Statistica Sinica 1(1991), 175-184

CONVERGENCE RATES FOR THE CRITICAL BRANCHING PROCESS WITH IMMIGRATION

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Abstract: For a critical branching process with immigration, $\{X_n\}$, $(\log X_n)/\log n$ is shown to converge almost surely to 1 when $\{X_n\}$ is transient. The rates of growth for $\Sigma_{i=0}^n X_i$ and $\Sigma_{i=0}^n (1+X_i)^{-1}$ are then derived and used to obtain convergence rates for the conditional weighted least squares estimators of the generation and immigration means.

Key words and phrases: Branching process with immigration, convergence rate, martingale, conditional least squares.

1. Introduction

Let $\{X_n\}$ be a branching process with immigration defined by

$$X_n = \sum_{j=1}^{X_{n-1}} Y_{n,j} + I_n, \quad n = 1, 2, \dots, \tag{1.1}$$

where $\{Y_{n,j}\}$ and $\{I_n\}$ are independent sequences of i.i.d. nonnegative, integer valued random variables and X_0 is a nonnegative, integer valued random variable which is independent of $\{Y_{n,j}\}$ and $\{I_n\}$. We can interpret X_n as the size of the nth generation of a population, where $Y_{n,j}$ is the offspring size of the jth individual in the $(n-1)$st generation and I_n is the number of immigrants contributing to the population's nth generation.

When the generation mean $m = E(Y_{1,1}) < \infty$, the process $\{X_n\}$ is referred to as subcritical if $m < 1$, critical if $m = 1$ and supercritical if $m > 1$. The study of $\{X_n\}$ dates back to Smoluchowski (1916), and there is a substantial literature on asymptotic behavior of functionals of $\{X_n\}$. It is now relatively well understood for the subcritical and supercritical cases. (See Athreya and Ney (1972) for the basic properties of $\{X_n\}$.) But for the critical case the path properties of $\{X_n\}$ and its functionals are quite intricate and remain unknown.

Let $\lambda = E(I_1) < \infty$, $0 < \sigma^2 = E(Y_{1,1} - m)^2 < \infty$ and $0 < b^2 = E(I_1 - \lambda)^2 < \infty$. For the critical case, it is known (Pakes (1972), Mellein (1982b, 1983a), Wei and Winnicki (1989)) that as a Markov process, $\{X_n\}$ is transient or recurrent according as $\tau = 2\lambda/\sigma^2 > 1$ or $\tau \leq 1$. It is also known (Kawazu and Watanabe (1971), Mellein (1982a, 1983b), Wei and Winnicki (1989)) that $X_{[nt]}/n$ converges weakly to a diffusion limit. For the path properties, when $\tau > 1$, i.e. $\{X_n\}$ is transient, it is obvious that $X_n \to \infty$ a.s. In this case, under the assumption that $EY_{1,1}^{4+\delta} < \infty$ for some $\delta > 0$, Wei and Winnicki (1989, Theorem 2.15) also show that

$$\limsup_{n\to\infty} |X_n - X_{n-1}|/(2\sigma^2 X_{n-1}\log X_{n-1})^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

However, these results do not provide any clue about the growth rate of $\{X_n\}$. It is the purpose of this paper to investigate this problem.

In Section 2, using the martingale convergence theorem, we obtain the following result.

Theorem 1.1. *Assume that for some $\delta > 0$,*

$$m = 1, \quad 0 < \sigma^2 < \infty, \quad 0 < b^2 < \infty, \quad \tau > 1 \quad \text{and} \quad E\big(Y_{1,1}^{2+\delta}\big) < \infty. \tag{1.2}$$

Then

$$\lim_{n\to\infty} (\log X_n)/\log n = 1 \quad \text{a.s.} \tag{1.3}$$

As a corollary of Theorem 1.1, we also obtain the following theorem concerning the functionals of $\{X_n\}$.

