CONDITIONAL MARGINAL TEST IN HIGH DIMENSIONAL QUANTILE REGRESSION

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Supplementary Material

The online Supplementary Material includes additional numerical results, a discussion of condition A5, and proofs of the main results.

S1 Additional numerical results

S1.1 Computing time and empirical size of Case 2

The simulation is done in the cluster with the configuration of each node similar to MacBook Pro 2.3 GHz Intel Core i5, 8 GB 2133 MHz LPDDR3. Table S.1 summarizes the average computing time of different methods for analyzing one data in Case 1 at $\tau = 0.5$ or the mean, where $T_{n,k}(\tau)$ is the sum of computing

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time for $T^E_{n,k}(\tau)$ and $T^B_{n,k}(\tau)$, for $k = 1, 2$; the average computing times are similar in Cases 2–3 and, thus, are omitted. Results show that the methods that do not require the estimation of $\mathbf{f}_\tau$, namely, RS, $T_{n,2}(\tau)$, and GC, are more efficient than that do, namely, $T_{n,1}(\tau)$, BON, and CCT. In addition, the resampling-bootstrap-based methods, QME and CAR, are computationally much more expensive than the other methods, even if double bootstrap is not used for tuning parameter selection.

Table S.2 summarizes the empirical sizes from Case 2, which is similar to Case 1.

Table S.1: The average computing time (seconds) of different methods for analyzing one data in Case 1.

	n=200				n=800			
method \ p_n	10	50	200	1000	10	50	200	1000
$T_{n,1}(\tau)$	2.10	9.05	34.63	180.01	3.68	16.50	64.72	354.41
$T_{n,2}(\tau)$	0.16	0.64	2.31	11.59	0.37	1.78	7.09	37.82
RS	0.00	0.00	0.00	0.00	0.01	0.01	0.04	0.00
QME	2.26	4.45	19.64	284.27	5.66	12.62	69.91	1128.14
BON	1.93	8.36	32.40	167.52	3.31	15.23	61.56	309.93
CCT	1.93	8.36	32.40	167.52	3.31	15.23	61.56	309.93
CAR	1.70	3.19	18.66	354.39	2.39	8.66	66.84	1582.43
GC	0.79	1.02	1.60	4.42	12.54	16.68	26.29	86.60

$T^E_{n,k}(\tau)$ and $T^B_{n,k}(\tau)$, $k = 1, 2$: four variations of the proposed test; RS: the rank score test of Park and He (2017); QME: the quantile marginal effect test of Wang et al. (2018); BON, Bonferroni adjustment on d_n individual P-values; CCT, Cauchy combination test of Liu and Xie (2019); CAR: the conditional adaptive resampling test of Tang et al. (2018); GC: the sum-squared-type test of Guo and Chen (2016).

Table S.2: Rejection percentages for Case 2 with $\mathbf{b}_0 = \mathbf{0}$. All scenarios correspond to the null model.

Case	location	method \ p_n	n=200: 10	50	200	1000	n=800: 10	50	200	1000
2	$\tau = 0.25$	$T^E_{n,1}(\tau)$	2.3	3.1	3.6	3.0	3.0	4.5	2.5	3.0
		$T^B_{n,1}(\tau)$	4.8	5.0	5.5	4.2	5.5	5.6	4.2	3.5
		$T^E_{n,2}(\tau)$	2.6	3.5	4.0	2.8	3.1	4.8	2.9	3.3
		$T^B_{n,2}(\tau)$	4.7	6.0	5.5	3.9	5.8	6.1	4.1	4.0
		RS	4.6	2.8	/	/	5.0	4.8	2.1	/
		QME	1.7	2.7	7.3	16.0	2.0	3.0	2.4	5.4
		BON	4.1	3.3	4.0	3.6	4.9	5.2	2.9	4.0
		CCT	2.2	1.9	2.0	1.6	2.6	2.8	1.5	1.4
	$\tau = 0.5$	$T^E_{n,1}(\tau)$	2.3	3.0	4.4	3.2	3.3	4.6	4.4	3.5
		$T^B_{n,1}(\tau)$	4.6	5.2	6.0	6.1	6.7	6.7	6.0	4.9
		$T^E_{n,2}(\tau)$	2.4	2.8	4.2	3.9	3.5	5.0	4.5	3.7
		$T^B_{n,2}(\tau)$	4.2	5.0	6.3	6.5	7.1	7.0	6.2	5.3
		RS	4.4	1.9	/	/	5.6	6.5	2.2	/
		QME	1.3	1.4	1.7	1.7	2.2	2.4	1.7	3.0
		BON	3.7	4.0	5.1	4.0	5.7	6.0	5.3	4.3
		CCT	2.5	2.3	2.4	2.1	3.1	3.3	3.1	2.0
	mean	CAR	3.9	4.3	3.6	2.5	4.3	3.7	3.5	3.0
		GC	3.7	4.9	5.4	5.8	4.8	5.0	4.6	5.9

$T^E_{n,k}(\tau)$ and $T^B_{n,k}(\tau)$, $k = 1, 2$: four variations of the proposed test; RS: the rank score test of Park and He (2017); QME: the quantile marginal effect test of Wang et al. (2018); BON, Bonferroni adjustment on d_n individual P-values; CCT, Cauchy combination test of Liu and Xie (2019); CAR: the conditional adaptive resampling test of Tang et al. (2018); GC: the sum-squared-type test of Guo and Chen (2016).

S1.2 Additional Case 4

We consider a Case 4 to mimic the motivating GFR study and generate $\mathbf{X}_{i\cdot}$ as multivariate Bernoulli variables that are correlated with $\mathbf{Z}_{i\cdot}$. Specifically, we generate $\mathbf{U}_{i\cdot}$ and $\mathbf{Z}_{i\cdot}$ as in Case 3, and let $X_{i,l-5} = 1 - 2I(U_{i,l} \leq 0)$ for $l = 6, \ldots, p_n - 1$. In addition, we let ε_i be standard exponential with median centered at zero. Table S.3 and Figure S.1 present the rejection rates under the null and the power curves of different methods in Case 4.

