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ON OPTIMALITY AND CONSTRUCTION OF CIRCULAR REPEATED-MEASUREMENTS DESIGNS

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Abstract: The aim of this paper is to characterize and construct universally optimal designs among the class of circular repeated-measurements designs when the parameters do not permit balance for carry-over effects. It is shown that some circular weakly neighbour balanced designs defined by Filipiak and Markiewicz (2012) are universally optimal repeated-measurements designs. These results extend the work of Magda (1980), Kunert (1984b), and Filipiak and Markiewicz (2012).

Key words and phrases: Circular weakly balanced design, repeated-measurements design, uniform design, universal optimality.

1. Introduction

The problem of universal optimality of repeated-measurements designs is widely studied in the literature. Most of the designs considered have the same number of periods as treatments; we also make this assumption.

For experiments without a pre-period, Hedayat and Afsarinejad (1978) and Cheng and Wu (1980) proved the universal optimality, for the estimation of direct as well as carry-over effects, of some balanced uniform repeated-measurements designs over a restricted class of competing designs. If the number n of subjects is at most twice the number t of treatments, Kunert (1984a) showed that, for the estimation of direct effects, balanced uniform designs are universally optimal over the class of all designs. Hedayat and Yang (2003) extended this by showing universal optimality of balanced uniform designs if $n \leq t(t-1)/2$. Kunert (1984a) also proved that if n is sufficiently large then a balanced uniform design is no longer optimal. Moreover, this design is not universally optimal for the estimation of carry-over effects when certain other special designs exist. Stufken (1991) constructed some universally optimal designs using orthogonal arrays of type I. Jones, Kunert and Wynn (1992) proved universal optimality of some balanced uniform designs under the model with random carry-over effects.

Kunert (1983) considered repeated-measurements designs with or without a pre-period. He proved the universal optimality of some special generalized latin squares and generalized Youden designs over particular classes of designs.

A repeated-measurements design is called *circular* if there is a pre-period and, for each subject, the treatment on the pre-period is the same as the treatment on the last period. Magda (1980) proved the universal optimality of circular strongly balanced uniform designs (uniform CSBDs) and circular balanced uniform designs (uniform CBDs) over appropriate subclasses of possible designs. Kunert (1984b) strengthened these results by showing the universal optimality of CBDs over all designs. Recent constructions of CSBDs and CBDs have been given by Iqbal and Tahir (2009) using cyclic shifts and by Mandal, Parsad and Gupta (2016) using integer programming.

Universal optimality of some CBDs is also studied assuming a model of repeated measurements designs in which period effects are negligible. This simpler model, in which carry-over effects play the role of left-neighbour effects, is known in the literature as an interference model. Druilhet (1999) considered optimality of CBDs for the estimation of direct as well as carry-over effects, while Bailey and Druilhet (2004) proved their optimality for the estimation of total effects. Filipiak and Markiewicz (2012) showed universal optimality of circular weakly balanced designs (CWBDs) for the estimation of direct effects only.

In this paper we consider circular repeated-measurements designs under the full model and under two simpler models. We show universal optimality, for the estimation of direct as well as carry-over effects, of CWBDs and we give methods of constructing some of them. For particular parameter sets, there exists a CWBD using fewer subjects than uniform CBDs. The idea of the possible reduction of number of subjects is suggested by the results of Filipiak and Markiewicz (2012).

2. Models and Designs

Let $\mathcal{D}_{t,n,t}$ be the set of circular designs with t treatments, n experimental subjects, and t periods, each subject being given one treatment during each period. By $d(\ell, u)$, for $1 \leq \ell \leq t$ and $1 \leq u \leq n$, we denote the treatment assigned to the uth subject in the ℓth period. Magda (1980) proposed a model associated with the design d in $\mathcal{D}_{t,n,t}$:

$$y_{d\ell u} = \alpha_\ell + \beta_u + \tau_{d(\ell,u)} + \rho_{d(\ell-1,u)} + \varepsilon_{\ell u}, \quad 1 \leq \ell \leq t,\ 1 \leq u \leq n, \tag{2.1}$$

where $y_{d\ell u}$ is the response of the uth subject in the ℓth period, and α_ℓ, β_u, $\tau_{d(\ell,u)}$, and $\rho_{d(\ell-1,u)}$ are, respectively, the ℓth period effect, the uth subject effect, the direct effect of treatment $d(\ell, u)$, and the carry-over effect of treatment $d(\ell-1, u)$, where $d(0,u) = d(t,u)$. The $\varepsilon_{\ell u}$ are uncorrelated random variables with common variance and zero mean.

In vector notation model (2.1) can be rewritten as

$$\mathbf{y} = \mathbf{P}\boldsymbol{\alpha} + \mathbf{U}\boldsymbol{\beta} + \mathbf{T}_d\boldsymbol{\tau} + \mathbf{F}_d\boldsymbol{\rho} + \boldsymbol{\varepsilon}. \tag{2.2}$$

Here $\mathbf{y}$ is the transpose of the vector $\mathbf{y}' = (y_{d11}, y_{d21}, \ldots, y_{dtn})$. Also, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\tau}$, and $\boldsymbol{\rho}$ are the vectors of period, experimental subject, direct, and carry-over effects, respectively. Moreover, $\boldsymbol{\varepsilon}$ is the vector of random errors, with $\boldsymbol{\varepsilon} \sim N(\mathbf{0}_{nt}, \sigma^2\mathbf{I}_{nt})$, where σ^2 is a positive constant, $\mathbf{I}_n$ denotes the identity matrix of order n, and $\mathbf{0}_n$ is the n-dimensional vector of zeros. The matrices $\mathbf{T}_d$ and $\mathbf{F}_d$ are the design matrices for direct and carry-over effects, respectively, while $\mathbf{P} = \mathbf{1}_n \otimes \mathbf{I}_t$ and $\mathbf{U} = \mathbf{I}_n \otimes \mathbf{1}_t$ are the incidence matrices for period and experimental subject effects, respectively, where $\mathbf{1}_n$ is the n-dimensional vector of ones and $\otimes$ denotes the Kronecker product. Let $\mathbf{H}_t = (h_{ij})$ be the circulant matrix of order t with $h_{ij} = 1$ if $j - i = 1$ or $i = 1, j = t$, and $h_{ij} = 0$ otherwise. Then $\mathbf{F}_d = (\mathbf{I}_n \otimes \mathbf{H}_t)\mathbf{T}_d$.

In this paper we also consider simpler models – model (2.2) without period effects,

$$\mathbf{y} = \mathbf{U}\boldsymbol{\beta} + \mathbf{T}_d\boldsymbol{\tau} + \mathbf{F}_d\boldsymbol{\rho} + \boldsymbol{\varepsilon}, \tag{2.3}$$

and model (2.2) without experimental subject effects,

$$\mathbf{y} = \mathbf{P}\boldsymbol{\alpha} + \mathbf{T}_d\boldsymbol{\tau} + \mathbf{F}_d\boldsymbol{\rho} + \boldsymbol{\varepsilon}. \tag{2.4}$$

In the context of experiments in agriculture and forestry, as discussed by Azaïs, Bailey and Monod (1993), periods correspond to rows, subjects correspond to columns, and the carry-over effect corresponds to the neighbour effect of the treatment to the North. The roles of rows and columns are frequently interchanged in such literature, and so model (2.3) is known as the interference model with left-neighbour effects; cf., Druilhet (1999), Filipiak and Markiewicz (2012).

Following Magda (1980), we say that a design d in $\mathcal{D}_{t,n,t}$ is:

(i) *uniform on periods* if all treatments occur equally often in each period;

(ii) *uniform on subjects* if each treatment occurs exactly once on each subject;

(iii) *uniform* if it is uniform on both periods and subjects;

(iv) *circular strongly balanced* (CSBD) if the collection of ordered pairs $(d(\ell-1,u), d(\ell,u))$, for $1 \le \ell \le t$ and $1 \le u \le n$, contains each ordered pair of treatments (distinct or not) λ_0 times, where $\lambda_0 = n/t$;

(v) *circular balanced* (CBD) if the collection of ordered pairs $(d(\ell-1,u), d(\ell,u))$, for $1 \le \ell \le t$ and $1 \le u \le n$, contains each ordered pair of distinct treatments λ_1 times, where $\lambda_1 = n/(t-1)$, and does not contain any pair of equal treatments.

