Statistica Sinica 3(1993), 261-275

POISSON CONVERGENCE IN RELIABILITY OF A LARGE LINEARLY CONNECTED SYSTEM AS RELATED TO COIN TOSSING

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Abstract: Reliability of a large linearly connected engineering system is closely associated with the probability of a certain pattern occurring in a sequence of coin tossing. In this paper a new method is developed to show that if the failure probabilities of components are very small then the reliability of the system can be approximated by a Poisson random variable. The proof of the result is essentially dependent on the Markov property of coin tossing. It is more direct and elementary than the standard tools such as Bonferroni inequalities and the Stein-Chen method. The result is also extended for the case that the failure probabilities of components are different. Necessary and sufficient conditions for Poisson convergence are also obtained. Numerical upper and lower bounds of reliabilities of linearly connected systems developed from the new method are obtained and compared with the bounds derived from the Stein-Chen method.

Key words and phrases: Reliability, linearly connected system, coin tossing, pattern, Poisson convergence.

1. Introduction

Reliabilities of engineering systems such as atomic power plants, airplanes, automobiles and computers are vital to the public. Currrently, a great deal of effort has been made to study the reliabilities of various large engineering systems. An engineering system is called linearly connected system if it can be imbedded in a Markov Chain $\{X_t\}$ on a finite state space $S = \{1, 2, \ldots, k, k+1\}$, with the state "$k + 1$" as an absorbing state. The index t is not restricted as time. For example, in a large linearly connected system, the index t can be interpreted as the tth component. Throughout this paper, unless otherwise specified, we shall interpret t as tth component. There are many important engineering systems such as series systems, m-standby systems, consecutive-k-out-of-n: F systems, and repairable systems, which can all be viewed as linearly connected systems. The reliabilities of consecutive-k-out-of-n: F systems have been studied by many authors; for example, Derman, Lieberman and Ross (1982), Fu (1985) and Papastavridis (1987, 1990). Recently, Chao and Fu (1989, 1991) obtained a general formula for the reliability of linearly connected system,

$$R_n = \pi_0 \prod_{t=1}^{n} M_t U_0', \tag{1.1}$$

where π_0 is the initial probability of the Markov chain, for given $t = 1, 2 \ldots, n$, M_t is a $(k+1) \times (k+1)$ transition probability matrix, and $U_0' = (1, \ldots, 1, 0)'$ is a $(k+1) \times 1$ matrix. To the author's knowledge, formula (1.1) is the simplest way of computing the exact reliability for those linearly connected systems mentioned above. Theoretically speaking, it could be used for any fixed k and n. For large n, if the failure probabilities of components are bounded away from zero, then the reliability of the system tends to zero exponentially in a large deviation sense: i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log R_n = -\beta, \tag{1.2}$$

where β is the Chernoff index (Bucklew (1990)). They also show that if the failure probabilities of components are functions of n and tend to zero at a certain rate then the limiting reliability of a large linearly connected system has the form

$$\lim_{n\to\infty} R_n = \exp\{-\lambda\} > 0, \tag{1.3}$$

where the λ is failure rate of the system given by

$$\lambda = \int_0^1 \langle v(\alpha), c(\alpha)\rangle d\alpha, \tag{1.4}$$

and $\langle \cdot, \cdot \rangle$ stands for the inner product of two vectors $v(\alpha) = (v_1(\alpha), \ldots, v_k(\alpha))$ and $c(\alpha) = (c_1(\alpha), \ldots, c_k(\alpha))$. The components of the vector $c(\alpha)$ can be viewed as instantaneous rates that probabilities escape from states $i = 1, \ldots, k$ to the absorbing state $k+1$; and the components of $v(\alpha)$ can be interpreted as the conditional equilibrium probabilities of the system staying at the first k states respectively, given the system survives at the time α $(0 \leq \alpha = \lim_{n\to\infty} t/n \leq 1)$.

