Statistica Sinica 3(1993), 1-7

A MOMENT INEQUALITY FOR HSU AND ROBBINS SERIES

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Abstract: The series involved in the Hsu and Robbins complete convergence theorem (1947) and its extensions by Katz (1963) and others are called Hsu and Robbins series. If $\mathcal{P}(|S_n| > 2n)$ is replaced by $nE(|S_n| - n)^+$ in the series, the new one is called the moment version of Hsu and Robbins series. We establish an inequality for such a series. Some results of Chow and Lai (1978) and Chow (1988) have been improved.

Key words and phrases: Moment inequality, complete convergence, Hsu and Robbins series.

1. Introduction

Let $X, X_1, X_2, \ldots$ be iid, $EX = 0$ and $S_n = \sum_1^n X_j$. Hsu and Robbins (1947) have proved that if $EX^2 < \infty$, then

$$\sum_1^\infty \mathcal{P}(|S_n| > n) < \infty.$$

Katz (1963) generalized this result as follows: If $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $E|X|^p < \infty$, then

$$\sum_1^\infty n^{\alpha p-2}\mathcal{P}(|S_n| > n^\alpha) < \infty. \tag{1}$$

In 1978, Chow and Lai showed that if $1 \le r \le 2$, $\alpha > \beta \ge 0$, $\alpha p > 1 > \beta p$ and $\alpha r > 1 > \beta r$, there exists an absolute constant $C = C(p, r, \alpha, \beta)$ such that

$$\sum_1^\infty n^{\alpha p-2}\mathcal{P}\Big(\max_{j\le n} j^{-\beta}S_j \ge n^{\alpha-\beta}\Big) \le C\Big\{EX_+^p + \mu^{\frac{\alpha p-1}{(\alpha r-1)}}\Big\}, \tag{2}$$

where $a_+ = \max(a, 0)$ and $\mu = E|X|^r$. The series in (1) and (2) are called Hsu and Robbins series.

In Chow (1988, Theorem 2.5) a moment version of (1) has been given:

If $p \geq 1$, $\alpha > \frac{1}{2}$, $\alpha p \geq 1$ and $E\{|X|^p + |X|\log(1+|X|)\} < \infty$, then

$$\sum n^{\alpha p-2-\alpha} E\Big(\max_{j\leq n}|S_j| - n^\alpha\Big)_+ < \infty. \tag{3}$$

In this paper, we prove the following theorem, which is a moment version of Inequality (2). A two-sided version of the theorem for $q = 1$, $\beta = 0$ and $r = p > 1$ improves (3) when $\alpha p > 1$ and $p > 1$.

In the following, for $t > 0$, let

$$S_t = S_{[t]},\ X_t = x_{[t]},\ X_0 = S_0 = 0, \tag{4}$$

$$\overline{S_{[t]}[t]^{-\beta}} = \overline{S_t t^{-\beta}} = \max\Big(0, S_1, 2^{-\beta}S_2, \ldots, [t]^{-\beta}S_{[t]}\Big),$$

and similarly for $\overline{X_t t^{-\beta}}$.

Theorem 1. *Let $\beta > 0$, $p \geq q > 0$, $1 \leq r \leq 2$, $\alpha r > 1 > \beta r$, $\alpha p > 1$,*

$$\theta_1 = \frac{\alpha p - 1 - (\beta p - 1)_+}{\alpha r - 1},\ \theta_2 = \frac{q}{r},\ \theta = \max(\theta_1, \theta_2), \tag{5}$$

and

$$\mathcal{K} = \int_1^\infty t^{\alpha p - 2 - (\alpha-\beta)q - (\beta p - 1)_+} E\Big(\overline{S_t t^{-\beta}} - t^{-\alpha-\beta}\Big)_+^q\, dt. \tag{6}$$

Assume that $\theta_1 \neq \theta_2$. Then there exist constants $C = C(p,q,r,\alpha,\beta)$ such that

