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CONSTRUCTION OF OPTIMAL MIXED-LEVEL SUPERSATURATED DESIGNS

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Abstract: Supersaturated designs (SSDs) offer a potentially useful way to investigate many factors with only a few experiments during the preliminary stages of experimentation. While the construction and analysis of symmetrical SSDs have been widely explored, asymmetrical (or mixed-level) SSDs deserve further investigation. Mixed-level SSDs can be judged by various criteria. But, justified by existing results, the χ^2 criterion proposed by Yamada and Lin (1999) is adopted here. Optimality results for mixed-level SSDs are provided. A new construction method for χ^2-optimal SSDs is proposed, and we discuss properties of the resulting designs. Many new designs are tabulated for practical use.

Key words and phrases: Balanced design, column juxtaposition, Kronecker sum, orthogonal array.

1. Introduction

A supersaturated design (SSD) is essentially a factorial design whose run size is insufficient for estimating all the main effects represented by the design matrix. In an industrial or scientific experiment, if many factors are to be investigated (e.g. in a screening study) and the experiment is expensive to conduct, economic considerations may compel the adoption of an SSD. The data collected by SSDs are typically analyzed under the assumption of effect sparsity, i.e., the response of interest depends mainly on a few dominant or active factors, and the interactions and the effects of the remaining factors are relatively negligible. SSDs were introduced by Box (1959), but not studied further until the appearance of the work by Lin (1991, 1993) and Wu (1993). Since then there has been a large number of papers on this subject, for example, Xu and Wu (2005), Georgiou, Koukouvinos and Mantas (2006), Yamada, Matsui, Matsui, Lin and Tahashi (2006) Zhang, Zhang and Liu (2007) Liu, Liu and Zhang (2007), Chen and Liu (2008) and Nguyen and Cheng (2008). Various fields of research may benefit from the use of SSDs, including computer and medical experiments (Lin (1995)), industrial and engineering experiments (Wu (1993), Lin (1999, 2003) and Nguyen (1996)).

To set the issues, consider a study conducted by Nguyen and Cheng (2008) to examine the factors affecting the thermal performance of project homes. They needed an SSD with 16 runs and 18 two-level factors. Motivated by their study, suppose the factors are as follows: (1) wall insulation (R1, R1.5 or R2); (2) roof insulation (R2.5, R3 or R3.5); (3) floor insulation (R0, R0.5 or R1); (4) floor type (timber, leather or tile); (5) wall type (brick veneer, cavity or concrete); (6) north glass (5% or 20%); (7) east glass (5% or 15%); (8) west glass (5% or 15%); (9) south glass (5% or 15%); (10) north blinds (yes or no); (11) east blinds (yes or no); (12) west blinds (yes or no); (13) south blinds (yes or no); (14) north eave overhang (20% or 100%); (15) east eave overhang (20% or 70%); (16) west eave overhang (20% or 70%); and (17) south eave overhang (20% or 100%). The number of homes that can be used for this study is 12. Then we need an SSD with 12 runs, 5 three-level factors and 12 two-level factors. For the purpose of screening the active factors and keeping the prices of these homes comparable, it is further asked that (i) any two-level factor and three-level factor be orthogonal to each other; (ii) each home have six factors at the low level and six at the high level for all 12 two-level factors; (iii) for any two homes, either they take the same level on each of the three-level factors and different levels on each of the two-level factors, or they take the same level on only one of the three-level factors and take the level combinations on the two-level factors equally often. These constraints make existing mixed-level SSDs inapplicable. See, for example, Deng, Lin and Wang (1996), Liu and Zhang (2001), Yamada and Matsui (2002), Yamada and Lin (2002), Fang, Lin and Liu (2003b), Li, Liu and Zhang (2004), Fang, Ge, Liu and Qin (2004a), Koukouvinos and Mantas (2005), Yamada, Matsui, Matsui, Lin and Tahashi (2006) and Chen and Liu (2008).

This paper attempts to provide further optimality results for mixed-level SSDs and to find a combinatorial solution to the problems exemplified above. Section 2 reviews the $\chi^2(D)$ criterion (Yamada and Lin (1999) and Yamada and Matsui (2002)) and other optimality criteria for mixed-level SSDs. In particular, the $\chi^2(D)$ is well justified by some existing results, and is adopted as the optimality criterion for evaluating mixed-level SSDs in this paper. Section 3 presents some optimality results for mixed-level SSDs. Especially, optimal mixed-level SSDs are shown to be periodic. These optimality results indicate a feasible way to construct (nearly) $\chi^2(D)$-optimal mixed-level SSDs. And a new method for constructing them is proposed in Section 4. Many designs constructed from this new method are tabulated in the Appendix.

2. Optimality Criteria

Some definitions and notation are necessary in order to review the optimality criteria. Thus, a mixed-level (or asymmetrical) design of n runs and m factors with levels $q_1, \ldots, q_m$, denoted by $D(n; q_1, \ldots, q_m)$, is an $n \times m$ matrix $D = (d_{ij})$ in which the jth column takes values from a set of q_j symbols $\{0, \ldots, q_j - 1\}$. A $D(n; q_1, \ldots, q_m)$ is called an *orthogonal array* (OA) of strength two, denoted by $L_n(q_1, \ldots, q_m)$, if all possible level combinations for any two factors appear equally often. When $\sum_{j=1}^{m}(q_j - 1) = n - 1$, the design $D(n; q_1, \ldots, q_m)$ is called a *saturated* design; when $\sum_{j=1}^{m}(q_j - 1) > n - 1$, the design is called a *supersaturated* design, denoted by $S(n; q_1, \ldots, q_m)$. When some q_j's are equal, we use the notations $D(n; q_1^{r_1} \cdots q_l^{r_l})$, $L_n(q_1^{r_1} \cdots q_l^{r_l})$, and $S(n; q_1^{r_1} \cdots q_l^{r_l})$, respectively, where $\sum_{j=1}^{l} r_j = m$. Two columns (or rows) are called *orthogonal* if they (or their transposes) form an OA of strength two, and called *fully aliased* if one can be obtained from the other by permuting levels. In a design, it is necessary that no columns are fully aliased.