A linearly connected system is called m-$\wedge^*$ linearly connected when it fails if, and only if, there are m or more non-overlapping $\wedge^*$ patterns have occurred. For example, a consecutive-k-out-of-n: F system is the case where the $m = 1$ and the pattern $\wedge^* = F \ldots F$ is consecutive k failures, and a k-standby system is the special case of $m = k+1$ and $\wedge^* = F$. This manuscript studies mainly the reliability of a large m-$\wedge^*$ linearly connected system when the failure probabilities of components are very small. If all the components of an m-$\wedge^*$ linearly connected system operate independently, the reliability of the system is equal to the probability that there are fewer than m non-overlapping $\wedge^*$ patterns which occur in tossing a coin n times. Hence, to study the reliability of a m-$\wedge^*$ linearly connected system, it is sufficient to study the distribution of the number of non-overlapping $\wedge^*$ patterns which occur in coin tossing.

Without loss of generality, unless otherwise specified, we define n-tuple $w = (SFSS \ldots F)$ as a realization of tossing a coin n times and $\wedge^* = F \ldots F$ as a pattern of consecutive k failures. Further, the coordinates throughout this paper will be referred to as "positions". Given w, let $N_{n,k}(w)$ be the number of non-overlapping failure runs of size k. For example, with $n = 8$, $k = 2$, and $w = (SFFSFFFF)$, the total number $N_{8,2}(w)$ of non-overlapping failure runs of size 2 in w is 3. Therefore, the reliability of a consecutive-k-out-of-n : F System is equal to the probability of $N_{n,k} = 0$ and the reliability of a k-out-of-n : F system is equal to the probability $N_{n,1} < k$. It is a well-known fact that if the failure probabilities of components are very small and n is large then the random variable $N_{n,k}$ converges to a Poisson random variable. Technically, such a Poisson convergence problem requires computing the tail probability for an unusual event (a pattern). The standard tool for computing the tail probability of this type was based on Bonferroni's inequalities (see, for example, Watson (1954), and Karlin and Ost (1987)). Recently, a popular alternative method to establish Poisson convergence for dependent events is to use the Stein-Chen method (Chen (1975), Stein (1986)), as utilized, for example, by Arratia, Goldstein and Gordon (1989), Arratia, Gordon and Waterman (1990), Barbour and Holst (1989), Godbole (1991), Smith (1988), and Arratia and Waterman (1989). Further, Chrysaphinou and Papastavridis (1990), and Barbour, Holst and Janson (1991), also used the Stein-Chen method to obtain the upper and lower bounds of the reliability of the consecutive-k-out-of-n: F System. Besides the two methods mentioned above, there are several other similar approaches to establish Poisson convergence and bounds, for example, the results of Nedelman and Walleniu (1986) and Wang (1989, 1991). The main advantage of the Stein-Chen method over Bonferroni's inequalities is that the Stein-Chen method only requires computing the first two moments of the process but needs no computation of higher-order moments.

In Section 2, based on a simple large deviation inequality from Fu (1985), this manuscript provides a new elementary and more direct method to show that the random variable $N_{n,k}$ converges to a Poisson random variable. Hence, the result yields the reliability for a large m-$\wedge^*$ linearly connected system when the failure probabilities of components are small. The method needs no computation of moments of any order. For the special case, $N_{n,1} = S_n$, the method yields some well-known results obtained by Von Mises (1921), Hoeffding (1956), Nedelman and Wallenius (1986), Samuels (1965), Arratia, Goldstein and Gordon (1989), Barbour and Holst (1989), and Wang (1989, 1991), on the Poisson convergence of S_n in n independent Bernoulli trials with unequal failure probabilities. Necessary and sufficient conditions for Poisson convergence of $N_{n,k}$ are also obtained.

Section 3 mainly studies the bounds and the reliabilities of large m-$\wedge^*$ linearly connected systems with very small failure probabilities of components. Numerical results show that upper and lower bounds derived from the new method perform extremely well even in the case of small n.

2. Poisson Convergence of n Independent Bernoulli Trials

Let $\{X_{in}\}$ be a double sequence of independent Bernoulli trials, and $EX_{in} = q_{in}$ for $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$ be failure probabilities. This section studies Poisson convergence of the random variable $N_{n,k}$ for the following two cases:

(a) independent and identical Bernoulli trials, i.e. for every given n, $q_n = EX_{in}$ for $i = 1, \ldots, n$

(b) independent but non-identical Bernoulli trials, i.e. $q_{in} = EX_{in}$ not all equal.