(i) *if $\beta p \neq 1$ and $p > q$, then $\mathcal{K} \leq C(EX_+^p + E^\theta|X|^r)$;*

(ii) *if $\beta p = 1$ and $p > q$, or if $\beta p \neq 1$ and $p = q$, then $\mathcal{K} \leq C(EX_+^p \log(e+X) + E^\theta|X|^r)$;*

(iii) *if $\beta p = 1$ and $p = q$, then $\mathcal{K} \leq C(EX_+^p \log^2(e+X) + E^\theta|X|^r)$.*

The proof of Theorem 1 will be given in Section 3. For $\beta p > 1$, part (i) yields the following new result, since in the previous works one always assumes that $\beta p \leq 1$.

Corollary 1. *Let $1 \leq r \leq 2$, $\alpha > \beta \geq 0$, $\beta p \geq 1$ and $\alpha r > 1 > \beta r$. Then there exist constants $C = C(p,q,r,\alpha,\beta)$ such that*

(i) *if $\beta p > 1$, then*

$$\begin{aligned} \mathcal{T} &\equiv \sum_1^\infty n^{\alpha p - \beta p - 1} P\Big(\overline{S_n n^{-\beta}} \geq 2n^{\alpha-\beta}\Big) \\ &\leq C\Big\{EX_+^p + (E|X|^r)^{(\alpha-\beta)p/(\alpha r-1)}\Big\}. \end{aligned} \tag{7}$$

(ii) *if $\beta p = 1$, then*

$$\mathcal{T} \leq C\Big\{EX_+^p \log(e+X) + (E|X|^r)^{(\alpha-\beta)p/(\alpha r-1)}\Big\}. \tag{8}$$

Proof. (i) Choose $q > 0$ so small that $q < p$ and

$$\theta_2 = q/r < (\alpha - \beta)p/(\alpha r - 1) = \theta_1. \tag{9}$$

By Theorem 1(i),

$$\begin{aligned} \mathcal{T} &\le \sum_1^\infty n^{\alpha p - 1 - (\alpha-\beta)q - \beta p} E\left(\overline{S_n n^{-\beta}} - n^{\alpha-\beta}\right)_+^q \\ &\le C\left\{EX_+^p + (E|X|^r)^{(\alpha-\beta)p/(\alpha r - 1)}\right\}. \end{aligned}$$

Similarly for part (ii).

2. Some Lemmas

Let $X, X_1, X_2, \ldots$ be iid, $S_n = \sum_1^n X_j$ and $a_+ = \max(a, 0)$.

Lemma 1. *Let $\alpha \ge \beta \ge 0$, $p \ge q > 0$, $\delta = \alpha p - 2 - (\alpha - \beta)q - (\beta p - 1)_+$, and*

$$\mathcal{I} = \int_1^\infty t^\delta E\left(\overline{X_t t^{-\beta}} - t^{\alpha-\beta}\right)_+^q dt. \tag{10}$$

Assume that $\alpha > \beta$ when $\beta p \ge 1$ or $\beta = 0$. Then for some $O(1)$, depending only on p, q, α, β,
(i) *if $\beta p \ne 1$ and $p > q$, then $\mathcal{I} = O(1) \cdot EX_+^p$;*
(ii) *if $\beta p \ne 1$ and $p = q$, then $\mathcal{I} = O(1) \cdot EX_+^p \log(1 + X)$;*
(iii) *if $\beta p = 1$ and $p > q$, then $\mathcal{I} = O(1) \cdot EX_+^p \log(1 + X)$;*
(iv) *if $\beta p = 1$ and $p = q$, then $\mathcal{I} = O(1) \cdot EX_+^p \log^2(1 + X)$.*

