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A NOTE ON MINIMAL SUFFICIENCY

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Abstract: This paper shows that the classes of sufficient and minimal sufficient σ-fields are closed under products. The results are used to construct several examples that throw some light on the study of the relationship between minimal sufficiency and invariance, a problem posed in Hall, Wijsman and Ghosh (1965).

Key words and phrases: Completeness, invariance, minimal sufficiency.

1. Stability of Minimal Sufficiency under Products

From now on $(\Omega, \mathcal{A}, \mathcal{P})$ will denote a statistical experiment, i.e., $\mathcal{P}$ is a family of probability measures on the measurable space $(\Omega, \mathcal{A})$.

Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Recall that $\mathcal{B}$ is said to be sufficient if $\cap_{P\in\mathcal{P}} P(A|\mathcal{B}) \neq \emptyset$, for every $A \in \mathcal{A}$, where $P(A|\mathcal{B})$, the conditional probability of A given $\mathcal{B}$, is the set of all real and $\mathcal{B}$-measurable functions $g : (\Omega, \mathcal{B}) \to \mathbb{R}$ such that $P(A \cap B) = \int_B g dP$, for every $B \in \mathcal{B}$.

$\mathcal{B}$ is said to be minimal sufficient if it is sufficient and it is $\mathcal{P}$-contained in any other sufficient σ-field $\mathcal{C}$, in the sense that, for every $\mathcal{B}$-measurable set B there exists a $\mathcal{C}$-measurable set C such that the symmetric difference $B\triangle C$ is a $\mathcal{P}$-null set. In the dominated case it is known that there exists a privileged dominating probability (i.e., a probability measure of the form $P^* = \sum_{n=1}^{\infty} c_n P_n$ such that $c_n \geq 0$, $\sum_n c_n = 1$, $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}$ and $P \ll P^*$, $\forall P \in \mathcal{P}$); in this case, a sub-$\sigma$-field is minimal sufficient if and only if it is the least σ-field making measurable some versions of the densities dP/dP^*, $P \in \mathcal{P}$.

The σ-field $\mathcal{B}$ is said to be complete if every real and $\mathcal{B}$-measurable statistic $g : (\Omega, \mathcal{B}) \to \mathbb{R}$ such that $E_P(g) = 0$, for all $P \in \mathcal{P}$, is $\mathcal{P}$-equivalent to 0 (i.e., $P(g \neq 0) = 0$, $\forall P \in \mathcal{P}$).

It is well known that every sufficient and complete σ-field is minimal sufficient. Landers and Rogge (1976) shows the stability of the class of complete σ-fields under products. These results can be found, for example, in Pfanzagl (1994) or Lehmann (1986), where the reader is referred for other concepts and results to be used below. Our theorem shows that the classes of sufficient and minimal sufficient σ-fields exhibit the same property.

Theorem 1. *For $1 \le i \le n$, let $\mathcal{B}_i$ be a sufficient (resp., minimal sufficient) σ-field in the statistical experiment $(\Omega_i, \mathcal{A}_i, \mathcal{P}_i)$. Then the product σ-field $\prod_{i=1}^n \mathcal{B}_i$ is sufficient (resp., minimal sufficient) for the product statistical experiment*

$$\Big(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i, \prod_{i=1}^n \mathcal{P}_i\Big).$$

Proof. We write $(\Omega, \mathcal{A}, \mathcal{P}) = (\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{A}_i, \prod_{i=1}^n \mathcal{P}_i)$ and $\mathcal{B} = \prod_{i=1}^n \mathcal{B}_i$.