We additionally define circular weakly balanced designs. Let $\mathbf{S}_d = \mathbf{T}_d'\mathbf{F}_d = (s_{dij})_{1\le i,j\le t}$. The entry s_{dij} is the number of appearances of treatment i preceded by treatment j in the design d. Thus the rows and columns of $\mathbf{S}_d$ sum to the vector of treatment replications. Filipiak et al. (2008) called the matrix $\mathbf{S}_d$ the *left-neighbouring matrix*. When the number of treatments is equal to the number of periods, Filipiak and Markiewicz (2012) called a design d in $\mathcal{D}_{t,n,t}$

(vi) *circular weakly balanced* (CWBD) if the collection of ordered pairs $(d(\ell-1,u), d(\ell,u))$, for $1 \le \ell \le t$ and $1 \le u \le n$, contains each ordered pair of distinct treatments λ or $\lambda - 1$ times, where $\lambda = \lceil n/(t-1)\rceil$, and

   (a) $\mathbf{S}_d\mathbf{1}_t = \mathbf{S}_d'\mathbf{1}_t = n\mathbf{1}_t$, so that each treatment has replication n;

   (b) $\mathbf{S}_d\mathbf{S}_d'$ is completely symmetric (all diagonal entries are equal and all off-diagonal entries are equal).

In this definition $\lceil x\rceil$ is the smallest integer greater than or equal to x.

Wilkinson et al. (1983) defined partially neighbour balanced designs as designs with $s_{dij} \in \{0,\ 1\}$ if $i \ne j$; however, their designs are not circular, and they consider neighbours in more than one direction. Some methods of constructing circular partially neighbour-balanced designs are given by Azaïs, Bailey and Monod (1993).

If d is a CSBD then $\mathbf{S}_d = \lambda_0\mathbf{J}_t$, where $\mathbf{J}_t = \mathbf{1}_t\mathbf{1}_t'$; if d is a CBD then $\mathbf{S}_d = \lambda_1(\mathbf{J}_t - \mathbf{I}_t)$. If d is a CWBD but not a CBD then $\mathbf{S}_d$ is not completely symmetric but $\mathbf{S}_d\mathbf{S}_d'$ is.

3. Existence Conditions

A necessary condition for the existence of a CBD with t periods is that $(t-1)$ divides n: see e.g., Druilhet (1999), while for the existence of a CWBD the expression $n(n-2\lambda+1)$ must be divisible by $t-1$; cf., Filipiak and Markiewicz (2012). Parameters satisfying the necessary condition for the existence of a CWBD with $t \le 19$ and $n < 3(t-1)$ are listed in Table 1 of Filipiak and Markiewicz (2012).

Let d be a CWBD in $\mathcal{D}_{t,n,t}$ which is not a CBD. Then $\lambda = \lceil n/(t-1)\rceil$. Put

$$k = n - (\lambda - 1)(t-1). \tag{3.1}$$

Since d is not a CBD, $1 \le k \le t-2$. Using this notation, the necessary condition for a CWBD given by Filipiak and Markiewicz (2012) is

$$t-1 \quad \text{divides} \quad k(k-2\lambda+1). \tag{3.2}$$

Filipiak and Różański (2009) showed that if $n = 1$, or if t is even and $n = 2$, then all designs are disconnected in the sense that it is not possible to estimate all contrasts between direct effects and all contrasts between carry-over effects without bias. If d is disconnected then it cannot be considered to be universally optimal; in fact, the proof of Theorem 3.1 of Filipiak and Markiewicz (2012) breaks down in this case. If $n = 2$ then (3.1) and (3.2) show that the only CWBD is a CBD for $t = 3$. From now on, we assume that $n \geq 3$ and d is connected.

Let $\mathbf{A}_d = \mathbf{S}_d' - (\lambda - 1)(\mathbf{J}_t - \mathbf{I}_t)$. Then $\mathbf{A}_d$ is a $t \times t$ matrix whose diagonal entries are all zero and whose other entries are all in $\{0, 1\}$. Moreover, each row and column of $\mathbf{A}_d$ has k non-zero entries. Hence $\mathbf{A}_d\mathbf{J}_t = \mathbf{J}_t\mathbf{A}_d = k\mathbf{J}_t$. Therefore

$$\begin{aligned}\mathbf{S}_d\mathbf{S}_d' &= \left[(\lambda - 1)(\mathbf{J}_t - \mathbf{I}_t) + \mathbf{A}_d'\right]\left[(\lambda - 1)(\mathbf{J}_t - \mathbf{I}_t) + \mathbf{A}_d\right] \\ &= (\lambda - 1)^2\left[(t-2)\mathbf{J}_t + \mathbf{I}_t\right] + 2(\lambda - 1)k\mathbf{J}_t + \mathbf{A}_d'\mathbf{A}_d - (\lambda - 1)(\mathbf{A}_d + \mathbf{A}_d').\end{aligned}$$

Thus $\mathbf{S}_d\mathbf{S}_d'$ is completely symmetric if and only if

$$\mathbf{A}_d'\mathbf{A}_d - (\lambda - 1)(\mathbf{A}_d + \mathbf{A}_d') \quad \text{is completely symmetric.} \tag{3.3}$$

If it satisfies (3.3), we shall say that design d has

Type I if $\mathbf{A}_d + \mathbf{A}_d'$ is completely symmetric;

Type II if $\mathbf{A}_d + \mathbf{A}_d'$ is not completely symmetric and $\lambda = 1$;

Type III if $\mathbf{A}_d + \mathbf{A}_d'$ is not completely symmetric and $\lambda > 1$.

If d has Type I or II then $\mathbf{A}_d'\mathbf{A}_d$ is completely symmetric. The off-diagonal entries in each row of $\mathbf{A}_d'\mathbf{A}_d$ sum to $k(k-1)$, so in this case $k(k-1)$ is divisible by $t-1$. If d has Type I then $k = (t-1)/2$ and $\mathbf{A}_d + \mathbf{A}_d' = \mathbf{J}_t - \mathbf{I}_t$. Then $t-1$ divides $(t-1)(t-3)/4$, and so $t \equiv 3 \bmod 4$. If $k = 1$ then $\mathbf{A}_d'\mathbf{A}_d = \mathbf{I}_t$: thus d cannot have Type III, and (3.1) shows that if d has Type II then $n = 1$, which we exclude.

Theorem 1. *Suppose that d is a CWBD in $\mathcal{D}_{t,n,t}$ and d' is a CBD in $\mathcal{D}_{t,m,t}$, for some values of n and m. Then the design d'' in $\mathcal{D}_{t,n+m,t}$ which juxtaposes d and d' is a CWBD if and only if d has Type I.*

Proof. If d' is a CBD then m is a multiple of $t-1$ and $\mathbf{S}_{d'}$ is completely symmetric. Hence $\mathbf{S}_{d''} = \mathbf{S}_d + \mathbf{S}_{d'}$ and so $\mathbf{A}_{d''} = \mathbf{A}_d$. Put $\lambda' = m/(t-1)$. Condition (3.3) for design d'' says that

$$\mathbf{A}_d'\mathbf{A}_d - (\lambda' + \lambda - 1)(\mathbf{A}_d + \mathbf{A}_d') \quad \text{is completely symmetric.} \tag{3.4}$$

If d has Type I then $\mathbf{A}_d'\mathbf{A}_d$ and $\mathbf{A}_d + \mathbf{A}_d'$ are both completely symmetric, and so condition (3.4) is satisfied and d'' is a CWBD. Conversely, if d'' is a CWBD then conditions (3.3) and (3.4) are both satisfied. Hence $\mathbf{A}_d + \mathbf{A}_d'$ is completely symmetric and so d has Type I.

Lemma 1. *Suppose that d is a CWBD in $\mathcal{D}_{t,n,t}$ which has Type III.*

(a) *If $k = (t-1)/2$ then $\lambda \leq k/2$.*

(b) *If $k < (t-1)/2$ then $\lambda \leq k$.*

(c) *If $k > (t-1)/2$ then $\lambda \leq t-k-1$.*

Proof. Put $m_0 = \max\{0, 2k-t\}$. If $i \neq j$ then $m_0 \leq (\mathbf{A}'_d\mathbf{A}_d)_{ij} \leq k-(\mathbf{A}_d+\mathbf{A}'_d)_{ij}$. Let m_1 and m_2 be the smallest and largest off-diagonal entries in $\mathbf{A}_d + \mathbf{A}'_d$. The entries in the corresponding positions of $\mathbf{A}'_d\mathbf{A}_d - (\lambda-1)(\mathbf{A}_d + \mathbf{A}'_d)$ lie in the intervals $[m_0 - m_1(\lambda-1), k - m_1\lambda]$ and $[m_0 - m_2(\lambda-1), k - m_2\lambda]$ respectively. If the latter entries are equal then $k - m_2\lambda \geq m_0 - m_1(\lambda-1)$.

(a) If $k = (t-1)/2$ but $\mathbf{A}_d + \mathbf{A}'_d$ is not completely symmetric then $m_0 = 0$, $m_1 = 0$ and $m_2 = 2$. Hence $k - 2\lambda \geq 0$.

(b) If $k < (t-1)/2$ then $m_0 = 0$, $m_1 = 0$ and $m_2 \geq 1$. Hence $k - \lambda \geq 0$.

(c) If $k > (t-1)/2$ then $k \geq t/2$ and so $m_0 = 2k-t$. Also, $m_2 = 2$ and $m_1 \leq 1$. Hence $k - 2\lambda \geq 2k - t - (\lambda-1)$.