Table S.3: Rejection percentages for Case 4 with $\mathbf{b}_0 = \mathbf{0}$. All scenarios correspond to the null model.

			n=200				n=800			
Case	location	method \ p_n	10	50	200	1000	10	50	200	1000
4	$\tau = 0.25$	$T^E_{n,1}(\tau)$	3.2	4.1	3.8	4.0	2.5	3.6	4.1	6.0
		$T^B_{n,1}(\tau)$	5.7	5.7	5.3	5.1	4.7	4.8	5.8	6.5
		$T^E_{n,2}(\tau)$	3.1	4.8	4.1	4.7	2.9	3.6	4.4	6.5
		$T^B_{n,2}(\tau)$	5.4	6.2	6.0	5.2	5.1	4.9	5.6	7.3
		RS	6.1	2.5	/	/	5.4	4.1	2.2	/
		QME	2.5	1.8	2.5	7.1	4.0	2.6	3.8	7.7
		BON	5.4	4.2	4.7	3.2	4.3	4.7	5.5	6.1
		CCT	2.9	2.6	2.1	1.8	2.3	2.6	3.4	3.2
	$\tau = 0.5$	$T^E_{n,1}(\tau)$	2.7	4.5	3.7	3.4	3.0	4.7	4.0	4.1
		$T^B_{n,1}(\tau)$	4.7	6.5	5.3	4.8	4.9	6.2	5.0	5.0
		$T^E_{n,2}(\tau)$	2.9	4.7	4.0	3.8	3.1	4.8	3.9	4.2
		$T^B_{n,2}(\tau)$	5.4	6.5	5.7	5.6	5.1	6.2	4.8	5.1
		RS	5.0	2.8	/	/	5.2	3.2	2.4	/
		QME	2.2	1.0	0.3	1.8	3.1	1.6	2.0	1.8
		BON	4.5	5.6	4.3	3.7	4.6	5.6	4.5	4.6
		CCT	2.4	2.9	1.8	2.1	2.9	3.0	1.5	2.4
	mean	CAR	5.0	3.8	5.2	5.1	6.2	6.4	6.2	5.8
		GC	5.5	7.0	5.7	6.1	6.2	5.1	5.7	4.9

$T^E_{n,k}(\tau)$ and $T^B_{n,k}(\tau)$, $k = 1, 2$: four variations of the proposed test; RS: the rank score test of Park and He (2017); QME: the quantile marginal effect test of Wang et al. (2018); BON, Bonferroni adjustment on d_n individual P-values; CCT, Cauchy combination test of Liu and Xie (2019); CAR: the conditional adaptive resampling test of Tang et al. (2018); GC: the sum-squared-type test of Guo and Chen (2016).

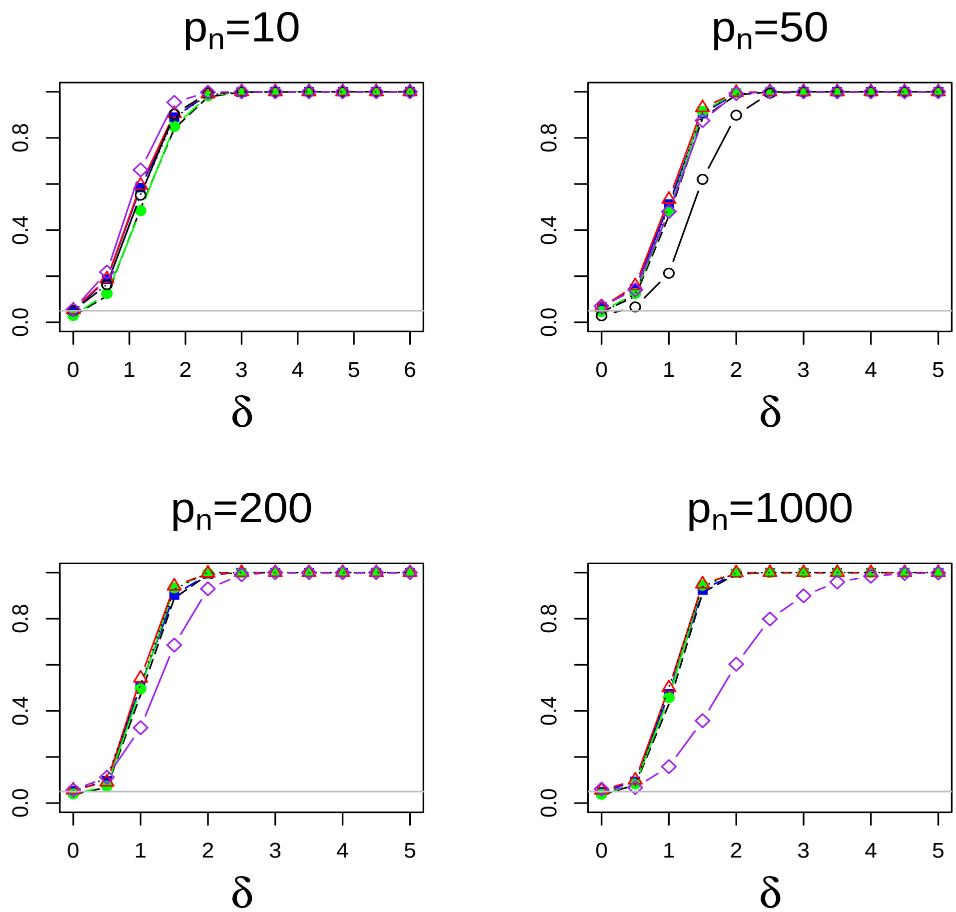

Figure S.1: Power curves of the methods in Case 4 with $n = 200$ and $\tau = 0.5$: $T^E_{n,1}(\tau)$ (dashed), $T^B_{n,1}(\tau)$ (line with solid square), $T^E_{n,2}(\tau)$ (line with solid dots), $T^B_{n,2}(\tau)$ (line with triangle), RS (line with open circle), CAR (dotted), GC (line with diamond). The gray horizontal line indicates the nominal level of 0.05.

