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AN EDGEWORTH EXPANSION FOR U-STATISTICS WITH WEAKLY DEPENDENT OBSERVATIONS

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Abstract. Under mild conditions, an Edgeworth expansion with remainder $o(N^{-1/2})$ is established for a U-statistic with a kernel h of degree two using weakly dependent observations. The ease of verifying these conditions is discussed in the context of three rather natural examples.

Key words and phrases: Edgeworth expansion, U-statistics, weak dependence.

1. Introduction

U-statistics based on independent and identically distributed observations were first discussed by Hoeffding (1948) who also showed that they are asymptotically normal under very mild conditions. The rate of convergence to normality was investigated by Grams and Serfling (1973) and Berry-Esseen type bounds were obtained under conditions of increasing generality by Bickel (1974), Chan and Wierman (1977), Callaert and Janssen (1978) and Helmers and van Zwet (1982).

Regarding the more involved problem of Edgeworth expansions for U-statistics with i.i.d. observations, Callaert, Janssen and Veraverbeke (1980) established sufficient conditions for a U-statistic to have a two-term Edgeworth expansion with remainder $o(N^{-1})$. This was followed by Bickel, Götze and van Zwet (1986) who gave more easily verifiable sufficient conditions for the validity of a one-term [two-term] Edgeworth expansion with remainder $o(N^{-1/2})$ [$o(N^{-1})$] respectively.

There has also been an impetus to obtain analogous results for U-statistics based on weakly dependent observations. Such results would, for example, be useful in the investigation of the robustness of U-statistics when the independence assumption is violated in the direction of some kind of weak dependence. Sen (1972), Yoshihara (1976), Denker and Keller (1983) and Harel and Puri (1989) among others obtained various sufficient conditions for the asymptotic normality of U-statistics with dependent observations. Berry-Esseen type bounds were obtained by Yoshihara (1984) for U-statistics generated by absolutely regular processes, Rhee (1988) for U-statistics based on m-dependent observations and Zhao and Chen (1987) for finite population U-statistics.

On the problem of Edgeworth expansions, Kokic and Weber (1990) established conditions for the validity of a one-term Edgeworth expansion for U-statistics based on samples from finite populations and Loh (1994) obtained an Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with an m-dependent shift under weak conditions.

The main aim of this paper is to generalize the result of Loh (1994) by establishing the validity of an Edgeworth expansion for a U-statistic with remainder $o(N^{-1/2})$ when the observations satisfy an absolutely regular condition and a Markov type condition. Before this result (Theorem 1) can be stated, some preliminaries are first needed.

Let $\{X_j : -\infty < j < \infty\}$ be a strictly stationary sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. We assume that there exists a sequence $\{\mathcal{A}_j : -\infty < j < \infty\}$ of sub σ-fields of $\mathcal{A}$ such that for all j, X_j is $\mathcal{A}_{j-m}^{j+m}$ measurable where m is a fixed nonnegative integer and $\mathcal{A}_a^b$ denotes the sub σ-field of $\mathcal{A}$ generated by $\{\mathcal{A}_j : a \le j \le b\}$. We further assume that the $\mathcal{A}_j$'s satisfy an absolutely regular condition and a Markov type condition, namely that there exists a constant $\lambda > 0$ such that for all $n \ge 1$, $p \ge 0$, $-\infty < j < \infty$ and $B \in \mathcal{A}_{j-p}^{j+p}$, we have

$$E\Big[\sup_{A \in \mathcal{A}_{j+n}^{\infty}} |P(A|\mathcal{A}_{-\infty}^{j}) - P(A)|\Big] \le \lambda^{-1} e^{-\lambda n}, \tag{1}$$

and

$$E\Big|P(B|\mathcal{A}_k : k \ne j) - P(B|\mathcal{A}_k : 0 < |j-k| \le n+p)\Big| \le \lambda^{-1} e^{-\lambda n}. \tag{2}$$

We denote the cumulative distribution function of X_j by $F(x)$, $\forall\, x \in R$. Next let $h : R^2 \to R$ be a measurable function symmetric in its two arguments. We assume throughout this paper that there exist constants $\gamma > 2$ and $M > 0$ such that

$$E|h(X_1, X_j)|^{\gamma} \ < \ M, \qquad \forall\, j > 1, \tag{3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y)|^{\gamma} dF(x) dF(y) \ < \ M \tag{4}$$

and, without loss of generality, that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) dF(x) dF(y) = 0. \tag{5}$$

Then $Eh(X_j, X_k)$ exists for all $j < k$. We write

$$h_{j,k}(x,y) = h(x,y) - Eh(X_j, X_k), \quad \forall\, x, y \in R,\ j < k,$$

and for $N \geq 2$, a U-statistic with a kernel h of degree two is defined as

$$U_N = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} h_{j,k}(X_j, X_k).$$