Theorem 2. *If d is a CWBD in $\mathcal{D}_{t,n,t}$ and d has Type II or III then d is not uniform on the periods.*

Proof. If d is uniform on the periods then t divides n. If d has Type II then $n = k \leq t-2$, and so this is not possible. If t divides n then equation (3.1) shows that t divides $k - \lambda + 1$. Lemma 1 shows that if d has Type III then $0 < \lambda \leq k < t-1$ and so $0 < k - \lambda + 1 < t$, thus t cannot divide $k - \lambda + 1$.

4. Optimality

4.1. Preliminaries

Kunert (1984b) showed that any CBD which is uniform on subjects is universally optimal for the estimation of direct as well as carry-over effects under model (2.3) over the class $\mathcal{D}_{t,n,t}$. Druilhet (1999) extended this to designs where the number of periods is any multiple of t. Filipiak and Markiewicz (2012) defined circular weakly neighbour balanced designs to be CWBDs for t periods which are uniform on subjects; they showed their universal optimality for the estimation of direct effects under model (2.3) over the class $\mathcal{D}_{t,n,t}$ with $n \leq t-1$, and over the class of equireplicated designs without self-neighbours if $n > t-1$. One aim of this paper is to prove universal optimality for the estimation of direct as well as carry-over effects of uniform CWBDs, CWBDs uniform on subjects, and CWBDs uniform on periods under models (2.2), (2.3) and (2.4), respectively.

We are interested in determining designs with minimal (in some sense) variance of the best linear unbiased estimator of the vector of parameters. Kiefer

(1975) formulated the universal optimality criterion in terms of the information matrix, which is the inverse of the variance-covariance matrix; cf., Pukelsheim (1993). Therefore, following Proposition 1 of Kiefer (1975), we suppose that a class $\mathcal{C} = \{\boldsymbol{C}_d : d \in \mathcal{D}_{t,n,t}\}$ of non-negative definite information matrices with zero row and column sums contains a matrix $\boldsymbol{C}_{d^*}$ which is completely symmetric and has maximal trace over $\mathcal{D}_{t,n,t}$. Then the design d^* is *universally optimal* in Kiefer's sense in the class $\mathcal{D}_{t,n,t}$.

For a $\kappa_1 \times \kappa_2$ matrix $\boldsymbol{K}$ define $\omega^{\perp}(\boldsymbol{K}) = \boldsymbol{I}_{\kappa_1} - \boldsymbol{K}(\boldsymbol{K}'\boldsymbol{K})^{-}\boldsymbol{K} = \boldsymbol{I}_{\kappa_1} - \omega(\boldsymbol{K})$ as the orthogonal projector onto the orthocomplement of the column space of $\boldsymbol{K}$, where $(\boldsymbol{K}'\boldsymbol{K})^{-}$ is a generalized inverse of $\boldsymbol{K}'\boldsymbol{K}$. Then the information matrix for the least squares estimate of $\boldsymbol{\tau}$ under model (2.g), $g = 2, 3, 4$, is given by

$$\boldsymbol{C}_d^{(g)} = \boldsymbol{T}_d'\omega^{\perp}(\boldsymbol{Z}^{(g)})\boldsymbol{T}_d$$

with zero row and column sums, where $\boldsymbol{Z}^{(g)}$ is a block matrix containing the design matrices of nuisance parameters, $\boldsymbol{Z}^{(2)} = (\boldsymbol{P} : \boldsymbol{U} : \boldsymbol{F}_d)$, $\boldsymbol{Z}^{(3)} = (\boldsymbol{U} : \boldsymbol{F}_d)$, and $\boldsymbol{Z}^{(4)} = (\boldsymbol{P} : \boldsymbol{F}_d)$; cf., e.g., Kunert (1983, 1984a,b). Since $\omega((\boldsymbol{A} : \boldsymbol{B})) = \omega(\boldsymbol{A}) + \omega(\omega^{\perp}(\boldsymbol{A})\boldsymbol{B})$, we may rewrite the matrix $\boldsymbol{C}_d^{(g)}$ as

$$\boldsymbol{C}_d^{(g)} = \boldsymbol{T}_d'\omega^{\perp}(\boldsymbol{F}_d)\boldsymbol{T}_d - \boldsymbol{T}_d'\omega(\omega^{\perp}(\boldsymbol{F}_d)\boldsymbol{W}^{(g)})\boldsymbol{T}_d,$$

with $\boldsymbol{W}^{(2)} = (\boldsymbol{P} : \boldsymbol{U})$, $\boldsymbol{W}^{(3)} = \boldsymbol{U}$, and $\boldsymbol{W}^{(4)} = \boldsymbol{P}$.

Similarly, Kunert (1984b) showed that the information matrix for the least squares estimate of $\boldsymbol{\rho}$ under model (2.g), $g = 2, 3, 4$, is

$$\widetilde{\boldsymbol{C}}_d^{(g)} = \boldsymbol{F}_d'\omega^{\perp}(\widetilde{\boldsymbol{Z}}^{(g)})\boldsymbol{F}_d = \boldsymbol{F}_d'\omega^{\perp}(\boldsymbol{T}_d)\boldsymbol{F}_d - \boldsymbol{F}_d'\omega(\omega^{\perp}(\boldsymbol{T}_d)\boldsymbol{W}^{(g)})\boldsymbol{F}_d,$$

with $\widetilde{\boldsymbol{Z}}^{(2)} = (\boldsymbol{P} : \boldsymbol{U} : \boldsymbol{T}_d)$, $\widetilde{\boldsymbol{Z}}^{(3)} = (\boldsymbol{U} : \boldsymbol{T}_d)$, and $\widetilde{\boldsymbol{Z}}^{(4)} = (\boldsymbol{P} : \boldsymbol{T}_d)$.

4.2. Optimality results

Filipiak and Markiewicz (2012) showed that for a CWBD, $\mathbf{S}_d\mathbf{S}_d' = \phi\mathbf{I}_t + \xi\mathbf{J}_t$ with $\phi = n(2\lambda-1)-\lambda(\lambda-1)t-n(n-2\lambda+1)/(t-1)$ and $\xi = \lambda(\lambda-1)+n(n-2\lambda+1)/(t-1)$. Since $\mathbf{S}_d$ is nonsingular and commutes with $\mathbf{J}_t$, pre-multiplying by $\mathbf{S}_d'$ and post-multiplying by $(\boldsymbol{S}_d')^{-1}$ we get $\mathbf{S}_d\mathbf{S}_d' = \mathbf{S}_d'\mathbf{S}_d$; cf., Raghavarao (1971, Theorem 5.2.1), Filipiak and Markiewicz (2016). Moreover, since for a CWBD $\boldsymbol{T}_d'\omega^{\perp}(\boldsymbol{F}_d)\boldsymbol{T}_d = n\mathbf{I}_t - n^{-1}\mathbf{S}_d\mathbf{S}_d' = \boldsymbol{F}_d'\omega^{\perp}(\boldsymbol{T}_d)\boldsymbol{F}_d$, the following holds.

Proposition 1. *Assume d is a CWBD. Under models* (2.2), (2.3) *or* (2.4), *if d is uniform, uniform on subjects, or uniform on periods, respectively, then d is universally optimal for the estimation of direct effects if and only if d is universally optimal for the estimation of carry-over effects.*

Let $\Lambda_{t,n,t}$ be the class of designs in $\mathcal{D}_{t,n,t}$ with no treatment preceded by itself. Using the above proposition we can extend Theorem 3.1 and Theorem 3.2 of Filipiak and Markiewicz (2012), in which optimality of CWBDs for the estimation of direct effects was shown, as follows.

Theorem 3. *If there exists a CWBD in $\mathcal{D}_{t,n,t}$ which is uniform on subjects, then it is universally optimal for the estimation of carry-over effects under model* (2.3) *over the collection of designs in $\mathcal{D}_{t,n,t}$ if $n \leq t-1$, and over the collection of equireplicated designs in $\Lambda_{t,n,t}$ otherwise.*

If a design is uniform on periods then t divides n and so $n > t-1$. The following theorem can be proved in the same way as Theorem 3.2 of Filipiak and Markiewicz (2012) using additionally Proposition 1 of this paper.

Theorem 4. *Assume that $t > 2$ and $n > t-1$. If there exists a CWBD in $\Lambda_{t,n,t}$ which is uniform on periods, then it is universally optimal for the estimation of direct as well as carry-over effects under model* (2.4) *over the collection of equireplicated designs in $\Lambda_{t,n,t}$.*

For the two models with subject effects, we now show optimality over a broader class of designs than in Theorem 4. Theorem 1 shows that if d is a CWBD which is not a CBD, then we can make larger CWBDs by juxtaposing d with one or more CBDs only if d has Type I. Therefore, we restrict attention to the case that $k = (t-1)/2$, where $t \equiv 3 \bmod 4$, and n is an odd multiple of $(t-1)/2$. It follows from Theorem 2 that a uniform CWBD which is not a CBD can only exist if, additionally, n is an odd multiple of $t(t-1)/2$. For such design parameters, $\lambda = n/(t-1) + 1/2$.