S2 Discussion on condition A5

Discussion on condition A5. The term $\omega^*_{j,l,\tau}$ in A5 measures the weighted partial correlation between $X^*_{i,j,\tau}$ and $X^*_{i,l,\tau}$ after accounting for the effect of $\mathbf{Z}$, where the weights are due to the heteroscedasticity. Condition A5 requires the maximum of the weighted coefficients to have a lower bound. If $X_j, j = 1, \ldots, d_n$ are uncorrelated across j, we have

$$\sum_{l=1}^{s_0(\tau)} b_{l,0}(\tau)\omega^*_{j,l,\tau} = \begin{cases} b_{j,0}(\tau)E\{f_{i,\tau}(0)X^{*2}_{i,j,\tau}\}/\{\tau(1-\tau)E(X^{*2}_{i,j,\tau})\}^{1/2}, & 1 \le j \le s_0(\tau), \\ 0, & j > s_0(\tau). \end{cases}$$

Thus condition A5 is equivalent to

$$\max_{1\le j\le s_0(\tau)} |b_{j,0}(\tau)|E\{f_{i,\tau}(0)X^{*2}_{i,j,\tau}\}/\{\tau(1-\tau)E(X^{*2}_{i,j,\tau})\}^{1/2} > \sqrt{2}+\epsilon. \quad \text{(S.1)}$$

Furthermore, if the errors are homoscedastic with $f_{i,\tau}(\cdot) \equiv f_\tau(\cdot)$, then A5 requires that

$$|b_{j_0,0}(\tau)| > \sqrt{2}\left\{\frac{\tau(1-\tau)}{f^2_\tau(0)}\right\}^{1/2}\frac{1}{\{E(X^{*2}_{i,j_0})\}^{1/2}},$$

where j_0 is the maxima of the left side of (S.1). This indicates that the larger the partial variance of X_{j_0} given $\mathbf{Z}$ is, the smaller signal is needed to achieve the desired power for testing.

S3 Proofs of Theorems 1-3

This section includes the proofs of Theorems 1-3.

S3.1 Some useful lemmas

Note that in the "bandwidth.rq" function of the R package *quantreg*,

$$h = n^{-1/5}[4.5\phi\{\Phi^{-1}(\tau)\}^4/\{2\Phi^{-1}(\tau)^2+1\}^2]^{1/5} \triangleq C_6 n^{-1/5},$$

where $\Phi(\cdot), \phi(\cdot)$ are the distribution and density functions of the standard normal distribution, respectively.

Lemma S.1. *Assume that conditions A.1–A.4 hold, and h in (2.6) of the main text satisfies $h \leq h_n^*$ and $h^{-1}(q+s_n)\sqrt{\log(p_n \vee n)/n} \to 0$, where $s_n = \max_{\nu\in[\tau-h_n^*,\tau+h_n^*]} \|\boldsymbol{\theta}_0(\nu)\|_0$. We have*

$$\delta_{\widehat{f}} = \max_{1\leq i\leq n} |\widehat{f}_{i,\tau}(0) - f_{i,\tau}(0)| = O_p\Big(h^2 + h^{-1}(q+s_n)\sqrt{\log(p_n \vee n)/n}\Big).$$

Especially, in our implementation, we have $h = C_6 n^{-1/5}$, thus

$$\delta_{\widehat{f}} = O_p\Big(n^{-2/5} + n^{-3/10}\sqrt{\log(p_n \vee n)}\Big) = O_p\Big(n^{-3/10}\sqrt{\log(p_n \vee n)}\Big).$$

Proof. Lemma S.1 is quite similar to Lemma 19 in the supplementary file of Belloni et al. (2019), and we present the detailed proof in the following.

Let $\widetilde{\mathbf{X}}_{i\cdot} = (\mathbf{Z}_{i\cdot}^\top, \mathbf{X}_{i\cdot}^\top)^\top$, then

$$\begin{aligned}
& \widehat{f}_{i,\tau}(0) \\
= \;& \frac{2h}{\widehat{Q}_{\tau+h}(Y_i \mid \mathbf{Z}_{i\cdot}, \mathbf{X}_{i\cdot}) - \widehat{Q}_{\tau-h}(Y_i \mid \mathbf{Z}_{i\cdot}, \mathbf{X}_{i\cdot})} = \frac{2h}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\widehat{\boldsymbol{\theta}}(\tau+h) - \widehat{\boldsymbol{\theta}}(\tau-h)\}} \\
= \;& \frac{2h}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\}} \Big/ \frac{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\widehat{\boldsymbol{\theta}}(\tau+h) - \widehat{\boldsymbol{\theta}}(\tau-h)\}}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\}} \\
= \;& \frac{2h}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\}} \Big/ \Big[1 + \frac{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\widehat{\boldsymbol{\theta}}(\tau+h) - \boldsymbol{\theta}_0(\tau+h)\} - \widetilde{\mathbf{X}}_{i\cdot}^\top\{\widehat{\boldsymbol{\theta}}(\tau-h) - \boldsymbol{\theta}_0(\tau-h)\}}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\}}\Big] \\
\triangleq \;& \frac{2h}{\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\}} / I_i. \qquad \text{(S.1)}
\end{aligned}$$

By assumption A4, we have $f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(y) = f_{i,\tau}(y - \widetilde{\mathbf{X}}_{i\cdot}^\top\boldsymbol{\theta}_0(\tau))$, thus it is easy to see that $f_{i,\tau}(0) = \frac{1}{Q'_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})}$, where $Q'_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})$ is the derivative of $Q_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})$ with respect to τ. By assumption A4, we get

$$\begin{aligned}
& (2h)^{-1}\widetilde{\mathbf{X}}_{i\cdot}^\top\{\boldsymbol{\theta}_0(\tau+h) - \boldsymbol{\theta}_0(\tau-h)\} = (2h)^{-1}\{Q_{\tau+h}(Y_i|\widetilde{\mathbf{X}}_{i\cdot}) - Q_{\tau-h}(Y_i|\widetilde{\mathbf{X}}_{i\cdot})\} \\
= \;& (2h)^{-1}[\{Q_{\tau+h}(Y_i|\widetilde{\mathbf{X}}_{i\cdot}) - Q_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})\} - \{Q_{\tau-h}(Y_i|\widetilde{\mathbf{X}}_{i\cdot}) - Q_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})\}] \\
= \;& (2h)^{-1}\Big[\{Q'_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h + \frac{1}{2}Q''_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h^2 + \frac{1}{6}Q'''_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h^3 + O(h^3)\} \\
& -\{-Q'_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h + \frac{1}{2}Q''_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h^2 - \frac{1}{6}Q'''_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot})h^3 + O(h^3)\}\Big] \\
= \;& Q'_\tau(Y_i|\widetilde{\mathbf{X}}_{i\cdot}) + O(h^2) = \frac{1}{f_{i,\tau}(0)} + O(h^2). \qquad \text{(S.2)}
\end{aligned}$$

