\phi) - E\ell_t(\phi)| = o_p(1)$.

Proposition B.3. $\sup_{\phi\in\Phi} |T^{-1}\widehat{\mathscr{L}_T}(\phi) - T^{-1}\widetilde{\mathscr{L}_T}(\phi)| = o_p(1)$.

Proposition B.4. $\sup_{\phi\in\Phi} |T^{-1}\widetilde{\mathscr{L}_T}(\phi) - T^{-1}\mathscr{L}_T(\phi)| = o_p(1)$

Proof of Proposition B.1. First, we can show that $G_t(\phi)$ is a symmetric positive definite matrix for $\phi \in \Phi$ by Assumption 7(ii). Second, we have

$$\begin{aligned}&E\ell_t(\phi) - E\ell_t(\phi_0)\\ =&E\mathrm{tr}(\Omega_\mathrm{t}^{-1}(\phi)\mathbf{y}_\mathrm{t}) + E\log\det[\Omega_t(\phi)] - E\log\det[\Omega_t(\phi_0)] - n\end{aligned}$$

$$=E\log\det[G_t^{-1}(\phi_0)G_t(\phi)] + E\mathrm{tr}(G_t^{-1}(\phi)G_t^{1/2}(\phi_0)E(\mathbf{e}_t|\mathcal{F}_{t-1})G_t^{1/2}(\phi_0)) - n$$

$$=E\log\det[G_t^{-1}(\phi_0)G_t(\phi)] + E\mathrm{tr}(G_t^{-1}(\phi)G_t(\phi_0)) - n$$

$$=E\sum_{i=1}^{n} -\log\lambda_i + \lambda_i - 1,$$

where $\lambda_i, i = 1, \cdots, n$ are the n eigenvalues of $G_t^{-1}(\phi)G_t(\phi_0)$. Using the inequality $x-1-\log(x) \geq 0$ for $x > 0$, we can obtain that $E\ell_t(\phi) - E\ell_t(\phi_0) \geq 0$, and the equality holds if and only if $\lambda_i = 1$, i.e., $G_t(\phi) = G_t(\phi_0)$ a.s., which implies $\phi = \phi_0$ by Assumption 7(iii). Hence, we have shown that ϕ_0 is the unique minimizer of $E\ell_t(\phi)$. $\square$

Proof of Proposition B.2. By Theorem 3.1 in Ling and McAleer (2003), it suffices to show that $E\sup_{\phi\in\Phi}\|\ell_t(\phi)\| < \infty$.

Under Assumption 5, $E\log\det(G_t(\phi)) = E\sum_{i=1}^n \log[\lambda_i(G_t(\phi))] < E\sum_{i=1}^n \lambda_i(G_t(\phi)) = E\mathrm{tr}(G_t(\phi))$ for all $\phi \in \Phi$, where $\lambda_i(G_t(\phi)) > 0$, $i = 1, \cdots, n$ are the n eigenvalues of $G_t(\phi)$. Hence, $E[\log\det(G_t(\phi))]^+ < \infty$. Obviously, $E[\log\det(G_t(\phi))]^- < \sup_{\phi\in\Phi}(-\log\det(I_n - AA' - BB'), 0) < \infty$, which follows that $E\sup_{\phi\in\Phi}|\log\det(G_t(\phi))| < \infty$. Since Σ_t is bounded, it implies that $E\sup_{\phi\in\Phi}|\log\det(\Omega_t(\phi))| < \infty$.

It remains to show that $E\sup_{\phi\in\Phi}\mathrm{tr}(\Omega_t(\phi)^{-1}\mathbf{y}_t) < \infty$. Since $G_t(\phi)$ is positive definite by Assumption 7(ii), its eigenvalues are positive, and then by using the compactness of the parameter space and the Wielandt-Hoffman theorem, we have

$$\min_{1\le t\le T}\inf_{\phi\in\Phi}\lambda_{min}(G_t(\phi))\ge\inf_{\phi\in\Phi}\lambda_{min}(I_n-AA'-BB')\ge\lambda_{min0}>0 \quad \text{(B.6)}$$

for some constant $\lambda_{min0}>0$. Hence, $\sup_{\phi\in\Phi}\|G_t(\phi)^{-1}\|<\infty$, which implies $\sup_{\phi\in\Phi}\|\Omega_t(\phi)^{-1}\|<\infty$ by the boundedness of Σ_t. By Hölder's inequality and Assumptions 1 and 5, it follows that

$$E\sup_{\phi\in\Phi}\mathrm{tr}(\Omega_t(\phi)^{-1}\mathbf{y}_t)\le E\sup_{\phi\in\Phi}\left\|\Omega_t(\phi)^{-1}\right\|\|\mathbf{y}_t\|\le C\Big(E\sup_{\phi\in\Phi}\left\|\Omega_t(\phi)^{-1}\right\|^2E\|\mathbf{y}_t\|^2\Big)^{1/2}<\infty.$$

This completes the proof. □

Proof of Proposition B.3. It suffices to prove

(i) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}|\mathrm{tr}(\widehat{G}_t(\phi)^{-1}\widehat{\mathbf{u}}_t-\widetilde{G}_t(\phi)^{-1}\mathbf{u}_t)|=o_p(1)$;

(ii) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}|\log\det\widetilde{G}_t(\phi)\widehat{G}_t(\phi)^{-1}|=o_p(1)$;

(iii) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}|\log\det\Sigma_t\widehat{\Sigma}_t^{-1}|=o_p(1)$.

We first show (i)–(iii) above without discontinuous points, and then we modify the proof to allow finite discontinuous points.

[Continuous case] For (i), by letting

$$S_t(\phi)\triangleq\widehat{G}_t^{-1}(\phi)-\widetilde{G}_t(\phi)^{-1}=-\widetilde{G}_t(\phi)^{-1}(\widehat{G}_t(\phi)-\widetilde{G}_t(\phi))\widehat{G}_t(\phi)^{-1}, \quad \text{(B.7)}$$

it is straightforward to show that $\mathrm{tr}(\widehat{G}_t^{-1}\mathbf{u}_t-\widetilde{G}_t^{-1}\mathbf{u}_t)=\mathrm{tr}\,(S_t\mathbf{u}_t)+\mathrm{tr}\big(\widetilde{G}_t^{-1}\Delta_t\big)+\mathrm{tr}\,(S_t\Delta_t)$.

Note that

$$\|\Delta_t\| \leq O(\kappa_T)\|\mathbf{u}_t\| \tag{B.8}$$

by Lemma B.2(iv), the boundedness of Σ_t and the fact that $\sup_t \|\widehat{\Sigma}_t - \Sigma_t\| = O(\kappa_T)$;

$$\begin{aligned}\widehat{G}_t - \widetilde{G}_t =& A\Delta_{t-1}A' + B[\widehat{G}_{t-1} - \widetilde{G}_{t-1}]B' \\ =& A\Delta_{t-1}A' + B[A\Delta_{t-2}A']B' + B^2[\widehat{G}_{t-2} - \widetilde{G}_{t-2}](B')^2 \\ =& \sum_{j=1}^{t-1} B^{j-1}[A\Delta_{t-j}A'](B')^{j-1};\end{aligned} \tag{B.9}$$

and

$$\rho_B \triangleq \rho(B) < 1 \tag{B.10}$$

by Proposition 1. Then, we can show

$$\begin{aligned}\|S_t\| \leq& n\lambda_{\min 0}^{-2}\|\widehat{G}_t - \widetilde{G}_t\| \leq Cn\lambda_{\min 0}^{-2}\sum_{j=1}^{t-1}\rho_B^{j-1}\|\Delta_{t-j}\| \\ \leq& Cn\lambda_{\min 0}^{-2}\sum_{j=1}^{t-1} O(\kappa_T)\rho_B^{j-1}\|\mathbf{u}_{t-j}\| = O(\kappa_T)\sum_{j=1}^{t-1} O(\rho_B^j)\|\mathbf{u}_{t-j}\|,\end{aligned} \tag{B.11}$$

where the first inequality holds by (B.6) and the inequality that $\|A\| \leq \sqrt{n}\rho(A)$, the second inequality holds by (B.9)–(B.10) and the fact that $\|AB\| \leq \|A\|\rho(B)$, and the third inequality holds by (B.8). Hence,

$$\begin{aligned}E\Big[\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\mathrm{tr}(S_t\mathbf{u}_t)|\Big] \leq& \frac{1}{T}\sum_{t=1}^{T} E\sup_{\phi\in\Phi}\|S_t\|\|\mathbf{u}_t\| \\ \leq& \frac{1}{T}\sum_{t=1}^{T} O(\kappa_T)\sum_{j=1}^{t-1} O(\rho_B^j)E\|\mathbf{u}_{t-j}\|\|\mathbf{u}_t\|\end{aligned}$$

$$\leq \frac{1}{T}\sum_{t=1}^{T} O(\kappa_T)\sum_{j=1}^{t-1}\rho_B^j E\|\mathbf{u}_t\|^2 = O(\kappa_T),$$

where the last inequality holds by Hölder's inequality. By Markov's inequality and (B.5), we have that $\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\text{tr}(S_t\mathbf{u}_t)| = o_p(1)$.

Similarly, we can show that $\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\text{tr}\big(\widetilde{G}_t^{-1}\Delta_t\big) + \text{tr}\big(S_t\Delta_t\big)| = o_p(1)$, hence the result (i) follows.

