Tests of Unit Root Hypothesis with Heavy-tailed Heteroscedastic Noises

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Supplementary Material

S1. Technical Proofs

Proof of Theorem 2.1. Denote

$$\xi_{nt} = n^{-1/2} \frac{y_{t-1}\Delta y_t}{(1+y_{t-1}^2)^{1/2}[1+(\Delta y_t)^2]^{1/2}}. \tag{S1.1}$$

Then T_n can be rewritten as $T_n = \sum_{t=1}^n \xi_{nt}$. Under H_0, by the symmetry of η_t and $\Delta y_t = \varepsilon_t = \eta_t h_t$, we can see that $E[\xi_{nt}|\mathcal{F}_{t-1}] = 0$, where $\mathcal{F}_i = \sigma(\eta_t, t \le i)$. Therefore, $\{\xi_{nt}\}$ is a martingale difference sequence. By Theorem 18.1 in Billingsley (1999), we only need to show that, as $n \to \infty$,

$$\sum_{t=1}^n E[\xi_{nt}^2 1_{(|\xi_{nt}|>\epsilon)}] \longrightarrow 0, \text{ for any } \epsilon > 0, \tag{S1.2}$$

$$\sum_{t=1}^n E[\xi_{nt}^2|\mathcal{F}_{t-1}] \xrightarrow{p} \sigma^2. \tag{S1.3}$$

Notice that $\sup_{t\leq n}|\xi_{nt}|\leq n^{-1/2}$, then (S1.2) obviously holds. For (S1.3), we have

$$\sum_{t=1}^{n}E[\xi_{nt}^2|\mathcal{F}_{t-1}]=\frac{1}{n}\sum_{t=1}^{n}E\left(\frac{\eta_t^2h_t^2}{1+\eta_t^2h_t^2}\Big|\mathcal{F}_{t-1}\right)-\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+y_{t-1}^2}E\left(\frac{\eta_t^2h_t^2}{1+\eta_t^2h_t^2}\Big|\mathcal{F}_{t-1}\right).$$

Since $h_t=h(\eta_{t-1},\eta_{t-2},\cdots)$ and $\{\eta_t\}$ is i.i.d., the ergodic theorem implies that

$$\frac{1}{n}\sum_{t=1}^{n}E\left(\frac{\eta_t^2h_t^2}{1+\eta_t^2h_t^2}\Big|\mathcal{F}_{t-1}\right)\xrightarrow{p}\sigma^2. \tag{S1.4}$$

Furthermore, it is obvious that

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+y_{t-1}^2}E\left(\frac{\eta_t^2h_t^2}{1+\eta_t^2h_t^2}\Big|\mathcal{F}_{t-1}\right)\leq\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+y_{t-1}^2}. \tag{S1.5}$$

For the right side in (S1.5), as $a_n\to\infty$, for any $\delta>0$, we have

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+y_{t-1}^2}=\int_0^1\frac{a_n^{-2}}{a_n^{-2}+S_n^2(\tau)}d\tau\leq\int_0^1\frac{\delta}{\delta+S_n^2(\tau)}d\tau. \tag{S1.6}$$

By Assumption 2.1 and the Skorohod representation theorem in Jakubowski (1997), there exists $\{\tilde{S}_n(\tau)\}(\tilde{S}(\tau))$ in $\mathbb{D}[0,1]$ such that $\tilde{S}_n(\tau)(\tilde{S}(\tau))$ has the same distribution with $S_n(\tau)(S(\tau))$, and $\tilde{S}_n(\tau)$ converges to $\tilde{S}(\tau)$ almost surely in S-topology. Then, by the properties of S-topology in Corollary 2.9 in Jakubowski (1997), we have

$$\int_0^1\frac{\delta}{\delta+\tilde{S}_n^2(\tau)}d\tau\xrightarrow{a.s}\int_0^1\frac{\delta}{\delta+\tilde{S}^2(\tau)}d\tau.$$

Therefore, we have

$$\int_0^1\frac{\delta}{\delta+S_n^2(\tau)}d\tau\xrightarrow{d}\int_0^1\frac{\delta}{\delta+S^2(\tau)}d\tau. \tag{S1.7}$$

By the dominated convergence theorem, it follows that

$$\int_0^1 \frac{\delta}{\delta + S^2(\tau)} d\tau \xrightarrow{a.s.} 0, \text{ as } \delta \to 0. \tag{S1.8}$$

Thus, by (S1.6)-(S1.8), we have shown that

$$\frac{1}{n} \sum_{t=1}^{n} \frac{1}{1 + y_{t-1}^2} \xrightarrow{p} 0. \tag{S1.9}$$

As a result, (S1.3) holds from (S1.4)-(S1.5) and (S1.9), which completes the proof for H_0.

On the other hand, under H_1, since $\Delta y_t = \varepsilon_t + (\phi - 1) y_{t-1}$ with $|\phi| < 1$, it is not hard to see that $\{\xi_{nt}\}$ is no longer a martingale difference sequence. Notice that

$$\frac{1}{\sqrt{n}} T_n = \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \xrightarrow{p} E\left(\frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \right),$$

as $n \to \infty$. Now, we further show that

$$E\left(\frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \right) < 0. \tag{S1.10}$$

Note that

$$E\left(\frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \right) = E\left\{ E\left(\frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \middle| \mathcal{F}_{t-1} \right) \right\}.$$

Let $F(x)$ be the distribution function of η_t. Since η_t is symmetric and independent with h_t and y_{t-1}, we have

$$E\left(\frac{y_{t-1} \Delta y_t}{(1 + y_{t-1}^2)^{1/2} [1 + (\Delta y_t)^2]^{1/2}} \middle| \mathcal{F}_{t-1} \right) = \frac{1}{(1 + y_{t-1}^2)^{1/2}}$$
$$\times \int_0^{\infty} \left\{ \frac{(x h_t + (\phi - 1) y_{t-1}) y_{t-1}}{[1 + (x h_t + (\phi - 1) y_{t-1})^2]^{1/2}} - \frac{(x h_t - (\phi - 1) y_{t-1}) y_{t-1}}{[1 + (x h_t - (\phi - 1) y_{t-1})^2]^{1/2}} \right\} dF(x).$$