Now we derive the term I_i in (S.1). By the definition of the conditional quantiles, we have

$$\int_{\widetilde{\mathbf{X}}_{i\cdot}^\top\boldsymbol{\theta}_0(\tau-h)}^{\widetilde{\mathbf{X}}_{i\cdot}^\top\boldsymbol{\theta}_0(\tau+h)} f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(y)dy = 2h.$$

Since $f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(y)$ is continuous in y by assumption A3, there exists $\xi_{i,\tau} \in [\widetilde{\mathbf{X}}_{i\cdot}^\top \boldsymbol{\theta}_0(\tau - h), \widetilde{\mathbf{X}}_{i\cdot}^\top \boldsymbol{\theta}_0(\tau + h)]$ such that

$$f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(\xi_{i,\tau}) = \frac{2h}{\widetilde{\mathbf{X}}_{i\cdot}^\top \{\boldsymbol{\theta}_0(\tau + h) - \boldsymbol{\theta}_0(\tau - h)\}}, \tag{S.3}$$

or equivalently,

$$\widetilde{\mathbf{X}}_{i\cdot}^\top \{\boldsymbol{\theta}_0(\tau + h) - \boldsymbol{\theta}_0(\tau - h)\} = \frac{2h}{f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(\xi_{i,\tau})}. \tag{S.4}$$

By theorem 1 of Belloni and Chernozhukov (2011), we have $\|\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}_0(\tau)\|_0 = O_p(q + s_n)$, and $\|\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}_0(\tau)\|_2 = O_p(\sqrt{(q + s_n)\log(p_n \vee n)/n})$, thus $\widetilde{\mathbf{X}}_{i\cdot}^\top \{\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}_0(\tau)\} = O_p\{(q + s_n)\sqrt{\log(p_n \vee n)/n}\}$. Thus, we use (S.4) to derive that

$$\begin{aligned} I_i &= 1 + O_p[(q + s_n)\sqrt{\log(p_n \vee n)/n}/\widetilde{\mathbf{X}}_{i\cdot}^\top \{\boldsymbol{\theta}_0(\tau + h) - \boldsymbol{\theta}_0(\tau - h)\}] \\ &= 1 + O_p\{(q + s_n)\sqrt{\log(p_n \vee n)/n} f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(\xi_{i,\tau})/(2h)\} \\ &= 1 + O_p\{h^{-1}(q + s_n)\sqrt{\log(p_n \vee n)/n}\}, \end{aligned} \tag{S.5}$$

where the last "=" is because of the boundedness of $f_{Y_i|\widetilde{\mathbf{X}}_{i\cdot}}(\cdot)$ implied by assumption A3.

Combining (S.1), (S.2) and (S.5), the proof of Lemma S.1 is completed.□

Lemma S.2. *Assume that conditions A.1–A.3 hold. Let* $\boldsymbol{\Theta}_n = \{\boldsymbol{\alpha} : \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\| \le C_7\sqrt{\log(d_n)/n}\}$*, where* C_7 *is some large enough positive constant.*

Under the null hypothesis $\boldsymbol{\beta}_{\mathbf{X},0}(\tau) = \mathbf{0}_{d_n}$, we have

$$\sup_{1\le j\le d_n,\ \boldsymbol{\alpha}\in\boldsymbol{\Theta}_n} \left|S_{\tau,j}(\boldsymbol{\alpha}) - S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\} - E[S_{\tau,j}(\boldsymbol{\alpha}) - S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}]\right| = O_p\left\{n^{-1/4}(\log n)^{3/4}\right\}, \tag{S.6}$$

where $S_{\tau,j}(\cdot)$ is either based on $\mathbf{f}_\tau$ or its estimate from the quotient method.

Lemma S.2 follows directly from the proof of lemma A.2 (expression (A.5)) of Wang and He (2007).

Lemma S.3. *Assume that conditions A.1–A.4 hold. Then, for any $x \in \mathbb{R}$, as $n \to \infty, d_n \to \infty$,*

$$P[\max_{1\le j\le d_n} S_j^2 - 2\log(d_n) + \log\{\log(d_n)\} \le x] \to \exp\{-\pi^{-1/2}\exp(-\frac{x}{2})\}, \tag{S.7}$$

$$P[\max_{1\le j\le d_n} S_j \le \sqrt{2\log(d_n) - \log\{\log(d_n)\} + x}] \to \exp\{-\frac{1}{2}\pi^{-1/2}\exp(-\frac{x}{2})\}, \tag{S.8}$$

where $S_j = n^{-1/2}\sum_{i=1}^n X^*_{i,j,\tau}\psi_\tau\{Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2}$, $j = 1, \ldots, d_n$.

Lemma S.3 is similar to Lemma 6 of Cai et al. (2014), while the difference lies in the *asymptotic* normality of S_j and the normality assumption required by Lemma 6 of Cai et al. (2014). We fill the theoretical gap by Theorem 1.1 in Zaïtsev (1987), similar to the proof of Theorem 6 of Cai et al. (2014).

Proof. We only prove (S.7), and the proof of (S.8) is similar.