Theorem 2.1. *For given k, if the failure probability $q_{in} = q_n$ for $1 \leq i \leq n$ and satisfies the condition*

$$nq_n^k \to \lambda \tag{2.1}$$

as $n \to \infty$, then the random variable $N_{n,k}$ converges to a Poisson random variable with mean λ; i.e., for every $x = 0, 1, \ldots$

$$\lim_{n\to\infty} P(N_{n,k} = x) = \frac{\lambda^x}{x!}e^{-\lambda}. \tag{2.2}$$

Given k, n, and x, define a subset

$$A = A(x, n, k) = \{w : N_{n,k}(w) = x\}. \tag{2.3}$$

Both the Bonferroni and Stein-Chen methods have one thing in common. They estimate the probability $P(A(x, n, k))$ of the event $N_{n,k} = x$, directly by obtaining the upper and lower bounds based on the moments of the process. Both methods require computation of moments. Unlike the methods mentioned above, our method is to find two suitable subsets A^* and A_* which have the following desired properties:

(1) $A_* \subseteq A \subseteq A^*$,

(2) the lower bound of $P(A_*)$ and the upper bound of $P(A^*)$ can be easily obtained,

(3) both bounds converge to the same limit.

A vital property of coin tossing important to the proofs of Theorem 2.1 and 2.2 is that, at any time t, the occurrence of a pattern $\wedge^*$ in the future is independent of the past.

Let $x \geq 1$. Select x positions randomly from the $n - x(k-1)$ positions and put the $\wedge^* = F \dots F$ patterns into the x positions respectively. Denote by $t_1^*, \dots, t_{x+1}^*$ the numbers of positions between the selected x of $\wedge^*$ patterns respectively (including the number of positions preceding the first selected pattern $\wedge^*$ and the number of positions after the last selected pattern $\wedge^*$). It follows that t_i^*, $i = 1, \dots, x+1$, are integers, $0 \leq t_i^*$ and $\sum_{i=1}^{x+1} t_i^* = n - xk$. Write $t^* = (t_1^*, \dots, t_{x+1}^*)$ and define

$$\Gamma_x^* = \Big\{(t_1^*, \dots, t_{x+1}^*) : \sum_{i=1}^{x+1} t_i^* = n - xk,\ t_i^* \geq 0\Big\}. \tag{2.4}$$

The total number of t^*'s in Γ_x^* is $\binom{n-(k-1)x}{x}$. For every given $t^* = (t_1^*, \dots, t_{x+1}^*) \in \Gamma_x^*$, we define subsets

$$A^*(x,n,k,t^*) = \Big\{w : w = \Big(SFS \underset{t_1^*}{\dots} F\wedge^*, \dots, \wedge^* S \underset{t_{x+1}^*}{\dots} \dots F\Big), \text{ and no } \wedge^* \text{ pattern}$$

$$\text{has occurred in all the subsequences } \Big(SS\underset{t_i^*}{\dots}FS\Big),\ i = 1, \dots, x+1\Big\}. \tag{2.5}$$

Denote the union of these subsets by

$$A^* = A^*(x,n,k) = \bigcup_{t^* \in \Gamma_x^*} A^*(x,n,k,t^*). \tag{2.6}$$

Define an auxiliary pattern $\wedge = SF \dots F$, i.e., an S followed by consecutive k failures. Similarly, select x positions randomly from the $n - xk$ positions and put x of the $\wedge$ patterns into the x positions. Denote by $t_1, \dots, t_{x+1}$ the numbers of positions between the patterns respectively. It follows that t_i, $i = 1, \dots, x+1$ are integers satisfying $0 \leq t_i \leq n - x(k+1)$ and $\sum_{i=1}^{x+1} t_i = n - x(k+1)$. Write $t = (t_1, \dots, t_{x+1})$ and define

$$\Gamma_x = \Big\{t = (t_1, \dots, t_{x+1}) : \sum_{i=1}^{x+1} t_i = n - x(k+1)\Big\}. \tag{2.7}$$