We denote by n_{diu} the number of times that treatment i appears in the uth subject ($\boldsymbol{T}_d'\boldsymbol{U} = \boldsymbol{F}_d'\boldsymbol{U} = (n_{diu})$), and by r_{di} the number of times that treatment i appears in the design. As shown by e.g., Kunert (1984b), if $g = 2$ or $g = 3$ then

$$\operatorname{tr} \boldsymbol{C}_d^{(g)} \leq \sum_{i=1}^{t} r_{di} - \frac{1}{t}\sum_{i=1}^{t}\sum_{u=1}^{n} n_{diu}^2 - \sum_{i=1}^{t}\sum_{j=1}^{t} \frac{(s_{dij} - (1/t)\sum_{u=1}^{n} n_{diu}n_{dju})^2}{r_{dj}}. \quad (4.1)$$

We begin with a technical lemma, and omit the straightforward proof.

Lemma 2. *If $x_1, x_2, \ldots, x_b$ satisfy $\sum_{i=1}^{b} x_i = c$ then $\sum_{i=1}^{b} x_i^2 \geq c^2/b$.*

Proposition 2. *If j is a treatment in design d in $\Lambda_{t,n,t}$ then*

$$\sum_{i=1}^{t}\Big(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju}\Big)^2 \geq \frac{r_{dj}^2}{t(t-1)}.$$

Proof. For all competing designs all $s_{djj} = 0$. Therefore,

$$\sum_{i=1}^{t}(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju}) = -\frac{1}{t}\sum_{u=1}^{n} n_{dju}^2 + \sum_{i\neq j}(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju}).$$

It follows that

$$\sum_{i\neq j}(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju}) = \frac{1}{t}\sum_{u=1}^{n} n_{dju}^2.$$

Applying Lemma 2, we conclude that

$$\begin{aligned}\sum_{i=1}^{t}(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju})^2 &= \frac{1}{t^2}\Big(\sum_{u=1}^{n} n_{dju}^2\Big)^2 + \sum_{i\neq j}(s_{dij} - \frac{1}{t}\sum_{u=1}^{n} n_{diu}n_{dju})^2 \\ &\geq \frac{1}{t^2}\Big(\sum_{u=1}^{n} n_{dju}^2\Big)^2 + \frac{1}{t-1}\frac{1}{t^2}\Big(\sum_{u=1}^{n} n_{dju}^2\Big)^2 \\ &= \frac{1}{t(t-1)}\Big(\sum_{u=1}^{n} n_{dju}^2\Big)^2 \geq \frac{r_{dj}^2}{t(t-1)}.\end{aligned}$$

Proposition 3. *For a design $d \in \Lambda_{t,n,t}$ define $a_d = \sum_{i=1}^{t}\sum_{u=1}^{n} \max\{n_{diu}-1, 0\}$. Then*

$$\sum_{i=1}^{t}\sum_{u=1}^{n} n_{diu}^2 \geq nt + 2a_d.$$

Proof. For $1 \leq i \leq t$ and $1 \leq u \leq n$, define $e_{diu} = n_{diu} - 1$. Then all e_{diu} are integers and, therefore, $e_{diu}^2 \geq |e_{diu}|$. Since $\sum_{i=1}^{t}\sum_{u=1}^{n} n_{diu} = nt$, we have $\sum\sum e_{diu} = 0$ and, since the sum of all positive e_{diu} equals a_d, we conclude that $\sum\sum |e_{diu}| = 2a_d$.

In all, we get

$$\begin{aligned}\sum_{i=1}^{t}\sum_{u=1}^{n} n_{diu}^2 &= \sum_{i=1}^{t}\sum_{u=1}^{n}(e_{diu}+1)^2 \\ &= nt + 2\sum_{i=1}^{t}\sum_{u=1}^{n} e_{diu} + \sum_{i=1}^{t}\sum_{u=1}^{n} e_{diu}^2 \geq nt + 2a_d.\end{aligned}$$

We immediately get a first bound for the trace of the information matrix which depends on a_d.

Proposition 4. *For any design $d \in \Lambda_{t,n,t}$ and $g = 2, 3$ we have*

$$\operatorname{tr} \boldsymbol{C}_d^{(g)} \leq n(t - 1 - \frac{1}{t-1}) - \frac{2a_d}{t}.$$

Proof. The bound is well-known; it was used by Kunert (1984a,b).

If a_d is small, we get a sharper bound, derived in the next proposition.

Proposition 5. *If $t \ge 5$ and design d has $a_d < (t-1)/2$ then, for $g = 2, 3$,*
$$\operatorname{tr} \boldsymbol{C}_d^{(g)} \le n\left(t-1-\frac{1}{t-1}\right) - \frac{2a_d}{t} - (t-2a_d)\left(\frac{t-1}{4n} - \frac{2a_d}{nt}\right).$$

Proof. There are at most $2a_d$ of the n_{diu} not equal to 1. Since $a_d < (t-1)/2$, we conclude that there must be at least $t-2a_d$ treatments j such that $n_{dju} = 1$ for $1 \le u \le n$. Define $\mathcal{J}_d^*$ as the set of all such treatments.

Assume without loss of generality that treatment t is in $\mathcal{J}_d^*$. Then $r_{dt} = n$. Since we consider the circular neighbour structure, $\sum_{i=1}^t s_{dit} = r_{dt} = n$. Without loss of generality we can assume that the treatments are labelled in such a way that $s_{d1t} \ge s_{d2t} \ge \cdots \ge s_{d,t-1,t}$.

Recall that $2k = t-1$ and $\lambda = n/(t-1) + 1/2$. If $s_{dkt} \le \lambda - 1$ then $\sum_{i=k+1}^{t-1} s_{dit} \le k s_{dkt} = k(\lambda-1) = (n-k)/2$. Since $\sum_{i=1}^{t-1} s_{dit} = n$, this implies that $\sum_{i=1}^k s_{dit} \ge (n+k)/2$. Otherwise, $s_{dkt} \ge \lambda$ and so $\sum_{i=1}^k s_{dit} \ge k\lambda = (n+k)/2$ again.

Now put $c = \sum_{i=1}^k (s_{dit} - r_{di}/t)$. The definition of a_d gives $\sum_{i=1}^k r_{di} \le kn + a_d$. Hence
$$c = \sum_{i=1}^k s_{dit} - \frac{1}{t}\sum_{i=1}^k r_{di} \ge \frac{n+k}{2} - \frac{1}{t}\left(\frac{(t-1)n}{2} + a_d\right) = \frac{n}{2t} + \frac{k}{2} - \frac{a_d}{t}. \qquad (4.2)$$

It follows that $c > n/(2t)$ because $a_d < k$ and $t > 2$. Furthermore,
$$0 = \sum_{i=1}^t \left(s_{dit} - \frac{r_{di}}{t}\right) = c + \sum_{i=k+1}^{t-1}\left(s_{dit} - \frac{r_{di}}{t}\right) + s_{dtt} - \frac{r_{dt}}{t} = c + \sum_{i=k+1}^{t-1}\left(s_{dit} - \frac{r_{di}}{t}\right) - \frac{n}{t}$$

and therefore $\sum_{i=k+1}^{t-1} (s_{dit} - r_{di}/t) = n/t - c$. Then Lemma 2 gives
$$\sum_{i=1}^t \left(s_{dit} - \frac{r_{di}}{t}\right)^2 \ge \frac{n^2}{t^2} + \frac{1}{k}\left[c^2 + \left(\frac{n}{t} - c\right)^2\right].$$

Since $c > n/(2t)$, this bound is increasing in c. Therefore (4.2) gives
$$\sum_{i=1}^t \left(s_{dit} - \frac{r_{di}}{t}\right)^2 \ge \frac{n^2}{t^2} + \frac{2}{k}\left(\frac{n^2}{4t^2} + \frac{k^2}{4} - \frac{ka_d}{t} + \frac{a_d^2}{t^2}\right) \ge \frac{n^2}{t(t-1)} + \frac{k}{2} - \frac{2a_d}{t}.$$

Since $n_{dtu} = 1$ for $1 \le i \le n$, this shows that
$$\sum_{i=1}^t \left(s_{dit} - \frac{1}{t}\sum_{u=1}^n n_{diu} n_{dtu}\right)^2 = \sum_{i=1}^t \left(s_{dit} - \frac{r_{di}}{t}\right)^2 \ge \frac{n^2}{t(t-1)} + \frac{k}{2} - \frac{2a_d}{t}.$$