Theorem 1.2. *Assume that* (1.2) *holds. Then*

$$\lim_{n\to\infty} \sum_{i=0}^{n} (1+X_i)^{-1}/\log n = (\lambda - \sigma^2/2)^{-1} \quad \text{a.s.} \tag{1.4}$$

The functional $\sum_{i=0}^{n}(1+X_i)^{-1}$ plays an important role in the estimation of m and λ. Recently, in an attempt (Wei and Winnicki (1990)) to solve a long standing estimation problem raised by Heyde and Seneta (1974), we proposed the conditional weighted least squares estimators $\hat{m}_n$ and $\hat{\lambda}_n$. Specifically, $\hat{m}_n$ and $\hat{\lambda}_n$ are defined by

$$\hat{m}_n = \left[\sum_{i=1}^{n} X_i \sum_{i=1}^{n}(1+X_i)^{-1} - n\sum_{i=1}^{n} X_i(1+X_i)^{-1}\right] \cdot \left[\sum_{i=1}^{n}(1+X_{i-1})\sum_{i=1}^{n}(1+X_{i-1})^{-1} - n^2\right]^{-1} \tag{1.5}$$

and

$$\hat{\lambda}_n = \left[\sum_{i=1}^{n} X_{i-1} \sum_{i=1}^{n} X_i(1+X_{i-1})^{-1} - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_{i-1}(1+X_{i-1})^{-1}\right] \cdot \left[\sum_{i=1}^{n}(1+X_{i-1}) \sum_{i=1}^{n}(1+X_{i-1})^{-1} - n^2\right]^{-1}. \tag{1.6}$$

In Wei and Winnicki (1990) under assumption (1.2), $\hat{m}_n$ and $\hat{\lambda}_n$ are shown to converge in probability to m and λ. The key tools are the weak convergence of the functional $\sum_1^n X_i$ and a weaker version of Theorem 1.2 in which the almost sure convergence is replaced by convergence in probability. In Section 2, using Theorems 1.1, 1.2 and an iterated logarithm type result for the martingale transform (Lemma 2.4), we are able to obtain a lower bound for $\sum_1^n X_i$ (Lemma 2.5). This lower bound not only enables us to establish the a.s. convergence for $\hat{m}_n$ and $\hat{\lambda}_n$ but also their convergence rates. This is the context of the following theorem.

Theorem 1.3. *Assume that* (1.2) *holds. Then*

$$\hat{m}_n - m = O\left(\left(\log\log n / \sum_{1}^{n} X_i\right)^{\frac{1}{2}}\right) = O(\log\log n/n) \quad \text{a.s.} \tag{1.7}$$

and

$$\hat{\lambda}_n - \lambda = O\left((\log\log\log n/\log n)^{\frac{1}{2}}\right) \quad \text{a.s.} \tag{1.8}$$

2. Proofs of Theorems 1.1, 1.2 and 1.3

Throughout this section, we assume that $X_0 = \kappa$ a.s. where $\kappa = \inf\{j \geq 0 : P[I_1 = j] > 0\}$. As shown by Lemma 2.8 of Wei and Winnicki (1989), for the case of general X_0, one can always find a branching process with immigration $\{Z_n\}$ such that $\{Z_n\}$ has the same transition probabilities as $\{X_n\}$, $Z_0 = \kappa$ a.s. and Z_n eventually coincide with X_n. Hence, as far as the asymptotic results are concerned, our assumption does not affect the generality of the proved results.

Lemma 2.1. *Assume that* (1.2) *holds. Then for any positive and increasing sequence* a_n *such that* $\sum_{n=1}^{\infty} n a_n^{-2} < \infty$, *we have*

$$\lim_{n\to\infty} X_n/a_n = 0 \quad \text{a.s.} \tag{2.1}$$

Proof. Note that

$$X_n = X_0 + \sum_{i=1}^{n} W_i + \sum_{i=1}^{n} I_i, \tag{2.2}$$

where $W_i = \sum_{j=1}^{X_{i-1}}(Y_{i,j} - 1)$ is a sequence of martingale differences with respect to the σ-field $\mathcal{F}_n = \sigma\{X_0, Y_{i,j}, I_i, 1 \le i \le n, 1 \le j\}$. By the strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} I_i = \lambda \quad \text{a.s.} \tag{2.3}$$

The assumed condition on $\{a_n\}$ implies that $\lim_{n\to\infty} n/a_n = 0$. Hence

$$\lim_{n\to\infty} \frac{1}{a_n}\sum_{i=1}^{n} I_n = 0 \quad \text{a.s.}$$

Now it is sufficient to show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} W_i}{a_n} = 0 \quad \text{a.s.}$$