Let $V_{i,j} = X^*_{i,j,\tau}\psi_\tau\{Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2}$, thus $S_j = n^{-1/2}\sum_{i=1}^n V_{i,j}$. Let $\check{V}_{i,j} = V_{i,j}I(|V_{i,j}| \le \zeta_n)$ for $i = 1, \ldots, n$ and $\check{S}_j =$

$n^{-1/2}\sum_{i=1}^{n}\check{V}_{i,j}$, where $\zeta_n = 2C_1^{-1/2}\sqrt{d_n+n}$, with C_1 defined in Assumption A1 (ii). Then

$$P\Big\{\max_{1\le j\le d_n}|S_j-\check{S}_j|\ge\frac{1}{\log(d_n)}\Big\}\le P(\max_{1\le j\le d_n}\max_{1\le i\le n}|V_{i,j}|\ge\zeta_n)$$
$$\le\ nd_n\max_{1\le j\le d_n}P(|V_{1,j}|\ge\zeta_n)=O(d_n^{-1}). \tag{S.9}$$

Note that

$$|\max_{1\le j\le d_n}S_j^2-\max_{1\le j\le d_n}\check{S}_j^2|\le 2\max_{1\le j\le d_n}|S_j|\max_{1\le j\le d_n}|S_j-\check{S}_j|+\max_{1\le j\le d_n}|S_j-\check{S}_j|^2. \tag{S.10}$$

By expression (S.9) and (S.10), it is enough to prove that, for any $x\in\mathbb{R}$, as $n\to\infty, d_n\to\infty$,

$$P[\max_{1\le j\le d_n}\check{S}_j^2-2\log(d_n)+\log\{\log(d_n)\}\le x]\to\exp\Big\{-\frac{1}{\sqrt{\pi}}\exp(-x/2)\Big\}.$$

Given t, we define $\mathcal{I}=\{1\le j_1<\ldots<j_t\le d_n:\max_{1\le k<l\le t}|\text{corr}(S_{j_k},S_{j_l})|\ge d_n^{-\gamma_0}\}$, where γ_0 is a sufficiently small number satisfying $\gamma_0<1/(2t)$; we omit t from the definition of $\mathcal{I}$ for notation ease. For $2\le g\le t-1$, define

$$\begin{aligned}\mathcal{I}_g\ =\ &\{1\le j_1<\ldots<j_t\le d_n:\text{card}(\Delta)=g, \text{where }\Delta\text{ is the largest subset of }\{j_1,\ldots,j_t\}\\ &\text{such that }\forall j_k\ne j_l\in\Delta, |\text{corr}(S_{j_k},S_{j_l})|<d_n^{-\gamma_0}\}.\end{aligned}$$

For $g=1$, define $\mathcal{I}_1=\{1\le j_1<\ldots<j_t\le d_n:|\text{corr}(S_{j_k},S_{j_l})|\ge d_n^{-\gamma_0}$ for every $1\le k<l\le t\}$. So we have $\mathcal{I}=\cup_{g=1}^{t-1}\mathcal{I}_g$.

It follows from lemma 1 of Cai et al. (2014) that, for any fixed $k\le[d_n/2]$,

$$\sum_{t=1}^{2k}(-1)^{t-1}E_t\le P(\max_{1\le j\le d_n}|\check{S}_j|\ge\sqrt{x_{d_n}})\le\sum_{t=1}^{2k-1}(-1)^{t-1}E_t, \tag{S.11}$$

where $x_{d_n} = 2\log(d_n) - \log\{\log(d_n)\} + x,\ E_t = \sum_{1\le j_1<\ldots<j_t\le d_n} P(|\check{S}_{j_1}| \geq \sqrt{x_{d_n}}, \ldots, |\check{S}_{j_t}| \geq \sqrt{x_{d_n}})$.

Then, by Theorem 1.1 of Zaïtsev (1987), we have

$$\begin{aligned} P(\min_{1\le l\le t} |\check{S}_{j_l}| \geq \sqrt{x_{d_n}}) \quad &\leq \quad P\{\min_{1\le l\le t} |S^*_{j_l}| \geq \sqrt{x_{d_n}} - \epsilon_n \log(d_n)^{-1/2}\} \\ &\quad + a_1 g^{5/2} \exp\Big\{ - \frac{n^{1/2}\epsilon_n}{a_2 g^3 \zeta_n \log(d_n)^{1/2}} \Big\}, \quad \text{(S.12)} \end{aligned}$$

where $a_1 > 0$ and $a_2 > 0$ are positive constants, $\epsilon_n \to 0$ will be specified later, and $(S^*_{j_1}, \ldots, S^*_{j_t})^\top$ is a t-dimensional normal vector, which is a sub vector of

$$\mathbf{S}^* = (S^*_1, \ldots, S^*_{d_n})^\top \sim N(\mathbf{0}, \mathbf{R}_{\tau,\mathbf{X}|\mathbf{Z}}).$$

Because $\log(d_n) = o\{n^{1/4}/\log(n)^{3/4}\}$, we can let $\epsilon_n \to 0$ sufficiently slowly such that

$$a_1 g^{5/2} \exp\Big\{ - \frac{n^{1/2}\epsilon_n}{a_2 g^3 \zeta_n \log(d_n)^{1/2}} \Big\} = O(d_n^{-M}) \qquad \text{(S.13)}$$

for any large $M > 0$. It follows from expressions (S.11), (S.12) and (S.13) that

$$\begin{aligned} &P(\min_{1\le j\le d_n} |\check{S}_j| \geq \sqrt{x_{d_n}}) \\ \leq\ & \sum_{t=1}^{2k-1} (-1)^{t-1} \sum_{1\le j_1<\ldots<j_t\le d_n} P\Big\{ \min_{1\le l\le t} |S^*_{j_l}| \geq \sqrt{x_{d_n}} - \epsilon_n \log(d_n)^{-1/2} \Big\} + o(1). \quad \text{(S.14)} \end{aligned}$$

Similarly, using Theorem 1.1 of Zaïtsev (1987) again, we can obtain

$$\begin{aligned} &P(\min_{1\le j\le d_n} |\check{S}_j| \geq \sqrt{x_{d_n}}) \\ \geq\ & \sum_{t=1}^{2k} (-1)^{t-1} \sum_{1\le j_1<\ldots<j_t\le d_n} P\Big\{ \min_{1\le l\le t} |S^*_{j_l}| \geq \sqrt{x_{d_n}} - \epsilon_n \log(d_n)^{-1/2} \Big\} + o(1). \quad \text{(S.15)} \end{aligned}$$