Since the function $y/(1+y^2)^{1/2}$ is strictly increasing function on the real line, then by the fact $\phi < 1$, we get that for any given $y_{t-1} \neq 0$,

$$\frac{(xh_t + (\phi-1)y_{t-1})y_{t-1}}{[1+(xh_t+(\phi-1)y_{t-1})^2]^{1/2}} - \frac{(xh_t - (\phi-1)y_{t-1})y_{t-1}}{[1+(xh_t-(\phi-1)y_{t-1})^2]^{1/2}} < 0, \text{ for } x \in \mathbb{R}.$$

Thus, we have

$$E\left(\frac{y_{t-1}\Delta y_t}{(1+y_{t-1}^2)^{1/2}[1+(\Delta y_t)^2]^{1/2}}\bigg|\mathcal{F}_{t-1}\right) \leq 0, \tag{S1.11}$$

where ' =' holds if and only if $y_{t-1} = 0$. Furthermore, by the fact that $y_t = \phi y_{t-1} + \eta_t h_t$ and $P(\eta_t \neq 0) > 0$ and h_t is positive, it is clear to see that $P(y_{t-1} \neq 0) > 0$, which implies that (S1.10) holds. Hence, under H_1, we have shown that

$$\frac{1}{\sqrt{n}} T_n \xrightarrow{p} E\left(\frac{y_{t-1}\Delta y_t}{(1+y_{t-1}^2)^{1/2}[1+(\Delta y_t)^2]^{1/2}}\right) < 0. \tag{S1.12}$$

This completes the whole proof. □

Proof of Theorem 3.1. Under H_0, following the proof in Owen (2001), we first consider the magnitude of the Lagrange multiplier λ, which satisfies

$$g(\lambda) \equiv \sum_{t=1}^{n} \frac{Z_t(1)}{1+\lambda Z_t(1)} = 0. \tag{S1.13}$$

Denote $\theta = \text{sign}(\lambda)$, then

$$\begin{aligned}
0 &= |\theta g(\lambda)| \\
&= \left|\theta \sum_{t=1}^{n} Z_t(1) - \sum_{t=1}^{n} \frac{|\lambda| Z_t^2(1)}{1+\lambda Z_t(1)}\right| \\
&\geq \sum_{t=1}^{n} \frac{|\lambda| Z_t^2(1)}{1+\lambda Z_t(1)} - \Big|\sum_{t=1}^{n} Z_t(1)\Big|. \qquad \text{(S1.14)}
\end{aligned}$$

By the fact that $\max_{t\leq n} |Z_t(1)| \leq 1$ and (S1.14) and $1 + \lambda Z_t(1) > 0$, it follows that

$$\frac{|\lambda|}{1+|\lambda|} \sum_{t=1}^{n} Z_t^2(1) \leq \Big|\sum_{t=1}^{n} Z_t(1)\Big|. \qquad \text{(S1.15)}$$

Notice that $Z_t(1) = \sqrt{n}\xi_{nt}$, where ξ_{nt} is defined in (S1.1). Furthermore, by (S1.3) and the bounded convergence theorem and Theorem 2.1, we get that

$$n^{-1} \sum_{t=1}^{n} Z_t^2(1) \xrightarrow{p} \sigma^2, \quad n^{-1/2} \sum_{t=1}^{n} Z_t(1) \xrightarrow{d} N(0, \sigma^2). \qquad \text{(S1.16)}$$

Thus, (S1.15)-(S1.16) implies that

$$\lambda = O_p(n^{-1/2}). \qquad \text{(S1.17)}$$

Let $\gamma_t = \lambda Z_t(1)$ and then we have

$$\max_{t\leq n} |\gamma_t| = O_p(n^{-1/2}). \qquad \text{(S1.18)}$$

Then, by (S1.13) and (S1.17)-(S1.18), it follows that

$$\begin{aligned}
0 &= n^{-1} \sum_{t=1}^{n} Z_t(1)\Big(1 - \gamma_t + \frac{\gamma_t^2}{1+\gamma_t}\Big) \\
&= n^{-1} \sum_{t=1}^{n} Z_t(1) - n^{-1} \sum_{t=1}^{n} \lambda Z_t^2(1) + O_p(n^{-1}).
\end{aligned}$$

Therefore, we have

$$\lambda = \Big[n^{-1}\sum_{t=1}^{n} Z_t^2(1)\Big]^{-1}\Big[n^{-1}\sum_{t=1}^{n} Z_t(1)\Big] + O_p(n^{-1}). \tag{S1.19}$$

For $l(1)$, by Taylor expansion, it is straightforward to show that

$$l(1) = 2\sum_{t=1}^{n}\gamma_t - \sum_{t=1}^{n}\gamma_t^2 + \sum_{t=1}^{n}\frac{2\gamma_t^3}{3(1+\lambda_t\gamma_t)^3}, \tag{S1.20}$$

where $\lambda_t \in [0,1]$ for $t = 1, \cdots, n$. By (S1.18)-(S1.20) and (S1.16), it follows that

$$\begin{aligned} l(1) &= 2\sum_{t=1}^{n}\gamma_t - \sum_{t=1}^{n}\gamma_t^2 + O_p(n^{-1/2}) \\ &= 2\lambda\sum_{t=1}^{n} Z_t(1) - \lambda^2\sum_{t=1}^{n} Z_t^2(1) + o_p(1) \\ &= \Big[n^{-1}\sum_{t=1}^{n} Z_t^2(1)\Big]^{-1}\Big[n^{-1/2}\sum_{t=1}^{n} Z_t(1)\Big]^2 + o_p(1) \\ &\xrightarrow{d} \chi_1^2, \end{aligned} \tag{S1.21}$$

as $n \to \infty$. This completes the proof for $l(1)$ under H_0.