For (ii) and (iii), by Lemma A.1(x) in Zhou et al. (2022), $|\log\det(\Sigma_t\widehat{\Sigma}_t^{-1})| \leq n\|\Sigma_t - \widehat{\Sigma}_t\|(\|\widehat{\Sigma}_t^{-1}\| + \|\Sigma_t^{-1}\|)$ and $|\log\det(\widetilde{G}_t\widehat{G}_t^{-1})| \leq n\|\widetilde{G}_t - \widehat{G}_t\|(\|\widehat{G}_t^{-1}\| + \|\widetilde{G}_t^{-1}\|)$. Then, the results (ii) and (iii) follow similarly as for the result (i).

[Discontinuous case] For simplicity, we only prove that $\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\text{tr}(S_t\mathbf{u}_t)| = o_p(1)$ in the case of one discontinuous point $u_d \in (0,1)$.

Define $\mathbb{E}_d = \{[Tu_d]-[Th], ..., [Tu_d]+[Th]\}$. Since $\sup_{t\in\mathbb{E}_d}\|\widetilde{\Sigma}_t - \Sigma_t\| < \infty$ and $\sup_{t\in\mathbb{E}_d}|\widetilde{\Sigma}_t - \widehat{\Sigma}_t| = \sup_{t\in\mathbb{E}_d}\left|\frac{1}{T}\sum_{s=1}^{T}K_h((t-s)/T)V_s(t/T)\right| \leq O(\sqrt{\frac{\log T}{Th}})$ by Lemma B.1(ii), it follows that

$$\|\Delta_t\| \leq C\|\mathbf{u}_t\| \text{ for } t \in \mathbb{E}_d \tag{B.12}$$

by using Lemma B.2(ii).

Next, for ease of proof, we introduce a truncation lag n_T such that $n_T = o(Th)$ and $n_T \to \infty$ as $T \to \infty$. Here, n_T is introduced such that the impact of the discontinuous point on S_t is small enough for t away from the $[Tu_d]+[Th]$. Then, for $t \geq [Tu_d]+[Th]+n_T$, the similar arguments as for

(B.11) entail

$$\|S_t\| \leq n\lambda_{min0}^{-2}\|\widehat{G}_t - \widetilde{G}_t\| \leq Cn\lambda_{min0}^{-2}\sum_{j=1}^{t-1}\rho_B^{j-1}\|\Delta_{t-j}\| = S_{1t} + S_{2t},$$

where

$$S_{1t} = Cn\lambda_{min0}^{-2}\sum_{j=1}^{t-([Tu_d]+[Th])}\rho_B^{j-1}\|\Delta_{t-j}\|,$$
$$S_{2t} = Cn\lambda_{min0}^{-2}\sum_{j=t-([Tu_d]+[Th])+1}^{t-1}\rho_B^{j-1}\|\Delta_{t-j}\|.$$

Since $t \geq [Tu_d]+[Th]+n_T$, Δ_{t-j} in S_{1t} behaves similarly as in the continuous case, and hence by (B.8), we have that $S_{1t} \leq C\kappa_T \sum_{j=1}^{t-([Tu_d]+[Th])}\rho_B^j\|\mathbf{u}_{t-j}\|$. On the other hand, $S_{2t} \leq C\sum_{j=t-([Tu_d]+[Th])+1}^{t}\rho_B^j\|\mathbf{u}_{t-j}\|$ by (B.12). Hence, it follows that for $t \geq [Tu_d]+[Th]+n_T$,

$$\|S_t\| \leq C\kappa_T \sum_{j=1}^{t-([Tu_d]+[Th])}\rho_B^j\|\mathbf{u}_{t-j}\| + C\sum_{j=t-([Tu_d]+[Th])+1}^{t}\rho_B^j\|\mathbf{u}_{t-j}\|. \quad \text{(B.13)}$$

Third, it is straightforward to see

$$E\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\mathrm{tr}(S_t\mathbf{u}_t)| \leq \frac{1}{T}\sum_{t=1}^{T}E\sup_{\phi\in\Phi}\|S_t\|\|\mathbf{u}_t\| = SU_1 + SU_2 + SU_3,$$

where $SU_1 = \frac{1}{T}\sum_{t=1}^{[Tu_d-Th]}E\sup_{\phi\in\Phi}\|S_t\|\|\mathbf{u}_t\|$, $SU_2 = \frac{1}{T}\sum_{t=[Tu_d-Th]}^{[Tu_d+Th]+n_T}E \sup_{\phi\in\Phi}\|S_t\|\|\mathbf{u}_t\|$, and $SU_3 = \frac{1}{T}\sum_{t=[Tu_d+Th]+n_T}^{T}E\sup_{\phi\in\Phi}\|S_t\|\|\mathbf{u}_t\|$.

For SU_1 and SU_2, we have $SU_1 \leq \frac{1}{T}\sum_{t=1}^{[Tu_d-Th]}O(\kappa_T) = O(\kappa_T)$ and $SU_2 \leq O\big(\frac{1}{T}\big)\sum_{t=[Tu_d-Th]}^{[Tu_d+Th]+n_T}\sum_{j=1}^{t-1}\rho_B^j E\|\mathbf{u}_{t-j}\|\|\mathbf{u}_t\| = O\big(\frac{2Th+n_T}{T}\big)$. For SU_3, by (B.13), we have

$$SU_3 \leq O\Big(\frac{\kappa_T}{T}\Big)\sum_{t=[Tu_d+Th]+n_T}^{T}\sum_{j=1}^{t-([Tu_d]+[Th])}\rho_B^j E\|\mathbf{u}_{t-j}\|\|\mathbf{u}_t\|$$

$$+ O\Big(\frac{1}{T}\Big) \sum_{t=[Tu_d+Th]+n_T}^{T} \sum_{j=t-([Tu_d]+[Th])+1}^{t} \rho_B^j E\|\mathbf{u}_{t-j}\|\|\mathbf{u}_t\|$$

$$= O(\kappa_T) + O\Big(\frac{1}{T}\Big) \sum_{t=[Tu_d+Th]+n_T}^{T} \rho_B^{n_T} = O(\kappa_T) + O(\rho_B^{n_T}).$$

Hence, $E\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\mathrm{tr}(S_t\mathbf{u}_t)| \leq O(\kappa_T) + O(\frac{2Th+n_T}{T}) + O(\rho_B^{n_T}) = o(1)$, and the result follows by Markov's inequality. This completes the proof. $\square$

Proof of Proposition B.4. It suffices to prove

(i) $\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\mathrm{tr}([G_t(\phi)^{-1} - \widetilde{G}_t(\phi)^{-1}]\mathbf{u}_t)| = o_p(1)$;

(ii) $\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\log\det\widetilde{G}_t(\phi)G_t(\phi)^{-1}| = o_p(1)$.

Note $\|G_t^{-1}\| \leq \|G_t^{-1/2}\|^2 = \mathrm{tr}(G_t^{-1}) \leq \mathrm{tr}(I_n - AA' - BB')$. Hence, by the compactness of Φ,

$$\sup_{\phi\in\Phi}\|G_t^{-1}\| < \infty, \tag{B.14}$$

and similarly,

$$\sup_{\phi\in\Phi}\|\widetilde{G}_t^{-1}\| < \infty. \tag{B.15}$$

In addition, similar to (B.9), given initial values $\widehat{\mathbf{u}}_0$, we can show $G_t(\phi) - \widetilde{G}_t(\phi) = B^{t-1}[A(\widehat{\mathbf{u}}_0 - \mathbf{u}_0)A' + B(\widehat{G}_0 - G_0)B'](B')^{t-1}$, and hence

$$\|G_t(\phi) - \widetilde{G}_t(\phi)\| \leq K\rho_B^t. \tag{B.16}$$

Observe that

$$\frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}|\mathrm{tr}([G_t(\phi)^{-1} - \widetilde{G}_t(\phi)^{-1}]\mathbf{u}_t)|$$

$$\leq \frac{1}{T}\sum_{t=1}^{T}\sup_{\phi\in\Phi}\|G_t(\phi)^{-1}\|\|G_t(\phi)-\widetilde{G}(\phi)_t\|\|\widetilde{G}_t(\phi)^{-1}\|\|\mathbf{u}_t\|.$$

By (B.14)–(B.16), the result (i) follows. Using $|\log\det(\widetilde{G}_tG_t^{-1})|\leq n\|\widetilde{G}_t-G_t\|(\|G_t^{-1}\|+\|\widetilde{G}_t^{-1}\|)$, the result (ii) follows similarly. □

In order to proceed the proof of Theorem 4(ii), we need Lemmas B.3–B.6 below. We note that the related assumptions for Lemmas B.3–B.6 are the same as those for Theorem 4(ii) unless stated otherwise.