Throughout the paper, we only consider *balanced* (with equal occurrence property) designs in which all levels appear equally often for any column.

2.1. $\chi^2(D)$ and $E(f_{NOD})$ criteria

Let $c_1, \ldots, c_m$ be the columns of an $S(n; q_1, \ldots, q_m)$ design D, and $n_{uv}^{(ij)}$ be the number of (u, v)-pairs in (c_i, c_j). Yamada and Lin (1999) defined an index between c_i and c_j, by analogy with the χ^2 statistic,

$$\chi^2(c_i, c_j) = \sum_{u=0}^{q_i-1} \sum_{v=0}^{q_j-1} \frac{[n_{uv}^{(ij)} - n/(q_i q_j)]^2}{n/(q_i q_j)},$$

to evaluate the dependency of the two columns. The $\chi^2(D)$ criterion defined by Yamada and Matsui (2002) is to minimize

$$\chi^2(D) = \sum_{1 \le i < j \le m} \chi^2(c_i, c_j).$$

Fang, Lin and Liu (2003b) proposed the $E(f_{NOD})$ criterion for comparing mixed-level SSDs from the viewpoint of orthogonality and uniformity: minimize

$$E(f_{NOD}) = \frac{2}{m(m-1)} \sum_{1 \le i < j \le m} \frac{\chi^2(c_i, c_j) n}{q_i q_j}.$$

Note that the $\chi^2(D)$ considers different weights for factors with different levels, while $E(f_{NOD})$ does not.

It is obvious that the $\chi^2(D)$ and $E(f_{NOD})$ criteria are equivalent in the symmetric case. It has been shown as well that they are extensions of existing criteria defined for symmetrical SSDs, see Fang, Lin and Liu (2003b) Xu (2003) and Li, Liu and Zhang (2004) for details.

2.2. Other optimality criteria and connections

There are several other optimality criteria for evaluating mixed-level SSDs. One is the *generalized minimum aberration* criterion developed by Ma and Fang

(2001) and Xu and Wu (2001). Based on the ANOVA decomposition model, for a design $D(n; q_1, \ldots, q_m)$, let $X_j = (x^j_{ik})$ be the matrix consisting of all j-factor contrast coefficients for $j = 0, \ldots, m$. If

$$A_j(D) = \frac{1}{n^2} \sum_k \Big| \sum_{i=1}^n x^j_{ik} \Big|^2,$$

the generalized minimum aberration criterion is to sequentially minimize $A_j(D)$ for $j = 1, \ldots, m$.

For a design $D = (d_{ij})$, let

$$\delta_{ij}(D) = \sum_{k=1}^m q_k \delta^{(k)}_{ij},$$

where $\delta^{(k)}_{ij} = 1$ if $d_{ik} = d_{jk}$, and 0 otherwise; $\delta_{ij}(D)$ is called the *natural weighted coincidence number* between the ith and jth rows of D. Define the tth power moment to be

$$M_t(D) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} [\delta_{ij}(D)]^t,$$

where t is a positive integer. The *minimum moment aberration* criterion proposed by Xu (2003) is to sequentially minimize $M_t(D)$ for $t = 1, \ldots, m$.

Hickernell and Liu (2002) developed the minimum projection uniformity criterion for a $D(n; q_1, \ldots, q_m)$ design D. Define the t-dimensional projection discrepancy $\mathcal{D}_{(t)}(D; K)$ as the non-negative square root of

$$\mathcal{D}^2_{(t)}(D; K) = \frac{1}{n^2} \sum_{i,j=1}^n \sum_{1 \le l_1 < \cdots < l_t \le m} \prod_{g=1}^t \left(-1 + q_{l_g} \delta^{(l_g)}_{ij} \right).$$

The *minimum projection uniformity* criterion is to sequentially minimize $\mathcal{D}_{(t)}(D; K)$ for $t = 1, \ldots, m$.

Recently, Liu, Fang and Hickernell (2006) generalized the $\chi^2(D)$ criterion to the so-called *minimum* χ^2 *criterion*, and investigated the connections among these four criteria. Especially, their Corollary 1 implies the following.

Lemma 1. *For any $S(n; q_1, \ldots, q_m)$ design D, $\mathcal{D}^2_{(1)}(D; K) = A_1(D) = 0$, $M_1(D)$ is minimized, and*

$$\mathcal{D}^2_{(2)}(D; K) = A_2(D) = \frac{\chi^2(D)}{n} = \frac{n-1}{2n} \left[M_2(D) - \gamma_1 \right],$$

where γ_1 is a constant depending on n, m, and the levels $q_1, \ldots, q_m$.

This result implies that, though $A_2(D), M_2(D), \mathcal{D}^2_{(2)}(D;K)$ and $\chi^2(D)$ arise from distinct considerations, they are strongly connected: an $S(n;q_1,\ldots,q_m)$ design that minimizes one of these criteria will minimize them all. This conclusion provides a strong justification for using $\chi^2(D)$ as an optimality criterion for choosing mixed-level SSDs, and we adopt it as the optimality criterion for assessing mixed-level SSDs.

3. Optimality Properties of the $\chi^2(D)$ Criterion

This section provides some optimality results on $\chi^2(D)$ for mixed-level SSDs.

3.1. $\chi^2(D)$ for the design obtained by column juxtaposition

For any $D(n;q_1^{r_1}\cdots q_l^{r_l})$ design D, it is obvious that

$$\sum_{i=1,i\neq j}^{n}\delta_{ij}(D)=\sum_{k=1}^{l}r_k(n-q_k). \tag{3.1}$$

Theorem 1 of Li, Liu and Zhang (2004) shows the following.