(i) (Sufficiency) Suppose that, for every $1 \le i \le n$, $\mathcal{B}_i$ is sufficient for $\mathcal{A}_i$. We show that $\mathcal{B}$ is sufficient for $\mathcal{A}$, i.e., for every $A \in \mathcal{A}$, $\cap_{P \in \mathcal{P}} P(A|\mathcal{B}) \neq \emptyset$. First, consider the case where A is a measurable rectangle $\prod_{i=1}^n A_i$. By hypothesis, for $1 \le i \le n$, there exists $f_{A_i} \in \cap_{P_i \in \mathcal{P}_i} P_i(A_i|\mathcal{B}_i)$. We show that the $\mathcal{B}$-measurable map $F_A(\omega_1, \dots, \omega_n) := \prod_{i=1}^n f_{A_i}(\omega_i)$ is in $\cap_{P \in \mathcal{P}} P(A|\mathcal{B})$, i.e., for every $B \in \mathcal{B}$,

$$P(A \cap B) = \int_B F_A dP, \quad \forall P \in \mathcal{P}. \tag{1}$$

Fubini's theorem readily shows that this is true when B is a measurable rectangle. The general case is obtained by proving that the class of all events $B \in \mathcal{B}$ that satisfy (1) is a Dynkin class containing the measurable rectangles. To extend (1) to any event $A \in \mathcal{A}$, we take

$$\mathcal{C} := \{A \in \mathcal{A} \;/\; \exists F_A : (\Omega, \mathcal{B}) \to \mathbb{R} \text{ such that (1) holds for every } B \in \mathcal{B}\}.$$

It is shown above that $\mathcal{C}$ contains the measurable rectangles. The proof that $\mathcal{C}$ is a Dynkin class is an easy consequence of the properties of conditional probability.

(ii) (Minimal sufficiency) Now suppose that $\mathcal{B}_i$ is minimal sufficient for $\mathcal{A}_i$, $1 \le i \le n$. Since $\mathcal{B}$ is sufficient, it is enough to prove that $\mathcal{B} \subset \overline{\mathcal{S}}^{\mathcal{P}}$ for every sufficient σ-field $\mathcal{S} \subset \mathcal{A}$, where $\overline{\mathcal{S}}^{\mathcal{P}}$ denotes the completion of $\mathcal{S}$ with the $\mathcal{P}$-null sets of $\mathcal{A}$.

Given $1 \le i \le n$, fix $P_j \in \mathcal{P}_j$ for $j \neq i$, and write $\mathcal{P}'_i := \{P_1\} \times \cdots \times \{P_{i-1}\} \times \mathcal{P}_i \times \{P_{i+1}\} \times \cdots \times \{P_n\}$. We consider any sub-$\sigma$-field $\mathcal{D}_i$ of $\mathcal{A}_i$ as a sub-σ-field of $\mathcal{A}$ by identifying it with $\prod_{i=1}^n \mathcal{C}_j$, where $\mathcal{C}_j = \{\emptyset, \Omega_j\}$ if $j \neq i$, and $\mathcal{C}_i = \mathcal{D}_i$ (in particular, the same notation $\mathcal{D}_i$ is used for both σ-fields). Consequently, an $\mathcal{A}_i$-measurable function $f(\omega_i)$ is identified with the map $F(\omega_1, \dots, \omega_n) := f(\omega_i)$.

Suppose that we have proved that

$$\mathcal{B}_i \subset \overline{\mathcal{S}}^{\mathcal{P}'_i} \cap \mathcal{A}_i. \tag{2}$$

Then $\mathcal{B} \subset \bigvee_{i=1}^n (\overline{\mathcal{S}}^{\mathcal{P}'_i} \cap \mathcal{A}_i) \subset \bigvee_{i=1}^n (\overline{\mathcal{S}}^{\mathcal{P}} \cap \mathcal{A}_i) \subset \overline{\mathcal{S}}^{\mathcal{P}} \bigcap \bigvee_{i=1}^n \mathcal{A}_i = \overline{\mathcal{S}}^{\mathcal{P}}$, where the symbol $\vee$ refers to the least σ-field containing the union. Hence it is enough to show (2). For the sake of simplicity, we prove the case $i = 1$. Now $P_j \in \mathcal{P}_j$, $j = 2, \ldots, n$, are supposed to be fixed.

For this, notice that $\mathcal{S}$, being sufficient for $\mathcal{P}$, is also sufficient for $\mathcal{P}'_1$, and hence $\overline{\mathcal{S}}^{\mathcal{P}'_1}$ is sufficient for $\mathcal{P}'_1$.