The total number of t in Γ_x is $\binom{n-xk}{x}$. For every $t \in \Gamma_x$, we define subsets

$$A_*(x,n,k,t) = \Big\{w : w = \Big(FS \underset{t_1}{\dots} F \wedge \dots \wedge FS \underset{t_{x+1}}{\dots} SF\Big), \text{ and no } \wedge \text{ patterns}$$

$$\text{have occurred in all the subsequences } \Big(S \underset{t_i}{\dots} FS\Big),\ i = 1, \dots, x+1\Big\} \tag{2.8}$$

and their union

$$A_* = A_*(x,n,k) = \bigcup_{t \in \Gamma_x} A_*(x,n,k,t). \tag{2.9}$$

Note that for all t, $t' \in \Gamma_x$ and $t' \neq t$,

$$A_*(x,n,k,t) \cap A_*(x,n,k,t') = \phi. \tag{2.10}$$

It follows from the definitions of $A^*(x,n,k)$ and $A_*(x,n,k)$ that

$$A_*(x,n,k) \subseteq A(x,n,k) \subseteq A^*(x,n,k). \tag{2.11}$$

Hence,

$$P(A_*(x,n,k)) \leq P(A(x,n,k)) \leq P(A^*(x,n,k)). \tag{2.12}$$

To prove the main result, we need the following lemmas.

Lemma 2.1. *For given k, n and $0 \leq m \leq n$,*

$$\left(1-q_n^k\right)^m \leq P\Big(A(0,m,k)\Big) \leq \left(1-q_n^k+q_n^{k+1}\right)^{m-k+1}. \tag{2.13}$$

Proof. See Fu (1985, 1986), Papastavridis & Koutras (1992) and Fu & Koutras (1992).

Lemma 2.2. *For any two fixed non-negative integers c_1 and c_2*

$$\binom{n-c_1}{x}\left(1-\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\right)^{n-c_2}\left(\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\right)^x \longrightarrow \frac{\lambda^x}{x!}e^{-\lambda} \tag{2.14}$$

as $n \longrightarrow \infty$.

Proof. It is trivial.

Proof of Theorem 2.1. Note that, at any stage t, the occurrence of the pattern $\wedge^*$ in the future is independent of the past. Hence it follows from the definitions of $A^*(x,n,k)$ and $A^*(x,n,k,t^*)$ and from Lemma 2.1 that, for $x = 0, 1, \ldots$, there exists c_2 such that

$$\begin{aligned} P(A^*(x,n,k)) &= P\Big(\bigcup_{t^*\in\Gamma_x^*} A^*(x,n,k,t^*)\Big) \\ &\leq \sum_{t^*\in\Gamma_x^*} P(A^*(x,n,k,t^*)) \\ &\leq \sum_{t^*\in\Gamma_x^*} \left(1-q_n^k+q_n^{k+1}\right)^{\sum_{i=1}^{x+1}(t_i^*-k+1)}(q_n^k)^x \\ &= \binom{n-(k-1)x}{x}\left(1-q_n^k+q_n^{k+1}\right)^{n-c_2}(q_n^k)^x. \end{aligned} \tag{2.15}$$

Similarly, for given x it follows from (2.10) and the definition of $A_*(x,n,k,t)$ that there exists c_3 such that

$$\begin{aligned} P\Big(A_*(x,n,k)\Big) &= P\Big(\bigcup_{t\in\Gamma_x} A_*(x,n,k,t)\Big) \\ &\geq \sum_{t\in\Gamma_x} \Big(1-q_n^k\Big)^{\sum_{i=1}^{x+1} t_i} (p_n q_n^k)^x \\ &= \binom{n-xk}{x}(1-q_n^k)^{n-c_3}(q_n^k)^x p_n^x. \end{aligned} \tag{2.16}$$

Condition (2.1) and inequalities (2.15) and (2.16) yield

$$\begin{aligned} &\binom{n-xk}{x}\Big(1-\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\Big)^{n-c_3}\Big(\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\Big)^x \\ \leq\ & P(N_{n,k}=x) \\ \leq\ & \binom{n-(k-1)x}{x}\Big(1-\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\Big)^{n-c_2}\Big(\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big)\Big)^x. \end{aligned} \tag{2.17}$$

Taking $n \to \infty$, the result follows immediately from the Lemma 2.2.

The proof of the main result, Theorem 2.1, is more direct and elementary than using the Bonferroni inequalities and the Stein-Chen method. The proof needs no computation of moments of any order. It is almost as elementary as the proof that a binomial random variable tends to a Poisson random variable under the condition $nq_n \to \lambda$, as $n \to \infty$.