Lemma 2. *For any $D(n;q_1^{r_1}\cdots q_l^{r_l})$ design D with $m=\sum_{k=1}^{l}r_k$,*

$$\chi^2(D)=\frac{\sum_{i,j=1,i\neq j}^{n}[\delta_{ij}(D)]^2}{2n}+\frac{1}{2}\Big\{\Big[\sum_{k=1}^{l}r_kq_k\Big]^2-n\Big[\sum_{k=1}^{l}r_kq_k+m(m-1)\Big]\Big\} \tag{3.2}$$

$$\geq\frac{n}{2(n-1)}\Big[\sum_{k=1}^{l}r_kq_k\Big]^2-\frac{n(n-1)+2mn}{2(n-1)}\sum_{k=1}^{l}r_kq_k+\frac{mn(m+n-1)}{2(n-1)}. \tag{3.3}$$

Equality holds if and only if $\delta_{ij}(D)$ is a constant for all $i\neq j$.

Further, if D is a saturated $L_n(q_1^{s_1}\cdots q_l^{s_l})$, then $\sum_{k=1}^{l}s_k(q_k-1)=n-1$ and, from Mukerjee and Wu (1995),

$$\delta_{ij}(D)=\sum_{k=1}^{l}s_k-1,\ \text{for } i\neq j, \tag{3.4}$$

which implies that D is $\chi^2(D)$ optimal.

Theorem 4 of Li, Liu and Zhang (2004) and Corollary 3 of Liu, Fang and Hickernell (2006) show the $\chi^2(D)$ optimality of mixed-level SSDs obtained by column juxtaposition of two or more SSDs with constant natural weighted coincidence numbers. The theorem below gives the change in $\chi^2(D)$ values when two designs are column juxtaposed, in particular when one of the two designs is a saturated OA.

Theorem 1. *Suppose D_0 is a $D(n; q_1^{r_1} \cdots q_l^{r_l})$ and D_1 is a $D(n; q_1^{s_1} \cdots q_l^{s_l})$. Let $D_0 \cup D_1$ be the column juxtaposition of D_0 and D_1. Then*

$$\chi^2(D_0 \cup D_1) = \chi^2(D_0) + \chi^2(D_1) + \frac{\sum_{i,j=1, i\neq j}^{n} \delta_{ij}(D_0)\delta_{ij}(D_1)}{n} + \Big[\sum_{k=1}^{l} r_k q_k\Big]\Big[\sum_{k=1}^{l} s_k q_k\Big] - n\sum_{k=1}^{l} r_k \sum_{k=1}^{l} s_k. \tag{3.5}$$

Further, if D_1 has constant $\delta_{ij}(D_1)$'s for $i \neq j$, then

$$\chi^2(D_0 \cup D_1) = \chi^2(D_0) + \gamma_2, \tag{3.6}$$

where γ_2 is a constant depending on n, q_i, r_i and s_i for $i = 1, \ldots, l$. In particular, if D_1 is a saturated $L_n(q_1^{s_1} \cdots q_l^{s_l})$, then

$$\chi^2(D_1) = 0, \text{ and} \tag{3.7}$$

$$\chi^2(D_0 \cup D_1) = \chi^2(D_0) + n\sum_{k=1}^{l} r_k(q_k - 1). \tag{3.8}$$

Proof. To derive (3.5), we first express $\chi^2(D_0 \cup D_1)$ in terms of $\delta_{ij}(D_0 \cup D_1)$ based on (3.2), then note that $\delta_{ij}(D_0 \cup D_1) = \delta_{ij}(D_0) + \delta_{ij}(D_1)$. Using the expressions for $\chi^2(D_0)$ and $\chi^2(D_1)$ in (3.2), (3.5) is obtained following lengthy but straightforward algebra.

Equation (3.6) follows from (3.5) by noting that $\chi^2(D_1)$ attains its lower bound in (3.3), and that (3.1) holds for D_0. Equations (3.7) and (3.8) follow directly since (3.4) holds for D_1.

Theorem 1 provides a method for constructing $\chi^2(D)$-optimal or nearly optimal SSDs by column-juxtaposing a design D_0 to a saturated OA, or an SSD D_1 with constant $\delta_{ij}(D_1)$'s for $i \neq j$. From (3.6) and (3.8), if D_0 is $\chi^2(D)$-optimal, then the resulting design D is $\chi^2(D)$-optimal among those designs obtained by column-juxtaposing a design to D_1, which is also an optimal design. Of course, optimality may not be achievable among $D(n; q_1^{(r_1+s_1)} \cdots q_l^{(r_l+s_l)})$'s; the resulting design does have a $\chi^2(D)$ value very close to the lower bound in Lemma 2, thus it is a nearly $\chi^2(D)$-optimal SSD. For example, the design D_0 can be selected to be a design with only one balanced column, or with two orthogonal (or two nearly orthogonal) columns c_1 and c_2, or more generally a $\chi^2(D)$-optimal design. The next subsection shows when the resulting design is optimal among designs with the same parameters.

Remark 1. Theorem 2 of Yamada and Matsui (2002) showed the $\chi^2(D)$ optimality of a design D obtained by column-juxtaposing several *symmetrical* saturated

OA's $D_1, \ldots, D_s$. The $\chi^2(D)$ value of this design can be easily obtained by using (3.8) recursively since $\chi^2(D_k) = 0$ for $k = 1, \ldots, s$, and the optimality of this design is ensured since $\delta_{ij}(D) = \sum_{k=1}^{s} \delta_{ij}(D_k)$ and (3.4) holds for each D_k. When the saturated OA's are asymmetrical, the resulting design is still $\chi^2(D)$-optimal based on Theorem 4 of Li, Liu and Zhang (2004), or Corollary 3 of Liu, Fang and Hickernell (2006). If not all the D_k's are saturated OA's, the $\chi^2(D)$ optimality of the resulting design is unclear, but see the subsection below.