But this follows from Kronecker's lemma, martingale convergence theorem and

$$\sum_{i=1}^{\infty} E(W_i^2)a_i^{-2} = \sum_{i=1}^{\infty} \sigma^2[\kappa + (i-1)\lambda]a_i^{-2} < \infty.$$

The last identity is ensured by the facts that

$$E(W_i^2) = E[E(W_i^2|\mathcal{F}_{i-1})] = \sigma^2 E(X_{i-1})$$

and

$$E(X_i) = E(X_{i-1}) + \lambda = E(X_0) + i\lambda.$$

Lemma 2.2. *Let $M_n = \max_{1\le i\le n} X_i$. Assume that* (1.2) *holds. Then*

$$\lim_{n\to\infty} \log M_n/\log n = 1 \quad \text{a.s.} \tag{2.4}$$

Proof. Applying Lemma 2.1 with $a_n = n\log n$, we have that $X_n = o(n\log n)$ a.s. This in turn implies that

$$\limsup_{n\to\infty} \log M_n/\log n \le 1 \quad \text{a.s.}$$

To prove (2.4), it is sufficient to show that

$$\liminf_{n\to\infty} \log M_n/\log n \ge 1 \quad \text{a.s.}$$

or, more strongly,

$$\liminf_{n\to\infty} n^{-1} M_n \log M_n > 0 \quad \text{a.s.} \tag{2.5}$$

By (2.2), (2.3) and the fact that

$$\lim_{n\to\infty} (X_n - X_0)/(M_n \log M_n) = 0 \quad \text{a.s.},$$

to prove (2.5) it suffices to show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} W_i/(M_n \log M_n) = 0 \quad \text{a.s.} \tag{2.6}$$

Since $M_n \uparrow \infty$ a.s., Kronecker's lemma can be used to obtain (2.6) if

$$\sum_{i=2}^{n} W_i/(\widetilde{M}_{i-1} \log \widetilde{M}_{i-1}) \quad \text{converges a.s.},$$

where $\widetilde{M}_i = M_i + 2$. But this follows from the local martingale convergence theorem (Chow (1965)) and

$$\begin{aligned}
&\sum_{i=2}^{\infty} E\left[W_i^2/(\widetilde{M}_{i-1} \log \widetilde{M}_{i-1})^2 | \mathcal{F}_{i-1}\right] \\
&\quad = \sigma^2 \sum_{i=2}^{n} X_{i-1}/(\widetilde{M}_{i-1} \log \widetilde{M}_{i-1})^2 \\
&\quad \leq \sigma^2 \sum_{i=2}^{n} (X_{i-1} + 2)^{-1} (\log(X_{i-1} + 2))^{-2} \\
&\quad < \infty \qquad \text{a.s.}
\end{aligned}$$

The last inequality is a simple corollary of Lemma 2.13 of Wei and Winnicki (1989).

Lemma 2.3. *Let* $T_n = \sum_{i=1}^{n} (1 + X_i)^{-1}$. *Assume that* (1.2) *holds. Then*

$$\lim_{n\to\infty} \log X_n / T_n = \lambda - \sigma^2/2 \quad \text{a.s.} \tag{2.7}$$

Proof. This is (2.24) of Wei and Winnicki (1989).

Proof of Theorem 1.1.

By the definition of M_n, for each n there is a random variable $j(n)$ such that $j(n) \le n$ a.s. and $M_n = X_{j(n)}$. Since $X_n \to \infty$ a.s., $j(n) \to \infty$ a.s. Observe that

$$\begin{aligned} 1 &\ge \log X_n / \log M_n \\ &= (\log X_n/T_n)(T_{j(n)}/\log X_{j(n)})(T_n/T_{j(n)}). \end{aligned} \tag{2.8}$$

But $T_n \uparrow$ as $n \uparrow$. Hence $T_n/T_{j(n)} \ge 1$ a.s. In view of (2.8) and Lemma 2.3,

$$\begin{aligned} 1 &\ge \limsup_{n\to\infty} \log X_n / \log M_n \\ &\ge \liminf_{n\to\infty} \log X_n / \log M_n \ge 1 \quad \text{a.s.} \end{aligned}$$

This and Lemma 2.2 complete our proof.