Lemma B.3. *Let m_T be a truncation lag such that*

$$m_T = O(T^{\lambda_m}) \text{ for some } \lambda_m > 0 \text{ and } \lambda_m+\lambda_h<1/2. \tag{B.17}$$

Then, $\max_{1\leq j\leq m_T}\max_{j+1\leq t\leq T}\|\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-1/4}-\Sigma_t^{-3/4}(\widehat{\Sigma}_t-\Sigma_t)\Sigma_t^{-1/4}\| = o\big(\frac{1}{\sqrt{T}}\big)$ *a.s.*

Proof of Lemma B.3. For $1\leq j\leq m_T$ and $j+1\leq t\leq T$, we have

$$\begin{aligned}
&\|\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-1/4}-\Sigma_t^{-3/4}(\widehat{\Sigma}_t-\Sigma_t)\Sigma_t^{-1/4}\|\\
=&\|\Sigma_{t-j}^{-3/4}\widehat{\Sigma}_{t-j}\Sigma_{t-j}^{-1/4}-\Sigma_t^{-3/4}\widehat{\Sigma}_t\Sigma_t^{-1/4}\|\\
\leq&\|\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\widehat{\Sigma}_t)\Sigma_{t-j}^{-1/4}\|+\|\Sigma_t^{-3/4}\widehat{\Sigma}_t\Sigma_t^{-1/4}-\Sigma_{t-j}^{-3/4}\widehat{\Sigma}_t\Sigma_{t-j}^{-1/4}\|\\
\triangleq& I_1+I_2.
\end{aligned}$$

We first consider I_1. Note that $|K(\frac{s-t}{Th})-K(\frac{s'-t}{Th})|\leq\frac{C|s-s'|}{Th}$ by Lipschitz condition. Then, since $K(x)=0$ for $|x|>1$, we have that for any $0<$

$s - s' < m_T$,

$$
\begin{aligned}
\|\widehat{\Sigma}_s - \widehat{\Sigma}_{s'}\| =& \frac{1}{Th}\Big\| \sum_{t=[s'-Th]}^{[s+Th]} \Big[K\Big(\frac{s-t}{Th}\Big) - K\Big(\frac{s'-t}{Th}\Big)\Big]\mathbf{y}_t \Big\| \\
\leq& \frac{1}{Th} \sum_{t=[s'-Th]}^{[s'+m_T+Th]} \Big|K\Big(\frac{s-t}{Th}\Big) - K\Big(\frac{s'-t}{Th}\Big)\Big| \|\mathbf{y}_t\| \\
\leq& C|s-s'|\frac{1}{T^2h^2} \sum_{t=[s'-Th]}^{[s'+m_T+Th]} c_u \|\mathbf{u}_t\| \\
=& C|s-s'|\frac{m_T+2Th}{T^2h^2}\frac{1}{m_T+2Th} \sum_{t=[s'-Th]}^{[s'+m_T+Th]} \|\mathbf{u}_t\| \\
\leq& C m_T \frac{m_T+2Th}{T^2h^2} E\|\mathbf{u}_t\| = o\Big(\frac{1}{\sqrt{T}}\Big) \quad \text{a.s.} \qquad \text{(B.18)}
\end{aligned}
$$

Hence, by the boundedness of Σ_t^{-1} and (B.18), it follows that $I_1 = o\big(\frac{1}{\sqrt{T}}\big)$.

Second, we consider I_2. Since Σ_t^{-1} is bounded and Σ_t is trice differentiable, we can show that $\big\|\Sigma_{t-j} - \Sigma_t\big\| \leq C\big(\frac{j}{T}\big) = o\big(\frac{1}{\sqrt{T}}\big)$ and $\big\|\Sigma_t^{-5/8}(\Sigma_{t-j} - \Sigma_t)\Sigma_t^{-5/8}\big\| \leq C\big(\frac{j}{T}\big) = o\big(\frac{1}{\sqrt{T}}\big)$. Then, by Taylor's expansion, we have

$$
\begin{aligned}
\Sigma_{t-j}^{-1/4} =& [\Sigma_t + \Sigma_{t-j} - \Sigma_t]^{-1/4} \\
=& \Sigma_t^{-1/8}\Big[I_n + \Sigma_t^{-1/2}(\Sigma_{t-j} - \Sigma_t)\Sigma_t^{-1/2}\Big]^{-1/4}\Sigma_t^{-1/8} \\
=& \Sigma_t^{-1/8}\Big[I_n - \frac{1}{4}\Sigma_t^{-1/2}(\Sigma_{t-j} - \Sigma_t)\Sigma_t^{-1/2}\Big]\Sigma_t^{-1/8} + o\Big(\frac{1}{T}\Big) \\
=& \Sigma_t^{-1/4} - \frac{1}{4}\Sigma_t^{-5/8}(\Sigma_{t-j} - \Sigma_t)\Sigma_t^{-5/8} + o\Big(\frac{1}{T}\Big) \\
=& \Sigma_t^{-1/4} + o\Big(\frac{1}{\sqrt{T}}\Big) + o\Big(\frac{1}{T}\Big),
\end{aligned}
$$

which implies that $\Sigma_{t-j}^{-1/4} = \Sigma_t^{-1/4} + o\big(\frac{1}{\sqrt{T}}\big)$. Similarly, $\Sigma_{t-j}^{-3/4} = \Sigma_t^{-3/4} + o\big(\frac{1}{\sqrt{T}}\big)$, and hence it is not hard to show that $I_2 = o\big(\frac{1}{\sqrt{T}}\big)$ in view of that

$\|\widehat{\Sigma}\| = O(1)$ a.s. by Lemma B.1. This completes the proof. $\square$

Lemma B.4. *Under the conditions in Proposition 1,* $\{z_t, \mathrm{vec}(G_t), \frac{\mathrm{vec}(G_t)}{\partial\phi'}\}$ *is strictly stationary and* β*-mixing with exponential decay.*

Proof of Lemma B.4. The proof is omitted, since it is similar to the proof of Proposition 1 by noting the recursive representation in (C.1). $\square$

Lemma B.5. *Let* $\{c_t\}_{t\in\mathbb{Z}}$ *be a sequence of stationary process, and* $\mathcal{F}_t^s = \sigma(c_i, t \le i \le s)$ *be the sigma-filed generated by* $\{c_i, t \le i \le s\}$*. Define*

$$S_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} b_t \Big\{ \frac{1}{Th} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K\Big(\frac{s-t}{Th}\Big) a_s \Big\},$$

where $a_t = f(c_t)$*,* $b_t = g(c_t, c_{t-k})$ *for some* $k \le n_T$*, and* $f(\cdot)$ *and* $g(\cdot,\cdot)$ *are two real-valued functions. Suppose the following conditions hold:*

(1) $Ea_t = 0$*,* $Eb_t = 0$*,* $E|a_t|^{6(1+2\delta)} < \infty$ *and* $E|b_t|^{3(1+2\delta)} < \infty$ *for some* $\delta > 0$*;*

(2) c_t *is* β*-mixing with mixing coefficients* $\beta(j)$ *satisfying* $\sum_{j=1}^{\infty} \beta(j)^{\delta/(1+\delta)} < \infty$*;*

(3) $K(\cdot)$ *satisfies Assumption 3 and* h *satisfies Assumption 9;*

(4) n_T *is a constant or* $n_T \to \infty$ *and* $n_T = o(\sqrt{Th^2})$ *as* $T \to \infty$*.*

Then,

$$\text{(i) } |ES_T| \le \frac{Cn_T}{\sqrt{T}h} \quad \textit{and} \quad \text{(ii) } ES_T^2 \le C \max\Big\{\frac{n_T}{\sqrt{Th}}, \frac{1}{Th^2}\Big\}.$$

Proof of Lemma B.5. The proof is omitted, since it is similar to the one of Proposition A.1 in Jiang et al. (2021) by some minor modifications.

□

Lemma B.6. *Under (3.1),*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} D_n z_t = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Xi_0 (G_t^{1/2})^{\otimes 2}\xi_t + o_p(1).$$

Proof of Lemma B.6. Write $\mathbf{u}_t - I_n = G_t^{1/2}(\mathbf{e}_t - I_n)G_t^{1/2} + G_t - I_n$. Then,

$$\begin{aligned}\mathbf{u}_t - I_n =& A_0(\mathbf{u}_{t-1} - I_n)A_0' + B_0(\mathbf{u}_{t-1} - I_n)B_0' \\ & - B_0 G_{t-1}^{1/2}(\mathbf{e}_{t-1} - I_n)G_{t-1}^{1/2}B_0' + G_t^{1/2}(\mathbf{e}_t - I_n)G_t^{1/2},\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} D_n z_t =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(A_0^{\otimes 2} + B_0^{\otimes 2})D_n z_{t-1} + \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(G_t^{1/2})^{\otimes 2}\mathrm{vec}(\mathbf{e}_t - I_n) \\ & - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(B_0^{\otimes 2})(G_{t-1}^{1/2})^{\otimes 2}\mathrm{vec}(\mathbf{e}_{t-1} - I_n).\end{aligned}$$

Recalling that $\Xi_0 = (I_{n^2} - A_0^{\otimes 2} - B_0^{\otimes 2})^{-1}(I_{n^2} - B_0^{\otimes 2})$, the result follows. □

Proof of Theorem 4(ii). By Assumption 8 and Taylor's expansion, we have

$$0 = \frac{1}{\sqrt{T}}\frac{\partial \widehat{\mathscr{L}_T}(\widehat{\phi})}{\partial \phi} = \frac{1}{\sqrt{T}}\frac{\partial \widehat{\mathscr{L}_T}(\phi_0)}{\partial \phi} + \frac{1}{T}\frac{\partial \widehat{\mathscr{L}_T}(\overline{\phi})}{\partial \phi \partial \phi'}\sqrt{T}(\widehat{\phi} - \phi_0),$$

where $\overline{\phi}$ lies between $\widehat{\phi}$ and ϕ_0. Hence,

$$\sqrt{T}(\widehat{\phi}-\phi_0) = -\Big[\frac{1}{T}\frac{\partial^2\widehat{\mathscr{L}_T}(\overline{\phi})}{\partial\phi\partial\phi'}\Big]^{-1}\frac{1}{\sqrt{T}}\frac{\partial\widehat{\mathscr{L}_T}(\phi_0)}{\partial\phi}.$$

