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A CENTRAL LIMIT THEOREM FOR THE NUMBER OF SUCCESS RUNS: AN EXAMPLE OF REGENERATIVE PROCESSES

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Abstract: For each $n \geq 1$, let $\{X_{n,j},\ j \geq 1\}$ be i.i.d. Bernoulli random variables with $P\{X_{n,1} = 1\} = p_n = 1 - q_n = 1 - P\{X_{n,1} = 0\}$, $0 < p_n < 1$. Define $W_{n,m_n} = \sum_{j=1}^{m_n}(R_{n,j} - a_n)$, where $R_{n,j} = \inf\{k \geq 0 : X_{n,j+k} = 0\}$ is the number of success runs starting at j, m_n is a sequences of positive integers, and $a_n = p_n/q_n$. We show that, under the condition $m_n p_n \to \infty$, the central limit theorem

$$\frac{W_{n,m_n}}{\sigma_{m_n}\sqrt{m_n q_n}} \xrightarrow{d} N(0,1)$$

holds if and only if $m_n q_n \to \infty$ as $n \to \infty$, where $\sigma_n^2 = 2a_n^4 + 5a_n^3 + 4a_n^2 + a_n$. A key observation here is that $\{R_{n,j},\ 1 \leq j \leq m_n\}$ forms a regenerative process so that some useful techniques from renewal theory can be utilized here.

Key words and phrases: Central limit theorem, number of success runs, regenerative process, renewal theory.

1. Introduction

For each $n \geq 1$, let $X_{n,1}, X_{n,2}, \ldots$ be independent identically distributed (i.i.d.) Bernoulli random variables with

$$P\{X_{n,1} = 1\} = p_n = 1 - q_n = 1 - P\{X_{n,1} = 0\}, \quad 0 < p_n < 1.$$

Associated with these are random variables $R_{n,j}$, $j \geq 1$, which are of paramount interest in this paper, where

$$R_{n,j} = \inf\{k \geq 0 : X_{n,j+k} = 0\}, \qquad j \geq 1,$$

or equivalently,

$$\{R_{n,j} = k\} = \{X_{n,i} = 1 \ \text{ for } \ j \leq i < j+k \text{ and } X_{n,j+k} = 0\}.$$

In the literature $R_{n,j}$ is called the *number of success runs* starting at j, and it has some applications in Sequential Analysis and studying the match of two sequences of DNA (see, for example, Aki (1985), Philippou and Makri (1986), Goldstein (1990), and Hirano and Aki (1993)).

For the special case of a single indexed sequence, in other words the case in which we simply have R_j, X_1, X_2, p, q instead of $R_{n,j}$, $X_{n,1}$, $X_{n,2}$, $p_{n,1}$, $q_{n,1}$, etc., the almost sure convergence result of R_j was first obtained by D.J. Neuman (see Chow and Teicher (1988), p. 61); and the Central Limit Theorem (CLT) and the Law of Iterated Logarithm (LIL) was proved in Chow (1992a). In this paper we consider the limiting behavior of $R_{n,j}$ for the case of double arrays; or rather, to be more precise, study the asymptotic distribution of the centered sum

$$W_{n,k} = \sum_{j=1}^{k} (R_{n,j} - a_n), \qquad 1 \le k \le m_n, \tag{1}$$

where $a_n = p_n/q_n$ and m_n is some sequence of positive integers.

If $m_n p_n \to$ some constant $\lambda \in (0, \infty)$ and $m_n \to \infty$, as $n \to \infty$, then Chow (1992b) gives the following result.

Theorem 1.1. *Suppose $m_n p_n \to \lambda \in (0, \infty)$ and $m_n \to \infty$, as $n \to \infty$. Then*

$$W_{n,m_n} \xrightarrow{d} W - \lambda, \tag{2}$$

where W is a Poisson random variable with mean λ.

Since the original proof is very short and is in Chinese, we shall reproduce it here for completeness.