So, by expression (S.14) and (S.15) and the proof of Theorem 1 (Lemma 6) of Cai et al. (2014), the lemma is proved.□

S3.2 Proof of Theorem 1

Recall that

$$S_{\tau,j}(\boldsymbol{\alpha}_{\mathbf{Z}}) = n^{-1/2}\sum_{i=1}^{n} X^*_{i,j,\tau}\psi_\tau(Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha}_{\mathbf{Z}})/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2},\ j = 1,\ldots,d_n.$$

Since the density function matrix $\mathbf{f}_\tau$ is estimated by $\widehat{\mathbf{f}}_\tau$, we further define

$$\begin{aligned}
\widehat{\mathbb{X}}^*_{\cdot j,\tau} &= \left\{\mathbf{I}_n - \widehat{\mathbf{f}}_\tau\mathbb{Z}(\mathbb{Z}^\top\widehat{\mathbf{f}}_\tau^2\mathbb{Z})^{-1}\mathbb{Z}^\top\widehat{\mathbf{f}}_\tau\right\}\mathbb{X}_{\cdot j} \doteq (\widehat{X}^*_{1,j,\tau},\ldots,\widehat{X}^*_{n,j,\tau})^\top,\\
S_{\tau,j}(\boldsymbol{\alpha}_{\mathbf{Z}};\widehat{\mathbb{X}}^*_{\cdot j,\tau}) &= n^{-1/2}\sum_{i=1}^{n}\widehat{X}^*_{i,j,\tau}\psi_\tau(Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha}_{\mathbf{Z}})/\{\tau(1-\tau)\|\widehat{\mathbb{X}}^*_{\cdot j,\tau}\|^2/n\}^{1/2},
\end{aligned}$$

$j = 1,\ldots,d_n$. Because we actually use $S_{\tau,j}(\boldsymbol{\alpha}_{\mathbf{Z}};\widehat{\mathbb{X}}^*_{\cdot j,\tau})$ to construct our test statistic, we prove Theorem 1 with $S_{\tau,j}(\boldsymbol{\alpha}_{\mathbf{Z}};\widehat{\mathbb{X}}^*_{\cdot j,\tau})$.

Under the null hypothesis $\boldsymbol{\beta}_{\mathbf{X},0}(\tau) = \mathbf{0}_{d_n}^\top$, it is easy to show that $E[S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau);\widehat{\mathbb{X}}^*_{\cdot j,\tau}\} \mid$

$\mathbb{Z}, \mathbb{X}] = 0, j = 1, \ldots, d_n$. Due to the fact that $\mathbb{Z}^\top \widehat{\mathbf{f}}_\tau \widehat{\mathbb{X}}^*_{\cdot j,\tau} = \mathbf{0}$, we have

$$\begin{aligned}
& E\{S_{\tau,j}(\boldsymbol{\alpha}; \widehat{\mathbb{X}}^*_{\cdot j,\tau}) \mid \mathbb{Z}, \mathbb{X}\} \cdot \{\tau(1-\tau)\|\widehat{\mathbb{X}}^*_{\cdot j,\tau}\|^2/n\}^{1/2} \\
= \; & n^{-1/2} \sum_{i=1}^n \widehat{X}^*_{i,j,\tau} \{\tau - P(Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha})\} \\
= \; & n^{-1/2} \sum_{i=1}^n \widehat{X}^*_{i,j,\tau} [P\{Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\} - P(Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha})] \\
= \; & n^{-1/2} \sum_{i=1}^n \widehat{X}^*_{i,j,\tau} \Big\{ - f_{i,\tau}(0) \mathbf{Z}_{i\cdot}^\top \{\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\} - O\Big(f'_{i,\tau}(0) [\mathbf{Z}_{i\cdot}^\top \{\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}]^2 \Big) \Big\} \\
= \; & n^{-1/2} \sum_{i=1}^n \widehat{X}^*_{i,j,\tau} \Big[\{\widehat{f}_{i,\tau}(0) - f_{i,\tau}(0)\} \mathbf{Z}_{i\cdot}^\top \{\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\} \Big] \\
& + O_p \Big\{ n^{-1/2} \sum_{i=1}^n X^*_{i,j,\tau} \left(f'_{i,\tau}(0) [\mathbf{Z}_{i\cdot}^\top \{\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}]^2 \right) \Big\} \\
= \; & O_p \Big(\sqrt{\log(d_n)} \max_{1 \le i \le n} |\widehat{f}_{i,\tau}(0) - f_{i,\tau}(0)| + n^{-1/2} \log(d_n) \Big) \\
= \; & O_p \Big(\delta_{\widehat{f}} \sqrt{\log(d_n)} + n^{-1/2} \log(d_n) \Big), \qquad \text{(S.16)}
\end{aligned}$$

uniformly over $\boldsymbol{\alpha} \in \boldsymbol{\Theta}_n = \{\boldsymbol{\alpha} : \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\| \le C_7 \sqrt{\log(d_n)/n}\}$. It is easy to show that $\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau) \in \boldsymbol{\Theta}_n$ with probability approaching 1. Thus, combined with Lemmas S.1–S.2 and assumption A1, we have

$$\begin{aligned}
& S_{\tau,j}\{\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\} \\
= \; & S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\} + E[S_{\tau,j}\{\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\}] + O_p\left\{n^{-1/4} (\log n)^{3/4}\right\} \\
= \; & S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\} + O_p\left\{ \delta_{\widehat{f}} \sqrt{\log(d_n)} + n^{-1/2}\log(d_n) + n^{-1/4}(\log n)^{3/4} \right\} \\
= \; & S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\} + o_p(1). \qquad \text{(S.17)}
\end{aligned}$$

Since we assume that $X_{i,j}$ is subGaussian, it is easy to prove that $S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau); \widehat{\mathbb{X}}^*_{\cdot j,\tau}\}$

is asymptotic normal. The proof of Theorem 1 follows by Lemma S.3.□

S3.3 Proof of Theorem 2

From the proof of Theorem 1, we find that the plug-in of $\widehat{\mathbf{f}}_\tau$ doesn't affect the proof, thus here we use $\mathbf{f}_\tau$ for notational ease.