Under H_1, we first show that $l(1) \xrightarrow{p} \infty$. Note that the Lagrange dual function of $l(1)$ is given by

$$d(\mu_1, \mu_2) = \inf_{p_t > 0}\Big\{-2\textstyle\sum_{t=1}^{n}\log(np_t) + \mu_1(\sum_{t=1}^{n} p_t - 1) + \mu_2\sum_{t=1}^{n} p_t Z_t(1)\Big\},$$

where μ_1, μ_2 are any real numbers. By definition, it is obvious that $d(\mu_1, \mu_2) \le$

$l(1)$. Then, if we choose $\mu_1 = 2n$ and $\mu_2 = 2n\lambda_1$, it is not hard to show that

$$\begin{aligned} n^{-1}l(1) &\geq n^{-1}d(2n, 2n\lambda_1) \\ &= 2n^{-1}\sum_{t=1}^{n}\log(1+\lambda_1 Z_t(1)) \\ &= 2\lambda_1 n^{-1}\sum_{t=1}^{n} Z_t(1) + O_p(\lambda_1^2), \end{aligned}$$

where we only need the restriction that $1 + \lambda_1 Z_t(1) > 0$. Then, by (S1.12) and $n^{-1/2}T_n = n^{-1}\sum_{t=1}^{n} Z_t(1)$, we can see that when $\lambda_1 = -n^{-1/2}$, it follows that

$$l(1) \xrightarrow{p} \infty. \tag{S1.22}$$

Next, we show that $l(\phi) \xrightarrow{d} \chi_1^2$ under H_1. In this case, it follows that

$$Z_t(\phi) = \frac{y_{t-1}\varepsilon_t}{(1+y_{t-1}^2)^{1/2}(1+\varepsilon_t{}^2)^{1/2}}. \tag{S1.23}$$

Notice that $\{Z_t(\phi)\}$ is still a martingale difference sequence. Then, by the similar argument for Theorem 2.1 and the ergodic theorem, it is not hard to show that

$$n^{-1}\sum_{t=1}^{n} Z_t^2(\phi) \xrightarrow{p} \sigma_1^2, \quad n^{-1/2}\sum_{t=1}^{n} Z_t(\phi) \xrightarrow{d} N(0, \sigma_1^2), \tag{S1.24}$$

where $\sigma_1^2 = E\{y_{t-1}^2\varepsilon_t^2/[(1+y_{t-1}^2)(1+\varepsilon_t^2)]\}$. Finally, using the same procedure for (S1.21), we can get the conclusion. This completes the proof. □

Proof of Theorem 3.2. Under H_0, using the same procedure in proof of

Theorem 3.1, it follows that

$$\frac{|\lambda|}{1+|\lambda|\max_{t\le n+1}|Z_t(1)|}\sum_{t=1}^{n+1} Z_t^2(1) \le \Big|\sum_{t=1}^{n+1} Z_t(1)\Big|. \tag{S1.25}$$

By (S1.16) and $b_n = o(n)$, we can directly show that

$$|Z_{n+1}(1)| = o_p(\sqrt{n}), \tag{S1.26}$$

and then

$$n^{-1}\sum_{t=1}^{n+1} Z_t^2(1) \xrightarrow{p} \sigma^2, \quad n^{-1/2}\sum_{t=1}^{n+1} Z_t(1) \xrightarrow{d} N(0,\sigma^2). \tag{S1.27}$$

Then, by (S1.25)-(S1.27), we have gotten that

$$\lambda = O_p(n^{-1/2}). \tag{S1.28}$$

Let $\gamma_t = \lambda Z_t(1)$ for $t = 1, \cdots, n+1$. By (S1.26), it follows that

$$\max_{t\le n+1}|\gamma_t| = o_p(1). \tag{S1.29}$$

Then, by the similar arguments for (S1.21), we have $l^a(1) \xrightarrow{d} \chi_1^2$, as $n \to \infty$.

Furthermore, under H_1, since $b_n/n + 1/b_n = o(1)$, we can choose the negative number λ_2 with the condition that $\lambda_2 = o(1)$ and $\lambda_2 n \to \infty$ and

$\lambda_2 b_n = o(1)$, then

$$\begin{aligned} l^a(1) &\geq 2\sum_{t=1}^{n+1}\log(1+\lambda_2 Z_t(1)) \\ &= 2\lambda_2\sum_{t=1}^{n+1} Z_t(1) + O_p(n\lambda_2^2) \\ &\xrightarrow{p} \infty. \end{aligned}$$

On the other hand, by (S1.24), we can show that $|Z_{n+1}(\phi)| = o_p(\sqrt{n})$ and

$$n^{-1}\sum_{t=1}^{n+1} Z_t^2(\phi) \xrightarrow{p} \sigma_1^2, \quad n^{-1/2}\sum_{t=1}^{n+1} Z_t(\phi) \xrightarrow{d} N(0,\sigma_1^2). \tag{S1.30}$$

Then, it is easy to get that $l^a(\phi) \xrightarrow{d} \chi_1^2$, as $n \to \infty$. This completes the proof. □

Proof of Corollary 3.1. Since the proof process is very close to those in Theorems 3.1-3.2, we only present some key points and the details are omitted.

Under H_0, by the fact that $\rho(x)$ is a bounded and odd function, we can easily show that

$$n^{-1}\sum_{t=1}^{n} Z_t^2(1) \xrightarrow{p} \sigma_\rho^2, \quad n^{-1/2}\sum_{t=1}^{n} Z_t(1) \xrightarrow{d} N(0,\sigma_\rho^2),$$

where $\sigma_\rho^2 = E(\rho^2(\varepsilon_t))$. Similarly, under H_1, we have

$$n^{-1}\sum_{t=1}^{n} Z_t^2(\phi) \xrightarrow{p} \sigma_\rho^2(1), \quad n^{-1/2}\sum_{t=1}^{n} Z_t(\phi) \xrightarrow{d} N(0,\sigma_\rho^2(1)),$$

where $\sigma_\rho^2(1) = E[y_{t-1}^2\rho^2(\varepsilon_t)/(1+y_{t-1}^2)]$.

Under H_1, it is clear to see that

$$n^{-1}\sum_{t=1}^{n} Z_t(1) \xrightarrow{p} \mu_\rho \equiv E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\Delta y_t)\Big].$$

Then, we have

$$\mu_\rho = E\Big\{E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\eta_t h_t + (\phi-1)y_{t-1})\Big|\mathcal{F}_{t-1}\Big]\Big\}.$$

Furthermore, it follows that

$$E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\eta_t h_t + (\phi-1)y_{t-1})\Big|\mathcal{F}_{t-1}\Big] = \frac{y_{t-1}}{1+y_{t-1}^2} \times \int_0^\infty [\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi-1)y_{t-1})]dF(x). \tag{S1.31}$$

Without loss of generality, $\rho(x)$ is assumed to be an increasing function.