Proof. Observe that since

$$\{X_{n,j} \ne R_{n,j}\} = \{X_{n,j} = 1 = X_{n,j+1}\}$$

we have

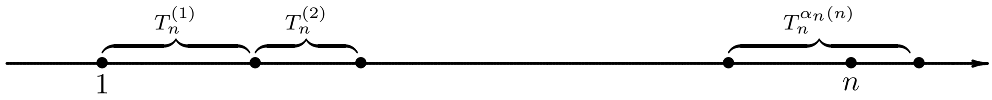
$$P\Big(\sum_{j=1}^{m_n} X_{n,j} \ne \sum_{j=1}^{m_n} R_{n,j}\Big) \le P\Big(\bigcup_{j=1}^{m_n} \{X_{n,j} \ne R_{n,j}\}\Big)$$
$$\le \sum_{j=1}^{m_n} P(X_{n,j} \ne R_{n,j}) = m_n P(X_{n,1} = 1 = X_{n,2}) = m_n p_n^2 \longrightarrow 0,$$

as $n \to \infty$. Hence,

$$\sum_{j=1}^{m_n} X_{n,j} - \sum_{j=1}^{m_n} R_{n,j} \xrightarrow{p} 0,$$

from which we have, by the classical Poisson theorem, $\sum_{j=1}^{m_n} R_{n,j} \xrightarrow{d} W$, and (2) follows.

We deal exclusively in this paper with the case $m_n p_n \to \infty$ as $n \to \infty$. An important observation here is that the summand in (1) can be split into many blocks so that the summation over each block forms a new i.i.d. sequence, upon which the asymptotic analysis may be built up. Indeed, if we set up the blocks by stopping times $T_n^{(1)} = \inf\{j \geq 1 : X_{n,j} = 0\}$ and its copies $T_n^{(2)}$, $T_n^{(3)}, \ldots$, then the summations within each block, $Y_{n,j} = W_{n,T_{n,j}} - W_{n,T_{n,j-1}}$, are i.i.d. random variables (see Lemma 3.1 and Lemma 3.2 below). A picture may illustrate the idea better.



In the above picture $\alpha_n(n) = \inf\{j \geq 1 : \sum_{i=1}^{j} T_n^{(i)} > m_n\}$. Expressing this in standard terminology, for each $n \geq 1$, $\{R_{n,j}, 1 \leq j \leq m_n\}$ forms a *regenerative process* with the regenerative cycle lengths $T_n^{(1)}$, $T_n^{(2)}$, etc. (we refer the reader to Asmussen (1987) for the background of regenerative processes).

Unfortunately, the standard results for the regenerative processes can *not* be applied directly to our case. Perhaps more to the point, we deal with a concrete example in an explicit and constructive fashion, giving exactly a necessary and sufficient condition for the CLT to hold, as well as obtaining a closed form of mean and variance in the CLT (cf. Theorem 2.1), whereas the literature on regenerative processes usually gives relatively abstract statements. Moreover, the double array setting also brings a few technical difficulties; for example, some analogies of classical renewal theory have to be established.

The rest of paper is organized as follows. In Section 2, a CLT is given along with the proof of its necessary part. Some explicit calculation regarding the distribution of regenerative cycles and the analogies of some classical renewal results are derived in Section 3. The last section is devoted to the proof of the sufficient part of the CLT.

2. Main Results

Theorem 2.1. *Suppose that $m_n p_n \to \infty$ as $n \to \infty$. Then the CLT*

$$\frac{W_{n,m_n}}{\sigma_{m_n}\sqrt{m_n q_n}} \xrightarrow{d} N(0,1) \tag{3}$$

holds if and only if $m_n q_n \to \infty$ as $n \to \infty$, where $\sigma_n^2 = 2a_n^4 + 5a_n^3 + 4a_n^2 + a_n$ and $a_n = p_n/q_n$.

Remark. To lighten our notation burden, we may, without loss of generality, take $m_n \equiv n$ from now on, as the arguments are the same.

Proof of the necessary part. We shall prove this by contradiction. Suppose $nq_n \not\to \infty$. Then there exists a subsequence $\{n'\} \subset \{n\}$, such that $n'q_{n'} \to \lambda, \lambda \in [0, \infty)$. We may replace $\{n'\}$ by n, again for notation simplicity.