Also we write

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} h(x,y) dF(y), \\ \psi(x,y) &= h(x,y) - g(x) - g(y), \\ \psi_{j,k}(x,y) &= h_{j,k}(x,y) - g(x) - g(y), \quad \forall\, j < k. \end{aligned}$$

Thus for $N \geq 2$, $U_N = (N-1)\sum_{j=1}^{N} g(X_j) + \sum_{a=1}^{N-1}\sum_{b=a+1}^{N} \psi_{a,b}(X_a, X_b)$. We further assume that

$$\sigma_g^2 = E[g^2(X_1) + 2\sum_{j=2}^{\infty} g(X_1)g(X_j)] > 0, \tag{6}$$

and

$$Eg^4(X_1) < \infty. \tag{7}$$

Let σ_N^2 denote the variance of $(N-1)\sum_{j=1}^{N} g(X_j)$. Then by the stationarity of the X_j's and Lemma 1 (see Appendix), we have $\sigma_N^2 = N^3\sigma_g^2 + O(N^2)$ as $N \to \infty$. Next let $\{X_j' : -\infty < j < \infty\}$ be an independent replicate of $\{X_j : -\infty < j < \infty\}$ and

$$\begin{aligned} \kappa_3 = \sigma_g^{-3} E\{ & g^3(X_1) + 3\sum_{j=2}^{\infty}[g^2(X_1)g(X_j) + g(X_1)g^2(X_j)] \\ &+ 6\sum_{j=2}^{\infty}\sum_{k=j+1}^{\infty} g(X_1)g(X_j)g(X_k) \\ &+ 3\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g(X_j)\psi(X_1, X_1')g(X_k')\}. \end{aligned} \tag{8}$$

Using Lemma 1, it can be seen that $-\infty < \kappa_3 < \infty$. We observe that if $E|h(X_j, X_k)|^3 < \infty$ whenever $j < k$, then $\kappa_3 N^{-1/2}$ is an asymptotic approximation [with error $o(N^{-1/2})$] for the third cumulant of $\sigma_N^{-1}U_N$. Define

$$F_N(x) = \Phi(x) - \phi(x)\frac{\kappa_3}{6}N^{-1/2}(x^2 - 1), \quad \forall\, x \in R, \tag{9}$$

where ϕ and Φ denote the standard normal density and distribution function respectively. The main result of this paper is as follows.

Theorem 1. *Suppose* $(1)-(7)$ *are satisfied and that for each* $d>0$*, there exists a constant* $0<\delta_d<1$ *such that*

$$E\Big|E\{e^{it[g(X_{j-m})+\cdots+g(X_{j+m})]}|\mathcal{A}_k : k\neq j\}\Big| < \delta_d, \quad \forall j>m, \tag{10}$$

whenever $|t|\geq d$*. Then*

$$\sup_x |P(\sigma_N^{-1}U_N\leq x)-F_N(x)| = o(N^{-1/2}), \quad \textit{as } N\to\infty.$$

The remainder of this paper is organized as follows. Section 2 examines the weak dependence assumptions (1), (2) and (10) more closely. In particular, three examples are given in which these conditions are shown to hold, the first of which is an m-dependent shift, the second is a homogeneous Markov chain and the third is a stationary Gaussian process. Due to the delicate dependence structure (in contrast to the independence case), the proof of Theorem 1 is rather long and is deferred to Section 3. The Appendix contains technical lemmas which are used in the previous section.

2. Examples

In this section, we shall examine the weak dependence conditions (1), (2) and (10) in the context of three examples.

2.1. On an m-dependent shift

Let $\{\xi_j : -\infty<j<\infty\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega,\mathcal{A},P)$. We suppose that ξ_1 has a probability density function π with respect to Lebesgue measure. Let $f: R^{m+1}\to R$ be a measurable function and define $X_j=f(\xi_j,\ldots,\xi_{j+m})$, $\forall -\infty<j<\infty$. The sequence $\{X_j : -\infty<j<\infty\}$ is said to be an m-dependent shift, and an immediate consequence is that $(\ldots,X_{j-1},X_j)$ and $(X_{j+m+1},X_{j+m+2},\ldots)$ are stochastically independent for all j. With g as in Section 1, assume that $g\circ f: R^{m+1}\to R$ is continuously differentiable such that there exist real numbers $y_1,\ldots,y_{2m+1}$ and an open set $\Theta\supset\{y_1,\ldots,y_{2m+1}\}$ satisfying $\pi(x)>0$ whenever $x\in\Theta$ and

$$\sum_{j=1}^{m+1}\frac{\partial}{\partial x_{m+1}} g\circ f(x_j,\ldots,x_{j+m})|_{(x_1,\ldots,x_{2m+1})=(y_1,\ldots,y_{2m+1})}\neq 0. \tag{11}$$

