Statistica Sinica: Supplement

# Hypothesis Testing for the Covariance Matrix in High-Dimensional Transposable Data with Kronecker Product Dependence Structure

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## Supplementary Material

It contains useful identities (S1), moment derivations (S2), computationally cheap expressions for the proposed test statistics (S3), proofs or sketches of proofs for Theorems 1-7 of the manuscript (S4–S8), additional simulation results (S9) and R code to run the analysis in Section 5 of the manuscript (S11). Herein, we assume that $\mathbf{M} = \mathbf{0}$ because the proposed test statistics are invariant to location transformations.

## S1 Useful identities

We list six identities (P1-P6) that hold under the matrix-valued nonparametric model (2.1). It is further assumed that the matrices $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ in P1-P6 are symmetric whose dimensions are meaningful for each of the operations considered.

P1: $E(\boldsymbol{Z}_i'\mathbf{B}_1\boldsymbol{Z}_i) = \text{tr}(\mathbf{B}_1)\mathbf{I}_c$

P2: $E(\boldsymbol{Z}_i\mathbf{B}_2\boldsymbol{Z}_i') = \text{tr}(\mathbf{B}_2)\mathbf{I}_r$

P3: $E[\text{tr}^2(\mathbf{B}_1\boldsymbol{Z}_i\mathbf{B}_2\boldsymbol{Z}_j')] = \text{tr}(\mathbf{B}_1^2)\text{tr}(\mathbf{B}_2^2)$

P4: $$E[\text{tr}(\boldsymbol{Z}_i'\mathbf{B}_1\boldsymbol{Z}_i\mathbf{B}_2\boldsymbol{Z}_i'\mathbf{B}_1\boldsymbol{Z}_i\mathbf{B}_3)] = \text{tr}^2(\mathbf{B}_1)\text{tr}(\mathbf{B}_2\mathbf{B}_3) + \text{tr}(\mathbf{B}_1^2)\text{tr}(\mathbf{B}_2)\text{tr}(\mathbf{B}_3) + \text{tr}(\mathbf{B}_1^2)\text{tr}(\mathbf{B}_2\mathbf{B}_3) + B\text{tr}(\boldsymbol{\Delta}_{\mathbf{B}_1}^2)\text{tr}(\mathbf{B}_2 \circ \mathbf{B}_3)$$

P5: $$E\left\{\text{tr}\left[(\mathbf{B}_1\boldsymbol{Z}_i\mathbf{B}_2\boldsymbol{Z}_i'\mathbf{B}_1) \circ (\mathbf{B}_1\boldsymbol{Z}_i\mathbf{B}_2\boldsymbol{Z}_i'\mathbf{B}_1)\right]\right\} = B\text{tr}(\boldsymbol{\Delta}_{\mathbf{B}_2}^2)\text{tr}\left[(\mathbf{B}_1 \circ \mathbf{B}_1)^2\right] + [2\text{tr}(\mathbf{B}_2^2) + \text{tr}^2(\mathbf{B}_2)]\text{tr}(\boldsymbol{\Delta}_{\mathbf{B}_1^2}^2)$$

P6: $$\begin{aligned} &E\left(X_{ir_1c_1}X_{ir_2c_2}X_{ir_3c_3}X_{ir_4c_4}\right) \\ &= \left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_1r_2}\left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_3r_4}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_1c_2}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_3c_4} + \left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_1r_3}\left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_2r_4}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_1c_3}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_2c_4} \\ &+ \left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_1r_4}\left(\boldsymbol{\Sigma}_{\text{R}}\right)_{r_2r_3}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_1c_4}\left(\boldsymbol{\Sigma}_{\text{C}}\right)_{c_2c_3} \\ &+ B\sum_{r_5=1}^{r}\left(\boldsymbol{\Sigma}_{\text{R}}^{1/2}\right)_{r_1r_5}\left(\boldsymbol{\Sigma}_{\text{R}}^{1/2}\right)_{r_2r_5}\left(\boldsymbol{\Sigma}_{\text{R}}^{1/2}\right)_{r_3r_5}\left(\boldsymbol{\Sigma}_{\text{R}}^{1/2}\right)_{r_4r_5} \\ &\times \sum_{c_5=1}^{c}\left(\boldsymbol{\Sigma}_{\text{C}}^{1/2}\right)_{c_1c_5}\left(\boldsymbol{\Sigma}_{\text{C}}^{1/2}\right)_{c_2c_5}\left(\boldsymbol{\Sigma}_{\text{C}}^{1/2}\right)_{c_3c_5}\left(\boldsymbol{\Sigma}_{\text{C}}^{1/2}\right)_{c_4c_5} \end{aligned}$$

Properties of the trace of the Kronecker and Hadamard product and of the matrix product of two or more matrices were also employed. These properties can be found, for example, in Harville (1997) and Seber (2008).

## S2 Derivation of moments

It can be shown that $E(Y_{1N}) = \text{tr}(\boldsymbol{\Sigma}_{\text{R}})$, $E(Y_{2N}) = \text{tr}(\boldsymbol{\Sigma}_{\text{R}}^2)$, $E(Y_{3N}) = \text{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\text{R}}}^2\right)$ and $E(Y_{4N}) = E(Y_{5N}) = E(Y_{6N}) = E(Y_{7N}) = E(Y_{8N}) = 0$.

For the variances, it can be shown that

$$\mathrm{Var}(Y_{1N}) = \frac{2}{N}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2) + \frac{B}{N}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}^2\right) ,$$

$$\begin{aligned}
\mathrm{Var}(Y_{2N}) = & \frac{8}{N}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4) + \frac{4}{N(N-1)}\frac{\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^4}\left[\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{R}}^2) + \mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4)\right] \\
& + \frac{4B}{N}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}^2}^2\right)\left[1 + \frac{2}{N-1}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\right] \\
& + \frac{2B^2}{N(N-1)}\frac{\mathrm{tr}^2\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^4}\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})^2\right] ,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left(Y_{3N}\right) = & \frac{8}{N}\frac{\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{R}}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\mathbf{\Sigma}_{\mathrm{R}}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\right) \\
& + \frac{8}{N(N-1)}\frac{\mathrm{tr}^2\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^4}\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})^2\right] \\
& + \frac{4B}{N}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left[\left(\mathbf{\Sigma}_{\mathrm{R}}^{1/2}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\mathbf{\Sigma}_{\mathrm{R}}^{1/2}\right) \circ \left(\mathbf{\Sigma}_{\mathrm{R}}^{1/2}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\mathbf{\Sigma}_{\mathrm{R}}^{1/2}\right)\right] \\
& + \frac{8B}{N(N-1)}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\frac{\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})\left(\mathbf{\Sigma}_{\mathrm{R}}^{1/2} \circ \mathbf{\Sigma}_{\mathrm{R}}^{1/2}\right)^2\right] \\
& + \frac{2B^2}{N(N-1)}\frac{\mathrm{tr}^2\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^4}\mathrm{tr}\left[\left(\mathbf{\Sigma}_{\mathrm{R}}^{1/2} \circ \mathbf{\Sigma}_{\mathrm{R}}^{1/2}\right)^4\right] ,
\end{aligned}$$

$$\mathrm{Var}(Y_{4N}) = \frac{2}{N(N-1)}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2) ,$$

$$\begin{aligned}
\mathrm{Var}(Y_{5N}) = & \frac{2}{N(N-1)}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\left[\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4) + \frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\frac{\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{R}}^2) + \mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4)}{(N-2)(N-3)}\right] \\
& + \frac{2B}{N(N-1)(N-2)}\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^2}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}^2}^2\right) ,
\end{aligned}$$

$$\mathrm{Var}(Y_{6N}) = \frac{\left[\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{C}}^2) + \mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^4)\right]\left[\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{R}}^2) + \mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4)\right] + 2\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^4)\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^4)}{c^4 N(N-1)(N-2)(N-3)} ,$$

$$\begin{aligned}
\mathrm{Var}(Y_{7N}) = & \frac{2}{N(N-1)}\frac{\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{R}}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\mathbf{\Sigma}_{\mathrm{R}}\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{R}}}\right) \\
& + \frac{4}{N(N-1)(N-2)}\frac{\mathrm{tr}^2\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^4}\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})^2\right] \\
& + \frac{2B}{N(N-1)(N-2)}\frac{\mathrm{tr}\left(\mathbf{\Delta}_{\mathbf{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\frac{\mathrm{tr}\left(\mathbf{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})\left(\mathbf{\Sigma}_{\mathrm{R}}^{1/2} \circ \mathbf{\Sigma}_{\mathrm{R}}^{1/2}\right)^2\right] ,
\end{aligned}$$

$$\mathrm{Var}(Y_{8N}) = \frac{8}{N(N-1)(N-2)(N-3)}\left[\frac{\mathrm{tr}^2(\mathbf{\Sigma}_{\mathrm{C}}^2)}{c^4} + 2\frac{\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{C}}^4)}{c^4}\right]\mathrm{tr}\left[(\mathbf{\Sigma}_{\mathrm{R}} \circ \mathbf{\Sigma}_{\mathrm{R}})^2\right] .$$

For the covariances, it can be shown that

$$\begin{aligned}
\mathrm{Cov}(Y_{1N}, Y_{4N}) =& \mathrm{Cov}(Y_{1N}, Y_{5N}) = \mathrm{Cov}(Y_{1N}, Y_{6N}) = 0\,,\\
\mathrm{Cov}(Y_{2N}, Y_{5N}) =& \mathrm{Cov}(Y_{2N}, Y_{6N}) = \mathrm{Cov}(Y_{2N}, Y_{7N}) = \mathrm{Cov}(Y_{2N}, Y_{8N}) = 0\,,\\
\mathrm{Cov}(Y_{3N}, Y_{5N}) =& \mathrm{Cov}(Y_{3N}, Y_{6N}) = \mathrm{Cov}(Y_{3N}, Y_{7N}) = \mathrm{Cov}(Y_{3N}, Y_{8N}) = 0\,,\\
\mathrm{Cov}(Y_{5N}, Y_{6N}) =& \mathrm{Cov}(Y_{5N}, Y_{8N}) = \mathrm{Cov}(Y_{6N}, Y_{7N}) = \mathrm{Cov}(Y_{7N}, Y_{8N}) = 0\,,\\
\mathrm{Cov}(Y_{1N}, Y_{2N}) =& \frac{4}{N}\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^2}\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^3) + \frac{2B}{N}\frac{\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2 \circ \boldsymbol{\Sigma}_{\mathrm{R}})\,,\\
\mathrm{Cov}\left(Y_{2N}, Y_{3N}\right) =& \frac{8}{N(N-1)}\frac{\mathrm{tr}^2\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^4}\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}^2}^2\right)\\
&+ \frac{8}{N}\frac{\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{R}}^3\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\right)\\
&+ \frac{4B}{N}\frac{\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}^2}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\right)\\
&+ \frac{2B^2}{N(N-1)}\frac{\mathrm{tr}^2\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2\right)}{c^4}\mathrm{tr}\left[\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\left(\boldsymbol{\Sigma}_{\mathrm{R}} \circ \boldsymbol{\Sigma}_{\mathrm{R}}\right)\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\right]\\
&+ \frac{4B}{N(N-1)}\frac{\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\frac{\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\mathrm{tr}\left[\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{3/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{3/2}\right)\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\right]\,,\\
\mathrm{Cov}\left(Y_{2N}, Y_{4N}\right) =& \frac{\left[E(Z_{ir_1c_1}^3)\right]^2}{N(N-1)}\frac{\mathrm{tr}^2(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}\boldsymbol{\Sigma}_{\mathrm{C}}^{1/2})}{c^3}\mathrm{tr}^2(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2})\,,\\
\mathrm{Cov}(Y_{5N}, Y_{7N}) =& \frac{2}{N(N-1)}\frac{\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{R}}^3\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\right)\\
&+ \frac{4}{N(N-1)(N-2)}\frac{\mathrm{tr}^2\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^4}\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{R}}^2 \circ \boldsymbol{\Sigma}_{\mathrm{R}}^2\right)\\
&+ \frac{2B}{N(N-1)(N-2)}\frac{\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2\right)}{c^2}\frac{\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}^2\right)}{c^2}\mathrm{tr}\left[\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\left(\boldsymbol{\Sigma}_{\mathrm{R}}^{3/2} \circ \boldsymbol{\Sigma}_{\mathrm{R}}^{3/2}\right)\right]\,,\\
\mathrm{Cov}(Y_{6N}, Y_{8N}) =& \frac{8}{N(N-1)(N-2)(N-3)}\frac{\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4}\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2 \circ \boldsymbol{\Sigma}_{\mathrm{R}}^2)\\
&+ \frac{16}{N(N-1)(N-2)(N-3)}\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^4)}{c^4}\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2 \circ \boldsymbol{\Sigma}_{\mathrm{R}}^2)
\end{aligned}$$