By letting $I_n - \widehat{\mathbf{u}}_t\widehat{G}_t^{-1} = I_n - \mathbf{u}_t\widetilde{G}_t^{-1} - \mathbf{u}_tS_t - \Delta_t\widehat{G}_t^{-1}$, $\frac{\partial\widehat{G}_t}{\partial\phi_i} = \big(\frac{\partial\widehat{G}_t}{\partial\phi_i} - \frac{\partial\widetilde{G}_t}{\partial\phi_i}\big) + \frac{\partial G_t}{\partial\phi_i}$ and $\widehat{G}_t^{-1} = \widetilde{G}_t^{-1} + S_t$, we can write

$$\frac{1}{\sqrt{T}}\frac{\partial\widehat{\mathscr{L}_T}(\phi_0)}{\partial\phi_i} = \sum_{j=1}^{12} R_j,$$

where

$$\begin{aligned}
R_1 =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n - \mathbf{u}_t\widetilde{G}_t^{-1})\frac{\partial\widetilde{G}_t}{\partial\phi_i}\widetilde{G}_t^{-1}\Big],\\
R_2 =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n - \mathbf{u}_t\widetilde{G}_t^{-1})\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i} - \frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)\widetilde{G}_t^{-1}\Big],\\
R_3 =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n - \mathbf{u}_t\widetilde{G}_t^{-1})\frac{\partial\widetilde{G}_t}{\partial\phi_i}S_t\Big],\\
R_4 =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n - \mathbf{u}_t\widetilde{G}_t^{-1})\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i} - \frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)S_t\Big],\\
R_5 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\mathbf{u}_tS_t\frac{\partial\widetilde{G}_t}{\partial\phi_i}\widetilde{G}_t^{-1}\Big],\\
R_6 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\mathbf{u}_tS_t\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i} - \frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)\widetilde{G}_t^{-1}\Big],\\
R_7 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\mathbf{u}_tS_t\frac{\partial\widetilde{G}_t}{\partial\phi_i}S_t\Big],\\
R_8 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\mathbf{u}_tS_t\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i} - \frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)S_t\Big],\\
R_9 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\Delta_t\widehat{G}_t^{-1}\frac{\partial\widetilde{G}_t}{\partial\phi_i}\widetilde{G}_t^{-1}\Big],
\end{aligned}$$

$$R_{10} = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\Delta_t\widehat{G}_t^{-1}\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i}-\frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)\widetilde{G}_t^{-1}\Big],$$
$$R_{11} = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\Delta_t\widehat{G}_t^{-1}\frac{\partial\widetilde{G}_t}{\partial\phi_i}S_t\Big],$$
$$R_{12} = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\Delta_t\widehat{G}_t^{-1}\Big(\frac{\partial\widehat{G}_t}{\partial\phi_i}-\frac{\partial\widetilde{G}_t}{\partial\phi_i}\Big)S_t\Big].$$

Here, Δ_t and S_t are defined in (B.4) and (B.7), respectively. Furthermore, by Propositions B.5–B.7 below and the central limit theory for mixing process (Hall and Heyde, 2014), we have that $-\frac{1}{\sqrt{T}}\frac{\partial\widehat{\mathscr{L}}_T(\phi_0)}{\partial\phi}\to_{\mathcal{L}} N(0,Q_{\phi_0})$, where

$$Q_{\phi_0} = \lim_{T\to\infty}\mathrm{Var}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_t\xi_t + \frac{F-E\eta_t}{\sqrt{T}}\sum_{t=1}^{T}[\Sigma_t^{1/4}\otimes\Sigma_t^{-1/4}]D_n z_t\Big).$$

In view of Proposition B.8 below, it remains to show

$$Q_{\phi_0} = N + \Psi + H + H'. \tag{B.19}$$

First, since $\rho_t\xi_t$ forms an m.d.s., we have

$$\lim_{T\to\infty}\mathrm{Var}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_t\xi_t\Big) = E[\rho_t\mathrm{Var}(\xi_t)\rho_t'] = N.$$

Second, by similar arguments as for Theorem 3.(ii), we can show

$$\lim_{T\to\infty}\mathrm{Var}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Upsilon(t/T)D_n z_t\Big) = \int_0^1\Upsilon(x)D_n\sum_{j=-\infty}^{\infty}Ez_t'z_{t-j}'D_n'\Upsilon(x)dx.$$

Since $D_n\sum_{j=-\infty}^{\infty}Ez_t'z_{t-j}'D_n' = \lim_{T\to\infty}\mathrm{Var}\big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}D_n z_t\big)$ and

$$\lim_{T\to\infty}\mathrm{Var}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}D_n z_t\Big) = \Xi_0 E\Big[(G_t^{1/2})^{\otimes 2}\mathrm{Var}(\xi_t)(G_t^{1/2})^{\otimes 2}\Big]\Xi_0'$$

by Lemma B.6, the expression of Ψ follows.

Third, by similar arguments we have

$$\begin{aligned}
H &\triangleq \lim_{T\to\infty} \mathrm{Cov}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_t\xi_t, \frac{F-E\eta_t}{\sqrt{T}}\sum_{t=1}^{T}[\Sigma_t^{1/4}\otimes\Sigma_t^{-1/4}]D_n z_t\Big)\\
&= \sum_{j=-\infty}^{\infty} E[\rho_t\xi_t z_{t-j}']D_n'\int_0^1 \Upsilon'(x)dx.
\end{aligned}$$

Using Lemma B.6, we obtain

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} E[\rho_t\xi_t z_{t-j}']D_n' &= \lim_{T\to\infty} \mathrm{Cov}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_t\xi_t, \frac{1}{\sqrt{T}}\sum_{t=1}^{T}D_n z_t\Big)\\
&= \lim_{T\to\infty} \mathrm{Cov}\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_t\xi_t, \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Xi_0(G_t^{1/2})^{\otimes 2}\xi_t\Big)\\
&=E\Big[\rho_t\mathrm{Var}(\xi_t)(G_t^{1/2})^{\otimes 2}\Big]\Xi_0'.
\end{aligned}$$

Hence, $H = E\Big[\rho_t\mathrm{Var}(\xi_t)(G_t^{1/2})^{\otimes 2}\Big]\Xi_0'\int_0^1\Upsilon'(x)dx$. Now, the result (B.19) follows. □

We note that Propositions B.5–B.8 are all proved under the conditions of Theorem 4(ii).

Proposition B.5. $R_j = o_p(1)$ *for* $j = 4, 6, 7, 8, 10, 11, 12$.

Proposition B.6. $R_j = o_p(1)$ *for* $j = 2, 3$.

Proposition B.7. *For each* i,

$$R_1 + R_5 + R_9 = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho_{t,i}\xi_t - \frac{F_i}{\sqrt{T}}\sum_{t=1}^{T}[\Sigma_t^{1/4}\otimes\Sigma_t^{-1/4}]D_n z_t$$

$$+\frac{E\eta_{t,i}}{\sqrt{T}}\sum_{t=1}^{T}[\Sigma_t^{1/4}\otimes\Sigma_t^{-1/4}]D_n z_t+o_p(1).$$

Proposition B.8. $\frac{1}{T}\frac{\partial^2\widehat{\mathscr{L}}_T(\bar{\phi})}{\partial\phi\partial\phi'}\to J_{\phi_0}$ *in probability.*

Proof of Proposition B.5. We only give the proof for R_{11}, since the proofs for other terms are similar. First, note $\|\Delta_t\|\leq O(\kappa_T)\|\mathbf{u}_t\|$ by (B.8), $\|\widehat{G}_t^{-1}\|\leq\lambda_{\min 0}^{-1}$ by (B.6), $\left\|\frac{\partial\widetilde{G}_t}{\partial\phi_i}\right\|\leq\sum_{m=0}^{t-1}\|O(\rho_B^m)+O(\rho_B^m)\mathbf{u}_{t-m}\|+O(\rho_B^t)\|\mathbf{u}_0\|$ as for (C.2), and $\|S_t\|\leq O(\kappa_T)\sum_{j=1}^{t-1}O(\rho_B^j)\|\mathbf{u}_{t-j}\|$ by (B.11). Then, it follows that

$$\begin{aligned}\|R_{11}\|\leq&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\|\Delta_t\|\|\widehat{G}_t^{-1}\|\left\|\frac{\partial G_t}{\partial\phi_i}\right\|\|S_t\|\\\leq&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}O(\kappa_T)\|\mathbf{u}_t\|\sqrt{n}\lambda_{\min}^{-1}\sum_{m=0}^{t}\|O(\rho_B^m)+O(\rho_B^m)\mathbf{u}_{t-m}\|\\&\times O(\kappa_T)\sum_{j=1}^{t-1}O(\rho_B^j)\|\mathbf{u}_{t-j}\|,\end{aligned}$$

which implies that $E\|R_{11}\|\leq O(\sqrt{T}\kappa_T^2)$ by Hölder's inequality and (B.10). Under Assumption 9, we have that $\sup_t(\widetilde{\Sigma}_t-\Sigma_t)=O(h^2)$, which entails $O(\kappa_T^2)=o(T^{-1/2})$ and hence $E\|R_{11}\|=o(1)$. Finally, the result follows by using Markov's inequality. □