Case 1: If $\rho(x)$ is a strictly increasing function, then by $\phi - 1 < 0$, we get that for any given $y_{t-1} \neq 0$,

$$y_{t-1}[\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi-1)y_{t-1})] < 0, \forall x \in \mathbb{R}.$$

Then, it follows that, $\forall y_{t-1} \neq 0$,

$$E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\eta_t h_t + (\phi-1)y_{t-1})\Big|\mathcal{F}_{t-1}\Big] < 0. \tag{S1.32}$$

Case 2: If $\rho(x) > \rho(y)$, $\forall x > 0, y < 0$ and the density of η_t is positive in a neighbourhood of zero denoted as $[-a, a]$ for some $a > 0$, then

(i). When $y_{t-1} > 0$, it follows that $\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi-1)y_{t-1}) \leq 0$, and $\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi-1)y_{t-1}) < 0$, if $|x| \leq$

$\min\{a, (1-\phi)y_{t-1}/h_t\}$, then

$$E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\eta_t h_t + (\phi-1)y_{t-1})\Big|\mathcal{F}_{t-1}\Big] < 0.$$

(ii). When $y_{t-1} < 0$, it follows that $\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi - 1)y_{t-1}) \geq 0$, and $\rho(xh_t + (\phi-1)y_{t-1}) - \rho(xh_t - (\phi-1)y_{t-1}) > 0$, if $|x| \leq \min\{a, (\phi-1)y_{t-1}/h_t\}$, then

$$E\Big[\frac{y_{t-1}}{1+y_{t-1}^2}\rho(\eta_t h_t + (\phi-1)y_{t-1})\Big|\mathcal{F}_{t-1}\Big] < 0.$$

Therefore, (S1.32) always holds for two cases. Then, by the fact that $y_t = \phi y_{t-1} + \eta_t h_t$ and $P(\eta_t \neq 0) > 0$ and h_t is positive, it is clear to see that $P(y_{t-1} \neq 0) > 0$, which implies that $\mu_\rho < 0$. In other words, we have

$$n^{-1}\sum_{t=1}^{n} Z_t(1) \xrightarrow{p} \mu_\rho < 0. \tag{S1.33}$$

This completes the proof. □

Now, following the standard procedures in Qin and Lawless (1994), we give the proofs for Theorem 4.1. Before that, we need the following two lemmas.

Lemma 1. *Suppose that y_t satisfies model (4.1) and the conditions in The-*

orem 4.1 hold, then under H_0, it follows that

$$n^{-1/2}\sum_{t=1}^{n}\tilde{Z}_{t,2}(1,\mu_0) = n^{-1/2}\sum_{t=1}^{n} w_t + o_p(1); \tag{S1.34}$$

$$n^{-1/2}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0) \xrightarrow{d} N(\mathbf{0},\boldsymbol{\Sigma}); \tag{S1.35}$$

$$n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0)\tilde{\mathbf{Z}}_t'(1,\mu_0) \xrightarrow{p} \boldsymbol{\Sigma}; \tag{S1.36}$$

$$n^{-1}\sum_{t=1}^{n}\frac{\partial\tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial\mu} \xrightarrow{p} \mathbf{a}, \tag{S1.37}$$

where the matrix $\boldsymbol{\Sigma} = diag\{\sigma^2, 1\}$ and the vector $\mathbf{a} = (-E(1+\varepsilon_t^2)^{-3/2}, 0)'$.

Proof of Lemma 1. For (S1.34), it is sufficient to show that, for any $\delta >$ 1/2

$$n^{-1/2}\sum_{t=1}^{n}\frac{y_{t-1}}{(1+y_{t-1}^2)^{\delta}}\tilde{Z}_{t,1}(1,\mu_0) \xrightarrow{p} 0. \tag{S1.38}$$

Notice that $\{n^{-1/2}\frac{y_{t-1}}{(1+y_{t-1}^2)^{\delta}}\tilde{Z}_{t,1}(1,\mu_0), t = 1,\cdots,n\}$ is a martingale difference sequence with respect to $\mathcal{F}_t$ and has the uniform bound $n^{-1/2}$, then by Theorem 18.1 Billingsley (1999), we only need to show that

$$n^{-1}\sum_{t=1}^{n}\frac{y_{t-1}^2}{(1+y_{t-1}^2)^{2\delta}} \xrightarrow{p} 0. \tag{S1.39}$$

By Cauchy inequality, it follows that

$$\begin{aligned} n^{-1}\sum_{t=1}^{n}\frac{y_{t-1}^2}{(1+y_{t-1}^2)^{2\delta}} &\le \sqrt{n^{-1}\sum_{t=1}^{n}\frac{y_{t-1}^4}{(1+y_{t-1}^2)^{2}}} \times \sqrt{n^{-1}\sum_{t=1}^{n}\frac{1}{(1+y_{t-1}^2)^{4\delta-2}}} \\ &\le \sqrt{n^{-1}\sum_{t=1}^{n}\frac{1}{(1+y_{t-1}^2)^{4\delta-2}}}. \end{aligned}$$

By Assumption 2.1 and $a_n/n \to c \in [0, \infty]$, and using the same procedure for (S1.9), it is straightforward to get that $n^{-1} \sum_{t=1}^{n} (1 + y_{t-1}^2)^{-4\delta+2} = o_p(1)$. This completes the proof for (S1.34). Furthermore, applying (S1.34), (S1.35)-(S1.37) can be directly proved by martingale central limit theorem and the ergodic theorem. □

Denote $\tilde{l}(\phi, \mu) = -2\log(\tilde{L}(\phi, \mu))$, then we have the next lemma.