Proof of Theorem 1.2.
Theorem 1.2 follows immediately from Theorem 1.1 and Lemma 2.3.

Now let us study the asymptotic properties of the conditional weighted least squares estimates $\hat{m}_n$ and $\hat{\lambda}_n$, defined by (1.5) and (1.6). For this, we need a lower bound for $\sum_{i=1}^n X_i$. Before stating such a result, we need a martingale result which can be found in Wei (1985, p. 1500).

Lemma 2.4. *Let $\{\varepsilon_n, \mathcal{G}_n\}$ be a martingale difference sequence satisfying*

$$\sup_n E(|\varepsilon_n|^{2+\delta} | \mathcal{G}_{n-1}) < \infty \quad \text{a.s. } \textit{for some} \quad \delta > 0. \tag{2.9}$$

Let $\{u_n\}$ be a sequence of random variables such that u_n is $\mathcal{G}_{n-1}$-measurable. Define $t_n^2 = \sum_{i=1}^n u_i^2$. If

$$u_n^2 = o(t_n^{2c}) \quad \text{a.s. } \textit{for some} \quad 0 < c < 1, \tag{2.10}$$

then

$$\sum_{i=1}^n u_i \varepsilon_i = O\left(t_n (\log\log t_n)^{\frac{1}{2}}\right) \quad \text{a.s.} \tag{2.11}$$

Lemma 2.5. *Let $S_n = \sum_{i=0}^n X_i$. Assume that* (1.2) *holds. Then*

$$\liminf_{n\to\infty} \frac{S_n \log\log n}{n^2} > 0 \quad \text{a.s.} \tag{2.12}$$

Proof. By (2.2),

$$S_n = (n+1)X_0 + \sum_{r=1}^n \sum_{i=1}^r W_i + \sum_{r=1}^n \sum_{i=1}^r I_i. \tag{2.13}$$

Observe that

$$W_i = \sum_{j=1}^{X_{i-1}} (Y_{i,j} - 1) = u_i \varepsilon_i,$$

where $u_i = \sqrt{X_{i-1}}$ and $\varepsilon_i = W_i/u_i$ if $u_i \neq 0$ and 0 otherwise. By Lemma 2.1 of Lai and Wei (1983), there is a constant C_δ such that

$$E\{|\varepsilon_i|^{2+\delta}|\mathcal{F}_{i-1}\} \leq C_\delta E|Y_{1,1} - 1|^{2+\delta} \quad \text{a.s.}$$

Hence (2.9) of Lemma 2.4 holds with $\mathcal{G}_i = \mathcal{F}_i$. Now by Lemma 2.1,

$$X_n = o(n^{1+\varepsilon}) \quad \text{a.s. for any } \varepsilon > 0.$$

But Theorem 1.1 ensures that

$$\liminf_{n\to\infty} S_n/n^\alpha > 0 \quad \text{a.s. for any } \alpha < 2.$$

Consequently,

$$X_n = o(S_n^c) \quad \text{a.s. for } c > \frac{1}{2}.$$

Thus (2.10) is satisfied with $t_n^2 = \sum_{i=1}^n u_i^2 = S_{n-1}$. Applying Lemma 2.4, we obtain

$$\sum_{i=1}^n W_i = O\left(S_{n-1}^{\frac{1}{2}}(\log\log S_{n-1})^{\frac{1}{2}}\right) \quad \text{a.s.} \tag{2.15}$$

Hence

$$\sum_{r=1}^n \sum_{i=1}^r W_i = O\left(nS_{n-1}^{\frac{1}{2}}(\log\log S_{n-1})^{\frac{1}{2}}\right) \quad \text{a.s.} \tag{2.16}$$

Dividing both sides of (2.13) by S_n, we then have that

$$\left(\sum_{r=1}^n \sum_{i=1}^r I_i\right)/S_n = -\frac{n+1}{S_n}X_0 + 1 + O\left(n\left(\frac{\log\log S_n}{S_n}\right)^{\frac{1}{2}}\right) \quad \text{a.s.}$$