It can also be seen from the proof that the pattern does not have to be restricted to the form $\wedge^* = F\ldots F$ of consecutive k failures. For example, if the pattern $\wedge^*$ is SFFSFFS and the condition $np_n^3 q_n^4 \to \lambda$ as $n \to \infty$ is satisfied, then the results remain true. Theorem 2.1 can be easily extended to any pattern $\wedge^* = SFF\ldots SFS$ with fixed length $(l+k)$ if the condition $np_n^l q_n^k \to \lambda$, as $n \to \infty$ is satisfied, where l is the number of S's and k $(k \geq 1)$ is the number of F's in the pattern $\wedge^*$ respectively. In view of Lemma 2.1, the following result is a by-product of the proof; for every $x = 0, 1, \ldots,$

$$\lim P(N_{n,k}=x) = \begin{cases} 1, & \text{if } np_n^l q_n^k \to 0, \quad \text{as } n \to \infty, \\ \dfrac{\lambda^x}{x!}e^{-\lambda} & \text{if } np_n^l q_n^k \to \lambda, \quad \text{as } n \to \infty, \\ 0, & \text{if } np_n^l q_n^k \to \infty, \quad \text{as } n \to \infty. \end{cases} \tag{2.18}$$

The condition $np_n^l q_n^k \to \lambda$ implies that either $np_n^l \to \lambda$ or $nq_n^k \to \lambda$ (since $p_n + q_n = 1$). If $nq_n^k \to \lambda$ then the above result does not depend on l (the number of S in the pattern $\wedge^*$) as long as l is finite and fixed. By the same token, if $np_n^l \to \lambda$ then the above result also holds and it does not depend on k. This yields an interesting phenomenon that if the failure probability q_n is very small, then one can always find a predetermined pattern $\wedge^*$ (in which l and k are fixed) in a long sequence of independent Bernoulli trials.

In the case of independent but non-identical Bernoulli trials with unequal failure probabilities, the Poisson convergence of the sum $S_n = \sum_{i=1}^n X_{in} = N_{n,1}$ have been studied by Hoeffding (1956), Nedelman and Wallenius (1986), Arratia, Goldstein and Gordon (1989), Barbour and Holst (1989), Wang (1989, 1991), and others. In the case of equal failure probability, a necessary and sufficient condition under which the random variable S_n converges to a Poisson random variable is

$$\sum_{i=1}^n q_{in} = nq_n \to \lambda > 0 \text{ as } n \to \infty. \tag{2.19}$$

For the case of unequal component failure probabilities, the condition

$$\sum_{i=1}^n q_{in} \to \lambda > 0 \text{ as } n \to \infty \tag{2.20}$$

is not sufficient for S_n to converge to a Poisson random variable, as illustrated by the following.

Example. Consider a series system of n components having failure probabilities $q_{in} = 1/2^i$, $i = 1, 2, \ldots, n$. Clearly, the condition (2.20), which reduces to $\sum_{i=1}^n 1/2^i \to 1$ as $n \to \infty$, is satisfied. Note that the random variable $N_{n,1} = S_n$ does not converge to a Poisson random variable with $\lambda = 1$, but to a non-Poisson random variable which has the following probability at 0:

$$\lim_{n\to\infty} P(N_{n,1} = 0) = e^{-\lambda^*} = 0.2894$$

where

$$\lambda^* = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{i=1}^n \left(\frac{1}{2^i}\right)^j = 1.24.$$

In order to generalize Theorem 2.1 to the case of unequal failure probabilities, we shall focus on the case $k = 1$ in Theorem 2.2 to show how our theorem works, and then consider the case of general k in Theorem 2.3.

Theorem 2.2. *The random variable $N_{n,1}$ converges to a Poisson random variable if and only if q_{in}, $i = 1, \ldots, n$, satisfy conditions* (2.20) *and*

$$\sum_{i=1}^n q_{in}^2 \to 0, \quad \text{as } n \to \infty. \tag{2.21}$$

The technique used to prove this result is based on the distribution of the number of non-overlapping patterns developed in Theorem 2.1 and illustrated in the example. Define

$$\lim_{n\to\infty}\sum_{i=1}^{n} q_{in}^{j} = \lambda_j, \quad j = 1, 2, \ldots \tag{2.22}$$

To prove the result we need the following lemmas.