## S3 Alternative formulae

Some algebraic manipulation shows that

$$
\begin{aligned}
T_{2N} =& Y_{2N} - 2Y_{5N} + Y_{6N} \\
=& \frac{1}{c^2 P_2^N} \sum_{i,j}^{*} \mathrm{tr}(\boldsymbol{X}_i \boldsymbol{X}_i' \boldsymbol{X}_j \boldsymbol{X}_j') - 2\frac{1}{c^2 P_3^N} \sum_{i,j,k}^{*} \mathrm{tr}(\boldsymbol{X}_i \boldsymbol{X}_i' \boldsymbol{X}_j \boldsymbol{X}_k') \\
& + \frac{1}{c^2 P_4^N} \sum_{i,j,k,l}^{*} \mathrm{tr}(\boldsymbol{X}_i \boldsymbol{X}_j' \boldsymbol{X}_k \boldsymbol{X}_l') \\
=& \frac{1}{c^2 P_2^N} Y_{2N}^{\star} - 2\frac{1}{c^2 P_3^N} Y_{5N}^{\star} + \frac{1}{c^2 P_4^N} Y_{6N}^{\star}\,, \\
T_{3N} =& Y_{3N} - 2Y_{7N} + Y_{8N} \\
=& \frac{1}{c^2 P_2^N} \sum_{i,j}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i \boldsymbol{X}_i') \circ (\boldsymbol{X}_j \boldsymbol{X}_j')\right] - 2\frac{1}{c^2 P_3^N} \sum_{i,j,k}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i \boldsymbol{X}_i') \circ (\boldsymbol{X}_j \boldsymbol{X}_k')\right] \\
& + \frac{1}{c^2 P_4^N} \mathrm{tr} \sum_{i,j,k,l}^{*} \left[(\boldsymbol{X}_i \boldsymbol{X}_j') \circ (\boldsymbol{X}_k \boldsymbol{X}_l')\right] \\
=& \frac{1}{c^2 P_2^N} Y_{3N}^{\star} - 2\frac{1}{c^2 P_3^N} Y_{7N}^{\star} + \frac{1}{c^2 P_4^N} Y_{8N}^{\star}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{4N} =& \frac{1}{P_2^N} \sum_{i,j}^{*} (\boldsymbol{x}_i' \boldsymbol{x}_j)^2 - 2\frac{1}{P_3^N} \sum_{i,j,k}^{*} \boldsymbol{x}_i' \boldsymbol{x}_j \boldsymbol{x}_i' \boldsymbol{x}_k + \frac{1}{P_4^N} \sum_{i,j,k,l}^{*} \boldsymbol{x}_i' \boldsymbol{x}_j \boldsymbol{x}_k' \boldsymbol{x}_l \\
=& \frac{N-1}{N(N-2)(N-3)} \left[(N-1)(N-2)\mathrm{tr}(\boldsymbol{S}^2) + \mathrm{tr}^2(\boldsymbol{S}) - NQ\right]
\end{aligned}
$$

where $\boldsymbol{x}_i = \mathrm{vec}(\boldsymbol{X}_i)$, $Q = \sum_{i=1}^N \left[(\boldsymbol{x}_i - \bar{\boldsymbol{x}})' (\boldsymbol{x}_i - \bar{\boldsymbol{x}})\right]^2 / (N-1)$, $\bar{\boldsymbol{x}} = \sum_{i=1}^N \boldsymbol{x}_i / N$, $\boldsymbol{S}$ is the sample covariance matrix of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, and where

$$
\begin{aligned}
Y^{\star}_{2N} &= \sum_{i,j}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_i'\boldsymbol{X}_j\boldsymbol{X}_j'),\\
Y^{\star}_{5N} &= N^2Y^{\star}_{51N} - (N-1)^2Y^{\star}_{52N} - Y^{\star}_{2N} + 2(N-1)Y^{\star}_{53N},\\
Y^{\star}_{51N} &= \sum_{i} \operatorname{tr}\left[(\boldsymbol{X}_i-\bar{\boldsymbol{X}})(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\boldsymbol{X}_i\boldsymbol{X}_i'\right],\\
\bar{\boldsymbol{X}} &= \sum_{i} \boldsymbol{X}_i/N,\\
Y^{\star}_{52N} &= \sum_{i} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_i'\boldsymbol{X}_i\boldsymbol{X}_i'),\\
Y^{\star}_{53N} &= \sum_{i,j}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_i'\boldsymbol{X}_i\boldsymbol{X}_j'),\\
Y^{\star}_{6N} &= \frac{1}{3}\left[(N-1)(N^2-3N+3)Y^{\star}_{52N} + (2N-3)(Y^{\star}_{2N}+Y^{\star}_{62N}+Y^{\star}_{63N})\right.\\
&\quad \left. + 2(N-3)(Y^{\star}_{5N}+Y^{\star}_{64N}+Y^{\star}_{65N}) - 4(N^2-3N+3)Y^{\star}_{53N} - N^3Y^{\star}_{61N}\right],\\
Y^{\star}_{61N} &= \sum_{i} \operatorname{tr}\left[(\boldsymbol{X}_i-\bar{\boldsymbol{X}})(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'(\boldsymbol{X}_i-\bar{\boldsymbol{X}})(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\right],\\
Y^{\star}_{62N} &= \sum_{i,j}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_j'\boldsymbol{X}_j\boldsymbol{X}_i'),\\
Y^{\star}_{63N} &= \sum_{i,j}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_j'\boldsymbol{X}_i\boldsymbol{X}_j'),\\
Y^{\star}_{64N} &= \sum_{i,j,k}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_j'\boldsymbol{X}_k\boldsymbol{X}_i')\\
&= N^2Y^{\star}_{641N} + 2(N-1)Y^{\star}_{53N} - (N-1)^2Y^{\star}_{52N} - Y^{\star}_{62N},\\
Y^{\star}_{641N} &= \sum_{i} \operatorname{tr}\left[(\boldsymbol{X}_i-\bar{\boldsymbol{X}})\boldsymbol{X}_i'\boldsymbol{X}_i(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\right],\\
Y^{\star}_{65N} &= \sum_{i,j,k}^{*} \operatorname{tr}(\boldsymbol{X}_i\boldsymbol{X}_j'\boldsymbol{X}_i\boldsymbol{X}_k')\\
&= N^2Y^{\star}_{651N} + 2(N-1)Y^{\star}_{53N} - (N-1)^2Y^{\star}_{52N} - Y^{\star}_{63N},\\
Y^{\star}_{651N} &= \sum_{i} \operatorname{tr}\left[\boldsymbol{X}_i(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\boldsymbol{X}_i(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\right],
\end{aligned}
$$

and where

$$Y^{\star}_{3N} = \sum_{i,j}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_i') \circ (\boldsymbol{X}_j\boldsymbol{X}_j')\right] ,$$

$$Y^{\star}_{7N} = N^2 Y^{\star}_{71N} - (N-1)^2 Y^{\star}_{72N} - Y^{\star}_{3N} + 2(N-1)Y^{\star}_{73N} ,$$

$$Y^{\star}_{71N} = \sum_{i} \mathrm{tr}\left\{\left[(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'\right] \circ (\boldsymbol{X}_i\boldsymbol{X}_i')\right\} ,$$

$$Y^{\star}_{72N} = \sum_{i} \mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_i') \circ (\boldsymbol{X}_i\boldsymbol{X}_i')\right] ,$$

$$Y^{\star}_{73N} = \sum_{i,j}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_i') \circ (\boldsymbol{X}_i\boldsymbol{X}_j')\right] ,$$

$$Y^{\star}_{8N} = \frac{1}{3}\left[(N-1)(N^2-3N+3)Y^{\star}_{72N} + (2N-3)(Y^{\star}_{3N} + 2Y^{\star}_{82N}) \right.$$
$$\left. + 2(N-3)(Y^{\star}_{7N} + 2Y^{\star}_{83N}) - 4(N^2-3N+3)Y^{\star}_{73N} - N^3 Y^{\star}_{81N}\right] ,$$

$$Y^{\star}_{81N} = \sum_{i} \mathrm{tr}\left\{\left[(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'\right] \circ \left[(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'\right]\right\} ,$$

$$Y^{\star}_{82N} = \sum_{i,j}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_j') \circ (\boldsymbol{X}_j\boldsymbol{X}_i')\right] ,$$

$$Y^{\star}_{83N} = \sum_{i,j,k}^{*} \mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_j') \circ (\boldsymbol{X}_k\boldsymbol{X}_i')\right]$$
$$= N^2 Y^{\star}_{831N} + 2(N-1)Y^{\star}_{73N} - (N-1)^2 Y^{\star}_{72N} - Y^{\star}_{82N} ,$$

$$Y^{\star}_{831N} = \sum_{i} \mathrm{tr}\left\{\left[(\boldsymbol{X}_i - \bar{\boldsymbol{X}})\boldsymbol{X}_i'\right] \circ \left[\boldsymbol{X}_i(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'\right]\right\} .$$

## S4 Proof of Theorem 1

*Proof.* From the moment derivations in S2, note that $E(T_{1N}) = \mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})$ and

$$\mathrm{Var}\left[\frac{T_{1N}}{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})}\right] = \frac{\mathrm{Var}(Y_{1N})}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}})} + \frac{\mathrm{Var}(Y_{4N})}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}})} \to 0 .$$

Hence, it follows that

$$\frac{T_{1N}}{\text{tr}(\boldsymbol{\Sigma}_{\text{R}})} \xrightarrow{P} 1\,.$$

Similarly, it can be shown that

$$\frac{T_{2N}}{\text{tr}(\boldsymbol{\Sigma}_{\text{R}}^2)} \xrightarrow{P} 1 \text{ and } \frac{T_{4N}}{\text{tr}(\boldsymbol{\Sigma}^2)} \xrightarrow{P} 1\,.$$

Combined these two results also prove the ratio-consistency of $T_{5N}$. Finally, using the variance and covariance expressions that involve $Y_{3N}, Y_{7N}$ and $Y_{8N}$ it can be shown that

$$\frac{\text{Var}\,(T_{3N})}{\text{tr}^2(\boldsymbol{\Sigma}_{\text{R}}^2)} \to 0\,.$$

