A MIXED-EFFECTS ESTIMATING EQUATION APPROACH TO NONIGNORABLE MISSING LONGITUDINAL DATA WITH REFRESHMENT SAMPLES

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Supplementary Material

S1 Notation and Regularity Conditions

Define the quadratic inference function:

$$Q_n(\boldsymbol{\beta}|\mathbf{b}) = (\bar{\mathbf{g}}_n^f)'(\bar{C}_n^f)^{-1}(\bar{\mathbf{g}}_n^f),$$

its first partial derivative:

$$\dot{Q}_n(\boldsymbol{\beta}|\mathbf{b}) = \frac{\partial}{\partial\beta}Q_n(\boldsymbol{\beta}|\mathbf{b}) = 2(\dot{\bar{\mathbf{g}}}_n^f)'(\bar{C}_n^f)^{-1}(\bar{\mathbf{g}}_n^f) + o(1),$$

and its second partial derivative:

$$\ddot{Q}_n(\boldsymbol{\beta}|\mathbf{b}) = \frac{\partial^2}{\partial\beta^2}Q_n(\boldsymbol{\beta}|\mathbf{b}) = 2(\dot{\bar{\mathbf{g}}}_n^f)'(\bar{C}_n^f)^{-1}(\dot{\bar{\mathbf{g}}}_n^f) + o(1).$$

Define $\dot{\mathbf{g}}_0 = \mathrm{E}(\dot{\mathbf{g}}_i^f|\mathbf{b}_0)$, and $C_0 = \mathrm{Var}(\mathbf{g}_i|\mathbf{b}_0)$.

We here provide the regularity conditions to prove Lemma 1 and Theorem 1.

(i) The response variables $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are i.i.d.

(ii) The fixed effect $\boldsymbol{\beta}$ is identifiable; that is, there exists a unique $\boldsymbol{\beta}_0$, such that $\mathrm{E}\{\mathbf{g}_i^f(\boldsymbol{\beta}_0|\mathbf{b}_0)\} = \mathbf{0}$.

(iii) The estimating function $\mathbf{g}_i(\boldsymbol{\beta}|\mathbf{b})$ is differentiable with respect to both $\boldsymbol{\beta}$ and $\mathbf{b}$, $i = 1, \ldots, n$.

(iv) $\text{Var}(\mathbf{g}_i|\mathbf{b}) < \infty$ in probability, for $i = 1, \ldots, n$.

(v) $\dot{\bar{\mathbf{g}}}_n^f(\boldsymbol{\beta}|\mathbf{b})$ is uniformly bounded in probability with respect to both $\boldsymbol{\beta}$ and $\mathbf{b}$ in an open bounded space containing $\boldsymbol{\beta}_0$ and $\mathbf{b}_0$, and conditional on $\mathbf{b}_0$, $\dot{\bar{\mathbf{g}}}_n^f \overset{a.s.}{\to} \dot{\mathbf{g}}_0$ as $n \to \infty$.

(vi) $\bar{C}_n^f(\boldsymbol{\beta}|\mathbf{b})$ is uniformly bounded in probability with respect to both $\boldsymbol{\beta}$ and $\mathbf{b}$ in an open bounded space containing $\boldsymbol{\beta}_0$ and $\mathbf{b}_0$, and conditional on $\mathbf{b}_0$, $\bar{C}_n^f \overset{a.s.}{\to} C_0$ as $n \to \infty$.