Now, we show that $\mathcal{A}_1$ is also sufficient for $\mathcal{P}'_1$, i.e., for all $A \in \mathcal{A}$ there exists an $\mathcal{A}_1$-measurable function f_A such that

$$\Big(\prod_{j=1}^{n} P_j\Big)(A \cap C) = \int_C f_A d\Big(\prod_{j=1}^{n} P_j\Big), \quad \forall C \in \mathcal{A}_1, \forall P_1 \in \mathcal{P}_1.$$

It is easy to show that, when A is the measurable rectangle $A_1 \times \cdots \times A_n$, the $\mathcal{A}_1$-measurable map $f_A := I_{A_1} \cdot P_2(A_2) \cdots P_n(A_n)$ works. The proof is extended to any event $A \in \mathcal{A}$ by showing that the class $\mathcal{C}$ of all events $A \in \mathcal{A}$ such that $\cap_{P' \in \mathcal{P}'_1} P'(A|\mathcal{A}_1) \neq \emptyset$ is a Dynkin class.

According to Heyer (1982, Theorem 5.5), $\overline{\mathcal{S}}^{\mathcal{P}'_1} \cap \mathcal{A}_1$ is sufficient for $(\Omega, \mathcal{A}, \mathcal{P}'_1)$ and, by the identification we are making throughout the proof, it is also sufficient for $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$.

Since $\mathcal{B}_1$ is minimal sufficient, we have that $\mathcal{B}_1 \subset \overline{\mathcal{S}}^{\mathcal{P}'_1} \cap \mathcal{A}_1$, as desired.

Remarks. (1) The previous result rests strongly upon Theorem 5.5 of Heyer (1982), whose proof is far from being trivial. For exponential statistical experiments it is possible to find a simple proof of the closure of minimal sufficiency under products, which is stated now in terms of statistics. First, we recall some facts about exponential families. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be an exponential statistical experiment. Hence, the densities of the probability measures of the family $\mathcal{P}$ with respect to some σ-finite measure admit the expression

$$f_P(x) = C(P) \cdot \exp\Big\{\sum_{i=1}^{m} Q_i(P) T_i(\omega)\Big\} \cdot h(\omega), \quad \omega \in \Omega,$$

where $Q_1, \ldots, Q_m : \mathcal{P} \to \mathbb{R}$ and $T_1, \ldots, T_m : (\Omega, \mathcal{A}) \to \mathbb{R}$. It is known (see, for example, Pfanzagl (1994)) that the m-dimensional statistic $T = (T_1, \ldots, T_m)$, which is sufficient in any case, is minimal sufficient if $Q_1, \ldots, Q_m$ are affinely independent (this means that $a_0 = a_1 = \cdots = a_m = 0$ if $a_0 + \sum_{i=1}^m a_i Q_i(P) = 0$, for all $P \in \mathcal{P}$); we refer to this as an exponential statistical experiment with affinely independent coefficients. Moreover, it is known that T is complete if the set $\{(Q_1(P), \ldots, Q_m(P)) \colon P \in \mathcal{P}\}$ has non-empty interior in $\mathbb{R}^m$, and that T is not complete if $Q_1, \ldots, Q_m$ are polynomial dependent and the distributions of T with respect to every $P \in \mathcal{P}$ are dominated by the Lebesgue measure in $\mathbb{R}^m$ (see Wijsman (1958)). Now we are prepared for the proof.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ and $(\Omega', \mathcal{A}', \mathcal{P}')$ be two exponential statistical experiments with affinely independent coefficients. With obvious notations, if

$$a_0 + \sum_{i=1}^{m} a_i Q_i(P) + \sum_{i'=1}^{m'} a'_{i'} Q'_{i'}(P') = 0, \quad \forall P \in \mathcal{P}, \forall P' \in \mathcal{P}',$$

then, for every $P \in \mathcal{P}$, we have $a_0 + \sum_{i=1}^{m} a_i Q_i(P) = 0$ and $a'_1 = \cdots = a'_{m'} = 0$ by the affine independence in the second experiment; from this and the affine independence in the first one, we also obtain $a_0 = a_1 = \cdots = a_m = 0$. This shows that the product experiment $(\Omega \times \Omega', \mathcal{A} \times \mathcal{A}', \mathcal{P} \times \mathcal{P}')$ is also an exponential statistical experiment with affinely independent coefficients.