In view of (2.3),

$$(n^2/S_n)\left(1 + O\left(\frac{1}{n}\right)\right) = O(1) + O\left(n/S_n^{\frac{1}{2}}(\log\log S_n)^{\frac{1}{2}}\right) \quad \text{a.s.}$$

This implies that

$$\begin{aligned} n/S_n^{\frac{1}{2}} &= O\left(1 + (\log\log S_n)^{\frac{1}{2}}\right) \quad \text{a.s.} \\ &= O\left((\log\log S_n)^{\frac{1}{2}}\right) \quad \text{a.s.} \end{aligned} \tag{2.17}$$

But Theorem 1.1 gives

$$\lim_{n\to\infty} \log\log S_n / \log\log n = 1 \quad \text{a.s.} \tag{2.18}$$

As a corollary of this and (2.17), we have that

$$n^2/S_n = O(\log\log n) \quad \text{a.s.}$$

Therefore, (2.12) of Lemma 2.5 is proved.

Remarks. (a) By Lemma 2.1, it is not difficult to obtain upper bounds for X_n and S_n:

$$X_n = o\big(n(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}+\alpha}\big) \quad \text{a.s.,}$$
$$S_n = o\big(n^2(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}+\alpha}\big) \quad \text{a.s. for all } \alpha > 0.$$

These bounds can be improved. First, in view of (2.13) and (2.16),

$$S_n = O\Big(nS_n^{\frac{1}{2}}(\log\log S_n)^{\frac{1}{2}}\Big) + O(n^2) \qquad \text{a.s.}$$

Consequently, by (2.18),

$$\begin{aligned} S_n^{\frac{1}{2}} &= O\Big(n(\log\log S_n)^{\frac{1}{2}}\Big) + O(n) \\ &= O\big(n(\log\log n)^{\frac{1}{2}}\big) \qquad \text{a.s.} \end{aligned}$$

Thus

$$S_n = O\big(n^2 \log\log n\big) \qquad \text{a.s.} \tag{2.19}$$

From (2.2), (2.3), (2.15) and (2.18)

$$\begin{aligned} X_n &= O\Big(S_{n-1}^{\frac{1}{2}}(\log\log S_{n-1})^{\frac{1}{2}}\Big) + O(n) \\ &= O(n\log\log n) \qquad \text{a.s.} \end{aligned} \tag{2.20}$$

(b) It is not known what the exact bounds are for X_n and S_n. To solve this problem, it seems that a functional law of iterated logarithm for $\{X_n\}$ is needed. For a related problem in random walk, see Jain (1982).

Proof of Theorem 1.3.

It is not difficult to see that

$$\begin{aligned} \hat{m}_n - m &= (X_n - X_0)T_n/\{(n + S_{n-1})T_n - n^2\} \\ &= \big[(X_n - X_0)/(n + S_{n-1})\big]\,[1 + o(1)] \qquad \text{a.s.} \end{aligned}$$

By (2.20) and (2.18),

$$\begin{aligned}\hat{m}_n - m &= O\left[S_{n-1}^{\frac{1}{2}}(\log\log S_{n-1})^{\frac{1}{2}}/(n+S_{n-1})\right]\\ &= O\Big((\log\log n/S_n)^{\frac{1}{2}}\Big)\\ &= O(\log\log n/n) \qquad \text{a.s.,}\end{aligned}$$

where the last identity is ensured by Lemma 2.5. Now, let us show (1.8). By Theorem 1.2, Lemma 2.5 and simple algebra, we have that

$$\begin{aligned}\hat{\lambda}_n - \lambda &= \left[S_{n-1}\sum_{i=1}^{n}\frac{X_i - X_{i-1}+\lambda}{1+X_{i-1}} - (X_n - X_0)\sum_{i=1}^{n}\frac{X_{i-1}}{1+X_{i-1}} - \lambda n T_n + \lambda n^2\right]\cdot\\ &\qquad \{(n+S_{n-1})T_n - n^2\}^{-1}\\ &= \left\{\frac{1}{T_n}\sum_{i=1}^{n}\frac{X_i - X_{i-1} - \lambda}{1+X_{i-1}} - \frac{X_n\cdot O(n)+O(n^2)}{S_{n-1}T_n}\right\}(1+o(1))\\ &= \left\{\frac{1}{T_n}\sum_{i=1}^{n}\frac{X_i - X_{i-1} - \lambda}{1+X_{i-1}}\right\}(1+o(1)) + O(\log\log n/\log n) \quad \text{a.s.,} \qquad (2.21)\end{aligned}$$