For any $\epsilon > 0$, we can find an integer $k \geq 0$, such that

$$\sum_{j=0}^{k} \frac{\lambda^j}{j!} e^{-\lambda} > 1 - \frac{\epsilon}{3}.$$

Introduce, for every $n \geq 1$, the sets

$$A_n = \{\text{the number of 0's in the sequence } X_{n,1}, \ldots, X_{n,n} \text{ is at least } (k+1)\}.$$

Then, by the classical Poisson theorem, we have for all sufficiently large n

$$P(A_n) = 1 - P(A_n^c) \leq 1 - \sum_{j=0}^{k} \frac{\lambda^j}{j!} e^{-\lambda} + \frac{\epsilon}{3} \leq \frac{2\epsilon}{3} < \epsilon. \tag{4}$$

Note that we take $k = 0$ if $\lambda = 0$; and in that case (4) still holds since $P(A_n) \leq nq_n \to 0$ as $n \to \infty$. Considering the sets

$$B_n = \{\exists\ 1 \leq j \leq n-1 \ \text{ such that } X_{n,j} = 0 \ \text{ and } \ X_{n,j+1} = 0\}, \qquad n \geq 1,$$

we get

$$P(B_n) \leq \sum_{j=1}^{n-1} P(X_{n,j} = 0,\ X_{n,j+1} = 0) = (n-1)q_n^2 \longrightarrow 0, \tag{5}$$

as $n \to \infty$. For all $x < \infty$, we can easily see that

$$\begin{aligned} P\Big(\frac{W_{n,n}}{\sigma_n \sqrt{nq_n}} > x\Big) &\geq P\Big(\Big\{\frac{W_{n,n}}{\sigma_n \sqrt{nq_n}} > x\Big\} \cap A_n^c \cap B_n^c\Big) \\ &\geq P\Big(\Big\{\frac{\min_{A_n^c \cap B_n^c} W_{n,n}}{\sigma_n \sqrt{nq_n}} > x\Big\} \cap A_n^c \cap B_n^c\Big) \\ &\geq 1 - P\Big(\frac{\min_{A_n^c \cap B_n^c} W_{n,n}}{\sigma_n \sqrt{nq_n}} \leq x\Big) - P(A_n) - P(B_n). \end{aligned} \tag{6}$$

Note, for all large n, the minimum in (6) is attained by $X_{n,i_1} = 0, X_{n,i_2} = 0, \ldots, X_{n,i_k} = 0$, $X_{n,j} = 1$, for all other $j \leq n$, and $X_{n,j} = 0$, $\forall\ j > n$, where

$$i_1 \sim n/(k+1),\ i_2 \sim 2n/(k+1), \ldots,\ i_k \sim kn/(k+1).$$

We have therefore, by the definition of $W_{n,n}$,

$$\begin{aligned} \frac{\min_{A_n^c \cap B_n^c} W_{n,n}}{\sigma_n \sqrt{nq_n}} &\sim \frac{(k+1)\{(1/2)(n/(k+1))(n/(k+1)+1)\} - np_n/q_n}{\sigma_n \sqrt{nq_n}} \\ &= \frac{n^2 q_n^2/(2k+2) + nq_n^2/2 - np_n q_n}{\sqrt{(\sigma_n^2 q_n^4) nq_n}} \\ &\geq -p_n \sqrt{\frac{nq_n}{\sigma_n^2 q_n^4}} \longrightarrow -\sqrt{\frac{\lambda}{2}}, \end{aligned} \tag{7}$$

as $n \to \infty$. Thus, we obtain

$$\liminf_{n\to\infty} P\Big(\frac{W_{n,n}}{\sigma_n\sqrt{nq_n}} > -\sqrt{\lambda}\Big) \geq 1-\epsilon,$$

in view of (4), (5), (6) and (7). We then arrive at a contradiction to (3) by letting $\epsilon \to 0$.

Remark. Since we shall deal exclusively in the remaining sections with the sufficient part of the theorem, we may assume $np_n \to \infty$ and $nq_n \to \infty$ from now on.