(2) Andersen (1967), in a different context and under much more restrictive hypotheses, also deals with the problem solved by the theorem above.

2. Minimal Sufficiency and Invariance

First, let us briefly recall the definitions of some well known concepts of invariance. A transformation on $(\Omega, \mathcal{A}, \mathcal{P})$ is a bimeasurable bijection from $(\Omega, \mathcal{A})$ onto itself. Let G be a group of transformations on $(\Omega, \mathcal{A})$; we say that the statistical experiment is G-invariant (resp., strongly G-invariant) if $P^g \in \mathcal{P}$ (resp., $P^g = P$) for every $P \in \mathcal{P}$ and every $g \in G$, where P^g denotes the probability distribution of g with respect to P, i.e., $P^g(A) := P(g^{-1}(A))$, for $A \in \mathcal{A}$. An event $A \in \mathcal{A}$ is said to be G-invariant (resp., almost-G-invariant) if $g^{-1}(A) = A$ (resp., if $g^{-1}(A)$ differs from A in a $\mathcal{P}$-null set); we write $\mathcal{A}_G$ (resp., $\mathcal{A}_A$) for the σ-field of all G-invariant (resp., almost-G-invariant) events. A sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ is said to be G-stable if $g^{-1}(\mathcal{B}) \subset \mathcal{B}$, for every $g \in G$.

For the study of the relationship between sufficiency and invariance we refer to Hall, Wijsman and Ghosh (1965) (see also Berk (1972), Nogales and Oyola (1996) and Berk, Nogales and Oyola (1996)), whose main result is the following.

Theorem 2. (Hall, Wijsman and Ghosh (1965)) *Let $\mathcal{B}$ be a G-stable and sufficient σ-field. If $\mathcal{B} \cap \mathcal{A}_A$ is $\mathcal{P}$-contained in $\mathcal{B} \cap \mathcal{A}_G$, then $\mathcal{B} \cap \mathcal{A}_G$ is sufficient for $\mathcal{A}_G$.*

They also consider the following question: if G is a group of transformations leaving invariant the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$, and $\mathcal{B}$ is a G-stable and minimal sufficient σ-field, is $\mathcal{B} \cap \mathcal{A}_G$ minimal sufficient for $\mathcal{A}_G$? They solve this question in the negative by means of the next example. It is noted there that the answer is positive for sufficiency and completeness.

Example 1. (Hall, Wijsman and Ghosh (1965)) The statistical experiment $(\mathbb{R}^n, \mathcal{R}^n, \{N(c\sigma, \sigma^2)^n \colon \sigma > 0\})$, corresponding to a sample of size n of a normal distribution with known coefficient of variation, remains invariant under the group $G := \{g_k \colon k > 0\}$, where $g_k(x_1, \ldots, x_n) := (kx_1, \ldots, kx_n)$. It is well known that $(\bar{X}, S)$ is a minimal sufficient statistic for this experiment and that $T := (X_1/X_n, \ldots, X_{n-1}/X_n, \operatorname{sign}(X_n))$ is a G-invariant maximal statistic. Hence, $\mathcal{A}_G$ is the induced σ-field $T^{-1}(\mathcal{R}^n)$ of T. It is readily shown that the σ-field $\mathcal{B}$ induced by $(\bar{X}, S)$ is G-stable and that the statistic $\bar{X}/S$ induces the σ-field $\mathcal{B} \cap \mathcal{A}_G$. But $\mathcal{B} \cap \mathcal{A}_G$ is not minimal sufficient for $\mathcal{A}_G$, as this σ-field is ancillary and, hence, the trivial σ-field $\{\emptyset, \mathbb{R}^n\}$ is sufficient for it.