To verify that Conditions (1), (2) and (10) hold in this case, choose $\mathcal{A}_j$ to be the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty<j<\infty$. Thus X_j is $\mathcal{A}_j^{j+m}$

measurable and it is clear that Conditions (1) and (2) are now trivially satisfied. Next consider the transformation

$$(x_1, \ldots, x_{2m+1}) \mapsto (x_1, \ldots, x_m, x_{m+2}, \ldots, x_{2m+1}, \sum_{j=1}^{m+1} g \circ f(x_j, \ldots, x_{j+m})).$$

From (11) observe that there exists an open set W of R^{2m+1} satisfying $(y_1, \ldots, y_{2m+1}) \in W$ such that the Jacobian of the above transformation is nonzero on W and that $(\xi_1, \ldots, \xi_{2m+1})$ takes values in W with positive probability. Consequently we conclude that the conditional distribution of $g(X_1) + \cdots + g(X_{m+1})$ given $(\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1})$ has a nonzero absolutely continuous component with positive probability. Hence it follows from the Riemann-Lebesgue lemma that (10) holds.

Remark. Due to the special structure of an m-dependent shift, the moment conditions of Theorem 1 can in fact be weakened slightly in this case and still retain the $o(N^{-1/2})$ rate. We refer the reader to Loh (1994) for a precise statement and proof of this result.

2.2. On a homogeneous Markov chain

Let $\{\xi_j : -\infty < j < \infty\}$ be a strictly stationary homogeneous Markov chain defined on a probability space $(\Omega, \mathcal{A}, P)$. Let Ξ and $\mathcal{F}$ denote its state space and the σ-field of measurable subsets of Ξ respectively. Assume that the transition kernel $P(x, A)$ of the Markov chain satisfies

$$\sup_{x,y \in \Xi, A \in \mathcal{F}} |P(x, A) - P(y, A)| = \delta < 1. \tag{12}$$

Let f be a real-valued measurable function defined on Ξ. Write $X_j = f(\xi_j)$, $\forall -\infty < j < \infty$. With g as in Section 1, further assume that $g(X_1)$ satisfies Cramér's condition, that is namely

$$\limsup_{|t| \to \infty} |E \exp[itg(X_1)]| < 1. \tag{13}$$

To verify that the Conditions (1), (2) and (10) are satisfied in this case, take $\mathcal{A}_j$ to be the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty < j < \infty$. Then X_j is $\mathcal{A}_j$ measurable and (2) is immediate from the Markov property. Let π denote the marginal distribution of ξ_1 and $P^n(x, A)$ the n-step transition kernel of the chain. Then from (12) and Nagaev (1961) page 62, we have $\sup_{x \in \Xi, A \in \mathcal{F}} |P^n(x, A) - \pi(A)| \leq \delta^n$, $\forall n \geq 1$, and hence for all $-\infty < j < \infty$ and $n \geq 1$,

$$\sup_{A \in \mathcal{A}_{j+n}^{\infty}} |P(A|\xi_j) - P(A)| \leq \delta^n.$$

This proves (1). Finally (10) follows from (12), (13) and Lemma 2 of Statulevičius (1969, pages 638-641).

2.3. On a stationary Gaussian process

Let $\{\xi_j : -\infty < j < \infty\}$ be a strictly stationary Gaussian process defined on a probability space $(\Omega, \mathcal{A}, P)$. As in Götze and Hipp (1983, page 215), suppose that this process has an absolutely continuous spectrum with a positive analytic spectral density. Let $f : R \to R$ be a function such that, with g as in Section 1, $g \circ f$ is a non-constant continuously differentiable function. Define $X_j = f(\xi_j)$, $\forall -\infty < j < \infty$. As in the previous example, let $\mathcal{A}_j$ denote the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty < j < \infty$. Then X_j is $\mathcal{A}_j$ measurable. Now (2) and (10) follow from Götze and Hipp (1983, pages 219-220) and (1) follows from Ibragimov (1962, page 1801) and Ibragimov and Solev (1969, page 374).

3. Proof of Theorem 1

Let α be a constant to be suitably chosen later for which $3/8 < \alpha < 1/2$. Define

$$T(x) = \begin{cases} x, & \text{if } |x| \leq N^\alpha, \\ xN^\alpha \hat{T}(|x|N^{-\alpha})/|x|, & \text{otherwise,} \end{cases} \tag{14}$$

where $\hat{T} \in C^\infty(0,\infty)$ satisfies $\hat{T}(x) = x$ if $x \leq 1$, $\hat{T}$ is increasing and $\hat{T}(x) = 2$ if $x \geq 2$. Write

$$Y_j = T[g(X_j)], \quad Z_j = Y_j - EY_j, \quad \forall j \geq 1. \tag{15}$$

Let $\hat{\sigma}_N^2$ denote the variance of $(N-1)\sum_{j=1}^N Z_j$, and with γ as in (3), let $\beta = \max\{2/(\gamma-2), 5/4\}$. Define