Lemma 2.3. (i) *If* $0 \le q_{in} \le 1$ *for all* n *and* $1 \le i \le n$, *then the product* $\prod_{i=1}^{n}(1 - q_{in})$ *and the sum* $\sum_{i=1}^{n} q_{in}$ *converge and diverge together.*

(ii) *If*

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=1}^{n} q_{in}^{j} = \lambda^* > 0 \tag{2.23}$$

then

$$\lim_{n\to\infty}\prod_{i=1}^{n}(1 - q_{in}) = e^{-\lambda^*} \tag{2.24}$$

Proof. Part (i) is proved in Apostol (1958, p.382). Part (ii) follows from (2.23).

Lemma 2.4. *If condition* (2.21) *is satisfied, then*

(i) $\lim_{n\to\infty}\sup_{1\le i\le n} q_{in} = 0$

(ii) $\lim_{n\to\infty}\sum_{i=1}^{n} q_{in}^{j} = \lambda_j = 0, \quad \forall\, j \ge 2$

(iii) *for every* $1 \le x \le n$,

$$\sum_{(a_1,\ldots,a_x)\in\zeta(n,x)}\prod_{i=1}^{x} q_{a_i,n} \to \frac{\lambda_1^x}{x!}$$

as $n \to \infty$, *where* $\zeta(n,x)$ *is the collection of all subsets of size* x; *i.e.*

$$\zeta(n,x) = \Big\{(a_1,\ldots,a_x); a_i \in (1,\ldots,n), \text{ and } a_i \neq a_j \text{ if } i \neq j\Big\}.$$

Proof. Since $0 < q_{in} < 1$ for all $i = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$, the results (i) and (ii) follow directly from the condition (2.21). For $x = 1$, it reduces to (2.20).

Define for $x = 2, \ldots$

$$H(x,n) = x!\sum_{(a_1,\ldots,a_x)\in\zeta(n,x)}\prod_{i=1}^{x} q_{a_in} \tag{2.25}$$

and

$$R(x,n) = \left(\sum_{i=1}^{n} q_{in}\right)^{x} - H(x,n). \tag{2.26}$$

To prove the result (iii), it is sufficient to prove $R(x,n) \to 0$, as $n \to \infty$. Mathematical induction is used to prove this result. For $x = 1$ and 2, the results are obvious. Suppose the result is true up to $x-1$. The result again follows from the condition (2.21) and the following general inequalities

$$H(x,n) \le \Big(\sum_{i=1}^{n} q_{in}\Big)^x \le R(x-1,n)\sum_{i=1}^{n} q_{in} + (x-1)H(x-2,n)\sum_{i=1}^{n} q_{in}^2 + H(x,n) \tag{2.27}$$

by taking $n \to \infty$. This completes the proof of part (iii).

Take $k = 1$. The same argument in the proof of Theorem 2.1 yields

$$P(N_{n,1} = x) = \prod_{i=1}^{n}(1-q_{in}) \sum_{(a_1,\dots,a_x)\in\zeta(n,x)} \prod_{i=1}^{x}(q_{a_i,n}/1-q_{a_in}). \tag{2.28}$$

If all the components have the same failure probabilities, ($q_{in} = q_n$ for all $1 \le i \le n$), then it reduces to classical i.i.d. Bernoulli case.

Proof of Theorem 2.2. If the conditions (2.20) and (2.21) are satisfied, it follows from the second part of Lemma 2.3 that

$$\lambda^* = \lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=1}^{n} q_{in}^j = \lim_{n\to\infty}\sum_{i=1}^{n} q_{in} = \lambda_1. \tag{2.29}$$

For any $x = 1, 2, \dots$, the following inequalities hold;

$$\begin{aligned}\sum_{(a_1,\dots,a_x)\in\zeta(n,x)}\prod_{i=1}^{x} q_{a_in} &\le \sum_{(a_1,\dots,a_x)\in\zeta(n,x)}\prod_{i=1}^{x}\frac{q_{a_in}}{1-q_{a_in}}\\ &\le \left(\frac{1}{1-\sup_{1\le i\le n} q_{in}}\right)^x \sum_{(a_1,\dots,a_x)\in\zeta(n,x)}\prod_{i=1}^{x}\frac{q_{a_in}}{1-q_{a_in}}.\end{aligned} \tag{2.30}$$

Hence, the result follows immediately from (2.28), (2.29), (2.30) and Lemma 2.4. This completes the sufficiency part.