Here we give an example showing that the answer to the above question in the discrete case is also negative.

Example 2. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{A}$ be the set of all subsets of Ω and $\mathcal{P} := \{P_1, P_2\}$, where $P_1 := (1/6)(\varepsilon_1 + \varepsilon_2) + (1/4)(\varepsilon_3 + \varepsilon_4) + (1/12)(\varepsilon_5 + \varepsilon_6)$ and $P_2 := (1/12)(\varepsilon_1 + \varepsilon_2) + (1/4)(\varepsilon_3 + \varepsilon_4) + (1/6)(\varepsilon_5 + \varepsilon_6)$, where ε_i stands for the probability measure degenerate at $\{i\}$. Let g be the permutation $(6, 5, 4, 3, 2, 1)$, Id be the identity map on Ω and $G := \{\mathrm{Id}, g\}$. The least σ-field $\mathcal{B}$ containing the sets $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ is minimal sufficient, since $P^* = (1/2)(P_1 + P_2)$ is a privileged dominating probability and

$$\frac{dP_1}{dP^*} = \frac{4}{3} I_{\{1,2\}} + I_{\{3,4\}} + \frac{2}{3} I_{\{5,6\}} \quad \text{and} \quad \frac{dP_2}{dP^*} = \frac{2}{3} I_{\{1,2\}} + I_{\{3,4\}} + \frac{4}{3} I_{\{5,6\}}.$$

$\mathcal{B}$ is also G-stable and non-complete (take a non-null function $f : \Omega \to \mathbb{R}$ such that $f(1) = f(2) = f(5) = f(6) = 1$ and $f(3) = f(4) = -1$). $\mathcal{A}_G$ is the least σ-field containing the sets $\{1, 6\}$, $\{3, 4\}$ and $\{2, 5\}$. $\mathcal{B} \cap \mathcal{A}_G$ is the least σ-field containing the set $\{3, 4\}$ but it is not minimal sufficient, since $\mathcal{A}_G$ is ancillary and, hence, the trivial σ-field $\{\emptyset, \Omega\}$ is sufficient for it.

In the light of the two examples below, we can pose the question if only non-trivial ancillary invariant σ-fields can be exhibited as counterexamples. That is, if $\mathcal{A}_G$ is not ancillary and $\mathcal{B}$ is minimal sufficient, is $\mathcal{B} \cap \mathcal{A}_G$ minimal sufficient for $\mathcal{A}_G$? To show that the answer remains negative, we use Theorem 1.

Example 3. Let $(\Omega', \mathcal{A}', \mathcal{P}')$ (resp., $(\Omega'', \mathcal{A}'', \mathcal{P}'')$) be a dominated statistical experiment invariant under the action of a group of transformations G' (resp., G'') and $\mathcal{B}'$ (resp., $\mathcal{B}''$) be a G'-stable and minimal sufficient (resp., G''-stable and minimal sufficient) sub-σ-field of it. Let us suppose that $\mathcal{B}' \cap \mathcal{A}'_{G'}$ is not minimal sufficient for $\mathcal{A}'_{G'}$, with obvious notations, because $\mathcal{A}'_{G'}$ is ancillary and $\mathcal{B}' \cap \mathcal{A}'_{G'}$ is not $\mathcal{P}'$-equivalent to the trivial σ-field.

Let $(\Omega, \mathcal{A}, \mathcal{P}) = (\Omega', \mathcal{A}', \mathcal{P}') \times (\Omega'', \mathcal{A}'', \mathcal{P}'')$, $\mathcal{B} = \mathcal{B}' \times \mathcal{B}''$ and $G = \{(g', g'') \colon g' \in G', g'' \in G''\}$, where $(g', g'')(\omega', \omega'') = (g'(\omega'), g''(\omega''))$. Since $(\Omega, \mathcal{A}, \mathcal{P})$ is G-invariant and $\mathcal{B}$ is G-stable and minimal sufficient by Theorem 1, it is enough to take the starred objects in such a way that the following propositions hold: (i) $(\mathcal{A}' \times \mathcal{A}'')_G = \mathcal{A}'_{G'} \times \mathcal{A}''_{G''}$, (ii) $\mathcal{B} \cap \mathcal{A}_G = (\mathcal{B}' \cap \mathcal{A}'_{G'}) \times (\mathcal{B}'' \cap \mathcal{A}''_{G''})$ and (iii) $\mathcal{A}''_{G''}$