Proof. Put $A_t = E(\overline{X_t t^{-\beta}} - t^{\alpha-\beta})_+^q$.
(a) Let $\beta = 0$. Then $A_t \le tE(X - t^\alpha)_+^q$ and

$$\mathcal{I} \le \int_1^\infty t^{\alpha p - \alpha q - 1} E(X - t^\alpha)_+^q dt \le E\left\{(X-1)_+^q \int_1^{X^{1/\alpha}} t^{\alpha p - \alpha q - 1} dt\right\}. \tag{11}$$

If $p > q$, then

$$\mathcal{I} \le \frac{E\{(X-1)_+^q X^{p-q}\}}{\alpha(p-q)} = O(1) EX_+^p,$$

and if $p = q$, then

$$\mathcal{I} = O(1) \cdot EX_+^p \log(1 + X).$$

Hence Lemma 1 holds for $\beta = 0$.

(b) Let $\beta > 0$. Then for $t \geq 1$ and $k = [t]$,

$$\begin{aligned} A_t &\leq E(X - t^{\alpha-\beta})_+^q + E(2^{-\beta}X - t^{\alpha-\beta})_+^q + \cdots + E(k^{-\beta}X - t^{\alpha-\beta})_+^q \\ &\leq \sum_1^k \int_{2j-1}^{2j} E(2^\beta y^{-\beta} X - t^{\alpha-\beta})_+^q \, dy. \end{aligned}$$

Hence

$$\begin{aligned} A_t &\leq \int_1^{2t} E(2^\beta y^{-\beta} X - t^{\alpha-\beta})_+^q \, dy \\ &= O(1) \cdot t^{(\alpha-\beta)(q-\beta^{-1})} \int_{t^{\alpha-\beta}}^{2^\beta t^\alpha} E(2^\beta X - u)_+^q u^{\beta^{-1}-1-q} \, du. \end{aligned}$$

By (10),

$$\mathcal{I} = O(1) \int_1^\infty t^{(\alpha p-2)-(\alpha-\beta)\beta^{-1}-(\beta p-1)_+} dt \cdot \int_{t^{\alpha-\beta}}^{2^\beta t^\alpha} E(2^\beta X - u)_+^q u^{\beta^{-1}-1-q} du.$$

Hence

$$\mathcal{I} = O(1) \int_1^\infty E(2^\beta X - u)_+^q u^{\beta^{-1}-1-q} du \int_{(2^{-\beta}u)^{1/\alpha}}^{u^{1/(\alpha-\beta)}} t^{\alpha p-1-\alpha/\beta-(\beta p-1)_+} dt. \tag{12}$$

Since $\alpha > \beta$ when $\beta p \geq 1$, for $u > 1$, it follows that

$$\int_{(2^{-\beta}u)^{1/\alpha}}^{u^{1/(\alpha-\beta)}} t^{\alpha(p-\beta^{-1})-1-(\beta p-1)_+} dt = \begin{cases} O(1) u^{p-\beta^{-1}} & \text{if } \beta p < 1, \\ O(1) \log u & \text{if } \beta p = 1, \\ O(1) u^{p-\beta^{-1}} & \text{if } \beta p > 1. \end{cases} \tag{13}$$

By (12) and (13), if $\beta p \neq 1$, then

$$\begin{aligned} \mathcal{I} &= O(1) \int_1^\infty E(2^\beta X - u)_+^q u^{p-q-1} du \\ &= O(1) E\Big\{ \int_1^{2^\beta X} (2^\beta X - 1)_+^q u^{p-q-1} du \Big\} \\ &= \begin{cases} O(1) E X_+^p, & \text{if } p > q, \\ O(1) E X_+^p \log(1+X), & \text{if } p = q, \end{cases} \end{aligned}$$

and if $\beta p = 1$, then

$$\begin{aligned} \mathcal{I} &= O(1) E\Big\{ \int_1^{2^\beta X} (2^\beta X - 1)_+^q u^{p-q-1} \log u du \Big\} \\ &= \begin{cases} O(1) E X_+^p \log(1+X), & \text{if } p > q, \\ O(1) E X_+^p \log^2(1+X), & \text{if } p = q. \end{cases} \end{aligned}$$

The following lemma is due to Chow and Lai (1978, Lemma 5) in a slightly different form.