Proof of Proposition B.6. We only give the proof for R_3, since the proof for R_2 is similar. By (B.7) and (B.16), it is not hard to see

$$R_3=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n-\mathbf{u}_tG_t^{-1})\frac{\partial G_t}{\partial\phi_i}S_t\Big]+o_p(1),$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[(I_n-\mathbf{u}_tG_t^{-1})\frac{\partial G_t}{\partial\phi_i}G_t^{-1}(\widehat{G}_t-\widetilde{G}_t)G_t^{-1}\Big]+o_p(1),\\
&= -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[G_t^{-1/2}(I_n-\mathbf{e}_t)G_t^{-1/2}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}(\widehat{G}_t-\widetilde{G}_t)\Big]+o_p(1).
\end{aligned}
$$

Let $\varphi_t = G_t^{-1/2}(I_n-\mathbf{e}_t)G_t^{-1/2}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}$. By using Lemma B.2(iv) and (B.9), we have

$$
\begin{aligned}
R_3 =& -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{j=1}^{t-1}\mathrm{tr}[B_0^{j-1}A_0\Delta_{t-j}A_0'(B_0')^{j-1}\varphi_t]+o_p(1)\\
=& -\frac{1}{\sqrt{T}}\sum_{j=1}^{T-1}\sum_{t=j+1}^{T}\mathrm{tr}[B_0^{j-1}A_0\Delta_{t-j}A_0'(B_0')^{j-1}\varphi_t]+o_p(1)\\
=& -\frac{1}{\sqrt{T}}\sum_{j=1}^{T-1}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0\otimes B_0^{j-1}A_0]\mathrm{vec}[\Delta_{t-j}]+o_p(1)\\
=& -\frac{1}{\sqrt{T}}\sum_{j=1}^{m_T}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0\otimes B_0^{j-1}A_0]\mathrm{vec}[\Delta_{t-j}]+R_{34}+o_p(1)\\
=& R_{31}+R_{32}+R_{33}+R_{34},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{31} =& -\frac{1}{2\sqrt{T}}\sum_{j=1}^{m_T}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0\otimes B_0^{j-1}A_0]\\
&\times\mathrm{vec}[\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-1/4}\mathbf{u}_{t-j}],\\
R_{32} =& -\frac{1}{2\sqrt{T}}\sum_{j=1}^{m_T}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0\otimes B_0^{j-1}A_0]\\
&\times\mathrm{vec}[\mathbf{u}_{t-j}\Sigma_{t-j}^{-1/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-3/4}],\\
R_{33} =& O(\kappa_T^2)\frac{1}{\sqrt{T}}\sum_{j=1}^{m_T}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0\otimes B_0^{j-1}A_0]\mathrm{vec}(\mathbf{y}_{t-j}),
\end{aligned}
$$

$$R_{34} = \frac{1}{\sqrt{T}} \sum_{j=m_t+1}^{T-1} \sum_{t=j+1}^{T} \mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{j-1}A_0]\mathrm{vec}[\Delta_{t-j}].$$

Here, m_T is defined in (B.17). The remaining is to show that (i) $R_{31} = o_p(1)$, (ii) $R_{32} = o_p(1)$, (iii) $R_{33} = o_p(1)$, and (iv) $R_{34} = o_p(1)$.

(i) Note that

$$\begin{aligned}
R_{31} =& -\frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{j-1}A_0] \\
&\times \mathrm{vec}[\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-1/4}\mathbf{u}_{t-j}] + o_p(1) \\
=& -\frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{j-1}A_0] \\
&\times [\mathbf{u}_{t-j}\Sigma_t^{-1/4} \otimes \Sigma_t^{-3/4}]\mathrm{vec}[(\widehat{\Sigma}_t - \Sigma_t)] + o_p(1),
\end{aligned}$$

where the first equality holds by Lemma B.3, and the second equality holds by the property of vec operator. Since $\mathrm{vec}(\widehat{\Sigma}_t - \Sigma_t) = \frac{1}{Th}\sum_{s=1}^{T} K\big(\frac{t-s}{Th}\big)(\Sigma_s^{1/2})^{\otimes 2}$ $\mathrm{vec}(\mathbf{u_s} - I_n) + O_p(\frac{\log T}{Th} + h^2)$ by Lemma B.1, it follows that

$$R_{31} = -\sum_{j=1}^{m_T} R_{31j} + o_p(1),$$

where

$$\begin{aligned}
R_{31j} =& -\frac{1}{2\sqrt{T}} \sum_{t=j+1}^{T} \mathrm{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{j-1}A_0] \\
&\times [\mathbf{u}_{t-j}\Sigma_t^{-1/4} \otimes \Sigma_t^{-3/4}]\frac{1}{Th}\sum_{s=1}^{T} K\Big(\frac{t-s}{Th}\Big)(\Sigma_s^{1/2})^{\otimes 2}\mathrm{vec}(\mathbf{u_s} - I_n).
\end{aligned}$$

Finally, we are going to show

$$\mathrm{Var}(R_{31j}) \le o(\rho_B^j), \tag{B.20}$$

where $o(1)$ holds uniformly in j. Note that $\|B_0^{j-1}A_0 \otimes B_0^{j-1}A_0\| = O(\rho_B^j)$, and $\sup_t \|\Sigma_t\| \leq c_u$ and $\sup_t \|\Sigma_t^{-1}\| \leq c_l^{-1}$ by Assumption 1. Then, (B.20) holds if

$$\mathrm{Var}\Big\{\frac{1}{\sqrt{T}}\sum_{t=j+1}^{T}\mathrm{vec}(\varphi_t)'[\mathbf{u}_{t-j}\otimes I_n]\frac{1}{Th}\sum_{s=1}^{T}K\big(\frac{t-s}{Th}\big)\mathrm{vec}(\mathbf{u_s}-I_n)\Big\} = o(1). \tag{B.21}$$

To prove (B.21), we apply Lemma B.5 with $a_t = \mathrm{vec}(\mathbf{u}_t - I_n)$, $b_t = \mathrm{vec}(\varphi_t)'[\mathbf{u}_{t-j} \otimes I_n]$ and $c_t = \{z_t, \mathrm{vec}(G_t), \frac{\partial \mathrm{vec}(G_t)}{\partial \phi'}\}$. It remains to verify Conditions (1)–(4) in Lemma B.5. First, by Hölder's inequality, (B.14) and the fact that $E\|\mathbf{u}_t\|^{6(1+2\delta)} < \infty$, we have

$$\begin{aligned}
&E\|b_t\|^{3(1+2\delta)}\\
\leq &CE\Big\|G_t^{-1/2}G_t^{-1/2}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}[\mathbf{u}_{t-j}\otimes I_n]\Big\|^{3(1+2\delta)}E\|I_n-\mathbf{e}_t\|^{3(1+2\delta)}\\
\leq &CE\Big\|\frac{\partial G_t}{\partial \phi_i}\Big\|^{6(1+2\delta)}E\Big\|\mathbf{u}_{t-j}-I_n\Big\|^{6(1+2\delta)}E\|I_n-\mathbf{e}_t\|^{3(1+2\delta)} < \infty,
\end{aligned}$$

and hence Condition (1) holds. Second, Lemma B.4 ensures Condition (2). Third, Conditions (3)–(4) hold by Assumption 3, 9 and (B.17). Therefore, (B.21) holds.

By the Cauchy-Schwarz inequality and (B.20),

$$\mathrm{Var}(\sum_{j=1}^{m_T}R_{31j}) \leq \sum_{j=1}^{m_T}[\mathrm{Var}^{1/2}(R_{31j})]^2 = o(1),$$

and hence Chebyshev's inequality implies that $R_{31} = o_p(1)$.

(ii) By the similar arguments as for R_{31}, we can show that $R_{32} = o_p(1)$.

(iii) Note that $E[\|\text{vec}(\varphi_t)\| \, \|\mathbf{y}_t\|] < \infty$ by Hölder's inequality. Since $\kappa_T^2 = o(T^{-1/2})$, by (B.10) we have

$$\begin{aligned} E|R_{33}| \leq & \frac{O(\kappa_T^2)}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \left\| B_0^{j-1} A_0 \otimes B_0^{j-1} A_0 \right\| E \left\| \text{vec}(\varphi_t) \right\| \left\| \mathbf{y}_t \right\| \\ \leq & \frac{O(\kappa_T^2)}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \rho_B^j E \left\| \text{vec}(\varphi_t) \right\| \left\| \mathbf{y}_t \right\| = o(1). \end{aligned}$$

Then, we obtain that $R_{33} = o_p(1)$ by Markov's inequality.

(iv) Note that $E[\|\text{vec}(\varphi_t)\| \sup_s \|\Delta_s\|] < \infty$ by Hölder's inequality. Then, we can show that $E|R_{34}| \leq O(\rho_B^{m_T}) O\big(\sqrt{\frac{\log T}{h}}\big) = o(1)$ as for (iii). Next, it follows that $R_{34} = o_p(1)$ by Markov's inequality. This completes the proof. □

Proof of Proposition B.7. First, we consider R_5. Write $R_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\big[\mathbf{u}_t \widetilde{G}_t^{-1} (\widehat{G}_t - \widetilde{G}_t) \widehat{G}_t^{-1} \frac{\partial \widetilde{G}_t}{\partial \phi_i} \widetilde{G}_t^{-1}\big]$ by (B.7), and then using similar arguments as for R_{11} in Proposition B.5, we can show

$$R_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\big[\mathbf{u}_t G_t^{-1} (\widehat{G}_t - \widetilde{G}_t) G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big] + o_p(1).$$

Further, by letting $\mathbf{u}_t G_t^{-1} = (\mathbf{u}_t G_t^{-1} - I_n) + I_n$, it follows that

$$R_5 = \widehat{R}_5 + o_p(1), \tag{B.22}$$

where $\widehat{R}_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\big[(\widehat{G}_t - \widetilde{G}_t) G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big]$, and we have used the fact that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\big[(\mathbf{u}_t G_t^{-1} - I_n)(\widehat{G}_t - \widetilde{G}_t) G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big] = o_p(1)$$

by Proposition B.6.