To prove the necessity, assume that the random variable $N_{n,1}$ converges to a Poisson random variable with mean $\lambda^* > 0$. For $x = 0$, we have $P(N_{n,1} = 0) \to \exp\{-\lambda^*\}$, where λ^* is defined by (2.23). Note that for $x = 1$,

$$\sum_{a_1\in\zeta(n,1)}\frac{q_{a_1n}}{1-q_{a_1n}} \to \sum_{j=1}^{\infty}\lambda_j, \quad \text{as } n \to \infty. \tag{2.31}$$

Hence, the following equation has to hold

$$\sum_{j=1}^{\infty}\left(1-\frac{1}{j}\right)\lambda_j = 0.$$

The above equation holds only if $\lambda_j = 0$ for all $j = 2, \ldots$ This completes the proof of the necessity.

For the more general case that the components may have different failure probabilities, let $\Lambda^* = F \ldots F$ be a pattern of k consecutive failures. Define

$$u_{in} = \prod_{j=0}^{k-1} q_{i+j,n}, \quad \text{for } i = 1, \ldots, n-k+1. \tag{2.32}$$

It is clear that the property $N_{n,1}$ is a sum of outcomes of n independent Bernoulli trials has never been used in the proof of the above theorem. Combining the results of Theorem 2.1 and Theorem 2.2, the following theorem gives necessary and sufficient conditions for the random variable $N_{n,k}$ to converge to a Poisson random variable in the case of independent Bernoulli trials with unequal failure probabilities.

Theorem 2.3. *If* $\limsup_{1 \le i \le n} q_{in} = 0$ *then the random variable* $N_{n,k}$ *converges to a Poisson random variable if and only if*

$$\sum_{i=1}^{n-k+1} u_{in} \to \lambda > 0 \tag{2.33}$$

and

$$\sum_{i=1}^{n-k+1} u_{in}^2 \to 0 \tag{2.34}$$

as $n \to \infty$, *where* u_{in} *are defined by* (2.32).

Proof. Replace q_{in} by u_{in}, this result can be proved along the lines of Theorems 2.1 and 2.2 with some simple modifications. We leave the details to the readers.

3. Reliability and Bounds

For convenience, let $\wedge^*$ be the pattern of k consecutive failures throughout this section. If the components operate independently and the failure probabilities of components are small, then the reliability of a large m-$\wedge^*$ ($m = 1, 2, \ldots$) linearly connected system can be approximated by the tail probability of Poisson random variable $N_{n,k}$.

Theorem 3.1. (i) *If all the components operate independently and have same failure probability* q_n *which satisfies the condition that* $nq_n^k \to \lambda$, *as* $n \to \infty$, *then*

$$\lim_{n \to \infty} R_n(m\text{-}\wedge^*) = \sum_{x=0}^{m-1} \frac{\lambda^x}{x!} e^{-\lambda}. \tag{3.1}$$

(ii) *If all the components operate independently and have different failure probabilities* $q_{in}, i = 1, \dots, n$, *which satisfy the conditions*
(a) $\limsup_{1 \le i \le n} q_{in} = 0$
(b) $\lim_{n\to\infty} \sum_{i=1}^{n} u_{in} = \theta$ *and*
(c) $\lim_{n\to\infty} \sum_{i=1}^{n} u_{in}^2 = 0$, *where* $u_{in} = \prod_{j=0}^{k-1} q_{i+j,n}$,
then,

$$\lim_{n\to\infty} R_n(m\text{-}\wedge^*) = \sum_{x=0}^{m-1} \frac{\theta^x}{x!} e^{-\theta}. \tag{3.2}$$

Proof. The reliability of an m-$\wedge^*$ linearly connected system has the following relationship with the random variable $N_{n,k}$: for every $m = 1, 2, \dots$,

$$\begin{aligned} & R_n(m\text{-}\wedge^*) \\ = \; & P(\text{less than } m \text{ non-overlapping } \wedge^* \text{ patterns have occurred in } n \text{ trials}) \\ = \; & P(N_{n,k} \le m-1). \end{aligned} \tag{3.3}$$

Both the above results (i) and (ii) are immediate consequences of (3.3), Theorem 2.1 and Theorem 2.3, respectively.