Lemma 2. *Let* $1 \le r \le 2$ *and* $1 > \beta r \ge 0$. *There exists a universal constant* $C = C(r,\beta)$ *such that if* $X, X_1, X_2, \ldots$ *are* iid *with* $EX = 0$, *then for* $t, y > 0$,

$$P\left(\overline{S_t\, t^{-\beta}} > 2ky\right) \;\le\; P\left(\overline{X_t\, t^{-\beta}} > y\right) + P^k\left(\overline{S_t\, t^{-\beta}} > y\right), \tag{14}$$

$$P\left(\overline{S_t\, t^{-\beta}} > y\right) \;\le\; Ct^{1-r\beta} y^{-r} E|X|^r. \tag{15}$$

Lemma 3. *Let* $\beta > 0$, $p \ge q > 0$, $EX = 0$, $\delta = \alpha p - 2 + (\beta p - 1)_+ - (\alpha - \beta)q$, $1 \le r \le 2$, $\alpha r > 1 > \beta r$, $\alpha p > 1$, $\theta_1 = \{\alpha p - 1 - (\beta p - 1)_+\}/(\alpha r - 1)$, $\theta_2 = q/r$. *Put* $\theta = \max(\theta_1, \theta_2)$ *and*

$$J = \int_1^\infty t^\delta dt \int_0^\infty P^k\left(\overline{S_t\, t^{-\beta}} > t^{\alpha-\beta} + y^{1/q}\right) dy \tag{16}$$

where $k > \theta$ *is an integer. If* $\theta_1 \ne \theta_2$, *then for some* $O(1)$ *depending only on* p, q, r, α, β

$$J = O(1) E^\theta |X|^r. \tag{17}$$

Proof. Put $Q(y,t) = P(\overline{S_t\, t^{-\beta}} > t^{\alpha-\beta} + y^{1/q})$ and $\mu = E|X|^r$. By Lemma 2,

$$Q(y,t) = O(1) t^{1-\beta r} (t^{\alpha-\beta} + y^{1/q})^{-r} \mu, \tag{18}$$

where $O(1)$ depends only on r, k and β.
Now by (18),

$$\int_0^\infty Q^k(y,t) dy = \left(\int_0^{t^{(\alpha-\beta)q}} + \int_{t^{(\alpha-\beta)q}}^\infty\right) Q^k(y,t) dy \tag{19}$$

$$= O(\mu^k)\left\{\int_0^{t^{(\alpha-\beta)q}} t^{(1-\alpha r)k}\, dy + \int_{t^{(\alpha-\beta)q}}^\infty t^{(1-\beta r)k} y^{-kr/q}\, dy\right\} \tag{20}$$

$$= O(\mu^k) t^{(1-\alpha r)k + (\alpha-\beta)q}. \tag{21}$$

On the other hand, for $W = (t^{1-\beta r}\mu)^{q/r}$ by (18),

$$\int_0^\infty Q^k(y,t) dy = O(1)\left\{\int_0^W dy + \int_W^\infty t^{(1-\beta r)k} \mu^k y^{-kr/q}\, dy\right\} \tag{22}$$

$$= O(W) = O(\mu^{q/r}) t^{(1-\beta r)q/r}. \tag{23}$$

By (21), for $V = \mu^{1/(\alpha r - 1)}$,

$$J_1 \equiv \int_V^\infty t^\delta\, dt \int_0^\infty Q^k(y,t)\, dy = O(\mu^k) \int_V^\infty t^{\delta + (1-\alpha r)k + (\alpha-\beta)q}\, dt$$

$$= O(\mu^k) \int_V^\infty t^{\alpha p - 2 - (\beta p - 1)_+ - (\alpha r - 1)k}\, dt.$$