Lemma 2. *Under H_0 and the same conditions in Lemma 1, as $n \to \infty$, with probability to one, the function $\tilde{l}(1, \mu)$ attains its minimum value at some point $\tilde{\mu}$ in the interior of the ball $|\mu - \mu_0| \le n^{-1/3}$, and $\tilde{\mu}$ and $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}(\tilde{\mu})$ satisfy*

$$Q_{1n}(\tilde{\mu}, \tilde{\boldsymbol{\lambda}}) = 0 \textit{ and } Q_{2n}(\tilde{\mu}, \tilde{\boldsymbol{\lambda}}) = 0,$$

where

$$Q_{1n}(\mu, \boldsymbol{\lambda}) = n^{-1} \sum_{t=1}^{n} \frac{\tilde{\mathbf{Z}}_t(1, \mu)}{1 + \boldsymbol{\lambda}' \tilde{\mathbf{Z}}_t(1, \mu)};$$
$$Q_{2n}(\mu, \boldsymbol{\lambda}) = n^{-1} \sum_{t=1}^{n} \frac{1}{1 + \boldsymbol{\lambda}' \tilde{\mathbf{Z}}_t(1, \mu)} \left\{ \frac{\partial \tilde{\mathbf{Z}}_t(1, \mu)}{\partial \mu} \right\}' \boldsymbol{\lambda}.$$

Proof of Lemma 2. Review that for any fixed μ, by Lagrange multiplier technique, we have

$$\tilde{l}(1, \mu) = 2 \sum_{t=1}^{n} \log[1 + \boldsymbol{\lambda}' \tilde{\mathbf{Z}}_t(1, \mu)], \tag{S1.40}$$

where $\boldsymbol{\lambda}$ is a function with respect to μ and satisfies $Q_{1n}(\mu, \boldsymbol{\lambda}) = 0$. Then, it is not hard to get that

$$\boldsymbol{\rho} n^{-1} \sum_{t=1}^{n} \tilde{\mathbf{Z}}_t(1,\mu) \geq \frac{\|\boldsymbol{\lambda}\|}{1 + \|\boldsymbol{\lambda}\| \max_t \|\tilde{\mathbf{Z}}_t(1,\mu)\|} n^{-1} \sum_{t=1}^{n} \tilde{\boldsymbol{\rho}'} \mathbf{Z}_t(1,\mu) \tilde{\mathbf{Z}}_t'(1,\mu) \boldsymbol{\rho}, \tag{S1.41}$$

where $\|\cdot\|$ is the Euclidean norm and $\boldsymbol{\rho} = \boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$.

Note that, by the definitions of $\tilde{\mathbf{Z}}_t(1,\mu)$, it is easy to see that there exists a constant C_0, such that

$$\sup_{\mu \in \mathbb{R}} \max_t \|\tilde{\mathbf{Z}}_t(1,\mu)\| \leq C_0, \ \sup_{\mu \in \mathbb{R}} \max_t \left\| \frac{\partial^k \tilde{\mathbf{Z}}_t(1,\mu)}{\partial \mu^k} \right\| \leq C_0, \tag{S1.42}$$

where $k = 1, 2, 3$. Furthermore, using Taylor expansion, in the domain $|\mu - \mu_0| \leq n^{-1/3}$, we uniformly have

$$\sup_{|\mu-\mu_0| \leq n^{-1/3}} n^{-1} \Big\| \sum_{t=1}^{n} \tilde{\mathbf{Z}}_t(1,\mu) - \sum_{t=1}^{n} \tilde{\mathbf{Z}}_t(1,\mu_0) \Big\| = O_p(n^{-1/3}); \tag{S1.43}$$

$$\sup_{|\mu-\mu_0| \leq n^{-1/3}} n^{-1} \Big\| \sum_{t=1}^{n} \tilde{\mathbf{Z}}_t(1,\mu) \tilde{\mathbf{Z}}_t'(1,\mu) - \sum_{t=1}^{n} \tilde{\mathbf{Z}}_t(1,\mu_0) \tilde{\mathbf{Z}}_t'(1,\mu_0) \Big\| = O_p(n^{-1/3}). \tag{S1.44}$$

Then, by (S1.41) and (S1.43)-(S1.44), and (S1.35)-(S1.36), we have

$$\sup_{|\mu-\mu_0| \leq n^{-1/3}} \frac{\|\boldsymbol{\lambda}\|}{1 + \|\boldsymbol{\lambda}\| C_0} = O_p(n^{-1/3}),$$

which implies that

$$\sup_{|\mu-\mu_0| \leq n^{-1/3}} \|\boldsymbol{\lambda}\| = O_p(n^{-1/3}). \tag{S1.45}$$

Then, it follows that

$$\boldsymbol{\lambda} = \Big[n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu)\tilde{\mathbf{Z}}_t'(1,\mu)\Big]^{-1}\Big[n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu)\Big] + O_p(n^{-2/3}). \quad \text{(S1.46)}$$

Now, consider the boundary $|\mu-\mu_0| = n^{-1/3}$, by (S1.40), (S1.42) and (S1.46), we have

$$\begin{aligned}\tilde{l}(1,\mu) = n\Bigg\{ &\Big[n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0) + n^{-1}\sum_{t=1}^{n}\frac{\partial\tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial\mu}(\mu-\mu_0) + O_p(n^{-2/3})\Big]' \\ &\times\Big[n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu)\tilde{\mathbf{Z}}_t'(1,\mu)\Big]^{-1} \\ &\times\Big[n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0) + n^{-1}\sum_{t=1}^{n}\frac{\partial\tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial\mu}(\mu-\mu_0) + O_p(n^{-2/3})\Big]\Bigg\} + O_p(1),\end{aligned}$$

Then, by the fact $|\mu-\mu_0| = n^{-1/3}$ and using (S1.36)-(S1.37) and (S1.43)-(S1.44), it follows that, with probability to one,

$$\inf_{|\mu-\mu_0|=n^{-1/3}} n^{-1/3}\tilde{l}(1,\mu) \geq \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}/2, \quad \text{(S1.47)}$$

On the other hand, when $\mu=\mu_0$, it is not hard to show that

$$\tilde{l}(1,\mu_0) = O_p(1). \quad \text{(S1.48)}$$

Then, with probability to one, the minimizer $\tilde{\mu}$ of $\tilde{l}(1,\mu)$ satisfies $|\tilde{\mu}-\mu_0| < n^{-1/3}$. Therefore, $\partial\tilde{l}(1,\tilde{\mu})/\partial\mu = 0$, which implies that $Q_{2n}(\tilde{\mu},\tilde{\boldsymbol{\lambda}}) = 0$. □

Based on the above two lemmas, we can prove Theorem 4.1 as follows.