The condition $q_n \sim (\lambda/n)^{1/k}$ is vital to the above results. In other words, to have nontrivial reliability for a large linearly connected system, the failure probabilities of components should be inversely proportional to the kth root of the size of the system. Theorem 3.1 yields many previous results about the reliabilities of large linearly connected systems, for example, the results of Chao & Fu (1989), Papastavridis (1987, 1990), and Fu and Lou (1991). Theorem 2.1 and Theorem 2.3 also yield some results of Poisson convergence in Godbole (1991), Barbour and Holst (1989), Wang (1989, 1991), and Arratia, Goldstein and Gordon (1989).

Lemma 2.1 is indispensable to our proofs of results. It also yields good upper and lower bounds for the reliability $R(k, n)$ of a consecutive-k-out-of-n: F systems:

$$\left(1 - q_n^k\right)^n \le R(k,n) \le \left(1 - q_n^k + q_n^{k+1}\right)^{n-k+1} \tag{3.4}$$

This fundamental inequality can be interpreted as follows: the reliability of a 1-$\wedge^*$ linearly connected system is bounded above by the reliability of a series system with failure probabilities $u_{in}^*(u_{in}^* = p_n q_n^k$ in the i.i.d. case), and is bounded below by the reliability of a series system with failure probabilities $u_{in}(u_{in} = q_n^k$ in the i.i.d. case). There are several other upper and lower bounds which have been proposed, for example, Derman et al. (1982) and Papastavridis (1987). Recently, the Stein-Chen method has become a very popular tool for studying Poisson convergence for sequences of dependent random variables. Using the Stein-Chen method, Chrysaphinou and Papastavridis (1990) proved that

$$\left|R(k,n) - e^{-\lambda_n}\right| \leq (2k-1)q_n^k + 2(k-1)q_n, \quad \text{where } \lambda_n = (n-k+1)q_n^k, \tag{3.5}$$

which also yields upper and lower bounds for $R(k,n)$.

Barbour, Holst and Janson (1992), also using the Stein-Chen method (along with a certain coupling), gave the following improved inequality

$$|R(k,n) - e^{-p_n\lambda_n}| \leq (2kp_n + 1)q_n^k. \tag{3.6}$$

In the next table, L and U are the lower and upper bounds of (3.4), and L_{CP}, U_{CP}, L_B, and U_B are the lower and upper bounds obtained by (3.5) and (3.6) respectively.

n	k	q_n	L_{CP}	L_B	L	U	U_B	U_{CP}
10	2	0.50	0.8703	0.9669	0.9777	0.9788	0.9909	1.0853
10	2	0.20	0.1777	0.5818	0.6925	0.7462	0.9180	1.2177
10	4	0.10	0.3986	0.9986	0.9993	0.9994	1.0002	1.6000
10	4	0.20	−0.2223	0.9792	0.9889	0.9911	1.0029	2.2001
50	2	0.05	0.7772	0.8781	0.8846	0.8900	0.9021	0.9922
50	2	0.10	0.3826	0.5674	0.6111	0.6421	0.6894	0.8426
50	4	0.05	0.6997	0.9997	0.9997	0.9997	0.9998	1.2998
50	4	0.10	0.3946	0.9950	0.9953	0.9958	0.9966	1.5960
100	2	0.05	0.6733	0.7785	0.7805	0.7903	0.8025	0.8883
100	2	0.10	0.1416	0.3642	0.3697	0.4086	0.4562	0.6016

It is surprising that the bounds developed by Stein-Chen method perform poorly, especially for n being small. The precise reason why the bounds given by Stein-Chen method perform poorly remains unknown. The Stein-Chen method is a general and useful tool for Poisson convergence, using only the first two moments but not the structure of the process. The bounds given by (3.4) depend on the Markov structure of the reliability system. Hence, the bounds derived from (3.4) are expected to perform well in this case. In view of the above numerical results for n reasonably large and q_n small, the bounds given by (3.4) provide an excellent approximation for the exact reliability of the linearly connected system.

Acknowledgement

This work is supported in part by the National Science and Engineering Research Council of Canada under grant NSERC A-9216.

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(Received August 1991; accepted January 1993).