(vii) There exists an open bounded parameter space $\mathcal{S} \subseteq \mathbb{R}^p$, such that $\boldsymbol{\beta}_0 \in \mathcal{S}$ and $Q_n(\boldsymbol{\beta}|\mathbf{b}_0)$ is uniformly convergent in probability in $\mathcal{S}$. Define:

$$Q(\boldsymbol{\beta}|\mathbf{b}_0) = \lim_{n\to\infty} Q_n(\boldsymbol{\beta}|\mathbf{b}_0),$$

and thus:

$$\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0) = \lim_{n\to\infty} \dot{Q}_n(\boldsymbol{\beta}|\mathbf{b}_0).$$

S2 Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1. Solving $\hat{\boldsymbol{\beta}} = \arg\min(\bar{\mathbf{g}}_n^f)'(\bar{C}_n^f)^{-1}(\bar{\mathbf{g}}_n^f)$ is equivalent to solving

$$\dot{Q}_n(\hat{\boldsymbol{\beta}}|\mathbf{b}_0) = \mathbf{0}.$$

By Taylor expansion, we have:

$$\mathbf{0} = \dot{Q}_n(\hat{\boldsymbol{\beta}}|\mathbf{b}_0) = \dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0) + \ddot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + o(\frac{1}{\sqrt{n}}),$$

By regularity conditions (ii), (v) and (vi), we have $\text{E}\{\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)\} = \mathbf{0}$. Then by regularity condition (iv) and the central limit theorem, we conclude that:

$$\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0) \sim O(\frac{1}{\sqrt{n}}) \text{ and } \sqrt{n}(\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)) \to \text{N}(\mathbf{0}, \Omega_0),$$

where

$$\begin{aligned}
\Omega_0 &= \lim_{n\to\infty} n\text{Var}(\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)) \\
&= 4\lim_{n\to\infty} (\dot{\bar{\mathbf{g}}}_n^f)'(\bar{C}_n^f)^{-1}\{\frac{1}{n}\sum_{i=1}^n \text{Var}(\mathbf{g}_i|\mathbf{b}_0)\}(\bar{C}_n^f)^{-1}(\dot{\bar{\mathbf{g}}}_n^f) \\
&= 4(\dot{\mathbf{g}}_0)'(C_0)^{-1}(\dot{\mathbf{g}}_0).
\end{aligned}$$

Since $\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = -\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\cdot\sqrt{n}\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)+o(1)$, we conclude that:

$$\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \to \mathrm{N}(\mathbf{0},\Sigma_0),$$

where $\Sigma_0 = \lim_{n\to\infty}\{\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\}\Omega_0\{\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\}' = \{(\dot{\mathbf{g}}_0)'(C_0)^{-1}(\dot{\mathbf{g}}_0)\}^{-1}$. □

Proof of Theorem 1. Solving $\hat{\boldsymbol{\beta}} = \arg\min(\bar{\mathbf{g}}_n^f)'(\bar{C}_n^f)^{-1}(\bar{\mathbf{g}}_n^f)$ is equivalent to finding $\hat{\boldsymbol{\beta}}$ such that $\dot{Q}_n(\hat{\boldsymbol{\beta}}|\hat{\mathbf{b}}) = \mathbf{0}$.

Based on regularity conditions (ii), (v), (vi) and (vii), we have $Q(\beta_0|\mathbf{b}_0) = 0$ and $\dot{Q}(\beta_0|\mathbf{b}_0) = \mathbf{0}$. And based on regularity conditions (v) and (vi) and the condition that $\frac{1}{n}\sum_{i=1}^{n}\mathbf{g}_i(\boldsymbol{\beta}_0|\hat{\mathbf{b}}) \to \mathbf{0}$ as $n\to\infty$, we have:

$$\lim_{n\to\infty}\dot{Q}_n(\boldsymbol{\beta}_0|\hat{\mathbf{b}}) = \mathbf{0} = \dot{Q}(\boldsymbol{\beta}_0|\mathbf{b}_0). \tag{S2.1}$$