The same bound applies when treatment t is replaced by any treatment j in $\mathcal{J}_d^*$. For all other treatments j, we use the bound in Proposition 2. Inserting these, and the bound in Proposition 3, into (4.1), we get, for $g = 2$, 3,

$$\begin{aligned}\operatorname{tr} \boldsymbol{C}_d^{(g)} &\leq nt - \frac{1}{t}(nt + 2a_d) - \sum_{j \notin \mathcal{J}_d^*} \frac{r_{dj}}{t(t-1)} - \sum_{j \in \mathcal{J}_d^*} \left(\frac{n}{t(t-1)} + \frac{t-1}{4n} - 2\frac{a_d}{nt} \right) \\ &= nt - \frac{1}{t}(nt + 2a_d) - \sum_{j=1}^{t} \frac{r_{dj}}{t(t-1)} - |\mathcal{J}_d^*| \left(\frac{t-1}{4n} - 2\frac{a_d}{nt} \right),\end{aligned}$$

where we have used the fact that $r_{dj} = n$ for all $j \in \mathcal{J}_d^*$, and where $|\mathcal{J}_d^*|$ is the number of elements of $\mathcal{J}_d^*$. Due to the restrictions that $a_d < (t-1)/2$ and $t \geq 5$, we observe that $(t-1)/(4n) - 2a_d/(nt) > (t-1)(t-4)/(4nt) > 0$. Since $|\mathcal{J}_d^*| \geq t - 2a_d$, it follows that, for $g = 2, 3$,

$$\operatorname{tr} \boldsymbol{C}_d^{(g)} \leq nt - \frac{1}{t}(nt + 2a_d) - \frac{n}{t-1} - (t - 2a_d) \left(\frac{t-1}{4n} - \frac{2a_d}{nt} \right),$$

which implies the desired inequality.

Now we can prove our main optimality result.

Theorem 5. *Assume that $t \geq 5$ and that $n \geq t(t-1)/2$. Assume that t is odd and that n is an odd multiple of $(t-1)/2$. If d^* is a uniform CWBD in $\Lambda_{t,n,t}$ then d^* is universally optimal for the estimation of direct as well as carry-over effects over the designs in $\Lambda_{t,n,t}$ under model (2.2). If d^* is a CWBD in $\Lambda_{t,n,t}$ which is uniform on subjects, then d^* is universally optimal for the estimation of direct as well as carry-over effects over the designs in $\Lambda_{t,n,t}$ under model (2.3).*

Proof. If the design d has $a_d = 0$, we get from Proposition 5 that, for $g = 2, 3$,

$$\operatorname{tr} \boldsymbol{C}_d^{(g)} \leq n \left(t - 1 - \frac{1}{t-1} \right) - \frac{t(t-1)}{4n},$$

which is the trace of the information matrix of the CWBD d^*. Considering the simple bound derived in Proposition 4, we see that any design $d \in \Lambda_{t,n,t}$ can only perform better than d^* if

$$\frac{t(t-1)}{4n} \geq \frac{2a_d}{t}.$$

Since we restrict to the case $n \geq t(t-1)/2$, the left-hand side is less than or equal to $1/2$. If, however, $a_d \geq (t-1)/2$, then the right-hand side is at least $(t-1)/t > 1/2$.

Therefore, we only have to consider designs with $a_d < (t-1)/2$ and the bound in Proposition 5 applies. Taking the derivative of

$$f(a) = n \left(t - 1 - \frac{1}{t-1} \right) - \frac{2a}{t} - (t - 2a) \left(\frac{t-1}{4n} - \frac{2a}{nt} \right)$$

with respect to a, we get

$$\begin{aligned}
f'(a) &= -\frac{2}{t} + 2\left(\frac{t-1}{4n} - \frac{2a}{nt}\right) - (t-2a)\left(-\frac{2}{nt}\right) \\
&= -\frac{2}{t} + \frac{2(t-1)}{4n} + \frac{2t}{nt} - \frac{4a}{nt} - \frac{4a}{nt} \\
&= \frac{-8n + 2t(t-1) + 8t - 32a}{4nt} \\
&\leq \frac{-4t(t-1) + 2t(t-1) + 8t}{4nt} = \frac{-(t-1)+4}{2n} \leq 0.
\end{aligned}$$

This, however, implies that the bound from Proposition 5 is largest for $a_d = 0$, and for any design $d \in \Lambda_{t,n,t}$ we have $\operatorname{tr} \boldsymbol{C}_d^{(g)} \leq \operatorname{tr} \boldsymbol{C}_{d^*}^{(g)}$, for $g = 2, 3$.

5. Constructions

In this section we suppose that d is a CWBD in $\mathcal{D}_{t,n,t}$ which is not a CBD. For each type of CWBD, we give constructions for a suitable matrix $\mathbf{A}$ and then search for a design d with $\mathbf{A}_d = \mathbf{A}$. By Theorem 2, only Section 5.1 includes uniform CWBDs.

5.1. Designs of Type I

For a design of Type I, we have $t \equiv 3 \bmod 4$ and $k = (t-1)/2$. We need a $t \times t$ matrix $\mathbf{A}$ which has zero entries on the diagonal, k entries equal to 1 in each row and column, and all other entries zero; it must also satisfy (a) $\mathbf{A} + \mathbf{A}' = \mathbf{J}_t - \mathbf{I}_t$ and (b) $\mathbf{A}'\mathbf{A} = \phi\mathbf{I}_t + \xi\mathbf{J}_t$ with $\phi = (t+1)/4$ and $\xi = (t-3)/4$.

The matrix $\mathbf{A}$ can be regarded as the adjacency matrix of a directed graph Γ on t vertices: there is an arc from vertex i to vertex j if and only if $\mathbf{A}_{ij} = 1$. This directed graph is called a *doubly regular tournament* precisely when the matrix $\mathbf{A}$ satisfies the foregoing conditions, see Reid and Brown (1972). For a design which is a CWBD, is uniform on subjects, and has $\lambda = 1$, we need a decomposition of a doubly regular tournament Γ into Hamiltonian cycles.

One construction of doubly regular tournaments uses finite fields. If t is a power of an odd prime then there is a finite field $\mathrm{GF}(t)$ of t elements. If t is prime then $\mathrm{GF}(t)$ is the same as $\mathbb{Z}_t$, which is the ring of integers modulo t. Let $\mathcal{S}$ be the set of non-zero squares in $\mathrm{GF}(t)$, and $\mathcal{N}$ the set of non-squares. If $t \equiv 3 \bmod 4$ then $-1 \in \mathcal{N}$; in this case, if we label the vertices of Γ by the elements of $\mathrm{GF}(t)$ and define the adjacency matrix $\mathbf{A}$ by putting $\mathbf{A}_{ij} = 1$ if and only if $j - i \in \mathcal{S}$, then Γ is a doubly regular tournament, see Lidl and Niederreiter (1997). By reversing all the edges of Γ, we obtain another doubly regular tournament, which can be made directly by using $\mathcal{N}$ in place of $\mathcal{S}$.

If t is itself prime, then there is an obvious Hamiltonian decomposition of Γ: the circular sequences have the form $(0, s, 2s, \ldots, (t-1)s)$ for s in $\mathcal{S}$.

Construction 1. Suppose that $t \equiv 3 \bmod 4$ and t is prime with $t > 3$. Put $n = (t-1)/2$. Label the t treatments and the t periods by the elements of $\mathbb{Z}_t$, and the n subjects by the elements of $\mathcal{S}$. Define the design d by $d(\ell, u) = \ell u$ for ℓ in $\mathbb{Z}_t$ and u in $\mathcal{S}$. Then d is a CWBD which is uniform on the subjects with $\lambda = 1$.

Example 1. When $t = 7$ we have $\mathcal{S} = \{1, 2, 4\}$. We obtain the design in Figure 1(a), where the entries are integers modulo 7. (In every figure, the rows denote periods and the columns denote subjects.)

Example 2. When $t = 11$ we have $\mathcal{S} = \{1, 3, 4, 5, 9\}$. This gives the design in Figure 1(b), where the entries are integers modulo 11.

For $n > 1$, Construction 1 deals with $t = 7$, 11, 19, 23 and 31 for $t < 35$.

Suitable matrices $\mathbf{A}$ also exist for many other values of t. Reid and Brown (1972) showed that the $(t+1) \times (t+1)$ matrix

$$\begin{bmatrix} 1 & \mathbf{1}_t' \\ \mathbf{1}_t & \mathbf{J}_t - 2\mathbf{A} \end{bmatrix}$$

is a skew-Hadamard matrix if and only if $\mathbf{A}$ is the adjacency matrix of a doubly regular tournament. Skew-Hadamard matrices of order $t+1$ are conjectured to exist whenever $t+1$ is divisible by 4. This has been verified for $t < 187$: see Craigen (1996).

Reid and Brown (1972) give the following doubling construction. If $\mathbf{A}_1$ is the adjacency matrix of a doubly regular tournament Γ_1 on t vertices and

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{A}_1' & \mathbf{0}_t & \mathbf{A}_1 + \mathbf{I}_t \\ \mathbf{1}_t' & 0 & \mathbf{0}_t' \\ \mathbf{A}_1 & \mathbf{1}_t & \mathbf{A}_1 \end{bmatrix}, \tag{5.1}$$

then $\mathbf{A}_2$ is the adjacency matrix of a doubly regular tournament Γ_2 on $2t+1$ vertices.