3.2. Periodicity of minimum $\chi^2(D)$

Given n and $q_1, \ldots, q_l$, let $f(r_1, \ldots, r_l) = \min\{\chi^2(D) : D$ is an $S(n; q_1^{r_1} \cdots q_l^{r_l})\}$, where designs may have fully aliased columns. The following result shows that for certain n, $f(r_1, \ldots, r_l)$ is periodic when the number of factors is sufficiently large.

Theorem 2. *Suppose a saturated design $L_n(q_1^{s_1} \cdots q_l^{s_l})$ exists. Then for $i = 1, \ldots, l$, there exist positive integers R_i such that for $r_i \geq R_i$, we have*

$$f(r_1 + s_1, \ldots, r_l + s_l) = f(r_1, \ldots, r_l) + n \sum_{k=1}^{l} r_k(q_k - 1). \tag{3.9}$$

Proof. Denote the right-hand side of (3.3) by $LB(n, q_1, \ldots, q_l, r_1, \ldots, r_l)$. Let

$$g(r_1, \ldots, r_l) = f(r_1, \ldots, r_l) - LB(n, q_1, \ldots, q_l, r_1, \ldots, r_l).$$

Inequality (3.3) implies that $g(r_1, \ldots, r_l) \geq 0$. From (3.8) we have

$$f(r_1 + s_1, \ldots, r_l + s_l) \leq f(r_1, \ldots, r_l) + n \sum_{k=1}^{l} r_k(q_k - 1).$$

Then we have $0 \leq g(r_1 + s_1, \ldots, r_l + s_l) \leq g(r_1, \ldots, r_l)$ after some straightforward algebra. Note that since $2n(n-1)f(r_1, \ldots, r_l)$ is an integer, so is $2n(n-1)g(r_1, \ldots, r_l)$. Therefore, for any $(t_1, \ldots, t_l)$ satisfying $1 \leq t_j \leq s_j$ for $j = 1, \ldots, l$, $2n(n-1)g(ks_1 + t_1, \ldots, ks_l + t_l)$ is a decreasing integer sequence in k and has a lower bound. There must exist a positive integer $k_0 = k_0(t_1, \ldots, t_l)$ such that, for $k \geq k_0$,

$$2n(n-1)g(ks_1 + t_1, \ldots, ks_l + t_l) = 2n(n-1)g(k_0 s_1 + t_1, \ldots, k_0 s_l + t_l).$$

Let $K = \max\{k_0(t_1, \ldots, t_l) : 1 \leq t_j \leq s_j$ for $j = 1, \ldots, l\}$, and $R_i = (K+1)s_i$, for $i = 1, \ldots, l$. Then for any $r_i \geq R_i$ with $i = 1, \ldots, l$, $g(r_1 + s_1, \ldots, r_l + s_l) = g(r_1, \ldots, r_l)$ or, equivalently, (3.9) holds.

Remark 2. The result of this theorem can be generalized to the case where the saturated $L_n(q_1^{s_1}\cdots q_l^{s_l})$ is replaced by a design D_1 with constant $\delta_{ij}(D_1)$'s for $i \neq j$.

This periodicity property of minimum $\chi^2(D)$ helps us understand mixed-level SSDs of large size; it shows how larger $\chi^2(D)$-optimal mixed-level SSDs are connected with smaller ones. From (3.8) and (3.9), when the number of factors is sufficient large, the column juxtaposition of a $\chi^2(D)$-optimal design and a saturated OA (as well as a design D_1 with constant $\delta_{ij}(D_1)$'s for $i \neq j$) is still a $\chi^2(D)$-optimal design. When the number of factors is not so large, the design obtained in this way will also be satisfactory according to the $\chi^2(D)$ criterion.

The optimal SSDs obtained through column juxtaposition may contain fully aliased columns; the next section presents an explicit construction method that produces optimal SSDs without them.

4. Construction of χ^2-Optimal Mixed-Level SSDs

Let G_l be an additive group of l elements, say $\{0,1,\ldots,l-1\}$. For a vector $A=(a_1,\ldots,a_u)'$ and a matrix B of order $v\times r$, both with entries from G_l, define their *Kronecker sum* to be the $uv \times r$ matrix

$$A \oplus_l B = \begin{bmatrix} B+a_1 \\ \vdots \\ B+a_u \end{bmatrix},$$

where $B+k$ is obtained from adding k, over G_l, to the elements of B. Let $\mathbf{0}_q$ denote a $q\times 1$ vector of 0's and $L_q=(0,1,\ldots,q-1)'$.

Theorem 3. *Suppose p,q,s,t,λ and m_0 are positive integers satisfying*

$$m_0(s-1)=\lambda(ps-1), \tag{4.1}$$

$$pm_0 = p\lambda + q^2 t. \tag{4.2}$$

Let $n_0=ps$ and $m_1=q^2t$. If there exist two designs D_0 and D_1 such that (i) *D_0 is an $S(n_0;p^{m_0})$ design with λ coincidence positions between any two distinct rows, and* (ii) *D_1 is the transpose of an $L_{m_1}(q^{n_0})$, then*

$$D=[\mathbf{0}_q \oplus_p D_0, L_q \oplus_q D_1] \tag{4.3}$$

is an $S(qn_0;p^{m_0}q^{m_1})$ design with the natural weighted coincidence number pm_0 between any two distinct rows, hence it is $\chi^2(D)$-optimal. For the symmetric case, D is a $\chi^2(D)$-optimal $S(qn_0;q^{m_0+m_1})$ design.

Proof. We need only prove that the resulting design has the natural weighted coincidence number pm_0 between any two distinct rows. For the ith and jth rows

Table 1. An $S(6;3^5)$.