$$\begin{aligned} \hat{\psi}_{j,k}(X_j, X_k) &= \psi_{j,k}(X_j,X_k) I\{|\psi_{j,k}(X_j,X_k)| \leq N^\beta\}, \quad \forall 1 \leq j < k \leq N, \\ \hat{\Delta}_N &= \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \hat{\psi}_{j,k}(X_j,X_k), \\ \Delta_N &= \hat{\Delta}_N - E\hat{\Delta}_N, \end{aligned}$$

where $I\{|\psi_{j,k}(X_j,X_k)| \leq N^\beta\}$ denotes the indicator function of the event $\{|\psi_{j,k}(X_j,X_k)| \leq N^\beta\}$. Observe that for all $x \in R$,

$$\begin{aligned} &|P(\sigma_N^{-1}U_N \leq x) - F_N(x)| \\ \leq\ & \Big|P(\hat{\sigma}_N^{-1}U_N \leq \sigma_N\hat{\sigma}_N^{-1}x) - P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Y_j + \hat{\Delta}_N] \leq \sigma_N\hat{\sigma}_N^{-1}x\}\Big| \\ &+ \Big|P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N] \leq y\} - F_N(y)\Big| + |F_N(y) - F_N(x)|, \end{aligned} \tag{16}$$

where $y = \sigma_N \hat{\sigma}_N^{-1} x - \hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N EY_j + E\hat{\Delta}_N]$. From the definitions of the Y_j's and $\hat{\Delta}_N$, it follows from (3), (7) and Markov's inequality that

$$\sup_x \Big| P(\hat{\sigma}_N^{-1} U_N \leq \sigma_N \hat{\sigma}_N^{-1} x) - P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Y_j + \hat{\Delta}_N] \leq \sigma_N \hat{\sigma}_N^{-1} x\}\Big|$$
$$= o(N^{-1/2}), \tag{17}$$

as $N \to \infty$. By choosing α sufficiently close to $1/2$, we observe from Lemma 3.30 of Götze and Hipp (1983) that $\sigma_N \hat{\sigma}_N^{-1} = 1 + o(N^{-\omega})$, for some constant $1/2 < \omega < 1$, and hence

$$\sup_x |F_N(y) - F_N(x)| = o(N^{-1/2}), \qquad \text{as } N \to \infty. \tag{18}$$

Thus from (16), (17) and (18) it remains only to prove

$$\sup_x \Big| P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N] \leq y\} - F_N(y)\Big| = o(N^{-1/2}), \qquad \text{as } N \to \infty.$$

Let $\phi_N(t) = E\exp\{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N]\}$, $\forall\, t \in R$, and for κ_3, as in (8), let $\phi_N^*(t) = (1 - i\kappa_3 N^{-1/2} t^3/6)\exp(-t^2/2)$, $\forall t \in R$, be the Fourier transform $\int \exp(itx)dF_N(x)$ of F_N in (9). By the smoothing lemma of Esseen (see for example, Feller (1971), page 538), it suffices to show that

$$\int_{-N^{1/2}\log N}^{N^{1/2}\log N} \left| \frac{\phi_N(t) - \phi_N^*(t)}{t} \right| dt = o(N^{-1/2}), \qquad \text{as } N \to \infty. \tag{19}$$

However (19) is an immediate consequence of Propositions 1 and 2 below. This concludes the proof of Theorem 1.

Proposition 1. *Let $0 < \varepsilon < 1/16$ be as in Lemma 3 (see Appendix). Then*

$$\int_{-N^\varepsilon}^{N^\varepsilon} \left| \frac{\phi_N(t) - \phi_N^*(t)}{t} \right| dt = o(N^{-1/2}), \qquad \textit{as } N \to \infty.$$

Proof. It is well known that for $r \geq 0$,

$$\Big| e^{ix} - \sum_{j=0}^r \frac{(ix)^j}{j\,!} \Big| \leq \min\Big\{ \frac{2}{r!}|x|^{r+\theta}, \frac{|x|^{r+1}}{(r+1)!} \Big\}, \qquad \forall\, \theta \in [0,1). \tag{20}$$