Example 3. Let $t = 15$. Take Γ_1 to be the doubly regular tournament used in Example 1. The doubling construction (5.1) gives the adjacency matrix $\mathbf{A}_2$ of a doubly regular tournament Γ_2 on 15 vertices. Label the vertices, in order, 0, 1, 2, 3, 4, 5, 6, ∞, $0'$, $1'$, $2'$, $3'$, $4'$, $5'$ and $6'$. For x in GF(7), there is an arc from ∞ to x and an arc from x' to ∞. For x and y in GF(7), there is an arc from x to y if $y - x \in \mathcal{N}$; an arc from x to y' if $x = y$ or $y - x \in \mathcal{S}$; an arc from x' to y' if $y - x \in \mathcal{S}$; and an arc from x' to y if $y - x \in \mathcal{S}$.

0	0	0
1	2	4
2	4	1
3	6	5
4	1	2
5	3	6
6	5	3

(a)

0	0	0	0	0
1	3	4	5	9
2	6	8	10	7
3	9	1	4	5
4	1	5	9	3
5	4	9	3	1
6	7	2	8	10
7	10	6	2	8
8	2	10	7	6
9	5	3	1	4
10	8	7	6	2

(b)

∞	∞	∞	∞	∞	∞	∞
0	1	2	3	4	5	6
$2'$	$3'$	$4'$	$5'$	$6'$	$0'$	$1'$
3	4	5	6	0	1	2
1	2	3	4	5	6	0
$5'$	$6'$	$0'$	$1'$	$2'$	$3'$	$4'$
$6'$	$0'$	$1'$	$2'$	$3'$	$4'$	$5'$
$1'$	$2'$	$3'$	$4'$	$5'$	$6'$	$0'$
5	6	0	1	2	3	4
4	5	6	0	1	2	3
$4'$	$5'$	$6'$	$0'$	$1'$	$2'$	$3'$
6	0	1	2	3	4	5
2	3	4	5	6	0	1
$3'$	$4'$	$5'$	$6'$	$0'$	$1'$	$6'$
$0'$	$1'$	$2'$	$3'$	$4'$	$5'$	$6'$

(c)

Figure 1. Three CWBDs for t treatments on n subjects in t periods which are uniform on the subjects: (a) $t = 7$ and $n = 3$; (b) $t = 11$ and $n = 5$; (c) $t = 15$ and $n = 7$.

To find a CWBD which is uniform on subjects, we used GAP (2014) to find a directed cycle $\boldsymbol{\varphi}$ of length 15 starting $(\infty, 0, \ldots)$ in Γ_2 with the extra property that if i is any non-zero element of GF(7), then the cycles $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi} + i$ have no arc in common. Here we use the conventions that if $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{15})$ then $\boldsymbol{\varphi} + i = (\varphi_1 + i, \ldots, \varphi_{15} + i)$, where $\infty + i = \infty$ and $x' + i = (x + i)'$ for x and i in GF(7). GAP (2014) found all such cycles. There are 120, and they come in groups of three because if $\boldsymbol{\varphi}$ is such a cycle and $s \in \mathcal{S}$ then $s\boldsymbol{\varphi}$ is also such a cycle (here the convention is that $s\boldsymbol{\varphi} = (s\varphi_1, \ldots, s\varphi_{15})$, where $s \times \infty = \infty$ and $s \times x' = (sx)'$ for s and x in GF(7)). For each such cycle $\boldsymbol{\varphi}$, the collection of cycles $\boldsymbol{\varphi}$, $\boldsymbol{\varphi} + 1$, …, $\boldsymbol{\varphi} + 6$ gives a CWBD d for 15 treatments on 7 subjects in 15 periods which is uniform on subjects and for which $\mathbf{A}_d = \mathbf{A}_2$. One of these is shown in Figure 1(c).

Alternatively, the function `FindHamiltonianCycles` in *Mathematica* 9.0 can be used to find a Hamiltonian decomposition of Γ_2.

For $t = 3$, Construction 1 gives a design with $n = 1$ that is disconnected. In order to obtain a connected CWBD which is not a CBD, we need to use one of the sequences $(0, 1, 2)$ and $(0, 2, 1)$ twice, and the other one once.

If design d is made by Construction 1 then a uniform design d' with $t(t-1)/2$ subjects may be obtained by replacing the sequence $\boldsymbol{\varphi}$ for each subject by the sequences $\boldsymbol{\varphi} + i$ for all i in GF(t). However, this has the effect that $\mathbf{S}_{d'} = t\mathbf{S}_d$, so d' is not a CWBD, because the off-diagonal entries of $\mathbf{S}_{d'}$ include both 0 and

t. Thus we need a different construction for uniform CWBDs. Again, we use $\mathrm{GF}(t)$, where $t \equiv 3 \bmod 4$. If x and y are both in $\mathcal{S}$ or $\mathcal{N}$ then $xy \in \mathcal{S}$; if one is in $\mathcal{S}$ and the other in $\mathcal{N}$ then $xy \in \mathcal{N}$: see Lidl and Niederreiter (1997).

If $\boldsymbol{\varphi}$ is any sequence $(\varphi_1, \ldots, \varphi_m)$ of elements of $\mathrm{GF}(t)$, we denote by $\boldsymbol{\varphi}^\delta$ the sequence $(\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \ldots, \varphi_m - \varphi_{m-1}, \varphi_1 - \varphi_m)$ of successive circular differences in $\boldsymbol{\varphi}$. Further, let $f_0(\boldsymbol{\varphi}^\delta)$, $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta)$ and $f_{\mathcal{N}}(\boldsymbol{\varphi}^\delta)$ be the number of entries of $\boldsymbol{\varphi}^\delta$ which are in $\{0\}$, $\mathcal{S}$ and $\mathcal{N}$, respectively.

Definition 1. Let $\boldsymbol{\varphi}$ be a sequence of length t whose entries are in $\mathrm{GF}(t)$, where t is a prime power congruent to 3 modulo 4. Then $\boldsymbol{\varphi}$ is *beautiful* if the entries in $\boldsymbol{\varphi}$ are all different and $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta) = f_{\mathcal{N}}(\boldsymbol{\varphi}^\delta) \pm 1$.

If all of the entries of $\boldsymbol{\varphi}$ are different then $f_0(\boldsymbol{\varphi}^\delta) = 0$. Thus if $\boldsymbol{\varphi}$ has length t then it is beautiful if and only if $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta) \in \{k, k+1\}$.

Construction 2. Given a beautiful sequence $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_t)$ of all the elements of $\mathrm{GF}(t)$, form the $t(t-1)/2$ sequences $s\boldsymbol{\varphi} + i$ for all s in $\mathcal{S}$ and all i in $\mathrm{GF}(t)$. Create the design d by using each of these sequences for one subject.

Theorem 6. *Suppose* $t \equiv 3 \bmod 4$ *and* t *is a prime power. If* $\boldsymbol{\varphi}$ *is beautiful then the design* d *given by Construction* 2 *is a uniform CWBD.*

Proof. The entries in $\boldsymbol{\varphi}$ are all different, so the entries in $s\boldsymbol{\varphi} + i$ are all different for each value of s and i. Therefore each treatment occurs once on each subject, so d is uniform on subjects and no treatment is preceded by itself.

For each fixed s in $\mathcal{S}$, every element of $\mathrm{GF}(t)$ occurs once in each period among the t sequences $s\boldsymbol{\varphi} + i$, as i varies in $\mathrm{GF}(t)$. Therefore d is uniform.

Consider period j. Put $\varphi_j^\delta = v$. Let $i \in \mathrm{GF}(t)$ and $s \in \mathcal{S}$. Treatment i occurs in period j of the sequence $s\boldsymbol{\varphi} + i - s\varphi_j$. The treatment in period $j+1$ of this sequence is $s\varphi_{j+1} + i - s\varphi_j = i + sv$. If $v \in \mathcal{S}$ then $\{sv : s \in \mathcal{S}\} = \mathcal{S}$, and so every ordered pair of treatments of the the form $(i, i+q)$, for i in $\mathrm{GF}(t)$ and q in $\mathcal{S}$, occurs exactly once in periods j and $j+1$. Otherwise, if $v \in \mathcal{N}$ then $\{sv : s \in \mathcal{S}\} = \mathcal{N}$, and so every ordered pair of treatments of the the form $(i, i+q)$, for i in $\mathrm{GF}(t)$ and q in $\mathcal{N}$, occurs exactly once in periods j and $j+1$.