Proof of Theorem 4.1. Taking derivatives of Q_{1n} and Q_{2n} with respect to μ and $\boldsymbol{\lambda}$, we have

$$\frac{\partial Q_{1n}(\mu_0,\mathbf{0})}{\partial \mu} = n^{-1}\sum_{t=1}^{n}\frac{\partial \tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial \mu}, \quad \frac{\partial Q_{1n}(\mu_0,\mathbf{0})}{\partial \boldsymbol{\lambda}'} = -n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0)\tilde{\mathbf{Z}}_t'(1,\mu_0);$$
$$\frac{\partial Q_{2n}(\mu_0,\mathbf{0})}{\partial \mu} = 0, \qquad \frac{\partial Q_{2n}(\mu_0,\mathbf{0})}{\partial \boldsymbol{\lambda}'} = n^{-1}\sum_{t=1}^{n}\left\{\frac{\partial \tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial \mu}\right\}'.$$

By the definitions of $\tilde{\mathbf{Z}}_t(1,\mu)$, there exists a constant C_0 such that

$$\sup_{\mu\in\mathbb{R}}\max_t \|\tilde{\mathbf{Z}}_t(1,\mu)\| \le C_0, \ \sup_{\mu\in\mathbb{R}}\max_t \left\|\frac{\partial^k \tilde{\mathbf{Z}}_t(1,\mu)}{\partial \mu^k}\right\| \le C_0, \tag{S1.49}$$

where $k = 1,2,3$. Expanding $Q_{1n}(\tilde{\mu},\tilde{\boldsymbol{\lambda}})$ and $Q_{2n}(\tilde{\mu},\tilde{\boldsymbol{\lambda}})$ at $(\mu_0,\mathbf{0})$, and by Lemma 2 and (S1.49), we can show that

$$\frac{\partial Q_{1n}(\mu_0,\mathbf{0})}{\partial \boldsymbol{\lambda}'}\tilde{\boldsymbol{\lambda}} + \frac{\partial Q_{1n}(\mu_0,\mathbf{0})}{\partial \mu}(\tilde{\mu}-\mu_0) = -Q_{1n}(\mu_0,\mathbf{0}) + o_p(n^{-1/2}), \tag{S1.50}$$
$$\frac{\partial Q_{2n}(\mu_0,\mathbf{0})}{\partial \boldsymbol{\lambda}'}\tilde{\boldsymbol{\lambda}} + \frac{\partial Q_{2n}(\mu_0,\mathbf{0})}{\partial \mu}(\tilde{\mu}-\mu_0) = o_p(n^{-1/2}). \tag{S1.51}$$

Denote

$$\mathbf{a}_n = n^{-1}\sum_{t=1}^{n}\frac{\partial \tilde{\mathbf{Z}}_t(1,\mu_0)}{\partial \mu} \text{ and } \boldsymbol{\Sigma}_n = n^{-1}\sum_{t=1}^{n}\tilde{\mathbf{Z}}_t(1,\mu_0)\tilde{\mathbf{Z}}_t'(1,\mu_0). \tag{S1.52}$$

It follows from Lemma 1 and (S1.50)-(S1.51) that

$$\sqrt{n}(\tilde{\mu}-\mu_0) = -(\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1}\mathbf{a}_n)^{-1}\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1} \times \sqrt{n}Q_{1n}(\mu_0,\mathbf{0}) + o_p(1), \tag{S1.53}$$
$$\sqrt{n}\tilde{\boldsymbol{\lambda}} = \boldsymbol{\Sigma}_n^{-1}\Big[\mathbf{I} - \mathbf{a}_n(\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1}\mathbf{a}_n)^{-1}\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1}\Big] \times \sqrt{n}Q_{1n}(\mu_0,\mathbf{0}) + o_p(1). \tag{S1.54}$$

On the other hand, by Lemma 2, we have $\tilde{l}(1) = \tilde{l}(1,\tilde{\mu})$. Then, by (S1.53)-

(S1.54), we can show that

$$\tilde{l}(1,\tilde{\mu}) = 2\sum_{t=1}^{n}\log[1+\tilde{\boldsymbol{\lambda}}'\tilde{\mathbf{Z}}_t(1,\tilde{\mu})]$$

$$= (\sqrt{n}\tilde{\boldsymbol{\lambda}})'\boldsymbol{\Sigma}_n(\sqrt{n}\tilde{\boldsymbol{\lambda}}) + o_p(1)$$

$$= [\sqrt{n}\boldsymbol{\Sigma}_n^{-1/2}Q_{1n}(\mu_0,\mathbf{0})]' \times [\mathbf{I}-\mathbf{A}_n] \times [\sqrt{n}\boldsymbol{\Sigma}_n^{-1/2}Q_{1n}(\mu_0,\mathbf{0})] + o_p(1),$$

where $\mathbf{A}_n = \boldsymbol{\Sigma}_n^{-1/2}\mathbf{a}_n(\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1}\mathbf{a}_n)^{-1}\mathbf{a}_n'\boldsymbol{\Sigma}_n^{-1/2}$. By Lemma 1, it is obvious that

$$\mathbf{A}_n \xrightarrow{p} \mathbf{A} = \boldsymbol{\Sigma}^{-1/2}\mathbf{a}(\mathbf{a}'\boldsymbol{\Sigma}^{-1}\mathbf{a})^{-1}\mathbf{a}'\boldsymbol{\Sigma}^{-1/2}.$$

Meanwhile, since $\mathbf{A}=\mathbf{A}'$ and $\mathbf{A}^2=\mathbf{A}$, the trace of $\mathbf{I}-\mathbf{A}$ is 1. Furthermore, $\sqrt{n}\boldsymbol{\Sigma}_n^{-1/2}Q_{1n}(\mu_0,\mathbf{0}) \xrightarrow{d} N(\mathbf{0},\mathbf{I})$ from Lemma 1, then we get that $\tilde{l}(1,\tilde{\mu}) \xrightarrow{d} \chi_1^2$.