□

## S5 Proof of Theorem 2

*Proof.* The essential step is to show

$$\frac{G_N - E(G_N)}{\text{Var}(G_N)} \xrightarrow{d} \text{N}(0,1)$$

where $G_N = Y_{2N}/\text{tr}(\boldsymbol{\Sigma}_{\text{R}}^2) - 2Y_{1N}/\text{tr}(\boldsymbol{\Sigma}_{\text{R}})$. To accomplish this, the martingale central limit theorem will be used. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k\}$ for $k = 1, \ldots, N$. Also let $E_k$ be the conditional expectation given $\mathcal{F}_k$, $D_{Nk} = E_k(G_N) - E_{k-1}(G_N)$ and $S_{Nm} = \sum_{k=1}^{m} D_{Nk} =$

$E_m(G_N) - E(G_N)$. Then

$$\begin{aligned} D_{Nk} =& \frac{2}{cN}\left[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Lambda}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right) - \mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{\Lambda}\right)\right] \\ &+ \frac{2}{c^2N(N-1)}\frac{1}{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)}\left[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{R}_{k-1}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right) - \mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{R}_{k-1}\right)\right] \end{aligned}$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}_{\mathrm{R}}^2/\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2) - \boldsymbol{\Sigma}_{\mathrm{R}}/\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})$, $\boldsymbol{Q}_k = \sum_{i=1}^k(\boldsymbol{X}_i\boldsymbol{X}_i' - c\boldsymbol{\Sigma}_{\mathrm{R}})$ and $\boldsymbol{R}_k = \boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{Q}_k\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}$. We need the following three lemmata:

**Lemma 1.** *For any $N$, $\{D_{Nk}, 1 \leq k \leq N\}$ is a martingale difference sequence with respect to the $\sigma$-fields $\{\mathcal{F}_k, 1 \leq k \leq N\}$.*

*Proof.* Note that $E(D_{Nk}) = 0$ and write $S_{Nq} = S_{Nm} + E_q(G_N) - E_m(G_N)$ for $q > m$. Then it can be shown that $E(S_{Nq}|\mathcal{F}_m) = S_{Nm}$ as desired. □

**Lemma 2.** *Let $\sigma_{Nk}^2 = E_{k-1}(D_{Nk}^2)$. Then*

$$\frac{\sum_{k=1}^N \sigma_{Nk}^2}{\mathrm{Var}(G_N)} \xrightarrow{P} 1\,.$$

*Proof.* First note that

$$\begin{aligned} \sum_{k=1}^N \sigma_{Nk}^2 =& \frac{8\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4N^2(N-1)^2\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2)}\sum_{k=1}^N \mathrm{tr}(\boldsymbol{R}_{k-1}^2) \\ &+ \frac{4B\mathrm{tr}(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2)}{c^4N^2(N-1)^2\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2)}\sum_{k=1}^N \mathrm{tr}(\boldsymbol{\Delta}_{\boldsymbol{R}_{k-1}}^2) \\ &+ \frac{16\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^3N^2(N-1)\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)}\sum_{k=1}^N \left[\frac{\mathrm{tr}(\boldsymbol{Q}_{k-1}\boldsymbol{\Sigma}_{\mathrm{R}}^3)}{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} - \frac{\mathrm{tr}(\boldsymbol{Q}_{k-1}\boldsymbol{\Sigma}_{\mathrm{R}}^2)}{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})}\right] \\ &+ \frac{8B\mathrm{tr}(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2)}{c^3N^2(N-1)\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)}\sum_{k=1}^N \mathrm{tr}(\boldsymbol{R}_{k-1}\circ\boldsymbol{\Lambda}) + H \\ =& H_{1N} + H_{2N} + H_{3N} + H_{4N} + H\,, \end{aligned}$$

where $H$ is a finite constant. To complete the proof, we need to show that $\mathrm{Var}(H_{mN}) = o\left\{[\mathrm{Var}(G_N)]^2\right\}$ for $m = 1, 2, 3, 4$. We will prove it for $m = 1$ since the arguments for $m = 2, 3, 4$ are similar. In this direction, first note that

$$\begin{aligned}\mathrm{Var}(G_N) =& 4\frac{\mathrm{Var}(Y_{1N})}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}})} + \frac{\mathrm{Var}(Y_{2N})}{\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{R}})} - 4\frac{\mathrm{Cov}(Y_{1N}, Y_{2N})}{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})\mathrm{tr}(\boldsymbol{\Sigma}^2_{\mathrm{R}})}\\ =& \frac{8\mathrm{tr}(\boldsymbol{\Sigma}^2_{\mathrm{C}})}{Nc^2}\mathrm{tr}(\boldsymbol{\Lambda}^2) + \frac{4B\mathrm{tr}\left(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}_{\mathrm{C}}}\right)}{Nc^2}\mathrm{tr}\left(\boldsymbol{\Delta}^2_{\boldsymbol{\Lambda}}\right) + \frac{4\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{C}})}{N^2c^4}\left\{1 + O(N^{-1})\right\}\end{aligned}$$

and hence for large $N$, there exists a constant $\lambda_1$ such that

$$[\mathrm{Var}(G_N)]^2 \geq \lambda_1 \max\left\{\frac{\mathrm{tr}^3(\boldsymbol{\Sigma}^2_{\mathrm{C}})}{N^3c^6}\mathrm{tr}(\boldsymbol{\Lambda}^2), \frac{\mathrm{tr}^4(\boldsymbol{\Sigma}^2_{\mathrm{C}})}{N^4c^8}, \frac{\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{C}})}{N^2c^4}\mathrm{tr}^2(\boldsymbol{\Lambda}^2)\right\}.$$

Write

$$\begin{aligned}\mathrm{tr}\left(\boldsymbol{R}^2_{k-1}\right) =& \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_j\boldsymbol{X}'_j - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\\ =& \sum_{i,j}^{*}\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_j\boldsymbol{X}'_j - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\\ &+ \sum_{i=1}^{k-1}\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\end{aligned}$$

and note that

$$\begin{aligned}\mathrm{Var}\left[\mathrm{tr}\left(\boldsymbol{R}^2_{k-1}\right)\right] =& (k-1)(k-2)\mathrm{Var}\left\{\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_j\boldsymbol{X}'_j - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\right\}\\ &+ (k-1)\mathrm{Var}\left\{\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\right\}.\end{aligned}$$

Next, algebraic manipulation shows that

$$\begin{aligned}&\mathrm{Var}\left\{\mathrm{tr}\left[\left(\boldsymbol{X}_i\boldsymbol{X}'_i - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{X}_j\boldsymbol{X}'_j - c\boldsymbol{\Sigma}_{\mathrm{R}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\right]\right\}\\ &= 2\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{C}})\mathrm{tr}^2(\boldsymbol{\Sigma}^4_{\mathrm{R}}) + 2\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{C}})\mathrm{tr}(\boldsymbol{\Sigma}^8_{\mathrm{R}}) + 4B\mathrm{tr}^2(\boldsymbol{\Sigma}^2_{\mathrm{C}})\mathrm{tr}(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}_{\mathrm{C}}})\mathrm{tr}(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}^4_{\mathrm{R}}})\\ &+ B^2\mathrm{tr}^2(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}_{\mathrm{C}}})\mathrm{tr}\left[(\boldsymbol{\Sigma}^2_{\mathrm{R}} \circ \boldsymbol{\Sigma}^2_{\mathrm{R}})^2\right]\end{aligned}$$

which implies that for large $N$ there exists constant $\lambda_2$ such that

$$\mathrm{Var}\left\{\mathrm{tr}\left[(\boldsymbol{X}_i\boldsymbol{X}_i' - c\boldsymbol{\Sigma}_\mathrm{R})\,\boldsymbol{\Sigma}_\mathrm{R}\left(\boldsymbol{X}_j\boldsymbol{X}_j' - c\boldsymbol{\Sigma}_\mathrm{R}\right)\boldsymbol{\Sigma}_\mathrm{R}\right]\right\} \le \lambda_2 \mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{C}^2)\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^4)\,.$$

Also for large $N$ there exists constant $\lambda_3$ such that

$$\begin{aligned}
&E\left\{\left[\sum_{r_1,r_2,r_3,r_4}(\boldsymbol{X}_i\boldsymbol{X}_i' - c\boldsymbol{\Sigma}_\mathrm{R})_{r_1r_2}(\boldsymbol{\Sigma}_\mathrm{R})_{r_2r_3}(\boldsymbol{X}_i\boldsymbol{X}_i' - c\boldsymbol{\Sigma}_\mathrm{R})_{r_3r_4}(\boldsymbol{\Sigma}_\mathrm{R})_{r_4r_1}\right]^2\right\} \\
&\le \lambda_3 \sum_{r_1,r_2}\left[E\left(\boldsymbol{X}_i\boldsymbol{X}_i' - c\boldsymbol{\Sigma}_\mathrm{R}\right)^2_{r_1r_2}\right]\sum_{r_1,r_2}(\boldsymbol{\Sigma}_\mathrm{R})^4_{r_1r_2} \\
&\le \lambda_3 \mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{C}^2)\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^2)\mathrm{tr}(\boldsymbol{\Sigma}_\mathrm{R}^4)\,.
\end{aligned}$$

Therefore for large $N$

$$\begin{aligned}
\mathrm{Var}(H_{1N}) &= \frac{64\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{C}^2)}{c^8N^4(N-1)^4\mathrm{tr}^4(\boldsymbol{\Sigma}_\mathrm{R}^2)}\sum_{k=1}^{N}\mathrm{Var}\left[\mathrm{tr}\left(\boldsymbol{R}_{k-1}^2\right)\right] \\
&\le \frac{64\lambda_2\mathrm{tr}^4(\boldsymbol{\Sigma}_\mathrm{C}^2)\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^4)}{c^8N^4\mathrm{tr}^4(\boldsymbol{\Sigma}_\mathrm{R}^2)} + \frac{64\lambda_3\mathrm{tr}^4(\boldsymbol{\Sigma}_\mathrm{C}^2)\mathrm{tr}(\boldsymbol{\Sigma}_\mathrm{R}^4)}{c^8N^6\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^2)}
\end{aligned}$$

and hence

$$\frac{\mathrm{Var}(H_{1N})}{[\mathrm{Var}(G_N)]^2} \le \frac{64\lambda_2}{\lambda_1}\frac{\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^4)}{\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^2)} + \frac{64\lambda_3}{\lambda_1N^2}\frac{\mathrm{tr}(\boldsymbol{\Sigma}_\mathrm{R}^4)}{\mathrm{tr}^2(\boldsymbol{\Sigma}_\mathrm{R}^2)} \to 0$$

as desired. □

**Lemma 3.** *It holds that*

$$\frac{\sum_{k=1}^{N}E(D_{Nk}^4)}{[\mathrm{Var}(G_N)]^2} \to 0\,.$$

*Proof.* There exists a constant $\lambda_4$ such that for large $N$,

$$
\begin{aligned}
E(D_{Nk}^4) \leq & \lambda_4 \left\{ \frac{1}{c^4 N^3} E[\operatorname{tr}(\boldsymbol{Z}_k' \boldsymbol{\Lambda} \boldsymbol{Z}_k \boldsymbol{\Sigma}_{\mathrm{C}}) - \operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{C}} \otimes \boldsymbol{\Lambda})]^4 \right. \\
& \left. + \frac{2}{c^8 N^4 (N-1)^4} \frac{1}{\operatorname{tr}^4(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} \sum_{k=1}^{N} E[\operatorname{tr}(\boldsymbol{Z}_k' \boldsymbol{R}_{k-1} \boldsymbol{Z}_k \boldsymbol{\Sigma}_{\mathrm{C}}) - \operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{C}} \otimes \boldsymbol{R}_{k-1})]^4 \right\} \\
\leq & \lambda_4 \left\{ \frac{1}{N^3} \frac{\operatorname{tr}^2(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4} \operatorname{tr}^2(\boldsymbol{\Lambda}^2) + \frac{2}{N^6} \frac{\operatorname{tr}^4(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^8} \left[ \frac{\operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^4)}{\operatorname{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} \right]^2 \right\}
\end{aligned}
$$