Thus if $w - i \in \mathcal{S}$ then (i, w) occurs $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta)$ times in the design, while if $w - i \in \mathcal{N}$ then (i, w) occurs $f_{\mathcal{N}}(\boldsymbol{\varphi}^\delta)$ times. If $\boldsymbol{\varphi}$ is beautiful then the off-diagonal entries of $\mathbf{S}_d$ are in $\{k, k+1\}$ and $\mathbf{A}_d$ is the adjacency matrix of one of the doubly regular tournaments defined by $\mathcal{S}$ or $\mathcal{N}$. Hence d is a CWBD.

Example 4. Let $t = 7$ and $\boldsymbol{\varphi} = (3, 1, 0, 2, 6, 4, 5)$, where the entries are the integers modulo 7. Then $\boldsymbol{\varphi}^\delta = (5, 6, 2, 4, 5, 1, 5)$. Here $\mathcal{S} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5, 6\}$, and so $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta) = 3$ and $f_{\mathcal{N}}(\boldsymbol{\varphi}^\delta) = 4$. Thus $\boldsymbol{\varphi}$ is beautiful. Hence Construction 2 gives a uniform CWBD for 7 treatments on 21 subjects in 7 periods.

Now let x be any primitive element of $\mathrm{GF}(t)$; that is, x is a generator of the cyclic group $(\mathrm{GF}(t)\setminus\{0\},\times)$. The even powers of x constitute $\mathcal{S}$, while the odd powers constitute $\mathcal{N}$. Let $\boldsymbol{\psi}$ be the sequence $(1,x,x^2,\ldots,x^{t-2})$. Then $\boldsymbol{\psi}$ contains each non-zero element of $\mathrm{GF}(t)$ exactly once. The entries in $\boldsymbol{\psi}^\delta$ are $x-1$, $x(x-1)$, …, $x^{t-3}(x-1)$ and $1-x^{t-2}$, which is $x^{t-2}(x-1)$. These are again all the non-zero elements of $\mathrm{GF}(t)$ exactly once, and so $f_{\mathcal{S}}(\boldsymbol{\psi}^\delta)=f_{\mathcal{N}}(\boldsymbol{\psi}^\delta)=k$.

Theorem 7. *Let t be a prime power congruent to* 3 *modulo* 4 *with $t>3$. If x is a primitive element of* $\mathrm{GF}(t)$ *and $\boldsymbol{\varphi}$ is obtained from $\boldsymbol{\psi}$ by replacing* $(1,x)$ *with* $(x,1,0)$*, then $\boldsymbol{\varphi}$ is beautiful.*

Proof. If $t>3$, the substitution removes $1-x^{-1}$, $x-1$ and x^2-x from $\boldsymbol{\psi}^\delta$, and replaces them in $\boldsymbol{\varphi}^\delta$ by $x-x^{-1}$, $1-x$, -1 and x^2. None of these is zero if $t>3$. Now, $-1\in\mathcal{N}$ and $x^2\in\mathcal{S}$. Since $x\in\mathcal{N}$, one of $x-1$ and $x(x-1)$ is in $\mathcal{S}$ and the other is in $\mathcal{N}$. Since $1-x^{-1}=(-x^{-1})(1-x)$ and $-x^{-1}\in\mathcal{S}$, the entries $1-x^{-1}$ and $1-x$ are either both in $\mathcal{S}$ or both in $\mathcal{N}$. Thus $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta)=f_{\mathcal{S}}(\boldsymbol{\psi}^\delta)+1=k+1$ if $x-x^{-1}\in\mathcal{S}$, while $f_{\mathcal{S}}(\boldsymbol{\varphi}^\delta)=f_{\mathcal{S}}(\boldsymbol{\psi}^\delta)=k$ if $x-x^{-1}\in\mathcal{N}$.

If $t=7$ then 3 is a primitive element. The construction in Theorem 7 gives the beautiful sequence $\boldsymbol{\varphi}$ in Example 4.

Theorems 6–7 show that there is a uniform CWBD for t treatments on $t(t-1)/2$ subjects in t periods whenever t is a prime power congruent to 3 modulo 4 and $t>3$. This covers $t=7$, 11, 19, 23, 27 and 31 for $t<35$.

5.2. Designs of Type II

For a design of Type II, we have $n=k$, where $2\leq k\leq t-2$. Also, condition (3.2) shows that $t-1$ divides $k(k-1)$. We need a $t\times t$ matrix $\mathbf{A}$ which has k entries equal to 1 in each row and column, and all other entries zero, in such a way that $\mathbf{A}'\mathbf{A}=\phi\mathbf{I}_t+\xi\mathbf{J}_t$ with $\phi=k(t-k)/(t-1)$ and $\xi=k(k-1)/(t-1)$. The matrix $\mathbf{A}$ can be regarded as the incidence matrix of a symmetric balanced incomplete-block design (BIBD) Δ: treatment i is in block j if and only if $\mathbf{A}_{ij}=1$. Given such a design Δ, Hall's Marriage Theorem (Bailey (2008), Cameron (1994), Hall (1935)) shows that the treatments and blocks can be labelled in such a way that the diagonal entries of $\mathbf{A}$ are all zero. Now our strategy is to find a known BIBD Δ of the appropriate size, label its blocks in such a way that the diagonal entries of $\mathbf{A}$ are all zero, and then try to find a CWBD which is uniform on subjects for which $\lambda=1$ and $\mathbf{S}'_d=\mathbf{A}_d=\mathbf{A}$.

The condition that $t-1$ divides $k(k-1)$ is not sufficient to guarantee the existence of a BIBD. The Bruck–Ryser–Chowla Theorem shows that some pairs (t,k) have no BIBD: see Cameron (1994). For $t<35$, the following pairs are excluded by this theorem: $(22,7)$, $(22,15)$, $(29,8)$, $(29,21)$, $(34,12)$ and $(34,22)$.

Some BIBDs can be constructed from difference sets, see Hall (1986). If $(\mathcal{G}, +)$ is a finite Abelian group and $\mathcal{P} \subset \mathcal{G}$, then $\mathcal{P}$ is called a *difference set* if every non-zero element of $\mathcal{G}$ occurs equally often among the differences $x - y$ for x and y in $\mathcal{P}$ with $x \neq y$. When $\mathcal{G}$ is the additive group of GF(t) and $t \equiv 3 \bmod 4$ then $\mathcal{S}$ and $\mathcal{N}$ are both difference sets. If $\mathcal{P}$ is a difference set then so is its complement $\overline{\mathcal{P}}$, and so is the set $\mathcal{P} + i = \{x + i : i \in \mathcal{P}\}$ for each i in $\mathcal{G}$. If $i \notin \mathcal{P}$ then $\mathcal{P} - i$ is a difference set that does not contain 0. In particular, when t is a prime power and $t \equiv 3 \bmod 4$ then $\overline{\mathcal{S}} - 1$ is a difference set with $k = (t+1)/2$ that does not contain 0.

To obtain a BIBD Δ from the difference set $\mathcal{P}$, label the treatments and blocks by the elements of $\mathcal{G}$, and put $\mathbf{A}_{ij} = 1$ if and only if $j - i \in \mathcal{P}$. As above, we can assume that $0 \notin \mathcal{P}$, and then the diagonal entries of $\mathbf{A}$ are all zero.

Difference sets give a generalization of Construction 1.

Construction 3. Suppose $\mathcal{P}$ is a difference set of size k in $\mathbb{Z}_t$, that $0 \notin \mathcal{P}$, and that all elements of $\mathcal{P}$ are coprime to t. Label the t treatments and the t periods by the elements of $\mathbb{Z}_t$, and the k subjects by the elements of $\mathcal{P}$. Define the design d by $d(\ell, u) = \ell u$ for ℓ in $\mathbb{Z}_t$ and u in $\mathcal{P}$. Then d is a CWBD which is uniform on subjects with $\lambda = 1$.

Example 5. When $t = 7$ and $k = 4$ we have the difference set $\overline{\mathcal{S}} - 1 = \{2, 4, 5, 6\}$. Then Construction 3 gives a CWBD for 4 subjects which is uniform on subjects.

Difference sets exist for many other values of t and k satisfying the divisibility conditions, see Baumert (1971) and Table 2 of Filipiak and Markiewicz (2012). For example, when $t = 13$ then $\{1, 2, 5, 7\}$ and $\{2, 3, 5, 7, 8, 9, 10, 11, 12\}$ are both difference sets in $\mathbb{Z}_{13}$. Construction 3 gives CWBDs that are uniform on subjects, one for 4 subjects and one for 9 subjects. When $t = 31$, $\{1, 2, 4, 9, 13, 19\}$ is a difference set in $\mathbb{Z}_{31}$. Thus Construction 3 gives CWBDs uniform on subjects, one for 6 subjects and one for 25 subjects.

A result of Mann (1964) shows that there is no difference set of size 9 or 16 for $\mathbb{Z}_{25}$. Theorems of Lander (1983) rule out difference sets of size k or $t - k$ for $\mathbb{Z}_t$ when (t, k) is $(16, 6)$, $(27, 13)$ or $(31, 10)$. There is a difference set of size 8 for $\mathbb{Z}_{15}$, but its elements are not all coprime to 15, so Construction 3 cannot be used. The same problem occurs for $k = 5$ and $k = 16$ when $t = 21$.