On the other hand, under H_1, by the ergodic theorem, it follows that

$$n^{-1}\sum_{t=1}^{n}\tilde{Z}_{t,2}(1,\mu_0) \xrightarrow{p} E\frac{y_{t-1}[\varepsilon_t+(\phi-1)y_{t-1}]}{(1+y_{t-1}^2)^{\delta}\{1+[\varepsilon_t+(\phi-1)y_{t-1}]^2\}^{1/2}}. \quad \text{(S1.55)}$$

Then, using the same arguments for (S1.10), it is direct to prove that

$$E\frac{y_{t-1}[\varepsilon_t+(\phi-1)y_{t-1}]}{(1+y_{t-1}^2)^{\delta}\{1+[\varepsilon_t+(\phi-1)y_{t-1}]^2\}^{1/2}} < 0. \quad \text{(S1.56)}$$

Meanwhile, for any fixed μ, we have

$$\tilde{l}(1,\mu) \geq d(\mu), \quad \text{(S1.57)}$$

where $d(\mu)$ is defined as

$$d(\mu) = \inf_{p_t>0}\left\{-2\textstyle\sum_{t=1}^{n}\log(np_t) + 2n(\sum_{t=1}^{n}p_t - 1) + 2n\lambda_3\sum_{t=1}^{n}p_t\tilde{Z}_{t,2}(1,\mu)\right\}$$

with $\lambda_3 = -n^{-1/2}$. Then, it is obvious that

$$\tilde{l}(1,\mu) \geq -2n^{-1/2}\sum_{t=1}^{n}\tilde{Z}_{t,2}(1,\mu) + O_p(1),$$

where $O_p(1)$ uniformly holds in the domain $|\mu - \mu_0| \leq n^{-1/3}$. Furthermore, by (S1.56) and (S1.49), we have shown that, with probability to one,

$$\tilde{l}(1) = \inf_{\mu}\tilde{l}(1,\mu) \geq -\sqrt{n}E\frac{y_{t-1}[\varepsilon_t + (\phi-1)y_{t-1}]}{(1+y_{t-1}^2)^{1/2}\{1+[\varepsilon_t+(\phi-1)y_{t-1}]^2\}^{1/2}} \longrightarrow \infty.$$

This completes the whole proof. □

Proof of Theorem 4.2. Like the proof of Lemma 1, and using Assumption 4.1, we can easily show that

$$\bar{\mathbf{Z}}_n = n^{-1/2}\sum_{t=1}^{n}\bar{\mathbf{Z}}_t(1,\boldsymbol{\theta}_0) \xrightarrow{d} N(\mathbf{0},\bar{\boldsymbol{\Sigma}});\ \bar{\boldsymbol{\Sigma}}_n = n^{-1}\sum_{t=1}^{n}\bar{\mathbf{Z}}_t(1,\boldsymbol{\theta}_0)\bar{\mathbf{Z}}_t'(1,\boldsymbol{\theta}_0) \xrightarrow{p} \bar{\boldsymbol{\Sigma}};$$
$$n^{-1}\sum_{t=1}^{n}\frac{\partial\bar{Z}_{t,1}(1,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'} \xrightarrow{p} \bar{\mathbf{b}}_1;\ n^{-1}\sum_{t=1}^{n}\frac{\partial\bar{Z}_{t,2}(1,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'} \xrightarrow{p} \mathbf{0};\ n^{-1}\sum_{t=1}^{n}\frac{\partial\bar{Z}_{t,2+j}(1,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'} \xrightarrow{p} \bar{\mathbf{b}}_{2+j}.$$

Meanwhile, for any $k = 1, 2, 3$, there exists some constant C_0 such that

$$\sup_{\boldsymbol{\theta}}\max_{t}\|\bar{\mathbf{Z}}_t(1,\boldsymbol{\theta})\| \leq C_0,\ \sup_{\boldsymbol{\theta}}\max_{t}\left\|\frac{\partial^k\bar{\mathbf{Z}}_t(1,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^k}\right\| \leq C_0, \tag{S1.58}$$

Then, it is not hard to show that

$$\bar{l}(1) = [\bar{\boldsymbol{\Sigma}}_n^{-1/2}\bar{\mathbf{Z}}_n]' \times [\mathbf{I} - \bar{\mathbf{A}}] \times [\bar{\boldsymbol{\Sigma}}_n^{-1/2}\bar{\mathbf{Z}}_n] + o_p(1),$$

where $\bar{\mathbf{A}} = \bar{\boldsymbol{\Sigma}}^{-1/2}\bar{\mathbf{B}}(\bar{\mathbf{B}}'\bar{\boldsymbol{\Sigma}}^{-1}\bar{\mathbf{B}})^{-1}\bar{\mathbf{B}}'\bar{\boldsymbol{\Sigma}}^{-1/2}$ with $\bar{\mathbf{B}} = (\bar{\mathbf{b}}_1, \mathbf{0}, \bar{\mathbf{b}}_3, \cdots, \bar{\mathbf{b}}_{r+2})'$. Then $\bar{l}(1) \xrightarrow{d} \chi_1^2$ since the trace of $\mathbf{I} - \bar{\mathbf{A}}$ is 1. Under H_1, it is not hard to prove that

$$n^{-1}\sum_{t=1}^{n}\bar{Z}_{t,2}(1,\boldsymbol{\theta}_0)\xrightarrow{p}\mu_0<0, \tag{S1.59}$$

where μ_0 is defined as

$$\mu_0 = E\frac{y_{t-1}[\varepsilon_t+(\phi-1)y_{t-1}]}{(1+y_{t-1}^2)^{\delta}[1+\sum_{j=1}^{r}(\Delta y_{t-j})^2]^{3/2}\{1+[\varepsilon_t+(\phi-1)y_{t-1}]^2\}^{1/2}}.$$

Then, the conclusion holds by the same arguments for Theorem 4.1. □

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S2. Simulation for Confidence Interval

In this section, we also examine the length of the 95% confidence interval for the regression parameter with a sample size $n = 100, 300$ and $\phi = 0.95, 0.9$ and 0.85. The results are summarized in Tables 1–4 below. Several observations can be deduced from these tables. First, the confidence interval length derived by $l^a(\phi)$ is very close to that by $l(\phi)$. Second, the heavier tail implies the shorter interval in the proposed methods. Third, both $l(\phi)$ and $l^a(\phi)$ provide much better inferences than that by ELT, especially when $E\eta_t^2 = \infty$.