Therefore,

$$
\frac{\sum_{k=1}^{N} E(D_{Nk}^4)}{[\operatorname{Var}(G_N)]^2} \leq \frac{\lambda_4}{\lambda_1} \left\{ \frac{1}{N} + \left[ \frac{\operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^4)}{\operatorname{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} \right]^2 \right\} \to 0\,.
$$

□

Combining the three lemmata it follows that

$$
\frac{G_N - E(G_N)}{\operatorname{Var}(G_N)} \xrightarrow{d} \mathrm{N}(0,1)\,.
$$

Next write

$$
\frac{\operatorname{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}})}{\operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} \frac{U_N + 1}{r} - 1 = \frac{G_N - \tilde{T}_{1N}^2}{\left(1 + \tilde{T}_{1N}\right)^2}
$$

where

$$
\tilde{T}_{1N} = \frac{T_{1N} - \operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})}{\operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}})}\,.
$$

By Theorem 1, it follows that $\tilde{T}_{1N} \xrightarrow{P} 0$ and $\sigma_{U_N}^{-1} \tilde{T}_{1N} \xrightarrow{P} 0$. Finally, moment derivations in S2 imply that $\operatorname{Var}(G_N) = \sigma_{U_N}^2 \{1 + o(1)\}$. It therefore holds that $\sigma_{U_N}^{-1} G_N \xrightarrow{d} \mathrm{N}(0,1)$, and hence

$$
\sigma_{U_N}^{-1} \left[ \frac{\operatorname{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}})}{\operatorname{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^2)} \frac{U_N + 1}{r} - 1 \right] \xrightarrow{d} \mathrm{N}(0,1)
$$

as desired. □

## S6 Sketch of the Proof of Theorem 4

*Sketch of the Proof.* The essential step is to show that

$$\frac{G_N - E(G_N)}{\text{Var}(G_N)} \xrightarrow{d} \text{N}(0,1)$$

where now $G_N = Y_{2N} - 2Y_{1N}$. Since $G_N$ is again a linear combination of $Y_{1N}$ and $Y_{2N}$, the proof of this fact is similar to that of the asymptotic normality of $G_N$ in the proof of Theorem 2. For this reason, we skip the details. Next write $V_N = G_N + 2Y_{3N} - 2Y_{4N} + Y_{5N}$ and note that $E(V_N) = \text{tr}\,[(\boldsymbol{\Sigma}_\text{R} - \mathbf{I}_r)^2]$. Moment derivations in S2 imply that $\text{Var}(V_N) = \sigma_{V_N}^2\,\{1 + o(1)\}$. Therefore

$$\frac{V_N - \text{tr}\,[(\boldsymbol{\Sigma}_\text{R} - \mathbf{I}_r)^2]}{\sigma_{V_N}} \xrightarrow{d} \text{N}(0,1)$$

as desired. □

## S7 Proof of Theorem 6

*Proof.* The essential step is to show that

$$\frac{G_N - E(G_N)}{\text{Var}(G_N)} \xrightarrow{d} \text{N}(0,1)$$

where now $G_N = Y_{2N} - Y_{3N}$. Although the proof of this fact is similar to that in Theorems 2 and 4, we provide some details because $G_N$ is no longer a linear combination of $Y_{1N}$ and $Y_{2N}$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k =$

$\sigma\{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_k\}$ for $k=1,\ldots,N$. Let

$$\begin{aligned}
D_{Nk} =& E_k(G_N)-E_{k-1}(G_N)\\
=& \frac{2}{c^2N(N-1)}\left[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{R}_{k-1}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{R}_{k-1}\right)\right]\\
&+\frac{2}{c^2N(N-1)}\left\{\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\left(\boldsymbol{\Sigma}_{\mathrm{R}}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\right]\right\}\\
&+\frac{2}{cN}\left[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^2\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{\Sigma}_{\mathrm{R}}^2\right)\right]\\
&+\frac{2}{cN}\left\{\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\left(\boldsymbol{\Sigma}_{\mathrm{R}}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\right)\right]\right\}.
\end{aligned}$$

It can be shown that for any $N$, $\{D_{Nk}, 1\leq k\leq N\}$ is a martingale difference sequence with respect to the $\sigma$-fields $\{\mathcal{F}_k, 1\leq k\leq N\}$. Next let $\sigma^2_{Nk}=E_{k-1}(D^2_{Nk})$ and write

$$\begin{aligned}
\sum_{k=1}^N\sigma^2_{Nk} =& \frac{8\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4N^2(N-1)^2}\sum_{k=1}^N\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{Q}_{k-1}-\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{Q}_{k-1}-\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\right]\\
&+\frac{4B\mathrm{tr}(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}_{\mathrm{C}}})}{c^4N^2(N-1)^2}\sum_{k=1}^N\mathrm{tr}\left[\left(\boldsymbol{R}_{k-1}-\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\circ\left(\boldsymbol{R}_{k-1}-\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\right]\\
&+\frac{16\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^3N^2(N-1)}\sum_{k=1}^N\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{Q}_{k-1}-\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{\Sigma}_{\mathrm{R}}-\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\right)\right]\\
&+\frac{8B\mathrm{tr}(\boldsymbol{\Delta}^2_{\boldsymbol{\Sigma}_{\mathrm{C}}})}{c^3N^2(N-1)}\sum_{k=1}^N\mathrm{tr}\left[\left(\boldsymbol{R}_{k-1}-\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\circ\left(\boldsymbol{\Sigma}_{\mathrm{R}}^2-\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right)\right]\\
&+F\\
=& F_{1N}+F_{2N}+F_{3N}+F_{4N}+F\,,
\end{aligned}$$

where $F$ is a finite constant. To prove that $\sum_{k=1}^N\sigma^2_{Nk}/\mathrm{Var}(G_N)\overset{P}{\to}1$ it suffices to show that $\mathrm{Var}(F_{mN})/[\mathrm{Var}(G_N)]^2\to 0$ for all $m=1,2,3,4$. In this direction, note that

$$\begin{aligned}\mathrm{Var}(G_N) =&\mathrm{Var}(Y_{2N}) + \mathrm{Var}(Y_{3N}) - 2\mathrm{Cov}(Y_{2N}, Y_{3N})\\ =&\frac{4}{N^2}\left[\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^2}\right]^2 \mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2) + \frac{8}{N}\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^2}\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{R}}(\boldsymbol{\Sigma}_{\mathrm{R}} - \boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}})\boldsymbol{\Sigma}_{\mathrm{R}}(\boldsymbol{\Sigma}_{\mathrm{R}} - \boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}})\right]\\ &+ \frac{4B}{N}\frac{\mathrm{tr}(\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{C}}}^2)}{c^2}\mathrm{tr}\left\{\left[\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}(\boldsymbol{\Sigma}_{\mathrm{R}} - \boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}})\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right] \circ \left[\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}(\boldsymbol{\Sigma}_{\mathrm{R}} - \boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}})\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\right]\right\}\\ &\times \{1 + O(1)\}\end{aligned}$$

and hence for large N, there exists a constant $\lambda_5$ such that

$$[\mathrm{Var}(G_N)]^2 \geq \frac{\lambda_5}{N^4}\left[\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^2}\right]^4 \mathrm{tr}^4(\boldsymbol{\Sigma}_{\mathrm{R}}^2)\,.$$

Expanding $F_{1N}$, we have that

$$\begin{aligned}&\sum_{k=1}^{N}\mathrm{tr}\left[\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{Q}_{k-1} - \boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\boldsymbol{\Sigma}_{\mathrm{R}}\left(\boldsymbol{Q}_{k-1} - \boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\right]\\ &\quad=\sum_{k=1}^{N}\mathrm{tr}\left(\boldsymbol{R}_{k-1}^2\right) + \sum_{k=1}^{N}\mathrm{tr}\left[\left(\boldsymbol{\Sigma}_{\mathrm{R}}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)^2\right] - 2\sum_{k=1}^{N}\mathrm{tr}\left(\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{Q}_{k-1}\right)\\ &\quad=F_{11N} + F_{12N} + F_{13N}\,.\end{aligned}$$

From the proof in S5, it follows that for large $N$ there exist constants $\lambda_6$ and $\lambda_7$ such that

$$\begin{aligned}&\mathrm{Var}\left[\frac{8\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4N^2(N-1)^2}\sum_{k=1}^{N}\mathrm{tr}\left(\boldsymbol{R}_{k-1}^2\right)\right]\\ &\quad\leq \frac{64\lambda_6\mathrm{tr}^4(\boldsymbol{\Sigma}_{\mathrm{C}}^2)\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^4)}{c^8N^4} + \frac{64\lambda_7\mathrm{tr}^4(\boldsymbol{\Sigma}_{\mathrm{C}}^2)\mathrm{tr}^2(\boldsymbol{\Sigma}_{\mathrm{R}}^2)\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{R}}^4)}{c^8N^6}\end{aligned}$$

and therefore

$$\frac{\mathrm{Var}\left[\frac{8\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4N^2(N-1)^2}F_{11N}\right]}{[\mathrm{Var}(G_N)]^2} \to 0\,.$$

Similarly, we can prove that

$$\frac{\mathrm{Var}\left[\frac{8\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{C}}^2)}{c^4N^2(N-1)^2}F_{1tN}\right]}{[\mathrm{Var}(G_N)]^2}\to 0\,.$$

for $t=2,3$. The above strateegy can be used to prove $\mathrm{Var}(F_{mN})/[\mathrm{Var}(G_N)]^2\to 0$ for $m=2,3,4$.