Recall that $S_{\tau,j}(\boldsymbol{\alpha}) = n^{-1/2}\sum_{i=1}^n X^*_{i,j,\tau}\psi_\tau(Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha})/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2}$, $j = 1,\ldots,d_n$. Under the local alternative $\boldsymbol{\beta}_{\mathbf{X},n}(\tau) = \mathbf{b}_0(\tau)\sqrt{\log(d_n)/n}$ with fixed $s_0(\tau) = \|\mathbf{b}_0(\tau)\|_0$, we assume without loss of generality that $b_{j,0}(\tau) \neq 0, j = 1,\ldots,s_0(\tau)$. For notational ease, we omit τ from $s_0(\tau)$, and for a vector $\mathbf{b}$, we use $\mathbf{b}_{1:s_0}$ to represent the first s_0 components. To derive the asymptotic property under the local alternative, we define

$$S^A_{\tau,j}(\boldsymbol{\alpha},\boldsymbol{\beta}_{1:s_0}) = n^{-1/2}\sum_{i=1}^n X^*_{i,j,\tau}\psi_\tau(Y_i - \mathbf{Z}_{i\cdot}^\top\boldsymbol{\alpha} - \mathbf{X}_{i,1:s_0}^\top\boldsymbol{\beta}_{1:s_0})/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2},$$

so that $S_{\tau,j}(\boldsymbol{\alpha}) = S^A_{\tau,j}(\boldsymbol{\alpha},\mathbf{0})$.

Recall that $\boldsymbol{\Theta}_n = \{\boldsymbol{\alpha} : \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\| \le C_7\sqrt{\log(d_n)/n}\}$. By Lemma S.2, we have

$$\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Theta}_n} \Big| S^A_{\tau,j}(\boldsymbol{\alpha},\mathbf{0}) - S^A_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau), \sqrt{\log(d_n)/n}\,\mathbf{b}_{1:s_0,0}(\tau)\} - E[S^A_{\tau,j}(\boldsymbol{\alpha},\mathbf{0}) - S^A_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau), \sqrt{\log(d_n)/n}\,\mathbf{b}_{1:s_0,0}(\tau)\}]\Big| = O_p\left\{n^{-1/4}(\log n)^{3/4}\right\}. \quad \text{(S.18)}$$

To derive the property of $S_{\tau,j}(\boldsymbol{\alpha}) = S^A_{\tau,j}(\boldsymbol{\alpha}, \mathbf{0})$, we first obtain

$$
\begin{aligned}
& E\{S^A_{\tau,j}(\boldsymbol{\alpha}, \mathbf{0}) \mid \mathbb{Z}, \mathbb{X}\} \times \{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2} \\
= \;& n^{-1/2} \sum_{i=1}^n X^*_{i,j,\tau}\{\tau - P(Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha})\} \\
= \;& n^{-1/2} \sum_{i=1}^n X^*_{i,j,\tau}[P\{Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha}_{\mathbf{Z},0}(\tau) + \sqrt{\log(d_n)/n}\mathbf{X}_{i\cdot}^\top \mathbf{b}_0(\tau)\} - P(Y_i < \mathbf{Z}_{i\cdot}^\top \boldsymbol{\alpha})] \\
= \;& n^{-1/2} \sum_{i=1}^n X^*_{i,j,\tau} \Big(f_{i,\tau}(0)[\mathbf{Z}_{i\cdot}^\top\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau) - \boldsymbol{\alpha}\} + \sqrt{\log(d_n)/n}\mathbf{X}_{i\cdot}^\top \mathbf{b}_0(\tau)] \\
& \qquad + f'_{i,\tau}(0)[\mathbf{Z}_{i\cdot}^\top\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau) - \boldsymbol{\alpha}\} + \sqrt{\log(d_n)/n}\mathbf{X}_{i\cdot}^\top \mathbf{b}_0(\tau)]^2 \\
& \qquad\qquad + O[\mathbf{Z}_{i\cdot}^\top\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau) - \boldsymbol{\alpha}\} + \sqrt{\log(d_n)/n}\mathbf{X}_{i\cdot}^\top \mathbf{b}_0(\tau)]^2 \Big) \\
= \;& n^{-1/2} \sum_{i=1}^n X^*_{i,j,\tau} \Big(f_{i,\tau}(0)[\mathbf{Z}_{i\cdot}^\top\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau) - \boldsymbol{\alpha}\} + \sqrt{\log(d_n)/n}\mathbf{X}_{i\cdot}^\top \mathbf{b}_0(\tau)] \Big) + O_p\{\log(d_n)/\sqrt{n}\} \\
= \;& \frac{1}{n}\sqrt{\log(d_n)} \sum_{i=1}^n X^*_{i,j,\tau} f_{i,\tau}(0) \sum_{l=1}^{s_0} X_{i,l} b_{l,0}(\tau) + O_p\{\log(d_n)/\sqrt{n}\} \\
= \;& \sqrt{\log(d_n)} \sum_{l=1}^{s_0} b_{l,0}(\tau) \frac{1}{n} \sum_{i=1}^n f_{i,\tau}(0) X^*_{i,j,\tau} X_{i,l} + O_p\{\log(d_n)/\sqrt{n}\} \\
= \;& \sqrt{\log(d_n)} \sum_{l=1}^{s_0} b_{l,0}(\tau) \omega^*_{j,l,\tau} \{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\}^{1/2} + O_p\{\log(d_n)/\sqrt{n}\},
\end{aligned} \tag{S.19}
$$

where the last but third equality is because $\sum_{i=1}^n X^*_{i,j,\tau} f_{i,\tau}(0)\mathbf{Z}_{i\cdot} = \mathbf{0}$, and the last equality is because the projection matrix $\mathbf{P}_{\mathbf{Z},\mathbf{f}} = \mathbf{f}_\tau \mathbb{Z}(\mathbb{Z}^\top \mathbf{f}_\tau^2 \mathbb{Z})^{-1}\mathbb{Z}^\top \mathbf{f}_\tau$ is idempotent, so that $\omega^*_{j,l,\tau} = E\{f_{i,\tau}(0) X^*_{i,j,\tau} X^*_{i,l,\tau}\}/\{\tau(1-\tau)E(X^{*2}_{i,j,\tau})\}^{1/2} = E\{f_{i,\tau}(0)X^*_{i,j,\tau}X_{i,l}\}/\{\tau(1-\tau)E(X^{*2}_{i,j,\tau})\}^{1/2}$ and $\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n = E(X^{*2}_{i,j,\tau})\{1 + O_p(n^{-1/2})\}$.