Define the boundary of a ball in $\mathcal{S}$ with center $\boldsymbol{\beta}_0$ and radius $\frac{1}{\sqrt{n}}$ as $\partial B_n(\boldsymbol{\beta}_0) = \{\boldsymbol{\beta} : \|\boldsymbol{\beta}-\boldsymbol{\beta}_0\| = \frac{1}{\sqrt{n}}\}$. Then for any $\boldsymbol{\beta}\in\partial B_n(\boldsymbol{\beta}_0)$, we have:

$$\mathbf{0} = Q(\boldsymbol{\beta}_0|\mathbf{b}_0) = Q(\boldsymbol{\beta}|\mathbf{b}_0) + \dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)(\boldsymbol{\beta}_0-\boldsymbol{\beta}) + o(\frac{1}{\sqrt{n}}).$$

Since $Q(\boldsymbol{\beta}|\mathbf{b}_0) > 0$ when $\boldsymbol{\beta}\neq\boldsymbol{\beta}_0$, we can find an $\epsilon > 0$, such that:

$$(\boldsymbol{\beta}-\boldsymbol{\beta}_0)\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0) = Q(\boldsymbol{\beta}|\mathbf{b}_0) + o(\frac{1}{\sqrt{n}}) > \epsilon > 0.$$

Then based on (S2.1), for such ϵ, there exists a large N, such that when $n > N$,

$$\begin{aligned} & \|\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}}) - \dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)\| \\ \leq\ & \|\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}}) - \dot{Q}_n(\boldsymbol{\beta}_0|\hat{\mathbf{b}})\| + \|\dot{Q}_n(\boldsymbol{\beta}_0|\hat{\mathbf{b}}) - \dot{Q}(\boldsymbol{\beta}_0|\mathbf{b}_0)\| + \|\dot{Q}(\boldsymbol{\beta}_0|\mathbf{b}_0) - \dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)\| \\ <\ & \epsilon \end{aligned}$$

for $\boldsymbol{\beta}\in\partial B_n(\boldsymbol{\beta}_0)$. This is because $\dot{\bar{\mathbf{g}}}_n^f(\boldsymbol{\beta}|\mathbf{b})$ and $\bar{C}_n^f(\boldsymbol{\beta}|\mathbf{b})$ are uniformly bounded and $\bar{\mathbf{g}}_n^f$ is continuous with respect to $\boldsymbol{\beta}$, so

$$\|\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}}) - \dot{Q}_n(\boldsymbol{\beta}_0|\hat{\mathbf{b}})\| < \frac{1}{3}\epsilon, \text{ and } \|\dot{Q}(\boldsymbol{\beta}_0|\mathbf{b}_0) - \dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)\| < \frac{1}{3}\epsilon$$

for a large N. And because of (S2.1),

$$\|\dot{Q}_n(\boldsymbol{\beta}_0|\hat{\mathbf{b}}) - \dot{Q}(\boldsymbol{\beta}_0|\mathbf{b}_0)\| < \frac{1}{3}\epsilon.$$

By the Cauchy-Schwarz Inequality:

$$
\begin{aligned}
|(\boldsymbol{\beta}-\boldsymbol{\beta}_0)[\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}})-\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)]| &\leq \|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|\cdot\|\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}})-\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)\| \\
&< \frac{1}{\sqrt{n}}\epsilon.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\boldsymbol{\beta}-\boldsymbol{\beta}_0)\dot{Q}_n(\boldsymbol{\beta}|\hat{\mathbf{b}}) &> (\boldsymbol{\beta}-\boldsymbol{\beta}_0)\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)-\frac{1}{\sqrt{n}}\epsilon \\
&> (\boldsymbol{\beta}-\boldsymbol{\beta}_0)\dot{Q}(\boldsymbol{\beta}|\mathbf{b}_0)-\epsilon>0.
\end{aligned}
$$

Then based on Theorem 6.3.4 of Ortega and Rheinboldt (1970, p(163)), there exists a $\hat{\boldsymbol{\beta}}_n \in B_n(\boldsymbol{\beta}_0)$, such that

$$
\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\hat{\mathbf{b}})=\mathbf{0}.
$$

This is a direct application of the p-dimensional intermediate value theorem. Since $\hat{\boldsymbol{\beta}}_n \in B_n(\boldsymbol{\beta}_0)$, we have $\hat{\boldsymbol{\beta}}_n = O(\frac{1}{\sqrt{n}})$ and $\hat{\boldsymbol{\beta}}_n \to \boldsymbol{\beta}_0$ as $n\to\infty$.