Hence it follows from Lemma 2 that

$$\phi_N(t) = Ee^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j}(1 + it\hat{\sigma}_N^{-1}\Delta_N) + O(t^2 N^{-1}), \tag{21}$$

as $N \to \infty$ uniformly in t. Observe from (21) and Lemma 3 that for α sufficiently close to 1/2,

$$\begin{aligned} &\phi_N(t) - e^{-t^2/2}(1 - \frac{i\hat{\kappa}_3}{6} N^{-1/2} t^3) \\ &= Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} + O(t^2 N^{-1}) + o[(|t|^3 + t^4)e^{-\varepsilon t^2} N^{-1/2}], \end{aligned} \tag{22}$$

as $N \to \infty$ uniformly over $|t| \leq N^{\varepsilon}$. It remains to approximate the term $Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j}$. First, observe that

$$\begin{aligned} &Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} \\ &= \sum_{a=1}^{N-1} \sum_{b=a+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} + O(|t| N^{-3/4}), \end{aligned} \tag{23}$$

as $N \to \infty$ uniformly in t. Next, define for $N \geq 2$, $u = \lceil K \log\ N \rceil$, where K is a positive constant to be suitably chosen later. Here, for all $x \in R$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Define for $1 \leq a < b \leq N$,

$$\begin{aligned} S_{a,b}^{(r)} &= \hat{\sigma}_N^{-1}(N-1) \sum_{1 \leq j \leq N, |j-a| \wedge |j-b| > ru} Z_j, \qquad \forall\, r \geq 1, \\ S_{a,b}^{(0)} &= \hat{\sigma}_N^{-1}(N-1) \sum_{j=1}^{N} Z_j. \end{aligned} \tag{24}$$

Using Lemmas 4, 5 and a method of Tikhomirov (1980), we have, for sufficiently large K,

$$\begin{aligned} &\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} \\ &= -\frac{i}{2}\sigma_g^{-3} t^3 e^{-t^2/2} N^{-1/2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1, X_1')g(X_k') \\ &\quad + O[(|t| + t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}], \end{aligned} \tag{25}$$

as $N \to \infty$ uniformly over $|t| \leq N^{\varepsilon}$ where $\{X_j' : -\infty < j < \infty\}$ denotes an independent replicate of $\{X_j : -\infty < j < \infty\}$. In a similar though less tedious way, we have, for sufficiently large K,

$$\begin{aligned} &\sum_{a=1}^{N-1} \sum_{b=a+1}^{(a+3u)\wedge N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} \\ &= O[(|t| + t^2)N^{-1}\log^2 N], \end{aligned} \tag{26}$$

as $N \to \infty$ uniformly in t. Thus, it follows from (23), (25) and (26) that

$$
\begin{aligned}
& E i t \hat{\sigma}_N^{-1} \Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} \\
= & -\frac{i}{2}\sigma_g^{-3} t^3 e^{-t^2/2} N^{-1/2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1, X_1')g(X_k') \\
& + O[(|t| + t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}],
\end{aligned}
$$

as $N \to \infty$ uniformly over $|t| \leq N^{\varepsilon}$. Consequently, we conclude from (8) and (22) that

$$
\begin{aligned}
& \phi_N(t) - \phi_N^*(t) \\
= & \, O[(|t| + t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}] + o[(|t|^3 + t^4)e^{-\varepsilon t^2}N^{-1/2}],
\end{aligned}
$$

as $N \to \infty$ uniformly over $|t| \leq N^{\varepsilon}$ and hence Proposition 1 follows.

Next, observe from (10) and Bhattacharya and Rao (1986, page 212) that for sufficiently large N, there exists a constant $0 < \delta < 1$ such that

$$
E\Big|E[e^{it(Z_{j-m}+\cdots+Z_{j+m})}|\mathcal{A}_k : k \neq j]\Big| \leq \delta, \qquad \forall\, j > m, \tag{27}
$$

whenever $|t| \geq 1/(2\sigma_g)$. Now it follows from Lemma 3.2 of Götze and Hipp (1983) that there exists a constant $\mu > 0$ such that

$$
E\Big|E[e^{it(Z_{j-m}+\cdots+Z_{j+m})}|\mathcal{A}_k : k \neq j]\Big| \leq e^{-\mu t^2}, \qquad \forall\, j > m, \tag{28}
$$

whenever $|t| \leq 3/(2\sigma_g)$.

Proposition 2. *Let ε be as in Proposition 1. Then*

$$
\int_{N^{\varepsilon} \leq |t| \leq N^{1/2}\log N} \left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}), \qquad \textit{as } N \to \infty.
$$

Proof. It is easy to see that

$$
\int_{|t| \geq N^{\varepsilon}} |\phi_N^*(t)/t| dt = o(N^{-1/2}),
$$

as $N \to \infty$. Hence it suffices only to show

$$
\int_{N^{\varepsilon} \leq |t| \leq N^{1/2}\log N} |\phi_N(t)/t| dt = o(N^{-1/2}), \qquad \text{as } N \to \infty.
$$

Let n and s be integer-valued functions of N satisfying $2m < s < n < N$ such that $s \to \infty$, $n \to \infty$ and $ne^{-\lambda s/2} \to 0$ as $N \to \infty$. Define

$$
\hat{\Delta}_N(n) = \sum_{j=1}^{n} \sum_{k=j+1}^{N} \hat{\psi}_{j,k}(X_j, X_k), \qquad \Delta_N(n) = \hat{\Delta}_N(n) - E\hat{\Delta}_N(n). \tag{29}
$$