We then obtain

$$E[S^A_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau), \sqrt{\log(d_n)/n}\mathbf{b}_{1:s_0,0}(\tau)\}] = 0,$$
$$S^A_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau), \sqrt{\log(d_n)/n}\mathbf{b}_{1:s_0,0}(\tau)\} = O_p(1), \tag{S.20}$$

which is straightforward under the local model.

Combining (S.18), (S.19) and (S.20), we have

$$\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Theta}_n}\Big|\frac{S_{\tau,j}(\boldsymbol{\alpha})}{\sqrt{\log(d_n)}} - \frac{\sum_{l=1}^{s_0} b_{l,0}(\tau)\omega^*_{j,l,\tau}\{E(X^{*2}_{i,j,\tau})\}^{1/2}}{(\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n)^{1/2}}\Big| = O_p\Big\{\frac{1}{\sqrt{\log(d_n)}}\Big\}.$$

Therefore,

$$P\Big[\sup_{\boldsymbol{\alpha}\in\boldsymbol{\Theta}_n}\Big|\frac{S_{\tau,j}(\boldsymbol{\alpha})}{\sqrt{\log(d_n)}} - \frac{\sum_{l=1}^{s_0} b_{l,0}(\tau)\omega^*_{j,l,\tau}\{E(X^{*2}_{i,j,\tau})\}^{1/2}}{(\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n)^{1/2}}\Big| \le \epsilon/4\Big] \to 1.$$

Under the local model (2.6), with s_0 being fixed, we can show that $\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau) \in \boldsymbol{\Theta}_n$ with probability approach 1. Therefore,

$$P\Big[\Big|\frac{S_{\tau,j}\{\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau)\}}{\sqrt{\log(d_n)}} - \sum_{l=1}^{s_0} b_{l,0}(\tau)\omega^*_{j,l,\tau}\{E(X^{*2}_{i,j,\tau})\}^{1/2}/(\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n)^{1/2}\Big| \le \epsilon/4\Big] \to 1.$$

Since $\max_{1\le j\le d_n}|\sum_{l=1}^{s_0} b_{l,0}(\tau)\omega^*_{j,l,\tau}| > \sqrt{2}+\epsilon$, we have

$$\max_{1\le j\le d_n}|\sum_{l=1}^{s_0} b_{l,0}(\tau)\omega^*_{j,l,\tau}|\{E(X^{*2}_{i,j,\tau})\}^{1/2}/(\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n)^{1/2} \ge \sqrt{2}+\epsilon/2.$$

Therefore,

$$P\Big[\max_{1\le j\le d_n}\Big|\frac{S_{\tau,j}\{\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau)\}}{\sqrt{\log(d_n)}}\Big| \ge \sqrt{2}+\epsilon/4\Big] \to 1,$$

which leads to

$$\begin{aligned} & P(\text{reject } H_0 | H_a) \\ = \ & P[T_{n,1}(\tau) - 2\log(d_n) + \log\{\log(d_n)\} \geq q_\gamma | H_a] \\ = \ & P\Big[\max_{1\leq j\leq d_n} \frac{S^2_{\tau,j}\{\widehat{\boldsymbol{\alpha}}_{\mathbf{Z}}(\tau)\}}{\log(d_n)} \geq 2 - \log\{\log(d_n)\}/\log(d_n) + q_\gamma/\log(d_n) | H_a \Big] \to 1. \end{aligned}$$

□

S3.4 Proof of Theorem 3

From the proof of Theorem 1, we find that the plug-in of $\widehat{\mathbf{f}}_\tau$ doesn't affect the proof, thus here we use $\mathbf{f}_\tau$ for notational ease.

Recall that

$$T_{n,1}(\tau)^* = \max_{1\leq j\leq d_n} \{n^{-1/2} \sum_{i=1}^n w_i X^*_{i,j,\tau} \psi(e_i)\}^2 / \{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\},$$

where $e_i \ i.i.d. \sim N(-\Phi^{-1}(\tau), 1)$, $w_i \ i.i.d. \sim P(w=1) = P(w=-1) = 1/2$. It is easy to verify that $\{n^{-1/2}\sum_{i=1}^n w_i X^*_{i,j,\tau}\psi(e_i)\}^2/\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2/n\} \sim \chi^2_1$. For dependence across j, we have

$$\begin{aligned} & \text{corr}\left[\frac{\sum_{i=1}^n w_i X^*_{i,j,\tau}\psi(e_i)}{\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|^2\}^{1/2}}, \frac{\sum_{i'=1}^n w_{i'} X^*_{i',j',\tau}\psi(e_{i'})}{\{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j',\tau}\|^2\}^{1/2}} \mid \mathbb{Z}, \mathbb{X} \right] \\ = \ & \sum_{i=1}^n E(w_i^2) E\{\psi(e_i)^2\} X^*_{i,j,\tau} X_{i,j',\tau} / \{\tau(1-\tau)\|\mathbb{X}^*_{\cdot j,\tau}\|\|\mathbb{X}^*_{\cdot j',\tau}\|\} \\ = \ & \sum_{i=1}^n X^*_{i,j,\tau} X_{i,j',\tau} / (\|\mathbb{X}^*_{\cdot j,\tau}\|\|\mathbb{X}^*_{\cdot j',\tau}\|) = r_{j,j'} + O_p(n^{-1/2}), \end{aligned}$$

which is asymptotically equivalent to the correlation of $S_{\tau,j}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}$ and $S_{\tau,j'}\{\boldsymbol{\alpha}_{\mathbf{Z},0}(\tau)\}$ given $\mathbb{Z}$ and $\mathbb{X}$. The proof follows similar steps as the proof of Theorem 1. $\square$

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