3. Preliminaries

Lemma 3.1. *For each $n \geq 1$, define*

$$T_n^{(1)} \equiv T_n = \inf\{j \geq 1, X_{n,j} = 0\}$$

and consider the copies of $T_n^{(1)}$, namely, $T_n^{(2)}, T_n^{(3)}, \ldots$, along with their partial sum $T_{n,m} = T_n^{(1)} + \cdots + T_n^{(m)}$ (we take $T_{n,0} \equiv 0$). Then we have:
(i) *for each $n \geq 1$, $T_n^{(1)}, T_n^{(2)}, \ldots$ are independent, identically distributed;*
(ii) *for each $n \geq 1$, T_n is geometrically distributed,*

$$P(T_n = j) = q_n p_n^{j-1}, \qquad j \geq 1 \tag{8}$$

and, therefore, $E(T_n) = a_n + 1 = 1/q_n$, $E(T_n^2) = 2a_n^2 + 3a_n + 1$, $E(T_n^3) = 6a_n^3 + 12a_n^2 + 7a_n + 1$, $E(T_n^4) = 24a_n^4 + 60a_n^3 + 50a_n^2 + 15a_n + 1$;
(iii) *$R_{n,j} = T_{n,m+1} - j$, for $T_{n,m} < j \leq T_{n,m+1}$.*

Proof. (i) follows from Lemma 5.3.3 in Chow and Teicher (1988), and an elementary computation leads to (ii). For (iii), if $T_{n,m+1} = k$, then $R_{n,j} = k - j$ via the definition of $R_{n,j}$.

Lemma 3.1 suggests that we may use the stopping times $\{T_n^{(j)},\ j \geq 1\}$ to separate the whole process $W_{n,n}$, and then consider the resulting random variables $Y_{n,j} = W_{n,T_{n,j}} - W_{n,T_{n,j-1}}$, $j \geq 1$, individually, where, recalling from Lemma 3.1, $T_{n,j} = T_n^{(1)} + \cdots + T_n^{(j)}$.

Lemma 3.2. *For each fixed $n \geq 1$, $\{Y_{n,j}, j \geq 1\}$ are independent, identically distributed random variables,*

$$Y_{n,j} = \frac{1}{2}(T_n^{(j)})^2 - (\frac{1}{2} + a_n)T_n^{(j)}, \tag{9}$$

with $E(Y_{n,1}) = 0$, $E(Y_{n,1}^2) = 2a_n^4 + 5a_n^3 + 4a_n^2 + a_n = \sigma_n^2$, $E(Y_{n,1}^4) = 864a_n^8 + 3888a_n^7 + 7046a_n^6 + 6517a_n^5 + 3197a_n^4 + 772a_n^3 + 71a_n^2 + a_n$.

Proof. By definition, for $j \geq 1$,

$$
\begin{aligned}
Y_{n,j} = W_{T_{n,j}} - W_{T_{n,j-1}} &= \sum_{m=T_{n,j-1}+1}^{T_{n,j}} R_{n,m} - a_n T_n^{(j)} \\
&= \sum_{m=T_{n,j-1}+1}^{T_{n,j}} (T_{n,j} - m) - a_n T_n^{(j)} \qquad \text{(by (iii) in Lemma 3.1)} \\
&= \frac{1}{2}(T_n^{(j)})^2 - (\frac{1}{2} + a_n) T_n^{(j)}.
\end{aligned}
$$

Therefore, $\{Y_{n,j}, j \geq 1\}$ are independent, identically distributed for each fixed $n \geq 1$; and a little algebra using (ii) in Lemma 3.1 gives the desired moments.

Remark. Lemma 3.1 and Lemma 3.2 tell us $\{R_{n,j}, 1 \leq j \leq m_n\}$ is a regenerative process with regeneration occurring at $T_{n,j}$, the points at which the process $R_{n,j}$ starts all over again, and $Y_{n,j}$ are the summations within regeneration cycles.