Run	1	2	3	4	5
1	0	0	0	0	0
2	0	1	1	1	1
3	1	0	2	2	1
4	1	2	0	1	2
5	2	1	2	0	2
6	2	2	1	2	0

Table 2. Transpose of an $L_{12}(2^6)$.

Run	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	1	1	1	0	0	0	1	1	1
3	0	1	1	0	0	1	0	0	1	0	1	1
4	1	0	0	0	1	1	0	1	1	0	0	1
5	0	0	1	1	1	0	1	0	1	0	0	1
6	0	1	1	0	1	0	0	1	0	1	0	1

Table 3. $S(12;3^5 2^{12})$ constructed from the two designs in Tables 1 and 2.

Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
2	0	1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
3	1	0	2	2	1	0	1	1	0	0	1	0	0	1	0	1	1
4	1	2	0	1	2	1	0	0	0	1	1	0	1	1	0	0	1
5	2	1	2	0	2	0	0	1	1	1	0	1	0	1	0	0	1
6	2	2	1	2	0	0	1	1	0	1	0	0	1	0	1	0	1
7	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0
8	0	1	1	1	1	1	1	1	0	0	0	1	1	1	0	0	0
9	1	0	2	2	1	1	0	0	1	1	0	1	1	0	1	0	0
10	1	2	0	1	2	0	1	1	1	0	0	1	0	0	1	1	0
11	2	1	2	0	2	1	1	0	0	0	1	0	1	0	1	1	0
12	2	2	1	2	0	1	0	0	1	0	1	1	0	1	0	1	0

of D, where $1 \le i < j \le qn_0$, if $j - i = 0 \mod n_0$, they have m_0 coincidence positions at the p-level factors, and no coincidence position at the q-level factors, so the natural weighted coincidence number between the two rows is pm_0; otherwise, they have λ coincidence positions at the p-level factors, and qt coincidence positions at the q-level factors, and then the natural weighted coincidence number between them is $p\lambda+q^2t$. Hence, from (4.2), the natural weighted coincidence number between any two distinct rows of D is pm_0.

Example 1. Here is an example of the construction method using (4.3). It can be verified that $p = 3, q = 2, s = 2, t = 3, \lambda = 1$ and $m_0 = 5$ satisfy (4.1) and (4.2). There exist designs D_0 and D_1, as shown in Tables 1 and 2 respectively, where D_0 is a $\chi^2(D)$-optimal $S(6;3^5)$ design obtained by Fang, Ge and Liu (2004b), and D_1 is the transpose of an $L_{12}(2^6)$ which is found at the website http://support.sas.com/techsup/technote/ts723_Designs.txt. From these two designs, an $S(12;3^5 2^{12})$ is constructed using(4.3); it is shown in Table 3. It has the natural weighted coincidence number 15 between any two distinct rows, and thus is a $\chi^2(D)$-optimal SSD.

Remark 3. Note that this optimal $S(12; 3^5 2^{12})$ design provides a solution for the motivating example in the Introduction, as all the constraints given in the example are satisfied.

Here are some properties of the designs constructed from Theorem 3.

Corollary 1. *If D is an $S(qn_0; p^{m_0} q^{m_1})$ constructed through* (4.3), *then any p-level and q-level columns in D are orthogonal to each other. Further, if there are no fully aliased columns in D_0 or D_1, then there are no fully aliased columns in D.*

Corollary 2. *If D is an $S(qn_0; p^{m_0} q^{m_1})$ constructed through* (4.3), *then*

(i) *each run has m_1/q q-level factors at each of the q levels;*

(ii) *for any two runs, either they take the same level on each of the p-level factors and different levels on each of the q-level factors, or they take the same level on each of some λ p-level factors and the level combinations on the q-level factors equally often.*

Based on Theorem 3, we can construct $\chi^2(D)$-optimal mixed-level SSDs that have the properties described in Corollaries 1 and 2. There are very rich results in the literature for multi-level SSDs with a constant number of coincidence positions between any two distinct rows. As for OAs, there is a library of over 200 OAs maintained by Dr. N.J.A. Sloane (http://www.research.att.com/~njas/oadir/). This library has been recently updated by Dr. W.F. Kuhfeld at his OA site (http://support.sas.com/techsup/technote/ts723.html). This site contains all OAs listed in the Appendix of Kuhfeld and Tobias (2005), as well as new ones contributed by other authors.

Appendix A displays optimal multi-level SSDs that can be constructed by the new method, while Appendix B tabulates optimal mixed-level SSDs which can be constructed from existing multi-level SSDs and OAs. Except for those designs marked with * in Appendix A, which can also be constructed by a method proposed by Georgiou, Koukouvinos and Mantas (2006), all other SSDs in these tables are apparently new. Note that there are no fully aliased columns in any of the initial SSDs used in the construction, thus if there are no fully aliased rows in the OAs, the resulting SSDs have no fully aliased columns. Further, any p-level and q-level columns are orthogonal to each other in any of the resulting SSDs, and these designs possess the properties listed in Corollary 2.

Remark 4. The construction method proposed in Theorem 3 can also be modified to construct $E(f_{NOD})$-optimal designs. For this case, we need only change the condition (4.2) to $m_0 = \lambda + qt$. Then many $E(f_{NOD})$-optimal SSDs can be generated through (4.3) from existing multi-level SSDs with λ coincidence positions between any two distinct rows, and OAs at Dr. Kuhfeld's OA website.