is not ancillary. In this case, $\mathcal{A}_G$ is not ancillary, because $\mathcal{A}''_{G''}$ is not; $\mathcal{B} \cap \mathcal{A}_G$ is not minimal sufficient since $(\{\emptyset, \Omega'\} \times \mathcal{B}'') \cap \mathcal{A}_G$ is sufficient for $\mathcal{A}_G$, as Theorems 1 and 2 show.

Take $(\Omega', \mathcal{A}', \mathcal{P}')$ as the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$ of Example 2, and let $\Omega'' = \{a, b, c\}$, $\mathcal{A}''$ be the set of all subsets of Ω'' and $\mathcal{P}'' = \{P''_1, P''_2\}$, where $P''_1 = (1/4)(\varepsilon_a + \varepsilon_b) + (1/2)\varepsilon_c$ and $P''_2 = (1/5)(\varepsilon_a + \varepsilon_b) + (3/5)\varepsilon_c$. Let $\mathcal{B}''$ be the smallest σ-field containing the set $\{a, b\}$ and $G'' := \{\mathrm{Id}'', g''\}$, where Id'' denotes the identity map on Ω'' and g'' the permutation (b, a, c). Then $\mathcal{A}''_{G''} = \mathcal{B}''$ is minimal sufficient and (i)–(iii) are readily verified:

(i) It is clear that $(\mathcal{A}' \times \mathcal{A}'')_G \supset \mathcal{A}'_{G'} \times \mathcal{A}''_{G''}$, since $(g', g'')(A' \times A'') \subset A' \times A''$ for every g', g'', A', A''. Moreover, as $\mathcal{A}'_{G'} \times \mathcal{A}''_{G''} = \sigma(\{1,6\} \times \{a,b\}, \{1,6\} \times \{c\}, \{3,4\} \times \{a,b\}, \{3,4\} \times \{c\}, \{2,5\} \times \{a,b\}, \{2,5\} \times \{c\})$, it is easy to verify that, if $A \in (\mathcal{A}' \times \mathcal{A}'')_G$ and $(x, y) \in A$, the atom of (x, y) in $\mathcal{A}'_{G'} \times \mathcal{A}''_{G''}$ is also contained in A, which shows that $A \in \mathcal{A}'_{G'} \times \mathcal{A}''_{G''}$.

(ii) It is easy to see that $\mathcal{B} \cap \mathcal{A}_G \supset (\mathcal{B}' \cap \mathcal{A}'_{G'}) \times (\mathcal{B}'' \cap \mathcal{A}''_{G''})$. For the reverse inclusion, it is enough to prove that, if $B \in \mathcal{B} \cap \mathcal{A}_G$ and $(x, y) \in B$ for $x \in \{1, 2, 5, 6\}$ and $y \in \{a, b\}$, then $\{1, 2, 5, 6\} \times \{a, b\} \subset B$; but this follows from $B \in \mathcal{B}' \times \mathcal{B}'' = \sigma(\{1,2\} \times \{a,b\}, \{3,4\} \times \{a,b\}, \{5,6\} \times \{a,b\}, \{1,2\} \times \{c\}, \{3,4\} \times \{c\}, \{5,6\} \times \{c\})$ and $B \in \mathcal{A}'_{G'} \times \mathcal{A}''_{G''} = \sigma(\{1,6\} \times \{a,b\}, \{3,4\} \times \{a,b\}, \{2,5\} \times \{a,b\}, \{1,6\} \times \{c\}, \{3,4\} \times \{c\}, \{2,5\} \times \{c\})$.