The following part shows the asymptotic normality of $\hat{\boldsymbol{\beta}}_n$.

From Lemma 1, we have:

$$
\sqrt{n}(\hat{\boldsymbol{\beta}}_0-\boldsymbol{\beta}_0)=-\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\cdot\sqrt{n}\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)+O(\frac{1}{\sqrt{n}}), \tag{S2.2}
$$

where $\hat{\boldsymbol{\beta}}_0$ is the solution of $\hat{\boldsymbol{\beta}}=\arg\min(\bar{\mathbf{g}}_n^f)'(\bar{C}_n^f)^{-1}(\bar{\mathbf{g}}_n^f)$ conditional on $\mathbf{b}_0$.

Since $\hat{\boldsymbol{\beta}}_n \in B_n(\boldsymbol{\beta}_0)$, for any $\epsilon>0$, we have $\|\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\hat{\mathbf{b}})-\dot{Q}(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)\|<\epsilon$, and hence $\|\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\hat{\mathbf{b}})-\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)\|<\epsilon$ for a large N and $n>N$. In addition,

$$
\begin{aligned}
\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0) &= \dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)-\dot{Q}_n(\hat{\boldsymbol{\beta}}_0|\mathbf{b}_0) \\
&= \ddot{Q}_n(\hat{\boldsymbol{\beta}}_0|\mathbf{b}_0)(\hat{\boldsymbol{\beta}}_n-\hat{\boldsymbol{\beta}}_0)+O(\frac{1}{n}).
\end{aligned}
$$

Thus, conditional on $\hat{\mathbf{b}}$,

$$
\sqrt{n}(\hat{\boldsymbol{\beta}}_n-\hat{\boldsymbol{\beta}}_0)=\ddot{Q}_n^{-1}(\hat{\boldsymbol{\beta}}_0|\mathbf{b}_0)\cdot\sqrt{n}\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)+o(1). \tag{S2.3}
$$

From (S2.2) and (S2.3), and because $\lim_{n\to\infty}\ddot{Q}_n(\hat{\boldsymbol{\beta}}_0|\mathbf{b}_0)=\lim_{n\to\infty}\ddot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)$, we have:

$$
\begin{aligned}
\sqrt{n}(\hat{\boldsymbol{\beta}}_n-\boldsymbol{\beta}_0) &= \sqrt{n}(\hat{\boldsymbol{\beta}}_n-\hat{\boldsymbol{\beta}}_0)+\sqrt{n}(\hat{\boldsymbol{\beta}}_0-\boldsymbol{\beta}_0) \\
&= \ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\{\sqrt{n}\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)-\sqrt{n}\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)\}+o(1).
\end{aligned}
$$

From the central limit theorem and the consistency of $\hat{\boldsymbol{\beta}}_n$, we know that $\sqrt{n}\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0)$ and $\sqrt{n}\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)$ are asymptotically normal. Therefore

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_0) \to \mathrm{N}(\mathbf{0}, \Sigma),$$

where $\Sigma = \{\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\}\Omega\{\ddot{Q}_n^{-1}(\boldsymbol{\beta}_0|\mathbf{b}_0)\}'$ and $\Omega = \lim_{n\to\infty} \mathrm{Var}\{\sqrt{n}\dot{Q}_n(\hat{\boldsymbol{\beta}}_n|\mathbf{b}_0) - \sqrt{n}\dot{Q}_n(\boldsymbol{\beta}_0|\mathbf{b}_0)\}$.
$\square$

References

Ortega, J. M. and Rheinboldt, W. C. (1970). *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York.