If $\mathbf{A}$ is symmetric then it can also be regarded as the adjacency matrix of an undirected graph. If $\mathbf{A}'\mathbf{A}$ is completely symmetric then every pair of distinct vertices have the same number of common neighbours. Such graphs were studied by Rudvalis (1971). If such a graph has a Hamiltonian decomposition then using each cycle once in each direction gives a CWBD which is uniform on subjects.

Example 6. The smallest such graph is the square lattice graph $L_2(4)$, which has 16 vertices and valency 6. Every pair of distinct vertices has exactly two common neighbours. The vertices form a 4×4 grid. There is an edge between i and j if $i \neq j$ but i and j are in the same row or i and j are in the same column.

Label the vertices row by row, so that the first row is $(1,2,3,4)$, and so on. Let π be the permutation $(2,3,4)(5,9,13)(6,11,16)(7,12,14)(8,10,15)$ of the vertices, which is an automorphism of $L_2(4)$. There is a Hamiltonian decomposition of $L_2(4)$ which is invariant under π. Using each of these cycles in both directions gives the design in Figure 2(a).

The Shrikhande graph is another graph with 16 vertices, valency 6, and the common-neighbour property, see Seidel (1968). Using GAP (2014), we found that it has a large number of Hamiltonian decompositions. Each gives a CWBD that cannot be obtained from the one in Figure 2(a) by renaming treatments.

Example 7. The Clebsch graph Ω is another such graph with 16 vertices, see Seidel (1968). It has valency 10, and every pair of distinct vertices has exactly 6 common neighbours. The vertices are the vectors of length 5 over GF(2) of even weight (equivalently, the treatments in the 2^{5-1} factorial design with defining contrast $ABCDE = I$); two vertices are joined if they differ in precisely two positions. The permutation π taking $(x_1, x_2, x_3, x_4, x_5)$ to $(x_2, x_3, x_4, x_5, x_1)$ is an automorphism of Ω.

Using GAP (2014), we found a very large number of Hamiltonian decompositions of Ω which are invariant under π (as in Example 6, it is sufficient to find a single Hamiltonian cycle which has no edges in common with any of its images under powers of π). For any one of these decompositions, using each cycle in both directions gives the required CWBD. One is shown in Figure 2(b), where vertex $(x_1, x_2, x_3, x_4, x_5)$ is identified as the integer $8x_1 + 4x_2 + 2x_3 + x_4 + 1$.

For $t > 16$, Rudvalis (1971) showed that the smallest value of t for which there exists a graph with the common-neighbour property is $t = 36$.

5.3. Designs of Type III

For a design of Type III, we consider $\mathbf{A}$ to be the adjacency matrix of a directed graph Ξ. Now $\lambda \neq 1$ and condition (3.3) is satisfied. However, neither $\mathbf{A}'\mathbf{A}$ nor $\mathbf{A} + \mathbf{A}'$ is completely symmetric, so at most one value of λ is possible for any given directed graph Ξ. As in Section 5.1, we build larger matrices from smaller ones.

Let $\mathbf{A}_1$ be the adjacency matrix of a doubly regular tournament Γ on r vertices, where $r = 4q+3$. Let $t = mr$, where $m \geq 2$, and put $\mathbf{A}_2 = \mathbf{J}_m \otimes (\mathbf{I}_r + \mathbf{A}_1) - \mathbf{I}_t$. Then $\mathbf{A}_2 + \mathbf{A}_2' = \mathbf{J}_m \otimes (\mathbf{J}_r + \mathbf{I}_r) - 2\mathbf{I}_t$ and $\mathbf{A}_2'\mathbf{A}_2 = (mq+m-1)\mathbf{J}_m \otimes (\mathbf{J}_r + \mathbf{I}_r) + \mathbf{I}_t$. Thus $\mathbf{A}_2$ satisfies condition (3.3) with $\lambda = m(q+1)$, $k = 2m(q+1) - 1$, and $n = m^2(4q+3)(q+1) - m(3q+2)$.

1	1	1	5	9	13
2	3	4	8	10	15
6	11	16	16	6	11
7	12	14	15	8	10
11	16	6	3	4	2
9	13	5	4	2	3
13	5	9	12	14	7
14	7	12	10	15	8
10	15	8	14	7	12
12	14	7	13	5	9
4	2	3	9	13	5
3	4	2	11	16	6
15	8	10	7	12	14
16	6	11	6	11	16
8	10	15	2	3	4
5	9	13	1	1	1

(a)

1	1	1	1	1	11	5	10	3	6
2	4	7	13	9	9	2	4	7	13
3	6	11	5	10	15	14	12	8	16
4	7	13	9	2	13	9	2	4	7
7	13	9	2	4	14	12	8	16	15
5	10	3	6	11	16	15	14	12	8
6	11	5	10	3	10	3	6	11	5
8	16	15	14	12	12	8	16	15	14
12	8	16	15	14	8	16	15	14	12
10	3	6	11	5	6	11	5	10	3
16	15	14	12	8	5	10	3	6	11
14	12	8	16	15	7	13	9	2	4
13	9	2	4	7	4	7	13	9	2
15	14	12	8	16	3	6	11	5	10
9	2	4	7	13	2	4	7	13	9
11	5	10	3	6	1	1	1	1	1

(b)

Figure 2. Two CWBDs for 16 treatments on n subjects in 16 periods which are uniform on the subjects: (a) $n = 6$; (b) $n = 10$.

Example 8. When $q = 0$ we may let $\mathbf{A}_1$ be the adjacency matrix of the doubly regular tournament defined by $\mathcal{S}$ in GF(3). When $m = 2$ then $t = 6$, $n = 8$, and $\mathbf{A}_2$ is $\mathbf{A}_d$ for the design d in Example 4.4 of Filipiak and Markiewicz (2012) with its treatments written in the order 1, 3, 5, 6, 2, 4.

Babai and Cameron (2000) give a doubling construction for what they call an *S-digraph*. Let $\mathbf{A}_1$ be the adjacency matrix of a doubly regular tournament Γ on r vertices, where $r = 4q + 3$. Put

$$\mathbf{A}_2 = \begin{bmatrix} 0 & \mathbf{1}_r' & 0 & \mathbf{0}_r' \\ \mathbf{0}_r & \mathbf{A}_1 & \mathbf{1}_r & \mathbf{A}_1' \\ 0 & \mathbf{0}_r' & 0 & \mathbf{1}_r' \\ \mathbf{1}_r & \mathbf{A}_1' & \mathbf{0}_r & \mathbf{A}_1 \end{bmatrix}$$

and $\mathbf{I}_{8q}^* = (\mathbf{J}_2 - \mathbf{I}_2) \otimes \mathbf{I}_{4q}$. Then the S-digraph Ξ has adjacency matrix $\mathbf{A}_2$. Now, $\mathbf{A}_2 + \mathbf{A}_2' = \mathbf{J}_{8q} - \mathbf{I}_{8q} - \mathbf{I}_{8q}^*$ and $\mathbf{A}_2'\mathbf{A}_2 = (4q+3)\mathbf{I}_{8q} + (2q+1)(\mathbf{J}_{8q} - \mathbf{I}_{8q} - \mathbf{I}_{8q}^*)$. Thus $\mathbf{A}_2$ satisfies condition (3.3) with $t = 8(q+1)$, $k = 4q+3$, $\lambda = 2(q+1)$, and $n = 16q^2 + 26q + 10$.

Example 9. If $q = 0$ and and $\mathbf{A}_1$ is as in Example 8, then this doubling construction gives a matrix $\mathbf{A}_2$ which, after relabelling of the treatments, is the matrix $\mathbf{A}_d$ for the design d in Example 4.3 of Filipiak and Markiewicz (2012).

If $t \notin \{4, 6\}$, then there is a CBD for t treatments with $n = t - 1$ and $\lambda = 1$, see Tillson (1980). Examples for $t = 3$, $t = 5$, and $7 \leq t \leq 16$ are given by

Azaïs, Bailey and Monod (1993). When $t = 4$ or $t = 6$ then there is a CBD with $n = 2(t-1)$, and there is no CWBD for $t = 4$ with $n \leq 5$. Thus Type III designs do not give a CWBD with fewer subjects than a CBD unless $t = 6$. However, in a situation like Example 9, the CWBD with 10 subjects gives lower variances of all treatment estimators than the CBD with 7 subjects, so there may still be some interest in constructing such designs. The methods in this section give two possible ways of constructing the matrix $\mathbf{A}_d$. A computer search should quickly find whether the corresponding digraph Ξ has a Hamiltonian decomposition. If so, this can be juxtaposed with $\lambda - 1$ copies of the relevant CBD to obtain a CWBD.

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