Since $\kappa_T^2 = o(T^{-1/2})$, by Lemma B.2(iv) and (B.9), we have

$$\widehat{R}_5 = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big[\sum_{j=1}^{t-1} B_0^{j-1}A_0\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-1/4}\mathbf{u}_{t-j}$$
$$\times A_0'(B_0')^{j-1}G_t^{-1}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}\Big] + o_p(1).$$

By interchanging the summations, we obtain

$$\widehat{R}_5 = -\frac{1}{\sqrt{T}}\sum_{j=1}^{T-1}\sum_{t=j+1}^{T}\mathrm{tr}\Big[B_0^{j-1}A_0\Sigma_{t-j}^{-3/4}(\widehat{\Sigma}_{t-j}-\Sigma_{t-j})\Sigma_{t-j}^{-1/4}\mathbf{u}_{t-j}$$
$$\times A_0'(B_0')^{j-1}G_t^{-1}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}\Big] + o_p(1). \tag{B.23}$$

Let m_T satisfy (B.17). By Lemma B.1, Lemma B.3 and similar arguments as for Proposition B.6, it is not hard to see

$$\widehat{R}_5 = -\frac{1}{\sqrt{T}}\sum_{j=1}^{m_T} R_{5j} + o_p(1),$$

where

$$\begin{aligned}R_{5j} =& \frac{1}{\sqrt{T}}\sum_{t=j+1}^{T}\mathrm{tr}\Big[B_0^{j-1}A_0\Sigma_t^{-3/4}(\widehat{\Sigma}_t-\Sigma_t)\Sigma_t^{-1/4}\mathbf{u}_{t-j}A_0'(B_0')^{j-1}G_t^{-1}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}\Big]\\ =& \frac{1}{\sqrt{T}Th}\sum_{t=j+1}^{T}\mathrm{vec}(B_0^{j-1}A_0)'\Big[\mathbf{u}_{t-j}\otimes G_t^{-1}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}\Big]\\ &\times[\Sigma_t^{1/4}\otimes B_0^{j-1}A_0\Sigma_t^{-1/4}]\sum_{s=1}^{T}K(\frac{t-s}{Th})D_n z_s + o_p(1).\end{aligned}$$

Decompose $R_{5j} = R_{5j1} + R_{5j2}$, where

$$R_{5j1} = \frac{1}{\sqrt{T}Th}\sum_{t=j+1}^{T}\mathrm{vec}(B_0^{j-1}A_0)'\Big(\Big[\mathbf{u}_{t-j}\otimes G_t^{-1}\frac{\partial G_t}{\partial \phi_i}G_t^{-1}\Big] - M_j^i\Big)$$

$$\times [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] \sum_{s=1}^{T} K(\frac{t-s}{Th}) D_n z_s,$$

$$R_{5j2} = \frac{1}{\sqrt{T}Th} \sum_{t=j+1}^{T} \text{vec}(B_0^{j-1} A_0)' M_j^i [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] \sum_{s=1}^{T} K(\frac{t-s}{Th}) D_n z_s,$$

and let $R_{51} = \sum_{j=1}^{m_T} R_{5j1}$ and $R_{52} = \sum_{j=1}^{m_T} R_{5j2}$. Then, by similar arguments as for Lemma B.5, we have that $R_{51} = o_p(1)$, and hence $R_5 = -R_{52} + o_p(1)$, where

$$\begin{aligned}
R_{52} =& \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{m_T} \frac{1}{Th} \sum_{t=j+1}^{T} K(\frac{t-s}{Th}) \text{vec}(B_0^{j-1} A_0)' M_j^i [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] D_n z_s \\
=& \frac{1}{\sqrt{T}} \sum_{s=1}^{T} D_n z_s \sum_{j=1}^{m_T} \frac{1}{Th} \sum_{t=j+1}^{T} K(\frac{t-s}{Th}) \text{vec}(B_0^{j-1} A_0)' M_j^i \\
& \times [\Sigma_s^{1/4} \otimes B_0^{j-1} A_0 \Sigma_s^{-1/4}] D_n z_s + o_p(1)
\end{aligned}$$

by the continuity of $\Sigma(x)$. Since $\frac{1}{Th} \sum_{t=j+1}^{T} K(\frac{t-s}{Th}) = 1 + O(\frac{m_T}{Th})$ for any fixed s, and $\frac{m_T}{\sqrt{T h^2}} \to 0$, we can show

$$R_{52} = \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{m_T} \text{vec}(B_0^{j-1} A_0)' M_j^i [\Sigma_s^{1/4} \otimes B_0^{j-1} A_0 \Sigma_s^{-1/4}] D_n z_s + o_p(1). \tag{B.24}$$

Moreover, since $\sqrt{T} \rho_B^{m_T} = o(1)$, it follows that

$$\begin{aligned}
R_5 =& -\frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{\infty} \text{vec}(B_0^{j-1} A_0)' M_j^i [I_n \otimes B_0^{j-1} A_0] [\Sigma_s^{1/4} \otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1) \\
=& -\frac{F_i}{\sqrt{T}} \sum_{s=1}^{T} [\Sigma_s^{1/4} \otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1). \qquad \text{(B.25)}
\end{aligned}$$

Next, we consider R_9. Write $R_9 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\big(\Delta_t G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big) -$

$\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\big(\Delta_t S_t \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big)$ by (B.7). Since $\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\big(\Delta_t S_t \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big) = o_p(1)$ by using similar arguments as for R_{11} in Proposition B.5, it follows that $R_9 = -\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\big(\Delta_t G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big) + o_p(1)$. Then, by the similar arguments as for (B.23), $R_9 = \frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\big[\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-1/4} G_t^{1/2}(\mathbf{e}_t - I_n) G_t^{-1/2} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big] + \widehat{R}_9 + o_p(1)$, where

$$\widehat{R}_9 = \frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\Big[\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-1/4} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\Big].$$

Moreover, since $\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathrm{tr}\big[\Sigma_t^{-3/4}(\widehat{\Sigma}_t - \Sigma_t)\Sigma_t^{-1/4} G_t^{1/2}(\mathbf{e}_t - I_n) G_t^{-1/2} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}\big] = o_p(1)$ by similar arguments as for Proposition B.6, it entails that $R_9 = \widehat{R}_9 + o_p(1)$.

Recall $\eta_{t,i} = \mathrm{vec}\big(G_t^{-1}\frac{\partial G_t}{\partial \phi_i}\big)'$. By Lemma B.1, we can show

$$\widehat{R}_9 = \frac{1}{\sqrt{T}}\sum_{s=1}^T \frac{1}{T}\sum_{t=1}^T K_h\Big(\frac{t-s}{T}\Big)\eta_{t,i}[\Sigma_t^{1/4}\otimes \Sigma_t^{-1/4}] D_n z_s + o_p(1),$$

using the property of trace operator. Since $\mathrm{Var}\big(\frac{1}{Th}\sum_{t=1,|t-s|\le Th}^T K\big(\frac{t-s}{Th}\big)\eta_{t,i}\big) = O\big(\frac{1}{Th}\big)$ by Davydov's inequality and $\int K(x)dx = 1$, it follows that

$$R_9 = \frac{E\eta_{t,i}}{\sqrt{T}}\sum_{s=1}^T [\Sigma_s^{1/4}\otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1), \tag{B.26}$$

by using similar arguments as for (B.24).

Finally, by (B.16), it is straightforward to see that $R_1 = -\frac{1}{\sqrt{T}}\sum_{t=1}^T \rho_{t,i}\xi_t + o_p(1)$, and then the conclusion follows by (B.25) and (B.26). □

Proof of Proposition B.8. By Theorem 3.1 in Ling and McAleer (2003), the conclusion holds by the following two arguments:

(i) $\sup_{\phi\in\Phi}|\frac{1}{T}\frac{\partial^2\widehat{\mathscr{L}}_T(\phi)}{\partial\phi_i\partial\phi_j}-\frac{1}{T}\frac{\partial^2\mathscr{L}_T(\phi)}{\partial\phi_i\partial\phi_j}|=o_p(1)$;

(ii) $E\sup_{\phi\in\Phi}\left|\frac{\partial^2\ell_t(\phi)}{\partial\phi_i\phi_j}\right|<\infty$.