Next, note that there exists a constant $\lambda_8$ such that

$$\begin{aligned}E(D_{Nk}^4)\leq&\lambda_8\left\{\frac{1}{c^8N^4(N-1)^4}\sum_{k=1}^N E[\mathrm{tr}\,(\boldsymbol{Z}_k'\boldsymbol{R}_{k-1}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}})-\mathrm{tr}\,(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{R}_{k-1})]^4\right.\\&+\frac{1}{c^8N^4(N-1)^4}\sum_{k=1}^N E\left[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{\Sigma}_{\mathrm{R}}\boldsymbol{\Delta}_{\boldsymbol{Q}_{k-1}}\right)\right]^4\\&+\frac{1}{c^4N^2}\sum_{k=1}^N E[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^2\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{\Sigma}_{\mathrm{R}}^2\right)]^4\\&\left.+\frac{1}{c^4N^2}\sum_{k=1}^N E[\mathrm{tr}\left(\boldsymbol{Z}_k'\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}\boldsymbol{\Sigma}_{\mathrm{R}}^{1/2}\boldsymbol{Z}_k\boldsymbol{\Sigma}_{\mathrm{C}}\right)-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{C}}\otimes\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}}^2\right)]^4\right\}\end{aligned}$$

Similar operations as in S5 imply that

$$\frac{\sum_{k=1}^N E(D_{Nk}^4)}{[\mathrm{Var}(G_N)]^2}\to 0\,.$$

Therefore, the martingale central limit theorem holds and

$$\frac{G_N-E(G_N)}{\mathrm{Var}(G_N)}\xrightarrow{d}\mathrm{N}(0,1)\,.$$

Next write $W_N=G_N-2(Y_{5N}-Y_{7N})+(Y_{6N}-Y_{8N})$. The results in S2 imply that $\mathrm{Var}(W_N)=\sigma_{W_N}^2\{1+o(1)\}$ and

$$\frac{W_N-\mathrm{tr}\,[(\boldsymbol{\Sigma}_{\mathrm{R}}-\boldsymbol{\Delta}_{\boldsymbol{\Sigma}_{\mathrm{R}}})^2]}{\sigma_{W_N}}\xrightarrow{d}\mathrm{N}(0,1)\,.$$

□

## S8 Sketch of the Proof of Theorems 3, 5 and 7

*Sketch of the Proofs.* The asymptotic power in Theorems 3, 5 and 7 follows from the inequalities for $\sigma^2_{U_N}$, $\sigma^2_{V_N}$ and $\sigma^2_{W_N}$ respectively and the results in Theorem 1. □

## S9 Simulation Tables

This section contains tables that display the simulation results for the proposed sphericity (Tables 1–5) and diagonality (Tables 6–10) test in the main manuscript. Each reported result comes from 1000 simulations.

It is worth mentioning that due to lack of alternative tests, we also checked whether estimation of $\mathrm{tr}(\mathbf{\Sigma}_\mathrm{C}^2)$ by $T_{5N}$ affected the performance of the proposed test statistics. To this end, we considered three alternative test statistics, one for each of the three hypotheses, calculated in the same manner as the corresponding proposed test statistic except that $\mathrm{tr}(\mathbf{\Sigma}_\mathrm{C}^2)$ was replaced by its true value (Tables 11–15 for the sphericity test and Tables 16–20 for the diagonality test). As desired, we did not observe any substantial difference in the empirical sizes and powers with those of the corresponding proposed testing procedures. This provides some assurance that $T_{5N}$ is an efficient estimator of $\mathrm{tr}(\mathbf{\Sigma}_\mathrm{C}^2)$ and does not lead to distorted sizes or to power losses.

Table 1: Empirical sizes of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.048 | 0.076 | 0.056 | 0.041 | 0.059 | 0.055 | 0.055 | 0.063 | 0.058 | 0.060 | 0.053 | 0.059 | 0.053 | 0.056 | 0.046 |
| 50 | 0.037 | 0.061 | 0.058 | 0.047 | 0.062 | 0.062 | 0.060 | 0.057 | 0.047 | 0.060 | 0.044 | 0.053 | 0.056 | 0.053 | 0.051 |
| 100 | 0.059 | 0.061 | 0.065 | 0.047 | 0.050 | 0.054 | 0.068 | 0.051 | 0.052 | 0.057 | 0.054 | 0.049 | 0.047 | 0.055 | 0.056 |
| 300 | 0.052 | 0.055 | 0.058 | 0.046 | 0.054 | 0.050 | 0.047 | 0.060 | 0.058 | 0.046 | 0.050 | 0.043 | 0.051 | 0.045 | 0.042 |
| 600 | 0.043 | 0.048 | 0.059 | 0.048 | 0.056 | 0.047 | 0.054 | 0.045 | 0.051 | 0.053 | 0.049 | 0.054 | 0.045 | 0.051 | 0.054 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.074 | 0.073 | 0.082 | 0.070 | 0.062 | 0.070 | 0.081 | 0.075 | 0.069 | 0.066 | 0.076 | 0.053 | 0.072 | 0.079 | 0.056 |
| 50 | 0.058 | 0.066 | 0.067 | 0.056 | 0.068 | 0.043 | 0.050 | 0.062 | 0.052 | 0.049 | 0.064 | 0.060 | 0.057 | 0.054 | 0.058 |
| 100 | 0.050 | 0.067 | 0.061 | 0.049 | 0.049 | 0.071 | 0.060 | 0.043 | 0.059 | 0.063 | 0.044 | 0.057 | 0.060 | 0.055 | 0.050 |
| 300 | 0.052 | 0.043 | 0.067 | 0.061 | 0.051 | 0.057 | 0.063 | 0.049 | 0.059 | 0.053 | 0.056 | 0.047 | 0.042 | 0.045 | 0.054 |
| 600 | 0.053 | 0.055 | 0.071 | 0.051 | 0.069 | 0.059 | 0.052 | 0.067 | 0.059 | 0.055 | 0.055 | 0.048 | 0.061 | 0.054 | 0.055 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.076 | 0.058 | 0.074 | 0.079 | 0.070 | 0.060 | 0.065 | 0.076 | 0.088 | 0.070 | 0.067 | 0.068 | 0.062 | 0.078 | 0.094 |
| 50 | 0.062 | 0.051 | 0.075 | 0.068 | 0.066 | 0.052 | 0.065 | 0.076 | 0.063 | 0.072 | 0.060 | 0.059 | 0.073 | 0.052 | 0.046 |
| 100 | 0.056 | 0.055 | 0.054 | 0.056 | 0.048 | 0.059 | 0.058 | 0.049 | 0.059 | 0.054 | 0.043 | 0.064 | 0.058 | 0.056 | 0.052 |
| 300 | 0.051 | 0.053 | 0.056 | 0.050 | 0.050 | 0.062 | 0.053 | 0.043 | 0.062 | 0.058 | 0.055 | 0.046 | 0.054 | 0.056 | 0.042 |
| 600 | 0.061 | 0.058 | 0.056 | 0.045 | 0.051 | 0.047 | 0.053 | 0.044 | 0.060 | 0.053 | 0.057 | 0.066 | 0.055 | 0.060 | 0.044 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.092 | 0.095 | 0.115 | 0.104 | 0.084 | 0.095 | 0.126 | 0.096 | 0.092 | 0.116 | 0.096 | 0.099 | 0.104 | 0.102 | 0.099 |
| 50 | 0.067 | 0.082 | 0.069 | 0.086 | 0.070 | 0.072 | 0.074 | 0.063 | 0.075 | 0.080 | 0.064 | 0.056 | 0.101 | 0.071 | 0.066 |
| 100 | 0.062 | 0.055 | 0.062 | 0.055 | 0.055 | 0.061 | 0.066 | 0.054 | 0.053 | 0.069 | 0.056 | 0.066 | 0.063 | 0.062 | 0.052 |
| 300 | 0.045 | 0.051 | 0.057 | 0.044 | 0.062 | 0.063 | 0.046 | 0.056 | 0.055 | 0.048 | 0.063 | 0.053 | 0.059 | 0.058 | 0.062 |
| 600 | 0.059 | 0.053 | 0.060 | 0.055 | 0.057 | 0.066 | 0.049 | 0.056 | 0.056 | 0.057 | 0.055 | 0.048 | 0.049 | 0.054 | 0.056 |