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Appendix A. Optimal $S(qn_0; q^{m_0+m_1})$ designs for $q > 2$.

q	n_0	m_0	m_1	initial SSD	[Source]	$L_{m_1}(q^{n_0})$	final SSD
3	6	15	36	$S(6;3^{15})$	[GK2006†]	$L_{36}(3^6)$	$S(18;3^{51})$
3	9	12	27	$S(9;3^{12})$	[Fang, Ge and Liu (2004b)]	$L_{27}(3^9)$	$S(27;3^{39})^*$
3	9	16	36	$S(9;3^{16})$	[Fang, Ge and Liu (2004b)]	$L_{36}(3^9)$	$S(27;3^{52})^*$
3	9	20	45	$S(9;3^{20})$	[Fang, Ge and Liu (2004b)]	$L_{45}(3^9)$	$S(27;3^{65})^*$
3	9	24	54	$S(9;3^{24})$	[Fang, Ge and Liu (2004b)]	$L_{54}(3^9)$	$S(27;3^{78})^*$
3	9	28	63	$S(9;3^{28})$	[Fang, Ge and Liu (2004b)]	$L_{63}(3^9)$	$S(27;3^{91})^*$
3	9	32	72	$S(9;3^{32})$	[GKM2006‡]	$L_{72}(3^9)$	$S(27;3^{104})^*$
3	9	36	81	$S(9;3^{36})$	[GKM2006]	$L_{81}(3^9)$	$S(27;3^{117})^*$
3	9	40	90	$S(9;3^{40})$	[GK2006]	$L_{90}(3^9)$	$S(27;3^{130})^*$
3	9	48	108	$S(9;3^{48})$	[GK2006]	$L_{108}(3^9)$	$S(27;3^{156})^*$
3	12	33	72	$S(12;3^{33})$	[GK2006]	$L_{72}(3^{12})$	$S(36;3^{105})$
3	18	51	108	$S(18;3^{51})$	[New]	$L_{108}(3^{18})$	$S(54;3^{159})$
3	27	39	81	$S(27;3^{39})$	[New]	$L_{81}(3^{27})$	$S(81;3^{120})$
3	27	52	108	$S(27;3^{52})$	[New]	$L_{108}(3^{27})$	$S(81;3^{160})$
3	27	65	135	$S(27;3^{65})$	[New]	$L_{135}(3^{27})$	$S(81;3^{200})$
4	8	14	48	$S(8;4^{14})$	[Fang, Ge and Liu (2002a)]	$L_{48}(4^8)$	$S(32;4^{62})$
4	8	28	96	$S(8;4^{28})$	[GK2006]	$L_{96}(4^8)$	$S(32;4^{124})$
4	8	42	144	$S(8;4^{42})$	[GK2006]	$L_{144}(4^8)$	$S(32;4^{186})$
4	16	20	64	$S(16;4^{20})$	[FGLQ2004c§]	$L_{64}(4^{16})$	$S(64;4^{84})^*$
4	16	30	96	$S(16;4^{30})$	[FGLQ2004c]	$L_{96}(4^{16})$	$S(64;4^{126})^*$
4	16	40	128	$S(16;4^{40})$	[GKM2006]	$L_{128}(4^{16})$	$S(64;4^{168})^*$
4	16	45	144	$S(16;4^{45})$	[GKM2006]	$L_{144}(4^{16})$	$S(64;4^{189})^*$
5	25	30	125	$S(25;5^{30})$	[GKM2006]	$L_{125}(5^{25})$	$S(125;5^{155})$

† GK2006: Georgiou and Koukouvinos (2006).
‡ GKM2006: Georgiou, Koukouvinos and Mantas (2006).
§ Fang, Ge, Liu and Qin (2004c).
* Designs can also be constructed via the method in Georgiou, Koukouvinos and Mantas (2006)

Appendix B. Optimal $S(qn_0; p^{m_0}q^{m_1})$ designs for $p = 2, 3$.