Table 1: Average lengths calculated for the confidence intervals based on the empirical likelihood methods with $\alpha \in (1, 2)$ and $n = 100$

		$\varepsilon_t \sim$model (5.1)			$\varepsilon_t \sim$model (5.2)		
$\eta_t \sim$	ϕ	$l(\phi)$	$l^a(\phi)$	ELT	$l(\phi)$	$l^a(\phi)$	ELT
$N(0,1)$	0.95	0.1778	0.1894	0.2799	0.1681	0.1794	0.2687
	0.90	0.2386	0.2549	0.3240	0.2248	0.2399	0.3142
	0.85	0.2794	0.2976	0.3547	0.2737	0.2911	0.3439
Laplace	0.95	0.1330	0.1424	0.3159	0.1327	0.1418	0.2913
	0.90	0.1882	0.1999	0.3772	0.1836	0.1969	0.3486
	0.85	0.2313	0.2476	0.4059	0.2251	0.2427	0.3868
t_3	0.95	0.1397	0.1495	0.3294	0.1357	0.1448	0.3046
	0.90	0.1939	0.2070	0.3921	0.1833	0.1944	0.3489
	0.85	0.2374	0.2551	0.4330	0.2198	0.2342	0.3943
t_2	0.95	0.1086	0.1157	0.3760	0.1116	0.1192	0.3467
	0.90	0.1540	0.1639	0.4396	0.1545	0.1653	0.4127
	0.85	0.1939	0.2068	0.5002	0.1893	0.2032	0.4934

Table 2: Average lengths calculated for the confidence intervals based on the empirical likelihood methods with $\alpha \in (1,2)$ and $n = 300$

		ε_t ~model (5.1)			ε_t ~model (5.2)		
$\eta_t \sim$	ϕ	$l(\phi)$	$l^a(\phi)$	ELT	$l(\phi)$	$l^a(\phi)$	ELT
$N(0,1)$	0.95	0.0878	0.0898	0.1602	0.0856	0.0876	0.1501
	0.90	0.1308	0.1342	0.1892	0.1250	0.1279	0.1773
	0.85	0.1618	0.1657	0.2049	0.1533	0.1576	0.1953
Laplace	0.95	0.0613	0.0627	0.1787	0.0648	0.0663	0.1672
	0.90	0.0954	0.0975	0.2153	0.0978	0.1000	0.1996
	0.85	0.1233	0.1264	0.2408	0.1254	0.1281	0.2280
t_3	0.95	0.0705	0.0723	0.1955	0.0694	0.0708	0.1751
	0.90	0.1042	0.1064	0.2330	0.0994	0.1017	0.2148
	0.85	0.1285	0.1319	0.2532	0.1215	0.1245	0.2328
t_2	0.95	0.0497	0.0507	0.2367	0.0523	0.0535	0.2177
	0.90	0.0802	0.0823	0.2868	0.0801	0.0822	0.2647
	0.85	0.1058	0.1084	0.3486	0.1019	0.1041	0.3047

Table 3: Average lengths calculated for the confidence intervals based on the empirical likelihood methods with $\alpha \in (0,1)$ and $n = 100$

		ε_t ~model (5.1)			ε_t ~model (5.2)		
$\eta_t \sim$	ϕ	$l(\phi)$	$l^a(\phi)$	ELT	$l(\phi)$	$l^a(\phi)$	ELT
$N(0,1)$	0.95	0.2086	0.2230	0.3218	0.1807	0.1929	0.2964
	0.90	0.2777	0.2949	0.3765	0.2516	0.2686	0.3524
	0.85	0.3276	0.3483	0.3995	0.3066	0.3267	0.3748
Laplace	0.95	0.1353	0.1441	0.3659	0.1327	0.1411	0.3398
	0.90	0.1955	0.2095	0.4181	0.1895	0.2032	0.3953
	0.85	0.2518	0.2696	0.4647	0.2373	0.2537	0.4379
t_2	0.95	0.1272	0.1359	0.4647	0.1154	0.1229	0.3856
	0.90	0.1772	0.1883	0.5278	0.1628	0.1732	0.4729
	0.85	0.2267	0.2428	0.6114	0.2065	0.2210	0.5178
Cauchy	0.95	0.0467	0.0495	1.1544	0.0418	0.0447	0.9664
	0.90	0.0783	0.0833	1.6059	0.0725	0.0777	1.3169
	0.85	0.1139	0.1210	2.3172	0.1048	0.1118	1.5173

Table 4: Average lengths calculated for the confidence intervals based on the empirical likelihood methods with $\alpha \in (0,1)$ and $n = 300$

		$\varepsilon_t \sim$model (5.1)			$\varepsilon_t \sim$model (5.2)		
$\eta_t \sim$	ϕ	$l(\phi)$	$l^a(\phi)$	ELT	$l(\phi)$	$l^a(\phi)$	ELT
$N(0,1)$	0.95	0.0991	0.1014	0.1894	0.0903	0.0926	0.1743
	0.90	0.1470	0.1510	0.2168	0.1362	0.1395	0.1991
	0.85	0.1860	0.1898	0.2333	0.1707	0.1742	0.2212
Laplace	0.95	0.0548	0.0559	0.2151	0.0555	0.0570	0.1921
	0.90	0.0966	0.0994	0.2494	0.0933	0.0956	0.2335
	0.85	0.1276	0.1306	0.2774	0.1256	0.1287	0.2543
t_2	0.95	0.0531	0.0546	0.2836	0.0536	0.0548	0.2444
	0.90	0.0918	0.0942	0.3528	0.0861	0.0881	0.3074
	0.85	0.1214	0.1241	0.3667	0.1124	0.1154	0.3569
Cauchy	0.95	0.0079	0.0081	1.0902	0.0081	0.0083	0.9136
	0.90	0.0212	0.0218	1.5566	0.0225	0.0230	1.2913
	0.85	0.0441	0.0450	1.9520	0.0412	0.0422	1.6862