Then it follows from (20) and Lemma 2 that

$$|\phi_N(t)| = \Big|Ee^{it\hat{\sigma}_N^{-1}[(N-1)\Sigma_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]\Big| + O(t^2 n N^{-2}), \quad (30)$$

as $N \to \infty$ uniformly in t. We shall now approximate the first term of the r.h.s. of (30). Let $1 \le a < b \le N$ and define $\mathcal{J}_{a,b} = \{1, \ldots, n\}\backslash\{a, b\}$. Divide $\mathcal{J}_{a,b}$ into blocks $B_0, A_1, B_1, \ldots, A_l, B_l$ as follows. Define $j_1, \ldots, j_l$ by

$$\begin{aligned} j_1 &= \inf\{j \in \mathcal{J}_{a,b} : [j-s, j+s] \subseteq \mathcal{J}_{a,b}\}, \\ j_{p+1} &= \inf\{j > j_p + 5s : [j-s, j+s] \subseteq \mathcal{J}_{a,b}\}, \qquad \forall\, 1 \le p \le l-1, \end{aligned}$$

where $l+1$ is the smallest integer for which the infimum is undefined. Write

$$\begin{aligned} A_p &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : |j - j_p| \le s\}, \qquad \forall\, 1 \le p \le l, \\ B_0 &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in \mathcal{J}_{a,b}, \quad 1 \le j \le j_1 - s - 1\}, \\ B_p &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in \mathcal{J}_{a,b}, \quad j_p + s + 1 \le j \le j_{p+1} - s - 1\}, \\ &\qquad \forall\, 1 \le p \le l-1, \\ B_l &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in \mathcal{J}_{a,b}, \quad j \ge j_l + s + 1\}, \end{aligned}$$

and

$$\begin{aligned} R_{a,b} = {} & it\hat{\sigma}_N^{-1}[\hat{\psi}_{a,b}(X_a, X_b) - E\hat{\psi}_{a,b}(X_a, X_b)]e^{it\hat{\sigma}_N^{-1}[\Delta_N - \Delta_N(n)]} \\ & \times \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : 1 \le j \le N, \quad j \notin \mathcal{J}_{a,b}\}. \end{aligned}$$

Using the convention that the product over an empty set is 1, we have

$$\begin{aligned} & it\hat{\sigma}_N^{-1}[\hat{\psi}_{a,b}(X_a, X_b) - E\hat{\psi}_{a,b}(X_a, X_b)]e^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]} \\ = {} & R_{a,b}B_0 \prod_{p=1}^{l} A_p B_p. \end{aligned}$$

Now observe from (2) that

$$E\Big|E(A_p|\mathcal{A}_j : j \ne j_p) - E(A_p|\mathcal{A}_j : 0 < |j - j_p| \le 2s)\Big| \le 4\lambda^{-1}e^{-\lambda(s-m)},$$

and hence

$$\begin{aligned} & \Big|E\Big[R_{a,b}B_0 \prod_{p=1}^{l} A_p B_p - R_{a,b}B_0 \prod_{p=1}^{l} B_p E(A_p|\mathcal{A}_j : 0 < |j - j_p| \le 2s)\Big]\Big| \\ \le {} & 8\lambda^{-1}|t|\hat{\sigma}_N^{-1} n N^{\beta} e^{-\lambda(s-m)}. \end{aligned}$$

We thus conclude that

$$\Big|ER_{a,b}B_0\prod_{p=1}^{l}A_pB_p\Big| \leq 2|t|\hat{\sigma}_N^{-1}N^{\beta}E\prod_{p=1}^{l}\Big|E(A_p|\mathcal{A}_j : 0<|j-j_p|\leq 2s)\Big| + 8\lambda^{-1}|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda(s-m)}. \tag{31}$$

By repeated use of Lemma 1 with $\nu=1$, observe that the r.h.s. of (31) is bounded by

$$\begin{aligned}&2|t|\hat{\sigma}_N^{-1}N^{\beta}\prod_{p=1}^{l}E\Big|E(A_p|\mathcal{A}_j : 0<|j-j_p|\leq 2s)\Big| + O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2})\\ \leq\ &2|t|\hat{\sigma}_N^{-1}N^{\beta}\prod_{p=1}^{l}E\Big|E(A_p|\mathcal{A}_j : j\neq j_p)\Big| + O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2}),\end{aligned} \tag{32}$$

as $N\to\infty$ uniformly in a,b and t.