p	q	n_0	m_0	m_1	initial SSD	[Source]	$L_{m_1}(q^{n_0})$	final SSD
2	3	12	33	36	$S(12; 2^{33})$	[Liu and Zhang (2000)]	$L_{36}(3^{12})$	$S(36; 2^{33}3^{36})$
2	3	12	66	72	$S(12; 2^{66})$	[Liu and Zhang (2000)]	$L_{72}(3^{12})$	$S(36; 2^{66}3^{72})$
2	3	12	99	108	$S(12; 2^{99})$	[Liu and Zhang (2000)]	$L_{108}(3^{12})$	$S(36; 2^{99}3^{108})$
2	3	12	132	144	$S(12; 2^{132})$	[Liu and Zhang (2000)]	$L_{144}(3^{12})$	$S(36; 2^{132}3^{144})$
2	3	16	135	144	$S(16; 2^{135})$	[EGMBT2004†]	$L_{144}(3^{16})$	$S(48; 2^{135}3^{144})$
2	3	18	68	72	$S(18; 2^{68})$	[Liu and Zhang (2000)]	$L_{72}(3^{18})$	$S(54; 2^{68}3^{72})$
2	3	18	102	108	$S(18; 2^{102})$	[Liu and Zhang (2000)]	$L_{108}(3^{18})$	$S(54; 2^{102}3^{108})$
2	3	18	136	144	$S(18; 2^{136})$	[EGMBT2004]	$L_{144}(3^{18})$	$S(54; 2^{136}3^{144})$
2	3	24	69	72	$S(24; 2^{69})$	[Liu and Zhang (2000)]	$L_{72}(3^{24})$	$S(72; 2^{69}3^{72})$
2	3	24	138	144	$S(24; 2^{138})$	[Liu and Zhang (2000)]	$L_{144}(3^{24})$	$S(72; 2^{138}3^{144})$
2	4	8	28	32	$S(8; 2^{28})$	[Liu and Zhang (2000)]	$L_{32}(4^{8})$	$S(32; 2^{28}4^{32})$
2	4	12	44	48	$S(12; 2^{44})$	[Liu and Zhang (2000)]	$L_{48}(4^{12})$	$S(48; 2^{44}4^{48})$
2	4	12	88	96	$S(12; 2^{88})$	[Liu and Zhang (2000)]	$L_{96}(4^{12})$	$S(48; 2^{88}4^{96})$
2	4	12	132	144	$S(12; 2^{132})$	[Liu and Zhang (2000)]	$L_{144}(4^{12})$	$S(48; 2^{132}4^{144})$
2	4	16	60	64	$S(16; 2^{60})$	[Liu and Zhang (2000)]	$L_{64}(4^{16})$	$S(64; 2^{60}4^{64})$
2	4	16	90	96	$S(16; 2^{90})$	[Liu and Zhang (2000)]	$L_{96}(4^{16})$	$S(64; 2^{90}4^{96})$
2	4	16	120	128	$S(16; 2^{120})$	[EGMBT2004]	$L_{128}(4^{16})$	$S(64; 2^{120}4^{128})$
2	4	16	135	144	$S(16; 2^{135})$	[EGMBT2004]	$L_{144}(4^{16})$	$S(64; 2^{135}4^{144})$
2	4	18	136	144	$S(18; 2^{136})$	[EGMBT2004]	$L_{144}(4^{18})$	$S(72; 2^{136}4^{144})$
2	4	24	138	144	$S(24; 2^{138})$	[Liu and Zhang (2000)]	$L_{144}(4^{24})$	$S(96; 2^{138}4^{144})$
2	5	20	95	100	$S(20; 2^{95})$	[Liu and Zhang (2000)]	$L_{100}(5^{20})$	$S(100; 2^{95}5^{100})$
2	8	16	120	128	$S(16; 2^{120})$	[EGMBT2004]	$L_{128}(8^{16})$	$S(128; 2^{120}8^{128})$
3	2	6	5	12	$S(6; 3^{5})$	[Fang, Ge and Liu (2004b)]	$L_{12}(2^{6})$	$S(12; 3^{5}2^{12})$
3	2	6	10	24	$S(6; 3^{10})$	[GK2006‡]	$L_{24}(2^{6})$	$S(12; 3^{10}2^{24})$
3	2	6	15	36	$S(6; 3^{15})$	[GK2006]	$L_{36}(2^{6})$	$S(12; 3^{15}2^{36})$
3	2	9	16	36	$S(9; 3^{16})$	[Fang, Ge and Liu (2004b)]	$L_{36}(2^{9})$	$S(18; 3^{16}2^{36})$
3	2	9	32	72	$S(9; 3^{32})$	[GK2006]	$L_{72}(2^{9})$	$S(18; 3^{32}2^{72})$
3	2	9	48	108	$S(9; 3^{48})$	[GK2006]	$L_{108}(2^{9})$	$S(18; 3^{48}2^{108})$
3	2	12	11	24	$S(12; 3^{11})$	[Lu, Hu and Zheng (2003)]	$L_{24}(2^{12})$	$S(24; 3^{11}2^{24})$
3	2	12	22	48	$S(12; 3^{22})$	[GK2006]	$L_{48}(2^{12})$	$S(24; 3^{22}2^{48})$
3	2	12	33	72	$S(12; 3^{33})$	[GK2006]	$L_{72}(2^{12})$	$S(24; 3^{33}2^{72})$
3	2	12	44	96	$S(12; 3^{44})$	[GK2006]	$L_{96}(2^{12})$	$S(24; 3^{44}2^{96})$
3	2	12	55	120	$S(12; 3^{55})$	[GK2006]	$L_{120}(2^{12})$	$S(24; 3^{55}2^{120})$
3	2	15	28	60	$S(15; 3^{28})$	[GK2006]	$L_{60}(2^{15})$	$S(30; 3^{28}2^{60})$
3	2	18	51	108	$S(18; 3^{51})$	[New in Appendix A]	$L_{108}(2^{18})$	$S(36; 3^{51}2^{108})$
3	2	27	52	108	$S(27; 3^{52})$	[Fang, Lin and Ma (2000)]	$L_{108}(2^{27})$	$S(54; 3^{52}2^{108})$
3	4	12	22	48	$S(12; 3^{22})$	[GK2006]	$L_{48}(4^{12})$	$S(48; 3^{22}4^{48})$
3	4	12	44	96	$S(12; 3^{44})$	[GK2006]	$L_{96}(4^{12})$	$S(48; 3^{44}4^{96})$

† EGMBT2004: Eskridge, Gilmour, Mead, Butler and Travnicek (2004).
‡ GK2006: Georgiou and Koukouvinos (2006)

Appendix B. Optimal $S(qn_0; p^{m_0} q^{m_1})$ designs for $p > 3$.