For (i), we first note that

$$\begin{aligned}\frac{\partial^2\widehat{\ell}_t(\phi)}{\partial\phi_i\partial\phi_j}=&\mathrm{tr}\Big[\frac{\partial^2\widehat{G}_t}{\partial\phi_i\partial\phi_j}\widehat{G}_t^{-1}-\widehat{\mathbf{u}}_t\widehat{G}_t^{-1}\frac{\partial^2\widehat{G}_t}{\partial\phi_i\partial\phi_j}\widehat{G}_t^{-1}-\frac{\partial\widehat{G}_t}{\partial\phi_i}\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_j}\widehat{G}_t^{-1}\\&+\widehat{\mathbf{u}}_t\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_i}\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_j}\widehat{G}_t^{-1}+\widehat{\mathbf{u}}_t\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_j}\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_i}\widehat{G}_t^{-1}\Big].\end{aligned}$$

By using the similar arguments as for Propositions B.2–B.4, we can show

(a) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}\left|\frac{\partial^2\widehat{G}_t}{\partial\phi_i\partial\phi_j}\widehat{G}_t^{-1}-\frac{\partial^2 G_t}{\partial\phi_i\partial\phi_j}G_t^{-1}\right|=o_p(1)$;

(b) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}\left|\widehat{\mathbf{u}}_t\widehat{G}_t^{-1}\frac{\partial^2\widehat{G}_t}{\partial\phi_i\partial\phi_j}\widehat{G}_t^{-1}-\mathbf{u}_tG_t^{-1}\frac{\partial^2 G_t}{\partial\phi_i\partial\phi_j}G_t^{-1}\right|=o_p(1)$;

(c) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}\left|\frac{\partial\widehat{G}_t}{\partial\phi_i}\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_j}\widehat{G}_t^{-1}-\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}\right|=o_p(1)$;

(d) $\frac{1}{T}\sum_{t=1}^T\sup_{\phi\in\Phi}\left|\widehat{\mathbf{u}}_t\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_i}\widehat{G}_t^{-1}\frac{\partial\widehat{G}_t}{\partial\phi_j}\widehat{G}_t^{-1}-\mathbf{u}_tG_t^{-1}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}\right|=o_p(1)$.

Hence, it follows that (i) holds.

For (ii), it suffices to prove that $E\sup_{\phi\in\Phi}\left|\frac{\partial^2\ell_t(\phi)}{\partial\phi_i\partial\phi_j}\right|<\infty$. Note that

$$\begin{aligned}\frac{\partial^2\ell_t(\phi)}{\partial\phi_i\partial\phi_j}=&\mathrm{tr}\Big[\frac{\partial^2 G_t}{\partial\phi_i\partial\phi_j}G_t^{-1}-\mathbf{u}_tG_t^{-1}\frac{\partial^2 G_t}{\partial\phi_i\partial\phi_j}G_t^{-1}-\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}\\&+\mathbf{u}_tG_t^{-1}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}+\mathbf{u}_tG_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\Big].\end{aligned}\tag{B.27}$$

To facilitate our proofs, we first claim that

$$E\sup_{\phi\in\Phi}\left\|\frac{\partial^2 G_t}{\partial\phi_i\partial\phi_j}\right\|^3<\infty\quad\text{and}\quad E\sup_{\phi\in\Phi}\left\|\frac{\partial G_t}{\partial\phi_i}\right\|^3<\infty,\tag{B.28}$$

where the preceding results hold by Minkowski's inequality, (B.10), (C.2)–(C.3), the fact that $E\|\mathbf{u}_t\|^3<\infty$, and some standard arguments.

Next, for the first term in (B.27), we have

$$E\sup_{\phi\in\Phi}\Big|\mathrm{tr}\Big[\mathbf{u}_t G_t^{-1}\frac{\partial G_t}{\partial\phi_j}G_t^{-1}\frac{\partial G_t}{\partial\phi_i}G_t^{-1}\Big]\Big| \le C\Big[E\sup_{\phi\in\Phi}\Big\|\mathbf{u}_t\Big\|\Big\|\frac{\partial G_t}{\partial\phi_j}\Big\|\Big\|\frac{\partial G_t}{\partial\phi_i}\Big\|\Big]$$
$$\le C\Big[E\sup_{\phi\in\Phi}\|\mathbf{u}_t\|^3\Big]^{1/3}\Big[E\sup_{\phi\in\Phi}\Big\|\frac{\partial G_t}{\partial\phi_j}\Big\|^3\Big]^{1/3}\Big[E\sup_{\phi\in\Phi}\Big\|\frac{\partial G_t}{\partial\phi_i}\Big\|^3\Big]^{1/3} < \infty,$$

where the first inequality holds by (B.6), Assumption 5 and the property that $\mathrm{tr}(AB) \le \|A\|\,\|B\|$, and the second inequality holds by Hölder's inequality and (B.28). Similarly, we can show the proofs for other terms in (B.27), and consequently, the result follows. $\square$

Proof of Theorem 5(i). Denote $l_t(\gamma) = \nu\ell_t(\phi) + c(y_t,\nu)$ and $\widehat{l_t}(\gamma) = \nu\widehat{\ell_t}(\phi) + c(y_t,\nu)$. Then, some straightforward calculations give

$$\frac{\partial l_t(\gamma)}{\partial\nu} = \mathrm{tr}(\Omega_t^{-1}(\phi)\mathbf{y}_t) + \log\det\Omega_t(\phi) - \log\det\mathbf{y}_t + n\log(2)$$
$$+\sum_{i=1}^{n}\psi\Big(\frac{\nu+1-i}{2}\Big) - n\log(\nu) - n.$$

From the proof of Theorem 4, we have shown that $\frac{1}{T}\sum_{t=1}^{T}[\mathrm{tr}(\widehat{\mathbf{e}}_t) - \mathrm{tr}(\mathbf{e}_t)] = o_p(1)$, and similarly, $\frac{1}{T}\sum_{t=1}^{T}[\log\det(\widehat{\mathbf{e}}_t) - \log\det(\mathbf{e}_t)] = o_p(1)$. Therefore, by Theorem 4.1.1 in Amemiya (1985) and the strong law of large numbers, it suffices to show that ν_0 is the unique solution to

$$E[\mathrm{tr}(\mathbf{e}_t)] - E\log\det(\mathbf{e}_t) + n\log(2) + \sum_{i=1}^{n}\psi\Big(\frac{\nu+1-i}{2}\Big) - n - n\log(\nu) = 0.$$

Note that by the property of Wishart distribution, we have $E[\mathrm{tr}(\mathbf{e}_t)] = n$ and $E\log\det(\mathbf{e}_t) = n\log(2) + \sum_{i=1}^{n}\psi\Big(\frac{\nu_0+1-i}{2}\Big) - n\log(\nu_0)$. So, clearly ν_0 is the solution of the above equation. Therefore, it suffices to show

$$f(\nu) \triangleq \sum_{i=1}^{n} \psi\Big(\frac{\nu+1-i}{2}\Big) - n\log(\nu)$$

is monotonic in $\nu > n$.

By Alzer and Batir (2007), we have that for $x > 0$,

$$\psi(x) - \log(x) + \frac{1}{2}\psi'(x) > 0 \text{ and } \log(x) - \frac{1}{2x} - \psi(x) > 0.$$

Therefore, when $\nu > n$,

$$\begin{aligned}
f'(\nu) &= \sum_{i=1}^{n} \Big[\frac{1}{2}\psi'\Big(\frac{\nu+1-i}{2}\Big) - \frac{1}{\nu}\Big] \\
&> \sum_{i=1}^{n} \Big[\log\Big(\frac{\nu+1-i}{2}\Big) - \psi\Big(\frac{\nu+1-i}{2}\Big) - \frac{1}{\nu}\Big] \\
&> \sum_{i=1}^{n} \Big[\frac{1}{\nu+1-i} - \frac{1}{\nu}\Big] \geq 0,
\end{aligned}$$

which implies the monotonicity of $f(\nu)$.

(ii) Note that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial \widehat{l}_t(\gamma_0)}{\partial \nu} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\gamma_0)}{\partial \nu} - P_1 + P_2, \tag{B.29}$$

where $P_1 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\log\det(\Omega_t\widehat{\Omega}_t^{-1})$ and $P_2 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\big((\widehat{G}_t^{-1} - G_t^{-1})\mathbf{u}_t + \widehat{G}_t^{-1}\Delta_t\big)$.

For P_1, we write it as $P_1 = P_{11} + P_{12}$, where $P_{11} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\log\det(\Sigma_t\widehat{\Sigma}_t^{-1})$ and $P_{12} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\log\det(G_t\widehat{G}_t^{-1})$. On one hand, we can show

$$P_{11} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\Big(\log[I_n + (\Sigma_t - \widehat{\Sigma}_t)\widehat{\Sigma}_t^{-1}]\Big)$$

$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}\big((\Sigma_t-\widehat{\Sigma}_t)\widehat{\Sigma}_t^{-1}+O_p(\kappa_T^2)\big)$$

$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}\big((\Sigma_t-\widehat{\Sigma}_t)\widehat{\Sigma}_t^{-1}\big)+o_p(1), \tag{B.30}$$

where the first equality holds by the identity $\log\det(I_n+\epsilon)=\operatorname{tr}\log(I_n+\epsilon)$, the second equality holds by Taylor's expansion that $\log(I_n+\epsilon)=\epsilon+O(\epsilon^2)$ and Lemma B.1, and the third equality holds since $\kappa_T^2=o(T^{-1/2})$. Further, by similar arguments as for Lemma B.2(iii), we have

$$\widehat{\Sigma}(x)^{-1}=\Sigma(x)^{-1}-\Sigma(x)^{-1}(\widehat{\Sigma}(x)-\Sigma(x))\Sigma(x)^{-1}+O(\kappa_T^2)$$

holds uniformly for all x, and then by (B.30), it is not hard to see

$$P_{11}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}\big((\Sigma_t-\widehat{\Sigma}_t)\Sigma_t^{-1}\big)+o_p(1). \tag{B.31}$$

On the other hand, by similar arguments as for (B.30), $P_{12}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}\big(G_t-\widehat{G}_t)\widehat{G}_t^{-1}\big)+o_p(1)$, and then

$$P_{12}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}\big((G_t-\widehat{G}_t)G_t^{-1}\big)+o_p(1) \tag{B.32}$$

by similar arguments as for Proposition B.5.