In order to study the asymptotic behavior of the regenerative process $w_{n,n}$ we need to control the overshoots

$$
\beta_n(n) = \sum_{j=1}^{\alpha_n(n)} T_n^{(j)} - n, \tag{10}
$$

where

$$
\alpha_n(n) = \inf \Big\{ j \geq 1 : \sum_{i=1}^{j} T_n^{(i)} > n \Big\}.
$$

The reader may want to look at the previous picture in Section 1 for a better intuition. Note that for each fixed $n \geq 1$, $E\alpha_n(n) < \infty$ and $\beta_n(n) > 0$.

Remark. Since $T_n^{(j)}$ are geometric random variables, the "memoryless" property tells us that the distribution of the overshot $\beta_n(n)$ is the same as that of T_n. In particular,

$$
\frac{E\beta_n(n)}{n} = \frac{a_n + 1}{n} \longrightarrow 0 \quad \text{and} \quad \frac{E(\beta_n(n))^2}{n^2} = \frac{2a_n^2 + 3a_n + 1}{n^2} \longrightarrow 0, \tag{11}
$$

as $n \to \infty$.

The following version of the elementary renewal theorem for $\alpha_n(n)$ will be used later.

Lemma 3.3. *As* $n \to \infty$,

$$
\frac{\alpha_n(n)}{n q_n} \xrightarrow{\mathcal{L}_2} 1.
$$

Proof. Wald's equation and (10) lead to

$$E(\alpha_n(n)) = E\Big(\sum_{i=1}^{\alpha_n(n)} T_n^{(j)}\Big)/E(T_n) = \{n + E(\beta_n(n))\}q_n. \tag{12}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}E(\alpha_n(n)/q_n - n)^2 &\le E\Big(\frac{\alpha_n(n)}{q_n} - \sum_{j=1}^{\alpha_n(n)} T_n^{(j)}\Big)^2 + E\Big(\sum_{j=1}^{\alpha_n(n)} T_n^{(j)} - n\Big)^2 \\ &= \mathrm{Var}(T_n)E(\alpha_n(n)) + E(\beta_n(n))^2 \\ &= \mathrm{Var}(T_n)q_n(n + E(\beta_n(n))) + E(\beta_n(n))^2,\end{aligned}$$

by (12) and the second order Wald's equation in Chow and Teicher (1988), p. 142. Note that (ii) in Lemma 3.1 yields

$$\frac{q_n \mathrm{Var}(T_n)}{n} \le \frac{q_n(2a_n^2 + 3a_n + 1)}{n} = \frac{2p_n^2}{nq_n} + \frac{3p_n + q_n}{n} \longrightarrow 0,$$

as $n \to \infty$. Therefore,

$$\frac{E(\alpha_n(n) - nq_n)^2}{n^2 q_n^2} \longrightarrow 0$$

as $n \to \infty$, via (11).

An immediate consequence of Lemma 3.3 is that it essentially enables us to replace $W_{n,T_{n,\alpha_n(n)}}$, the summation of the regenerative cycles right after $W_{n,n}$, by a relative simple process $W_{n,T_{n,[nq_n]}}$ in an asymptotic sense.

Lemma 3.4. *For the random time change, we have*

$$\frac{W_{n,T_{n,\alpha_n(n)}} - W_{n,T_{n,[nq_n]}}}{\sigma_n\sqrt{nq_n}} \xrightarrow{\mathcal{L}_2} 0 \qquad \textit{as} \quad n \to \infty,$$

where $[x]$ is the integer part of x.

Proof. Again, the second order Wald's equation yields

$$\begin{aligned}E(W_{n,T_{n,\alpha_n(n)}} - W_{n,T_{n,[nq_n]}})^2 &= E\Big(\sum_{j=1}^{\alpha_n(n)} Y_{n,j} - \sum_{j=1}^{[nq_n]} Y_{n,j}\Big)^2 \\ = E\Big(\sum_{j=\min(\alpha_n(n),[nq_n])+1}^{\max(\alpha_n(n),[nq_n])} Y_{n,j}\Big)^2 &= \sigma_n^2 E|\alpha_n(n) - [nq_n]| = o(1)\sigma_n^2 nq_n,\end{aligned}$$

via Lemma 3.3.