Case I. Suppose that $N^{1/2}\leq|t|\leq N^{1/2}\log N$. For sufficiently large N, take $n=\lceil K_1\log^2 N\rceil$ and $s=\lceil K_1\log N\rceil$ where K_1 and K_2 are positive constants to be suitably chosen later. Observe from (27), (31) and (32) that

$$\Big|ER_{a,b}B_0\prod_{p=1}^{l}A_pB_p\Big| \leq 2|t|\hat{\sigma}_N^{-1}N^{\beta}\delta^{l} + O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2}),$$

as $N\to\infty$ uniformly over $|t|\geq N^{1/2}$ and $1\leq a<b\leq N$. Note that $|l-n/(5s+1)|=O(1)$, as $N\to\infty$ uniformly over $1\leq a<b\leq N$. By choosing K_1 and K_2 so that $K_1K_2^{-1}$ and K_2 are both sufficiently large, we have $|ER_{a,b}B_0\prod_{p=1}^{l}A_pB_p| = O(|t|N^{-5/2})$, as $N\to\infty$ uniformly over $|t|\geq N^{1/2}$ and $1\leq a<b\leq N$. From the definitions of $R_{a,b}$, B_0, A_p and B_p, with $1\leq p\leq l$, we conclude that

$$\Big|Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}[(N-1)\Sigma_{j=1}^{N}Z_j+\Delta_N-\Delta_N(n)]}\Big| = O(|t|nN^{-3/2}), \tag{33}$$

as $N\to\infty$ uniformly over $|t|\geq N^{1/2}$. In a similar way it can be shown that

$$\Big|Ee^{it\hat{\sigma}_N^{-1}[(N-1)\Sigma_{j=1}^{N}Z_j+\Delta_N-\Delta_N(n)]}\Big| = O(N^{-1}), \tag{34}$$

as $N\to\infty$ uniformly over $|t|\geq N^{1/2}$. Thus, we conclude from (30), (33) and (34) that $|\phi_N(t)|=O(N^{-1}+|t|nN^{-3/2}+t^2nN^{-2})$, as $N\to\infty$ uniformly over $|t|\geq N^{1/2}$ and hence

$$\int_{N^{1/2}\leq|t|\leq N^{1/2}\log N}|\phi_N(t)/t|dt = o(N^{-1/2}), \qquad \text{as } N\to\infty. \tag{35}$$

Case II. Suppose that $N^\varepsilon \le |t| \le N^{1/2}$. Now for sufficiently large N, take $n = \lceil K_1 t^{-2} N \log^2 N \rceil$ and $s = \lceil K_2 \log N \rceil$ where K_1 and K_2 are positive constants to be suitably chosen later. Observe from (28), (31) and (32) that

$$\Big| ER_{a,b} B_0 \prod_{p=1}^{l} A_p B_p \Big| \le 2|t|\hat{\sigma}_N^{-1} N^\beta e^{-\mu l [t \hat{\sigma}_N^{-1}(N-1)]^2} + O(|t|\hat{\sigma}_N^{-1} n N^\beta e^{-\lambda s/2}),$$

as $N \to \infty$ uniformly over $|t| \le N^{1/2}$ and $1 \le a < b \le N$. Note that $|l - n/(5s+1)| = O(1)$, as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $N^\varepsilon \le |t| \le N^{1/2}$. Thus, we conclude that, by choosing K_1 and K_2 such that $K_1 K_2^{-1}$ and K_2 are both sufficiently large, $|ER_{a,b} B_0 \prod_{p=1}^{l} A_p B_p| \le O(|t| N^{-5/2})$, as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$ and $1 \le a < b \le N$. Thus

$$\Big| Eit\hat{\sigma}_N^{-1} \Delta_N(n) e^{it\hat{\sigma}_N^{-1}[(N-1)\Sigma_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]} \Big| = O(|t| n N^{-3/2}), \tag{36}$$

as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$. Similarly it can be shown that

$$\Big| Ee^{it\hat{\sigma}_N^{-1}[(N-1)\Sigma_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]} \Big| = O(N^{-1}), \tag{37}$$

as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$. We conclude from (30), (36) and (37) that $|\phi_N(t)| = O(N^{-1} + |t| n N^{-3/2} + t^2 n N^{-2})$, as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$ and hence

$$\int_{N^\varepsilon \le |t| \le N^{1/2}} |\phi_N(t)/t| dt = o(N^{-1/2}), \tag{38}$$

as $N \to \infty$. Proposition 2 now follows from (35) and (38).

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Appendix

Let $\{X_j : -\infty < j < \infty\}$ be as in Section 1 and for $j_1 < \cdots < j_l$, write $P_p^{(l)}(A^{(p)} \times A^{(l-p)}) = P[(X_{j_1}, \ldots, X_{j_p}) \in A^{(p)}]P[(X_{j_{p+1}}, \ldots, X_{j_l}) \in A^{(l-p)}]$ for all $1 \le p < l$, and $P_0^{(l)}(A^{(l)}) = P[(X_{j_1}, \ldots, X_{j_l}) \in A^{(l)}]$, whenever $A^{(k)}$ is a measurable subset of R^k with $1 \le k \le l$.