p	q	n_0	m_0	m_1	initial SSD	[Source]	$L_{m_1}(q^{n_0})$	final SSD
4	2	8	7	24	$S(8;4^{7})$	[Fang, Ge and Liu (2002a)]	$L_{24}(2^8)$	$S(16;4^{7}2^{24})$
4	2	8	14	48	$S(8;4^{14})$	[Fang, Ge and Liu (2002a)]	$L_{48}(2^8)$	$S(16;4^{14}2^{48})$
4	2	8	21	72	$S(8;4^{21})$	[GK2006†]	$L_{72}(2^8)$	$S(16;4^{21}2^{72})$
4	2	8	28	96	$S(8;4^{28})$	[GK2006]	$L_{96}(2^8)$	$S(16;4^{28}2^{96})$
4	2	8	35	120	$S(8;4^{35})$	[GK2006]	$L_{120}(2^8)$	$S(16;4^{35}2^{120})$
4	2	8	42	144	$S(8;4^{42})$	[GK2006]	$L_{144}(2^8)$	$S(16;4^{42}2^{144})$
4	2	12	11	36	$S(12;4^{11})$	[FGLQ2003a‡]	$L_{36}(2^{12})$	$S(24;4^{11}2^{36})$
4	2	12	22	72	$S(12;4^{22})$	[GK2006]	$L_{72}(2^{12})$	$S(24;4^{22}2^{72})$
4	2	12	33	108	$S(12;4^{33})$	[GK2006]	$L_{108}(2^{12})$	$S(24;4^{33}2^{108})$
4	2	16	10	32	$S(16;4^{10})$	[FGLQ2003a]	$L_{32}(2^{16})$	$S(32;4^{10}2^{32})$
4	2	16	15	48	$S(16;4^{15})$	[FGLQ2003a]	$L_{48}(2^{16})$	$S(32;4^{15}2^{48})$
4	2	16	20	64	$S(16;4^{20})$	[FGLQ2003a]	$L_{64}(2^{16})$	$S(32;4^{20}2^{64})$
4	2	16	25	80	$S(16;4^{25})$	[FGLQ2003a]	$L_{80}(2^{16})$	$S(32;4^{25}2^{80})$
4	2	16	30	96	$S(16;4^{30})$	[FGLQ2003a]	$L_{96}(2^{16})$	$S(32;4^{30}2^{96})$
4	2	16	35	112	$S(16;4^{35})$	[FGLQ2003a]	$L_{112}(2^{16})$	$S(32;4^{35}2^{112})$
4	2	16	40	128	$S(16;4^{40})$	[GKM2006]	$L_{128}(2^{16})$	$S(32;4^{40}2^{128})$
4	2	16	45	144	$S(16;4^{45})$	[GKM2006]	$L_{144}(2^{16})$	$S(32;4^{45}2^{144})$
4	2	20	19	60	$S(20;4^{19})$	[LFXY2002§]	$L_{60}(2^{20})$	$S(40;4^{19}2^{60})$
4	2	24	23	72	$S(24;4^{23})$	[LFXY2002]	$L_{72}(2^{24})$	$S(48;4^{23}2^{72})$
4	3	8	21	72	$S(8;4^{21})$	[GK2006]	$L_{72}(3^8)$	$S(24;4^{21}3^{72})$
4	3	8	42	144	$S(8;4^{42})$	[GK2006]	$L_{144}(3^8)$	$S(24;4^{42}3^{144})$
4	3	12	11	36	$S(12;4^{11})$	[FGLQ2003a]	$L_{36}(3^{12})$	$S(36;4^{11}3^{36})$
4	3	12	22	72	$S(12;4^{22})$	[GK2006]	$L_{72}(3^{12})$	$S(36;4^{22}3^{72})$
4	3	12	33	108	$S(12;4^{33})$	[GK2006]	$L_{108}(3^{12})$	$S(36;4^{33}3^{108})$
4	3	16	45	144	$S(16;4^{45})$	[GKM2006]	$L_{144}(3^{16})$	$S(48;4^{45}3^{144})$
4	3	24	23	72	$S(24;4^{23})$	[LFXY2002]	$L_{72}(3^{24})$	$S(72;4^{23}3^{72})$
4	8	16	40	128	$S(16;4^{40})$	[GK2006]	$L_{128}(8^{16})$	$S(128;4^{40}8^{128})$
5	2	10	9	40	$S(10;5^{9})$	[Fang, Ge and Liu (2002b)]	$L_{40}(2^{10})$	$S(20;5^{9}2^{40})$
5	2	10	18	80	$S(10;5^{18})$	[GK2006]	$L_{80}(2^{10})$	$S(20;5^{18}2^{80})$
5	2	10	27	120	$S(10;5^{27})$	[GK2006]	$L_{120}(2^{10})$	$S(20;5^{27}2^{120})$
5	2	15	14	60	$S(15;5^{14})$	[Fang, Ge and Liu (2004b)]	$L_{60}(2^{15})$	$S(30;5^{14}2^{60})$
5	2	15	28	120	$S(15;5^{28})$	[Fang, Ge and Liu (2004b)]	$L_{120}(2^{15})$	$S(30;5^{28}2^{120})$
5	2	20	19	80	$S(20;5^{19})$	[LFXY2002]	$L_{80}(2^{20})$	$S(40;5^{19}2^{80})$
5	2	25	24	100	$S(25;5^{24})$	[Fang, Lin and Ma (2000)]	$L_{100}(2^{25})$	$S(50;5^{24}2^{100})$
5	2	30	29	120	$S(30;5^{29})$	[LFXY2002]	$L_{120}(2^{30})$	$S(60;5^{29}2^{120})$
5	3	15	21	90	$S(15;5^{21})$	[Fang, Ge and Liu (2004b)]	$L_{90}(3^{15})$	$S(45;5^{21}3^{90})$
5	4	10	18	80	$S(10;5^{18})$	[GK2006]	$L_{80}(4^{10})$	$S(40;5^{18}4^{80})$
6	2	12	11	60	$S(12;6^{11})$	[Lu, Hu and Zheng (2003)]	$L_{60}(2^{12})$	$S(24;6^{11}2^{60})$
6	2	12	22	120	$S(12;6^{22})$	[GK2006]	$L_{120}(2^{12})$	$S(24;6^{22}2^{120})$
6	2	24	23	120	$S(24;6^{23})$	[Lu, Hu and Zheng (2003)]	$L_{120}(2^{24})$	$S(48;6^{23}2^{120})$
6	3	18	17	90	$S(18;6^{17})$	[Lu, Hu and Zheng (2003)]	$L_{90}(3^{18})$	$S(54;6^{17}3^{90})$
7	2	14	13	84	$S(14;7^{13})$	[Fang, Ge and Liu (2002b)]	$L_{84}(2^{14})$	$S(28;7^{13}2^{84})$
7	2	28	9	56	$S(28;7^{9})$	[Fang, Ge and Liu (2002b)]	$L_{56}(2^{28})$	$S(56;7^{9}2^{56})$
8	2	64	18	128	$S(64;8^{18})$	[GKM2006]	$L_{128}(2^{64})$	$S(128;8^{18}2^{128})$

† GK2006: Georgiou and Koukouvinos (2006).

‡ FGLQ2003a: Fang, Ge, Liu and Qin (2003a).

§ LFXY2002: Lu, Fang, Xu and Yin (2002).

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