4. Proof of the Sufficiency in Theorem 2.1

We want first to show that the CLT holds for the cycle sum $Y_{n,j}$:

$$\frac{\sum_{j=1}^{[nq_n]} Y_{n,j}}{\sigma_n\sqrt{nq_n}} \xrightarrow{d} N(0,1). \tag{13}$$

Lemma 3.2 and a little algebra yield

$$\frac{E(Y_{n,1}^4)}{\sigma_n^4} = \frac{432a_n^4 + 864a_n^3 + 499a_n^2 + 67a_n + 1}{\sigma_n^2} \le 864 + \frac{1}{\sigma_n^2}.$$

Therefore,

$$\Big(\frac{1}{nq_n}\Big)\frac{E(Y_{n,1}^4)}{\sigma_n^4} \le \frac{864}{nq_n} + \frac{1}{nq_n\sigma_n^2} \le \frac{864}{nq_n} + \frac{1}{np_n} \longrightarrow 0,$$

as $n \to \infty$; and (13) follows by checking the Lindeberg condition (cf. Chow and Teicher (1988), p. 308, Excercise 9.1.2). Indeed, for any $\epsilon > 0$,

$$\begin{aligned}\frac{1}{\sigma_n^2[nq_n]}\sum_{j=1}^{[nq_n]} E\Big(Y_{n,j}^2 I_{(|Y_{n,j}|>\epsilon\sigma_n\sqrt{[nq_n]})}\Big) &= \frac{1}{\sigma_n^2}E\Big(Y_{n,1}^2 I_{(|Y_{n,1}|>\epsilon\sigma_n\sqrt{[nq_n]})}\Big)\\ &\le \Big(\frac{1}{\epsilon^2[nq_n]}\Big)\frac{E(Y_{n,1}^4)}{\sigma_n^4} \longrightarrow 0,\end{aligned}$$

as $n \to \infty$.

By the definition of $Y_{n,j}$, (13) implies

$$\frac{W_{n,T_{n,[nq_n]}}}{\sigma_n\sqrt{nq_n}} \xrightarrow{d} N(0,1).$$

Hence, in view of Lemma 3.4,

$$\frac{W_{n,T_{n,\alpha_n(n)}}}{\sigma_n\sqrt{nq_n}} \xrightarrow{d} N(0,1).$$

We can also write

$$W_{n,n} = W_{n,T_{n,\alpha_n(n)}} - \sum_{j=1}^{\beta_n(n)} M_{n,n+j},$$

where $M_{n,j} = R_{n,j} - a_n$. Therefore,

$$\frac{W_{n,n}}{\sigma_n\sqrt{nq_n}} \xrightarrow{d} N(0,1),$$

if we are able to show

$$\frac{1}{\sigma_n\sqrt{nq_n}}\sum_{j=1}^{\beta_n(n)} M_{n,n+j}\xrightarrow{p}0 \quad \text{as} \quad n\to\infty. \tag{14}$$

It remains to verify (14). The definition of $R_{n,j}$ gives

$$\sum_{j=1}^{\beta_n(n)} M_{n,n+j} = \sum_{j=1}^{\beta_n(n)} (\beta_n(n)-j) - a_n\beta_n(n) = \frac{1}{2}(\beta_n(n))^2 - (\frac{1}{2}+a_n)\beta_n(n),$$

which, by (9), has the same distribution as $Y_{n,1}$. Since

$$\frac{E(Y_{n,1})^2}{\sigma_n^2 nq_n} = \frac{1}{nq_n}\longrightarrow 0,$$

as $n\to\infty$, we get

$$\frac{Y_{n,1}}{\sigma_n\sqrt{nq_n}} \xrightarrow{p} 0$$

as $n\to\infty$, and (14) follows.

Remark. Both Theorem 1.1 and Theorem 2.1 are also valid for the number of runs on finitely many Bernoulli variables $R'_{n,j} = \min(R_{n,j}, n-j)$, or more precisely, for $W'_{n,k} = \sum_{j=1}^{k}(R'_{n,j} - a_n)$. For example, since

$$0 \le W_{n,n} - W'_{n,n} \le \sum_{j=1}^{\beta_n(n)} M_{n,n+j},$$

(3) follows by (14). This extension is indicated by the referee.

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