Table 2: Empirical powers of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} \sim \mathrm{U}(0.5, 1.5)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | | | | | | | Normal Instances | | | | | | | | |
| 10 | 0.482 | 1.000 | 1.000 | 0.951 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.355 | 1.000 | 1.000 | 0.932 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.568 | 1.000 | 1.000 | 0.984 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.513 | 1.000 | 1.000 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.532 | 1.000 | 1.000 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances I | | | | | | | | |
| 10 | 0.472 | 1.000 | 1.000 | 0.903 | 1.000 | 1.000 | 0.990 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.388 | 1.000 | 1.000 | 0.920 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.578 | 1.000 | 1.000 | 0.977 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.542 | 1.000 | 1.000 | 0.978 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.527 | 1.000 | 1.000 | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances II | | | | | | | | |
| 10 | 0.243 | 0.989 | 1.000 | 0.834 | 1.000 | 1.000 | 0.986 | 1.000 | 1.000 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.460 | 1.000 | 1.000 | 0.951 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.628 | 1.000 | 1.000 | 0.942 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.459 | 1.000 | 1.000 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.526 | 1.000 | 1.000 | 0.973 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances III | | | | | | | | |
| 10 | 0.157 | 0.726 | 1.000 | 0.551 | 1.000 | 1.000 | 0.960 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.385 | 1.000 | 1.000 | 0.819 | 1.000 | 1.000 | 0.989 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.414 | 1.000 | 1.000 | 0.930 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.509 | 1.000 | 1.000 | 0.973 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.498 | 1.000 | 1.000 | 0.969 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 3: Empirical powers of the sphericity test when $\boldsymbol{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\boldsymbol{\Sigma}_{\mathrm{R}})_{r_1 r_1} = 1 + I(r_1 \le 0.9r)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | | | | | | | Normal Instances | | | | | | | | |
| 10 | 0.124 | 0.978 | 1.000 | 0.245 | 1.000 | 1.000 | 0.455 | 1.000 | 1.000 | 0.813 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 |
| 50 | 0.120 | 0.981 | 1.000 | 0.289 | 1.000 | 1.000 | 0.510 | 1.000 | 1.000 | 0.846 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.133 | 0.988 | 1.000 | 0.271 | 1.000 | 1.000 | 0.513 | 1.000 | 1.000 | 0.870 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.135 | 0.993 | 1.000 | 0.276 | 1.000 | 1.000 | 0.492 | 1.000 | 1.000 | 0.898 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.132 | 0.990 | 1.000 | 0.313 | 1.000 | 1.000 | 0.521 | 1.000 | 1.000 | 0.889 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances I | | | | | | | | |
| 10 | 0.151 | 0.927 | 1.000 | 0.292 | 1.000 | 1.000 | 0.443 | 1.000 | 1.000 | 0.746 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 50 | 0.143 | 0.980 | 1.000 | 0.292 | 1.000 | 1.000 | 0.503 | 1.000 | 1.000 | 0.843 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.147 | 0.972 | 1.000 | 0.294 | 1.000 | 1.000 | 0.495 | 1.000 | 1.000 | 0.845 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 300 | 0.133 | 0.988 | 1.000 | 0.306 | 1.000 | 1.000 | 0.504 | 1.000 | 1.000 | 0.883 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.145 | 0.983 | 1.000 | 0.289 | 1.000 | 1.000 | 0.507 | 1.000 | 1.000 | 0.889 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances II | | | | | | | | |
| 10 | 0.146 | 0.917 | 1.000 | 0.249 | 1.000 | 1.000 | 0.408 | 1.000 | 1.000 | 0.693 | 1.000 | 1.000 | 0.990 | 1.000 | 1.000 |
| 50 | 0.146 | 0.978 | 1.000 | 0.282 | 1.000 | 1.000 | 0.502 | 1.000 | 1.000 | 0.846 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 100 | 0.145 | 0.986 | 1.000 | 0.319 | 1.000 | 1.000 | 0.505 | 1.000 | 1.000 | 0.865 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.120 | 0.984 | 1.000 | 0.286 | 1.000 | 1.000 | 0.493 | 1.000 | 1.000 | 0.844 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.119 | 0.976 | 1.000 | 0.293 | 1.000 | 1.000 | 0.490 | 1.000 | 1.000 | 0.854 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances III | | | | | | | | |
| 10 | 0.161 | 0.863 | 1.000 | 0.268 | 0.999 | 1.000 | 0.391 | 1.000 | 1.000 | 0.619 | 1.000 | 1.000 | 0.962 | 1.000 | 1.000 |
| 50 | 0.148 | 0.965 | 1.000 | 0.312 | 1.000 | 1.000 | 0.463 | 1.000 | 1.000 | 0.778 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 100 | 0.156 | 0.973 | 1.000 | 0.296 | 1.000 | 1.000 | 0.514 | 1.000 | 1.000 | 0.828 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 300 | 0.124 | 0.979 | 1.000 | 0.271 | 1.000 | 1.000 | 0.470 | 1.000 | 1.000 | 0.846 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.143 | 0.985 | 1.000 | 0.270 | 1.000 | 1.000 | 0.508 | 1.000 | 1.000 | 0.867 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 4: Empirical powers of the sphericity test when $\boldsymbol{\Sigma}_{\text{R}}$ is a tridiagonal matrix with elements $(\boldsymbol{\Sigma}_{\text{R}})_{r_1 r_2} = 0.1^{|r_1 - r_2|} I(|r_1 - r_2|)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.094 | 0.782 | 1.000 | 0.183 | 0.987 | 1.000 | 0.296 | 1.000 | 1.000 | 0.557 | 1.000 | 1.000 | 0.928 | 1.000 | 1.000 |
| 50 | 0.105 | 0.900 | 1.000 | 0.219 | 1.000 | 1.000 | 0.375 | 1.000 | 1.000 | 0.682 | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 |
| 100 | 0.112 | 0.907 | 1.000 | 0.202 | 1.000 | 1.000 | 0.386 | 1.000 | 1.000 | 0.693 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.125 | 0.947 | 1.000 | 0.215 | 1.000 | 1.000 | 0.362 | 1.000 | 1.000 | 0.729 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.115 | 0.915 | 1.000 | 0.244 | 1.000 | 1.000 | 0.395 | 1.000 | 1.000 | 0.732 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.136 | 0.758 | 1.000 | 0.228 | 0.992 | 1.000 | 0.318 | 1.000 | 1.000 | 0.557 | 1.000 | 1.000 | 0.918 | 1.000 | 1.000 |
| 50 | 0.124 | 0.886 | 1.000 | 0.214 | 1.000 | 1.000 | 0.380 | 1.000 | 1.000 | 0.698 | 1.000 | 1.000 | 0.992 | 1.000 | 1.000 |
| 100 | 0.130 | 0.909 | 1.000 | 0.208 | 1.000 | 1.000 | 0.365 | 1.000 | 1.000 | 0.705 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 300 | 0.112 | 0.921 | 1.000 | 0.242 | 1.000 | 1.000 | 0.386 | 1.000 | 1.000 | 0.730 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 600 | 0.124 | 0.919 | 1.000 | 0.218 | 1.000 | 1.000 | 0.394 | 1.000 | 1.000 | 0.723 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.126 | 0.731 | 1.000 | 0.179 | 0.984 | 1.000 | 0.276 | 1.000 | 1.000 | 0.482 | 1.000 | 1.000 | 0.912 | 1.000 | 1.000 |
| 50 | 0.121 | 0.893 | 1.000 | 0.217 | 1.000 | 1.000 | 0.361 | 1.000 | 1.000 | 0.677 | 1.000 | 1.000 | 0.988 | 1.000 | 1.000 |
| 100 | 0.118 | 0.886 | 1.000 | 0.236 | 1.000 | 1.000 | 0.385 | 1.000 | 1.000 | 0.704 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.105 | 0.920 | 1.000 | 0.213 | 1.000 | 1.000 | 0.363 | 1.000 | 1.000 | 0.687 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.098 | 0.907 | 1.000 | 0.217 | 1.000 | 1.000 | 0.370 | 1.000 | 1.000 | 0.705 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.144 | 0.676 | 1.000 | 0.225 | 0.980 | 1.000 | 0.314 | 1.000 | 1.000 | 0.474 | 1.000 | 1.000 | 0.847 | 1.000 | 1.000 |
| 50 | 0.141 | 0.876 | 1.000 | 0.247 | 1.000 | 1.000 | 0.360 | 1.000 | 1.000 | 0.649 | 1.000 | 1.000 | 0.982 | 1.000 | 1.000 |
| 100 | 0.118 | 0.883 | 1.000 | 0.237 | 1.000 | 1.000 | 0.389 | 1.000 | 1.000 | 0.670 | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 |
| 300 | 0.108 | 0.930 | 1.000 | 0.205 | 1.000 | 1.000 | 0.348 | 1.000 | 1.000 | 0.704 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.126 | 0.921 | 1.000 | 0.206 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.670 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |

Table 5: Empirical powers of the sphericity test when $\boldsymbol{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\boldsymbol{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.15^{|r_1-r_2|} I(|r_1-r_2|)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | | | | | | | Normal Instances | | | | | | | | |
| 0 | 0.190 | 0.996 | 1.000 | 0.458 | 1.000 | 1.000 | 0.726 | 1.000 | 1.000 | 0.970 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.231 | 1.000 | 1.000 | 0.588 | 1.000 | 1.000 | 0.865 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.212 | 1.000 | 1.000 | 0.581 | 1.000 | 1.000 | 0.896 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.265 | 1.000 | 1.000 | 0.620 | 1.000 | 1.000 | 0.909 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.258 | 1.000 | 1.000 | 0.645 | 1.000 | 1.000 | 0.908 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances I | | | | | | | | |
| 10 | 0.242 | 0.996 | 1.000 | 0.480 | 1.000 | 1.000 | 0.715 | 1.000 | 1.000 | 0.946 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.258 | 1.000 | 1.000 | 0.598 | 1.000 | 1.000 | 0.883 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.250 | 1.000 | 1.000 | 0.622 | 1.000 | 1.000 | 0.882 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.246 | 1.000 | 1.000 | 0.626 | 1.000 | 1.000 | 0.908 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.254 | 1.000 | 1.000 | 0.613 | 1.000 | 1.000 | 0.898 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances II | | | | | | | | |
| 10 | 0.211 | 0.995 | 1.000 | 0.425 | 1.000 | 1.000 | 0.662 | 1.000 | 1.000 | 0.930 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.233 | 1.000 | 1.000 | 0.574 | 1.000 | 1.000 | 0.854 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.248 | 1.000 | 1.000 | 0.623 | 1.000 | 1.000 | 0.881 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.235 | 1.000 | 1.000 | 0.619 | 1.000 | 1.000 | 0.901 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.230 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.891 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances III | | | | | | | | |
| 10 | 0.230 | 0.990 | 1.000 | 0.449 | 1.000 | 1.000 | 0.630 | 1.000 | 1.000 | 0.906 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 50 | 0.256 | 1.000 | 1.000 | 0.587 | 1.000 | 1.000 | 0.836 | 1.000 | 1.000 | 0.989 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.248 | 1.000 | 1.000 | 0.598 | 1.000 | 1.000 | 0.886 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.237 | 1.000 | 1.000 | 0.598 | 1.000 | 1.000 | 0.877 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.253 | 1.000 | 1.000 | 0.627 | 1.000 | 1.000 | 0.924 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 6: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.042 | 0.058 | 0.050 | 0.038 | 0.053 | 0.043 | 0.052 | 0.064 | 0.059 | 0.064 | 0.049 | 0.048 | 0.050 | 0.048 | 0.050 |
| 50 | 0.037 | 0.053 | 0.054 | 0.042 | 0.066 | 0.056 | 0.062 | 0.055 | 0.044 | 0.060 | 0.047 | 0.038 | 0.054 | 0.055 | 0.050 |
| 100 | 0.057 | 0.055 | 0.066 | 0.047 | 0.049 | 0.053 | 0.062 | 0.059 | 0.048 | 0.052 | 0.052 | 0.048 | 0.037 | 0.053 | 0.058 |
| 300 | 0.051 | 0.058 | 0.060 | 0.045 | 0.054 | 0.052 | 0.045 | 0.063 | 0.059 | 0.047 | 0.048 | 0.047 | 0.049 | 0.051 | 0.040 |
| 600 | 0.044 | 0.047 | 0.056 | 0.047 | 0.060 | 0.045 | 0.056 | 0.045 | 0.050 | 0.052 | 0.050 | 0.050 | 0.045 | 0.051 | 0.052 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.047 | 0.054 | 0.049 | 0.047 | 0.054 | 0.057 | 0.054 | 0.056 | 0.052 | 0.062 | 0.046 | 0.049 | 0.052 | 0.054 | 0.049 |
| 50 | 0.045 | 0.053 | 0.071 | 0.057 | 0.063 | 0.052 | 0.050 | 0.057 | 0.048 | 0.050 | 0.058 | 0.058 | 0.048 | 0.050 | 0.064 |
| 100 | 0.050 | 0.059 | 0.054 | 0.050 | 0.048 | 0.071 | 0.055 | 0.038 | 0.060 | 0.054 | 0.041 | 0.052 | 0.057 | 0.047 | 0.049 |
| 300 | 0.050 | 0.045 | 0.064 | 0.058 | 0.045 | 0.057 | 0.056 | 0.047 | 0.058 | 0.055 | 0.055 | 0.044 | 0.041 | 0.047 | 0.058 |
| 600 | 0.046 | 0.056 | 0.071 | 0.049 | 0.067 | 0.059 | 0.046 | 0.063 | 0.059 | 0.053 | 0.054 | 0.050 | 0.054 | 0.050 | 0.053 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.039 | 0.044 | 0.056 | 0.049 | 0.052 | 0.051 | 0.048 | 0.055 | 0.048 | 0.041 | 0.049 | 0.061 | 0.044 | 0.058 | 0.058 |
| 50 | 0.052 | 0.046 | 0.061 | 0.054 | 0.057 | 0.042 | 0.053 | 0.063 | 0.058 | 0.060 | 0.058 | 0.044 | 0.051 | 0.052 | 0.043 |
| 100 | 0.048 | 0.055 | 0.055 | 0.053 | 0.051 | 0.052 | 0.056 | 0.045 | 0.048 | 0.057 | 0.038 | 0.057 | 0.053 | 0.045 | 0.048 |
| 300 | 0.044 | 0.053 | 0.060 | 0.054 | 0.043 | 0.056 | 0.055 | 0.044 | 0.060 | 0.064 | 0.052 | 0.050 | 0.055 | 0.049 | 0.042 |
| 600 | 0.061 | 0.058 | 0.064 | 0.043 | 0.050 | 0.047 | 0.050 | 0.046 | 0.057 | 0.047 | 0.057 | 0.062 | 0.051 | 0.061 | 0.040 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.043 | 0.037 | 0.051 | 0.048 | 0.043 | 0.046 | 0.050 | 0.047 | 0.049 | 0.046 | 0.043 | 0.049 | 0.038 | 0.045 | 0.052 |
| 50 | 0.049 | 0.071 | 0.057 | 0.052 | 0.054 | 0.046 | 0.047 | 0.054 | 0.057 | 0.058 | 0.048 | 0.050 | 0.057 | 0.058 | 0.049 |
| 100 | 0.053 | 0.051 | 0.050 | 0.043 | 0.052 | 0.056 | 0.049 | 0.050 | 0.040 | 0.053 | 0.051 | 0.057 | 0.041 | 0.051 | 0.053 |
| 300 | 0.048 | 0.053 | 0.062 | 0.044 | 0.052 | 0.058 | 0.037 | 0.052 | 0.059 | 0.044 | 0.062 | 0.047 | 0.048 | 0.054 | 0.054 |
| 600 | 0.047 | 0.054 | 0.064 | 0.060 | 0.059 | 0.052 | 0.044 | 0.054 | 0.044 | 0.054 | 0.057 | 0.047 | 0.052 | 0.050 | 0.052 |