where the last identity is ensured by Theorem 1.2, Lemma 2.5, (2.20) and (2.18). Note that $X_i - X_{i-1} - \lambda = W_i + (I_i - \lambda)$. Now

$$\sum_{i=1}^{\infty}\frac{E[(I_i-\lambda)^2|\mathcal{F}_{i-1}]}{(1+X_{i-1})^2} = b^2\sum_{i=1}^{\infty}\frac{1}{(1+X_{i-1})^2} < \infty \quad \text{a.s.,}$$

by Lemma 2.13 of Wei and Winnicki (1989). Hence, in view of the local martingale convergence theorem (Chow (1965)),

$$\sum_{i=1}^{n}(I_i - \lambda)/(1+X_{i-1}) \qquad \text{converges a.s.} \tag{2.22}$$

Furthermore, by Lemma 2.4 and arguments similar to the proof of (2.15),

$$\sum_{i=1}^{n} W_i/(1+X_{i-1}) = O\big(V_n(\log\log V_n)^{\frac{1}{2}}\big) \quad \text{a.s.} \tag{2.23}$$

where $V_n^2 = \sum_{i=1}^{n} X_{i-1}/(1+X_{i-1})^2$. In view of (2.21), (2.22), (2.23), Theorem 1.2 and the fact that $V_n^2/T_n \to 1$ a.s.,

$$\begin{aligned}\hat{\lambda}_n - \lambda &= O\Big((\log\log T_n/T_n)^{\frac{1}{2}}\Big) + O(\log\log n/\log n)\\ &= O\Big(\big(\frac{\log\log\log n}{\log n}\big)^{\frac{1}{2}}\Big) \qquad \text{a.s.}\end{aligned}$$

References

Athreya, K. B. and Ney, P. E. (1972). *Branching Processes.* Springer-Verlag, New York.

Chow, Y. S. (1965). Local convergence of martingales and the law of large numbers. *Ann. Math. Statist.* **36**, 552-558.

Heyde, C. C. and Seneta, E. (1974). Notes on "Estimation theory for growth and immigration rates in a multiplicative process". *J. Appl. Probab.* **11**, 572-577.

Jain, N. C. (1982). A Donsker-Varadhan type of invariance principle. *Z. Wahrsch. verw. Gebiete* **59**, 117-138.

Kawazu, K. and Watanabe, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36-54.

Lai, T. L. and Wei, C. Z. (1983). Lacunary systems and generalized linear processes. *Stochastic Process. Appl.* **14**, 187-199.

Mellein, B. (1982a). Diffusion limits of conditioned critical Galton-Watson processes. *Review Colomb. Mat.* **16**, 125-140.

Mellein, B. (1982b). Local limit theorems for the critical Galton-Watson process with immigration. *Review Colomb. Mat.* **16**, 31-56.

Mellein, B. (1983a). Green function behaviour of critical Galton-Watson process with immigration. *Bol. Soc. Bras. Mat.* **14**, 17-25.

Mellein, B. (1983b). Kac functionals of diffusion processes approximating critical branching processes. *Stochastic Processes Applied to Physics and Other Related Fields* (Edited by B. Gomez et al.), World Sci. Pub., Singapore.

Pakes, A. G. (1972). Further results on the critical Galton-Watson process with immigration. *J. Austral. Math. Soc. Ser.A* **13**, 277-290.

Smoluchowski, M. (1916). Drei vortrage uber diffusion Brownsche bewegung und wagulation von kolloidtelchen. *Physik. Zeits.* **17**, 557-585.

Wei, C. Z. (1985). Asymptotic properties of least-squares estimates in stochastic regression models. *Ann. Statist.* **13**, 1498-1508.

Wei, C. Z. and Winnicki, J. (1989). Some asymptotic results for the branching process with immigration. *Stochastic Process. Appl.* **31**, 261-282.

Wei, C. Z. and Winnicki, J. (1990). Estimation of the means in the branching process with immigration. To appear in the *Ann. Statist.*

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(Received September 1989; accepted June 1990)