(iii) $\mathcal{A}''_{G''}$ is not ancillary since it is sufficient and $\mathcal{P}''$ is not a singleton.

The next example shows a positive and non-trivial situation where minimal sufficiency is inherited after an invariance reduction, i.e., a case where $\mathcal{B}$ is minimal sufficient, G-stable, non-complete while $\mathcal{B} \cap \mathcal{A}_G$ is minimal sufficient for $\mathcal{A}_G$.

Example 4. Let $(\Omega, \mathcal{A}, \mathcal{P})$, $\mathcal{B}$ and Id be as in Example 2. Consider the group $\Gamma := \{\mathrm{Id}, \gamma\}$, γ being the permutation $(2, 1, 3, 4, 6, 5)$. It is clear that Γ leaves invariant this statistical experiment. Moreover $\mathcal{B}$ is minimal sufficient, Γ-stable and non-complete, and we have that $\mathcal{A}_\Gamma$ is the σ-field generated by the sets $\{1, 2\}$, $\{3\}$, $\{4\}$ and $\{5, 6\}$. Since $\mathcal{B} \subset \mathcal{A}_\Gamma$, $\mathcal{B} = \mathcal{B} \cap \mathcal{A}_\Gamma$ is minimal sufficient for $\mathcal{A}_\Gamma$.

Remark. The example above is, in fact, more general. It is known that if $(\Omega, \mathcal{A}, \mathcal{P})$ is dominated and strongly G-invariant and there exists a minimal sufficient sub-σ-field $\mathcal{B}$, then $\mathcal{B} \subset \mathcal{A}_A$; see, for example, Ghosh (1988, Chap.VIII). Hence, in this case, minimal sufficiency is trivially inherited after an invariance reduction (when it is understood as restricting to the almost invariant σ-field, instead of to the invariant one).

Theorem 1 above allows us to construct a positive and non-trivial example where the condition $\mathcal{B} \subset \mathcal{A}_G$ is not verified.

Example 5. Let $(\Omega, \mathcal{A}, \mathcal{P})$, $\mathcal{B}$ and Γ be as in the previous example. Let $\Omega' = \{a, b, c\}$, $\mathcal{A}'$ be the σ-field of all its subsets, $P'_1 = \varepsilon_a$, $P'_2 = \varepsilon_b$ and $P'_3 = \varepsilon_c$. $\mathcal{B}' = \mathcal{A}'$ is a sufficient and complete σ-field and hence is minimal sufficient. Let $\Gamma' = \{\mathrm{Id}', \gamma'\}$, where Id' is the identity map on Ω' and γ' is the permutation (b, a, c). Then $\mathcal{A}'_{\Gamma'} = \sigma(\{a, b\})$. The product experiment remains invariant under the action of the group $\Gamma \times \Gamma'$ and the σ-field of the invariant events is $(\mathcal{A} \times \mathcal{A}')_{\Gamma \times \Gamma'} = \mathcal{A}_\Gamma \times \mathcal{A}'_{\Gamma'}$. $\mathcal{B} \times \mathcal{B}'$ is a $\Gamma \times \Gamma'$-stable, minimal sufficient and non-complete σ-field that does not contain nor is contained in $(\mathcal{A} \times \mathcal{A}')_{\Gamma \times \Gamma'}$. Moreover, it holds that $(\mathcal{B} \times \mathcal{B}') \cap (\mathcal{A} \times \mathcal{A}')_{\Gamma \times \Gamma'} = (\mathcal{B} \cap \mathcal{A}_\Gamma) \times (\mathcal{B}' \cap \mathcal{A}'_{\Gamma'})$. Since $\mathcal{B}' \cap \mathcal{A}'_{\Gamma'}$ is minimal sufficient, Theorem 1 shows that $(\mathcal{B} \times \mathcal{B}') \cap (\mathcal{A} \times \mathcal{A}')_{\Gamma \times \Gamma'}$ is minimal sufficient for $(\mathcal{A} \times \mathcal{A}')_{\Gamma \times \Gamma'}$.

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