Table 7: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} \sim \mathrm{U}(0.5, 1.5)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.041 | 0.059 | 0.056 | 0.042 | 0.054 | 0.041 | 0.054 | 0.052 | 0.050 | 0.061 | 0.046 | 0.053 | 0.053 | 0.048 | 0.048 |
| 50 | 0.032 | 0.061 | 0.049 | 0.043 | 0.058 | 0.046 | 0.055 | 0.051 | 0.038 | 0.057 | 0.044 | 0.049 | 0.051 | 0.057 | 0.056 |
| 100 | 0.051 | 0.062 | 0.063 | 0.044 | 0.065 | 0.058 | 0.052 | 0.056 | 0.049 | 0.048 | 0.058 | 0.052 | 0.040 | 0.048 | 0.057 |
| 300 | 0.054 | 0.062 | 0.060 | 0.042 | 0.056 | 0.047 | 0.043 | 0.050 | 0.048 | 0.052 | 0.053 | 0.048 | 0.049 | 0.044 | 0.044 |
| 600 | 0.040 | 0.053 | 0.060 | 0.049 | 0.044 | 0.056 | 0.057 | 0.048 | 0.042 | 0.052 | 0.042 | 0.064 | 0.041 | 0.053 | 0.064 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.052 | 0.047 | 0.054 | 0.048 | 0.050 | 0.053 | 0.051 | 0.047 | 0.045 | 0.061 | 0.047 | 0.047 | 0.055 | 0.052 | 0.043 |
| 50 | 0.048 | 0.054 | 0.064 | 0.049 | 0.063 | 0.036 | 0.052 | 0.046 | 0.044 | 0.050 | 0.054 | 0.055 | 0.049 | 0.044 | 0.059 |
| 100 | 0.057 | 0.051 | 0.055 | 0.043 | 0.046 | 0.062 | 0.054 | 0.051 | 0.060 | 0.061 | 0.052 | 0.055 | 0.055 | 0.048 | 0.049 |
| 300 | 0.050 | 0.046 | 0.067 | 0.055 | 0.045 | 0.055 | 0.057 | 0.038 | 0.063 | 0.051 | 0.050 | 0.051 | 0.049 | 0.052 | 0.052 |
| 600 | 0.048 | 0.037 | 0.074 | 0.058 | 0.072 | 0.058 | 0.057 | 0.065 | 0.060 | 0.040 | 0.057 | 0.044 | 0.050 | 0.046 | 0.039 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.040 | 0.048 | 0.054 | 0.044 | 0.048 | 0.057 | 0.048 | 0.059 | 0.047 | 0.041 | 0.052 | 0.054 | 0.047 | 0.057 | 0.056 |
| 50 | 0.045 | 0.049 | 0.055 | 0.058 | 0.063 | 0.047 | 0.053 | 0.061 | 0.056 | 0.056 | 0.047 | 0.041 | 0.053 | 0.048 | 0.039 |
| 100 | 0.053 | 0.056 | 0.046 | 0.057 | 0.047 | 0.053 | 0.054 | 0.050 | 0.048 | 0.054 | 0.054 | 0.044 | 0.053 | 0.044 | 0.055 |
| 300 | 0.049 | 0.046 | 0.052 | 0.053 | 0.045 | 0.064 | 0.053 | 0.047 | 0.059 | 0.052 | 0.051 | 0.042 | 0.056 | 0.059 | 0.046 |
| 600 | 0.047 | 0.063 | 0.065 | 0.044 | 0.047 | 0.053 | 0.050 | 0.055 | 0.060 | 0.051 | 0.061 | 0.050 | 0.061 | 0.053 | 0.048 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.050 | 0.042 | 0.052 | 0.041 | 0.043 | 0.042 | 0.037 | 0.045 | 0.051 | 0.038 | 0.042 | 0.046 | 0.047 | 0.050 | 0.046 |
| 50 | 0.041 | 0.061 | 0.055 | 0.059 | 0.050 | 0.047 | 0.042 | 0.049 | 0.056 | 0.064 | 0.055 | 0.053 | 0.055 | 0.061 | 0.056 |
| 100 | 0.062 | 0.060 | 0.044 | 0.041 | 0.048 | 0.050 | 0.048 | 0.038 | 0.049 | 0.053 | 0.062 | 0.056 | 0.056 | 0.047 | 0.057 |
| 300 | 0.037 | 0.060 | 0.064 | 0.037 | 0.054 | 0.050 | 0.042 | 0.054 | 0.048 | 0.044 | 0.059 | 0.055 | 0.049 | 0.045 | 0.054 |
| 600 | 0.050 | 0.061 | 0.056 | 0.062 | 0.055 | 0.060 | 0.055 | 0.052 | 0.060 | 0.046 | 0.054 | 0.050 | 0.058 | 0.048 | 0.060 |

Table 8: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with the diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} = 1 + I(r_1 \leq 0.9r)$ and $\alpha = 0.05$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.045 | 0.050 | 0.055 | 0.045 | 0.060 | 0.044 | 0.049 | 0.059 | 0.056 | 0.060 | 0.048 | 0.056 | 0.060 | 0.051 | 0.046 |
| 50 | 0.038 | 0.048 | 0.056 | 0.046 | 0.062 | 0.056 | 0.062 | 0.055 | 0.043 | 0.050 | 0.043 | 0.043 | 0.054 | 0.056 | 0.048 |
| 100 | 0.055 | 0.059 | 0.061 | 0.051 | 0.047 | 0.050 | 0.060 | 0.047 | 0.056 | 0.053 | 0.054 | 0.050 | 0.045 | 0.046 | 0.063 |
| 300 | 0.055 | 0.060 | 0.057 | 0.044 | 0.055 | 0.048 | 0.050 | 0.059 | 0.057 | 0.047 | 0.057 | 0.041 | 0.050 | 0.046 | 0.048 |
| 600 | 0.045 | 0.051 | 0.061 | 0.045 | 0.057 | 0.056 | 0.050 | 0.046 | 0.051 | 0.050 | 0.047 | 0.062 | 0.048 | 0.052 | 0.046 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.049 | 0.048 | 0.045 | 0.048 | 0.051 | 0.056 | 0.054 | 0.053 | 0.051 | 0.049 | 0.041 | 0.052 | 0.053 | 0.053 | 0.049 |
| 50 | 0.046 | 0.059 | 0.068 | 0.066 | 0.056 | 0.056 | 0.050 | 0.057 | 0.046 | 0.054 | 0.057 | 0.059 | 0.055 | 0.056 | 0.061 |
| 100 | 0.051 | 0.059 | 0.061 | 0.051 | 0.049 | 0.066 | 0.056 | 0.042 | 0.059 | 0.057 | 0.060 | 0.058 | 0.059 | 0.048 | 0.053 |
| 300 | 0.051 | 0.051 | 0.066 | 0.056 | 0.048 | 0.061 | 0.059 | 0.053 | 0.055 | 0.050 | 0.046 | 0.043 | 0.039 | 0.048 | 0.050 |
| 600 | 0.040 | 0.059 | 0.075 | 0.051 | 0.062 | 0.062 | 0.045 | 0.063 | 0.055 | 0.057 | 0.059 | 0.046 | 0.048 | 0.051 | 0.048 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.045 | 0.043 | 0.056 | 0.039 | 0.048 | 0.049 | 0.046 | 0.053 | 0.054 | 0.044 | 0.058 | 0.063 | 0.041 | 0.055 | 0.058 |
| 50 | 0.047 | 0.061 | 0.064 | 0.057 | 0.063 | 0.048 | 0.056 | 0.062 | 0.059 | 0.055 | 0.048 | 0.049 | 0.055 | 0.056 | 0.049 |
| 100 | 0.054 | 0.053 | 0.058 | 0.060 | 0.051 | 0.052 | 0.049 | 0.041 | 0.054 | 0.051 | 0.044 | 0.051 | 0.053 | 0.045 | 0.045 |
| 300 | 0.052 | 0.048 | 0.054 | 0.047 | 0.047 | 0.057 | 0.053 | 0.047 | 0.057 | 0.059 | 0.055 | 0.050 | 0.046 | 0.055 | 0.044 |
| 600 | 0.045 | 0.051 | 0.060 | 0.041 | 0.048 | 0.047 | 0.052 | 0.045 | 0.050 | 0.048 | 0.056 | 0.060 | 0.064 | 0.060 | 0.052 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.047 | 0.046 | 0.054 | 0.042 | 0.048 | 0.046 | 0.044 | 0.038 | 0.048 | 0.046 | 0.038 | 0.047 | 0.044 | 0.048 | 0.051 |
| 50 | 0.044 | 0.065 | 0.055 | 0.060 | 0.048 | 0.050 | 0.051 | 0.054 | 0.057 | 0.049 | 0.040 | 0.049 | 0.056 | 0.060 | 0.048 |
| 100 | 0.053 | 0.055 | 0.056 | 0.043 | 0.051 | 0.048 | 0.053 | 0.039 | 0.051 | 0.050 | 0.052 | 0.058 | 0.044 | 0.055 | 0.057 |
| 300 | 0.036 | 0.041 | 0.056 | 0.048 | 0.048 | 0.055 | 0.046 | 0.052 | 0.057 | 0.051 | 0.064 | 0.063 | 0.057 | 0.056 | 0.048 |
| 600 | 0.049 | 0.050 | 0.056 | 0.047 | 0.056 | 0.044 | 0.045 | 0.052 | 0.044 | 0.056 | 0.060 | 0.058 | 0.048 | 0.048 | 0.060 |