Lemma 1. *Let $1 \le p < l$ and $f : R^l \to R$ be a measurable function such that there exist positive constants ν and C satisfying*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_l)|^{1+\nu} dP_k^{(l)} < C, \qquad k = 0, p.$$

Then, for $j_{p+1} - j_p > 2m$, we have

$$\left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_l) dP_0^{(l)} - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_l) dP_p^{(l)} \right|$$
$$\le 4C^{1/(1+\nu)} \{\lambda^{-1} \exp[-\lambda(j_{p+1} - j_p - 2m)]\}^{\nu/(1+\nu)}.$$

Proof. Since X_j is $\mathcal{A}_{j-m}^{j+m}$ measurable, the result follows directly from Lemma 1 of Yoshihara (1976) and (1).

Lemma 2. *Let $\Delta_N(n)$ be defined as in (29) with $1 < n < N$. Then $E\Delta_N^2(n) = O(nN)$, as $N \to \infty$. In particular, we have $E\Delta_N^2 = O(N^2)$, as $N \to \infty$.*

Proof. Since $E\Delta_N^2(n) \le E\hat{\Delta}_N^2(n)$, it suffices to show that $E\hat{\Delta}_N^2(n) = O(nN)$, as $N \to \infty$. By Hölder's inequality, observe from (3) and the definition of β that

$$E\Big|\psi_{a,b}(X_a, X_b)\psi_{j,k}(X_j, X_k) I\{|\psi_{a,b}(X_a, X_b)| > N^{\beta}\}\Big| = O(N^{-2}), \tag{39}$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $1 \le j < k \le N$. Using the techniques introduced by Yoshihara (1976) in the proof of his Lemma 2, observe that

$$E \sum_{a=1}^{n} \sum_{b=a+1}^{N} \sum_{j=1}^{n} \sum_{k=j+1}^{N} \psi_{a,b}(X_a, X_b)\psi_{j,k}(X_j, X_k) = O(nN), \tag{40}$$

as $N \to \infty$. Lemma 2 now follows from (39), (40) and the definition of $\hat{\Delta}_N(n)$.

Lemma 3. *Let $S_{a,b}^{(r)}$, α, Z_j be as in (24), (14), (15) respectively and*

$$\begin{aligned} \hat{\kappa}_3 &= \sigma_g^{-3} E\{g^3(X_1) + 3\sum_{j=2}^{\infty}[g^2(X_1)g(X_j) + g(X_1)g^2(X_j)] \\ &\quad + 6\sum_{j=2}^{\infty}\sum_{k=j+1}^{\infty} g(X_1)g(X_j)g(X_k)\}. \end{aligned}$$

Then, for α sufficiently close to $1/2$, there exists a constant $0 < \varepsilon < 1/16$ such that

$$Ee^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} = e^{-t^2/2}\Big(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3\Big) + o[(|t|^3 + t^4)e^{-\varepsilon t^2}N^{-1/2}],$$

and

$$Ee^{itS_{a,b}^{(r)}} = e^{-t^2/2} + O(|t|N^{-1/2}\log N), \qquad \forall\, 1 \le r \le 2,$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$ and $1 \le a < b \le N$.

Proof. The proof of the first statement follows from (3.36) and Lemma 3.30 of Götze and Hipp (1983). The second statement now follows from the first statement and the observation that

$$\left| E[e^{it\hat{\sigma}_N^{-1}(N-1)\Sigma_{j=1}^N Z_j} - e^{itS_{a,b}^{(r)}}] \right| = O(|t|N^{-1/2}\log N),$$

as $N \to \infty$ uniformly in a, b and t.

Lemma 4. *With the notation of Proposition* 1, *for sufficiently large K we have*

$$\begin{aligned}
&\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[e^{it(S_{a,b}^{(0)} - S_{a,b}^{(1)})} - 1]e^{itS_{a,b}^{(2)}} \\
= &-\frac{i}{2}e^{-t^2/2}t^3\sigma_g^{-3}N^{-1/2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1, X_1')g(X_k') \\
&+ O[(|t| + t^4)N^{-1}\log N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}],
\end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$, where $\mathcal{P}(|t|)$ is a linear combination (not depending on N) of non-negative powers of $|t|$ and $\{X_j' : -\infty < j < \infty\}$ denotes an independent replicate of $\{X_j : -\infty < j < \infty\}$.

Lemma 5. *With the notation of Proposition* 1, *we have, for sufficiently large K,*

$$\begin{aligned}
&\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} \left| Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)\prod_{l=1}^{2}[e^{it(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]e^{itS_{a,b}^{(2)}} \right| \\
&= O[(|t| + t^4)N^{-1}\log^3 N],
\end{aligned}$$

as $N \to \infty$ uniformly in t.

The proofs of Lemma 4 and 5 are rather long and we refer the reader to Loh (1991) for the proofs.

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