For P_2, we write it as $P_2=P_{21}+P_{22}+P_{23}$, where $P_{21}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}(S_t\mathbf{u}_t)$, $P_{22}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}(G_t^{-1}\Delta_t)$, and $P_{23}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}(S_t\Delta_t)$. Here, Δ_t and S_t are defined as in (B.4) and (B.7), respectively. By similar arguments as for Proposition B.6, we can show that $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\operatorname{tr}[G_t(\widehat{G}_t-G_t)(\mathbf{e}_t-I_n)]=o_p(1)$, and hence by (B.32), it follows that

$$P_{21} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\big(G_t^{-1}(G_t - \widehat{G}_t)\big) - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\big(G_t(\widehat{G}_t - G_t)(\mathbf{e}_t - I_n)\big)$$
$$= P_{12} + o_p(1).$$

Further, by Lemma B.2(iv), similar arguments as for P_{21}, and (B.31), it is not hard to see

$$\begin{aligned} P_{22} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\big(G_t^{-1/2}\Sigma_t^{-3/4}(\Sigma_t - \widehat{\Sigma}_t)\Sigma_t^{-1/4}G_t^{1/2}\mathbf{e}_t\big) + o_p(1) \\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathrm{tr}\big(G_t^{-1/2}\Sigma_t^{-3/4}(\Sigma_t - \widehat{\Sigma}_t)\Sigma_t^{-1/4}G_t^{1/2}\big) + o_p(1) \\ &= P_{11} + o_p(1). \end{aligned}$$

Note that $P_{23} = o_p(1)$ by using similar arguments as for Proposition B.5. Therefore, it follows that $P_2 = P_1 + o_p(1)$, and then by (B.29),

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial \widehat{l}_t(\gamma_0)}{\partial \nu} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\gamma_0)}{\partial \nu} + o_p(1). \tag{B.33}$$

By similar arguments as for Proposition B.8,

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 \widehat{l}_t(\widehat{\gamma})}{\partial \nu^2} = E\Big[\frac{\partial^2 l_t(\gamma_0)}{\partial \nu^2}\Big] + o_p(1) = J_{\nu_0} + o_p(1), \tag{B.34}$$

where we have used the fact that $E\big[\frac{\partial l_t(\gamma_0)}{\partial \nu}\big]^2 = 2J_{\nu_0}$ and $E\big[\frac{\partial^2 l_t(\gamma_0)}{\partial \nu^2}\big] = \frac{1}{2}E\big[\frac{\partial l_t(\gamma_0)}{\partial \nu}\big]^2$ by standard arguments for Fisher Information. Finally, by (B.33)–(B.34), the conclusion follows. $\square$

C Derivatives

Let J_{ij} be an $n \times n$ matrix zeros everywhere except for a one at the (i,j)th entry. Then,

$$\begin{aligned}
\frac{\partial G_t}{\partial A_{ij}} =& J_{ij}(\mathbf{u}_{t-1} - I_n)A' + A(\mathbf{u}_{t-1} - I_n)J_{ji} + B\frac{\partial G_{t-1}}{\partial A_{ij}}B', \\
\frac{\partial G_t}{\partial B_{ij}} =& J_{ij}(G_{t-1} - I_n)B' + B(G_{t-1} - I_n)J_{ji} + B\frac{\partial G_{t-1}}{\partial B_{ij}}B'.
\end{aligned} \tag{C.1}$$

Therefore, the first order derivative of $G_t(\phi)$ w.r.t ϕ is given by

$$\begin{aligned}
\frac{\partial G_t}{\partial A_{ij}} &= \sum_{m=0}^{\infty} B^m\{J_{ij}(\mathbf{u}_{t-m-1} - I_n)A' + A(\mathbf{u}_{t-m-1} - I_n)J_{ji}\}(B')^m, \\
\frac{\partial G_t}{\partial B_{ij}} &= \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}}(I_n - AA' - BB' + A\mathbf{u}_{t-m-1}A')(B')^m \\
&\quad + B^m(I_n - AA' - BB' + A\mathbf{u}_{t-m-1}A')\frac{\partial (B')^m}{\partial B_{ij}} \\
&\quad - B^m J_{ij}(B')^{m+1} - B^{m+1}J_{ji}(B')^m,
\end{aligned} \tag{C.2}$$

where $\frac{\partial B^m}{\partial B_{ij}} = \sum_{n=0}^{m-1} B^n J_{ij} B^{m-1-n}$. The second order derivative of $G_t(\phi)$ is given by

$$\begin{aligned}
\frac{\partial^2 G_t}{\partial A_{ij}\partial A_{kl}} =& N_1 + N_1'; \\
\frac{\partial^2 G_t}{\partial A_{ij}\partial B_{kl}} =& N_2 + N_2'; \\
\frac{\partial^2 G_t}{\partial B_{ij}\partial B_{kl}} =& \sum_{q=3}^{8}(N_q + N_q'),
\end{aligned} \tag{C.3}$$

where

$$N_1 = \sum_{m=0}^{\infty} B^m[J_{ij}(\mathbf{u}_{t-m-1} - I_n)J_{lk}](B')^m,$$

$$N_2 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{kl}} [J_{ij}(\mathbf{u}_{t-m-1} - I_n)A' + A(\mathbf{u}_{t-m-1} - I_n)J_{ji}](B')^m,$$

$$N_3 = \sum_{m=0}^{\infty} \frac{\partial^2 B^m}{\partial B_{ij} \partial B_{kl}} (I_n - AA' - BB' + A\mathbf{u}_{t-m-1}A')(B')^m,$$

$$N_4 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}} (I_n - AA' - BB' + A\mathbf{u}_{t-m-1}A') \frac{\partial (B')^m}{\partial B_{kl}},$$

$$N_5 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}} (-J_{kl}B' - BJ_{lk})(B')^m,$$

$$N_6 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{kl}} (-J_{ij}B' - BJ_{ji})(B')^m,$$

$$N_7 = \sum_{m=0}^{\infty} -\frac{\partial B^m}{\partial B_{kl}} J_{ij}(B')^{m+1},$$

$$N_8 = \sum_{m=0}^{\infty} -B^m J_{ij} \frac{\partial (B')^{m+1}}{\partial B_{kl}}.$$

D Some numerical evidences

In this appendix, we generate one data sample from model (2.1) with sample size $T = 5000$, where $\mathbf{u}_t$ follows model (3.1) with

$$A_0 = \begin{pmatrix} 0.5 & 0.4 \\ 0 & 0.2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0.6 & 0 \\ 0.2 & 0.3 \end{pmatrix}, \quad \text{and} \quad \Sigma(x) = \begin{pmatrix} 1 + 1.5x^2 & 1.1x^2 \\ 1.1x^2 & 1 + 1.5x^3 \end{pmatrix},$$

and $\mathbf{e}_t$ is a sequence of independent and identically distributed (i.i.d.) Wishart random matrices from $\nu_0^{-1}\text{Wishart}(\nu_0, I_2)$ with $\nu_0 = 10$. To make a comparison, we also generate another data sample from model (2.1) under the same settings except $\Sigma(x) = I_2$. Fig D.1 plots the $\rho_{rs}(j)$ for each generated data sample, where $\rho_{rs}(j)$ is the autocorrelation function of $\mathbf{y}_{rs,t}$ at lag j, and $\mathbf{y}_{rs,t}$ is the (r, s)th element of $\mathbf{y}_t$. From this figure, we find that when Σ_t is time variant (or invariant), $\rho_{rs}(j)$ decays slowly (or fast) with respect to j, exhibiting long memory (or short memory) patterns. This implies that the data sample of $\mathbf{y}_t$ may exhibit a spurious long memory phenomenon, resulting from the structural change.

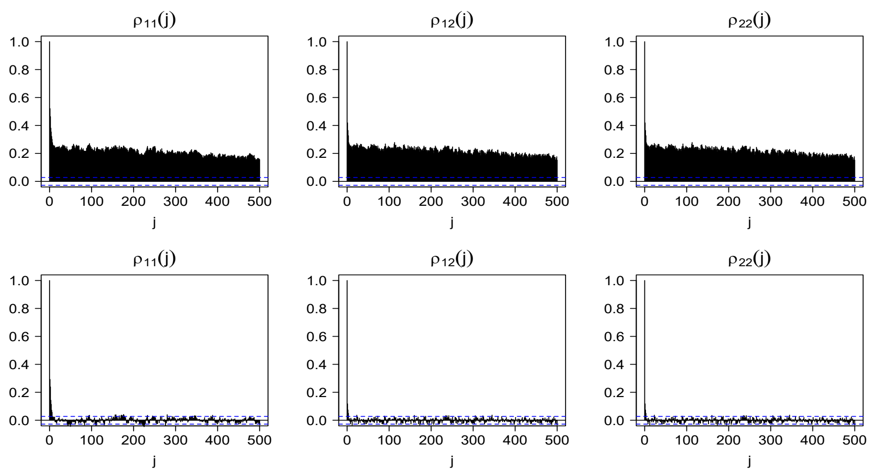

Figure D.1: Top panels: the plot of $\rho_{rs}(j)$ when Σ_t is time variant. Bottom panels: the plot of $\rho_{rs}(j)$ when Σ_t is time invariant.

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