Hence

$$J_1 = O(\mu^k) V^{\alpha p - 1 - (\beta p - 1)_+ - (\alpha r - 1)k} = O(\mu^{\theta_1}). \tag{24}$$

If $V \le 1$, this completes the proof. Otherwise, assume that $V > 1$ and $\theta_1 > \theta_2$. By (23)

$$\begin{aligned} J_2 &\equiv \int_1^V t^\delta \, dt \int_0^\infty Q^k(y,t)\, dy = \theta(\mu^{q/r}) \int_1^V t^{\delta + (1-\beta r) q/r}\, dt \\ &= \theta(\mu^{q/r}) \int_1^V t^{\alpha p - 2 - (\beta p - 1)_+ - (\alpha r - 1)q/r}\, dt. \end{aligned}$$

Since $\theta_1 > \theta_2$,

$$J_2 = O(\mu^{q/r}) V^{\alpha p - 1 - (\beta p - 1)_+ - (\alpha r - 1)q/r} = O(\mu^{\theta_1}). \tag{25}$$

From (24) and (25), if $\theta_1 > \theta_2$ then

$$J = J_1 + J_2 = O(\mu^{\theta_1}), \tag{26}$$

and if $\theta_1 < \theta_2$, then by (23)

$$\begin{aligned} J &= O(\mu^{q/r}) \int_1^\infty t^{\alpha p - 2 - (\alpha - \beta) q - (\beta p - 1)_+ + q/r - \beta q}\, dt \\ &= O(\mu^{q/r}) \int_1^\infty t^{\alpha p - 2 - (\alpha r - 1) q/r - (\beta p - 1)_+}\, dt \\ &= O(\mu^{q/r}) = O(\mu^{\theta_2}). \end{aligned}$$

3. Proof of Theorem 1

Put $M_t = \overline{S_t \, t^{-\beta}}$ and $m_t = \overline{X_t \, t^{-\beta}}$. Note that for any positive integer k,

$$(2k)^{-q} E\left(M_t - \frac{t^{\alpha-\beta}}{2k}\right)_+^q = \int_0^\infty P\left\{M_t > 2k\left(\frac{t^{\alpha-\beta}}{2k} + y^{1/q}\right)\right\} dy \tag{27}$$

$$\le \int_0^\infty P\left(m_t > \frac{t^{\alpha-\beta}}{2k} + y^{1/q}\right) dy + \int_0^\infty P^k\left(M_t > \frac{t^{\alpha-\beta}}{2k} + y^{1/q}\right) dy \tag{28}$$

by Lemma 2. Put $\delta = \alpha p - 2 - (\alpha - \beta) q - (\beta p - 1)_+$ and $k = [\theta] + 1$. Then

$$\begin{aligned} (2k)^{-q} K &\le \int_1^\infty t^\delta E\left(m_t - \frac{t^{\alpha-\beta}}{2k}\right)_+^q dt + \int_1^\infty t^\delta dt \int_0^\infty P^k\left(M_t > \frac{t^{\alpha-\beta}}{2k} + y^{1/q}\right) dy \\ &= K_1 + K_2. \end{aligned} \tag{29}$$

Assume that $\beta p = 1$ and $p > q$, or that $\beta p \ne 1$ and $p = q$. Then for some $O(1)$ depending only on p, q, r, α, β and by Lemma 1,

$$K_1 = O(1) E X_+^p \log(e + X), \tag{30}$$

and by Lemma 3,

$$K_2 = O(1)E^{\theta}|X|^r. \tag{31}$$

Hence for some constant $C = C(p, q, r, \alpha, \beta)$

$$K \le (2k)^q (K_1 + K_2) \le C\{EX_+^p \log(e + X) + E^{\theta}|X|^r\}, \tag{32}$$

yielding part (ii). Similarly for parts (i) and (iii).

Acknowledgement

This work has been done at National Central University with the support of N.S.C. of R.O.C.

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(Received April 1992; accepted July 1992)