Table 9: Empirical powers of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.10^{|r_1-r_2|} I(|r_1-r_2|)$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.093 | 0.821 | 1.000 | 0.177 | 0.991 | 1.000 | 0.317 | 1.000 | 1.000 | 0.586 | 1.000 | 1.000 | 0.945 | 1.000 | 1.000 |
| 50 | 0.101 | 0.908 | 1.000 | 0.215 | 1.000 | 1.000 | 0.366 | 1.000 | 1.000 | 0.690 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 100 | 0.111 | 0.908 | 1.000 | 0.202 | 1.000 | 1.000 | 0.382 | 1.000 | 1.000 | 0.703 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.126 | 0.949 | 1.000 | 0.218 | 1.000 | 1.000 | 0.357 | 1.000 | 1.000 | 0.731 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.110 | 0.917 | 1.000 | 0.241 | 1.000 | 1.000 | 0.399 | 1.000 | 1.000 | 0.732 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.102 | 0.808 | 1.000 | 0.223 | 0.996 | 1.000 | 0.338 | 1.000 | 1.000 | 0.605 | 1.000 | 1.000 | 0.952 | 1.000 | 1.000 |
| 50 | 0.114 | 0.904 | 1.000 | 0.216 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.720 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 100 | 0.128 | 0.919 | 1.000 | 0.225 | 1.000 | 1.000 | 0.381 | 1.000 | 1.000 | 0.699 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 300 | 0.110 | 0.923 | 1.000 | 0.241 | 1.000 | 1.000 | 0.382 | 1.000 | 1.000 | 0.733 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 600 | 0.123 | 0.916 | 1.000 | 0.226 | 1.000 | 1.000 | 0.394 | 1.000 | 1.000 | 0.723 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.100 | 0.807 | 1.000 | 0.202 | 0.996 | 1.000 | 0.315 | 1.000 | 1.000 | 0.573 | 1.000 | 1.000 | 0.957 | 1.000 | 1.000 |
| 50 | 0.103 | 0.898 | 1.000 | 0.236 | 1.000 | 1.000 | 0.373 | 1.000 | 1.000 | 0.687 | 1.000 | 1.000 | 0.991 | 1.000 | 1.000 |
| 100 | 0.115 | 0.896 | 1.000 | 0.238 | 1.000 | 1.000 | 0.377 | 1.000 | 1.000 | 0.705 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.106 | 0.927 | 1.000 | 0.217 | 1.000 | 1.000 | 0.361 | 1.000 | 1.000 | 0.689 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.098 | 0.903 | 1.000 | 0.209 | 1.000 | 1.000 | 0.371 | 1.000 | 1.000 | 0.711 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.107 | 0.814 | 1.000 | 0.212 | 0.996 | 1.000 | 0.324 | 1.000 | 1.000 | 0.585 | 1.000 | 1.000 | 0.956 | 1.000 | 1.000 |
| 50 | 0.107 | 0.904 | 1.000 | 0.238 | 1.000 | 1.000 | 0.376 | 1.000 | 1.000 | 0.710 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 100 | 0.113 | 0.908 | 1.000 | 0.203 | 1.000 | 1.000 | 0.375 | 1.000 | 1.000 | 0.690 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.101 | 0.929 | 1.000 | 0.212 | 1.000 | 1.000 | 0.356 | 1.000 | 1.000 | 0.708 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.113 | 0.919 | 1.000 | 0.212 | 1.000 | 1.000 | 0.387 | 1.000 | 1.000 | 0.640 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |

Table 10: Empirical powers of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.15^{|r_1-r_2|} I(|r_1-r_2|)$.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.203 | 0.997 | 1.000 | 0.493 | 1.000 | 1.000 | 0.774 | 1.000 | 1.000 | 0.975 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.231 | 1.000 | 1.000 | 0.602 | 1.000 | 1.000 | 0.876 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.211 | 1.000 | 1.000 | 0.590 | 1.000 | 1.000 | 0.894 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.270 | 1.000 | 1.000 | 0.628 | 1.000 | 1.000 | 0.905 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.253 | 1.000 | 1.000 | 0.653 | 1.000 | 1.000 | 0.908 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.233 | 0.996 | 1.000 | 0.520 | 1.000 | 1.000 | 0.767 | 1.000 | 1.000 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.245 | 1.000 | 1.000 | 0.612 | 1.000 | 1.000 | 0.908 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.259 | 1.000 | 1.000 | 0.628 | 1.000 | 1.000 | 0.881 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.248 | 1.000 | 1.000 | 0.636 | 1.000 | 1.000 | 0.905 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.248 | 1.000 | 1.000 | 0.614 | 1.000 | 1.000 | 0.902 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.209 | 0.998 | 1.000 | 0.493 | 1.000 | 1.000 | 0.765 | 1.000 | 1.000 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.235 | 1.000 | 1.000 | 0.602 | 1.000 | 1.000 | 0.886 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.246 | 1.000 | 1.000 | 0.624 | 1.000 | 1.000 | 0.895 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.230 | 1.000 | 1.000 | 0.628 | 1.000 | 1.000 | 0.899 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.230 | 1.000 | 1.000 | 0.607 | 1.000 | 1.000 | 0.889 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.224 | 0.999 | 1.000 | 0.517 | 1.000 | 1.000 | 0.771 | 1.000 | 1.000 | 0.974 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.234 | 1.000 | 1.000 | 0.632 | 1.000 | 1.000 | 0.876 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.231 | 1.000 | 1.000 | 0.627 | 1.000 | 1.000 | 0.922 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.240 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.883 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.249 | 1.000 | 1.000 | 0.638 | 1.000 | 1.000 | 0.932 | 1.000 | 1.000 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |

Table 11: Empirical sizes of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\alpha = 0.05$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.051 | 0.066 | 0.050 | 0.045 | 0.055 | 0.054 | 0.059 | 0.059 | 0.056 | 0.060 | 0.053 | 0.058 | 0.053 | 0.057 | 0.045 |
| 50 | 0.052 | 0.060 | 0.053 | 0.053 | 0.063 | 0.059 | 0.064 | 0.055 | 0.048 | 0.066 | 0.042 | 0.052 | 0.056 | 0.053 | 0.050 |
| 100 | 0.068 | 0.060 | 0.060 | 0.050 | 0.054 | 0.051 | 0.069 | 0.058 | 0.056 | 0.060 | 0.058 | 0.050 | 0.047 | 0.056 | 0.057 |
| 300 | 0.066 | 0.058 | 0.056 | 0.054 | 0.054 | 0.050 | 0.055 | 0.060 | 0.057 | 0.049 | 0.050 | 0.045 | 0.052 | 0.046 | 0.040 |
| 600 | 0.056 | 0.051 | 0.048 | 0.051 | 0.055 | 0.045 | 0.056 | 0.047 | 0.051 | 0.056 | 0.051 | 0.054 | 0.047 | 0.051 | 0.049 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.084 | 0.075 | 0.073 | 0.077 | 0.063 | 0.070 | 0.087 | 0.076 | 0.069 | 0.071 | 0.074 | 0.054 | 0.071 | 0.077 | 0.057 |
| 50 | 0.064 | 0.059 | 0.061 | 0.060 | 0.065 | 0.043 | 0.054 | 0.059 | 0.052 | 0.052 | 0.063 | 0.059 | 0.058 | 0.054 | 0.057 |
| 100 | 0.060 | 0.061 | 0.056 | 0.057 | 0.047 | 0.071 | 0.064 | 0.045 | 0.058 | 0.066 | 0.043 | 0.056 | 0.063 | 0.056 | 0.051 |
| 300 | 0.059 | 0.047 | 0.057 | 0.068 | 0.054 | 0.052 | 0.066 | 0.050 | 0.056 | 0.053 | 0.059 | 0.045 | 0.044 | 0.046 | 0.052 |
| 600 | 0.060 | 0.054 | 0.063 | 0.056 | 0.070 | 0.059 | 0.052 | 0.067 | 0.058 | 0.057 | 0.058 | 0.046 | 0.062 | 0.054 | 0.049 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.084 | 0.063 | 0.075 | 0.084 | 0.073 | 0.064 | 0.070 | 0.073 | 0.087 | 0.073 | 0.069 | 0.072 | 0.064 | 0.078 | 0.095 |
| 50 | 0.067 | 0.053 | 0.075 | 0.075 | 0.065 | 0.052 | 0.071 | 0.075 | 0.065 | 0.075 | 0.060 | 0.061 | 0.074 | 0.052 | 0.048 |
| 100 | 0.067 | 0.051 | 0.047 | 0.062 | 0.044 | 0.060 | 0.060 | 0.048 | 0.059 | 0.057 | 0.044 | 0.065 | 0.059 | 0.058 | 0.052 |
| 300 | 0.057 | 0.057 | 0.053 | 0.055 | 0.051 | 0.061 | 0.057 | 0.039 | 0.063 | 0.063 | 0.059 | 0.047 | 0.056 | 0.055 | 0.042 |
| 600 | 0.071 | 0.059 | 0.068 | 0.050 | 0.051 | 0.047 | 0.057 | 0.044 | 0.060 | 0.053 | 0.057 | 0.064 | 0.056 | 0.062 | 0.046 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.098 | 0.096 | 0.118 | 0.105 | 0.085 | 0.096 | 0.126 | 0.100 | 0.089 | 0.115 | 0.096 | 0.098 | 0.102 | 0.102 | 0.099 |
| 50 | 0.080 | 0.086 | 0.073 | 0.093 | 0.071 | 0.068 | 0.077 | 0.064 | 0.081 | 0.082 | 0.064 | 0.059 | 0.102 | 0.071 | 0.067 |
| 100 | 0.073 | 0.054 | 0.062 | 0.061 | 0.058 | 0.062 | 0.070 | 0.055 | 0.053 | 0.068 | 0.058 | 0.064 | 0.065 | 0.061 | 0.054 |
| 300 | 0.064 | 0.052 | 0.059 | 0.050 | 0.059 | 0.060 | 0.051 | 0.057 | 0.059 | 0.051 | 0.065 | 0.054 | 0.059 | 0.058 | 0.060 |
| 600 | 0.066 | 0.059 | 0.064 | 0.058 | 0.059 | 0.058 | 0.055 | 0.057 | 0.056 | 0.058 | 0.055 | 0.049 | 0.050 | 0.054 | 0.050 |

Table 12: Empirical powers of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} \sim \mathrm{U}(0.5, 1.5)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.474 | 1.000 | 1.000 | 0.945 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.367 | 1.000 | 1.000 | 0.929 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.562 | 1.000 | 1.000 | 0.976 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.506 | 1.000 | 1.000 | 0.976 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.518 | 1.000 | 1.000 | 0.973 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.462 | 1.000 | 1.000 | 0.893 | 1.000 | 1.000 | 0.991 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.391 | 1.000 | 1.000 | 0.917 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.574 | 1.000 | 1.000 | 0.976 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.539 | 1.000 | 1.000 | 0.975 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.512 | 1.000 | 1.000 | 0.962 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.237 | 0.989 | 1.000 | 0.827 | 1.000 | 1.000 | 0.985 | 1.000 | 1.000 | 0.991 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.464 | 1.000 | 1.000 | 0.945 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.619 | 1.000 | 1.000 | 0.941 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.454 | 1.000 | 1.000 | 0.973 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.527 | 1.000 | 1.000 | 0.971 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.155 | 0.729 | 1.000 | 0.548 | 1.000 | 1.000 | 0.960 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.387 | 1.000 | 1.000 | 0.812 | 1.000 | 1.000 | 0.987 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.424 | 1.000 | 1.000 | 0.928 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.488 | 1.000 | 1.000 | 0.971 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.506 | 1.000 | 1.000 | 0.965 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 13: Empirical powers of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} = 1 + I(r_1 \leq 0.9r)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | | | | | | | Normal Instances | | | | | | | | |
| 10 | 0.135 | 0.976 | 1.000 | 0.242 | 1.000 | 1.000 | 0.451 | 1.000 | 1.000 | 0.802 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 |
| 50 | 0.139 | 0.982 | 1.000 | 0.289 | 1.000 | 1.000 | 0.504 | 1.000 | 1.000 | 0.845 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.146 | 0.989 | 1.000 | 0.278 | 1.000 | 1.000 | 0.512 | 1.000 | 1.000 | 0.866 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.142 | 0.994 | 1.000 | 0.285 | 1.000 | 1.000 | 0.491 | 1.000 | 1.000 | 0.893 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.143 | 0.990 | 1.000 | 0.314 | 1.000 | 1.000 | 0.522 | 1.000 | 1.000 | 0.889 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances I | | | | | | | | |
| 10 | 0.159 | 0.925 | 1.000 | 0.297 | 1.000 | 1.000 | 0.445 | 1.000 | 1.000 | 0.747 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 50 | 0.156 | 0.982 | 1.000 | 0.302 | 1.000 | 1.000 | 0.502 | 1.000 | 1.000 | 0.840 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.157 | 0.979 | 1.000 | 0.296 | 1.000 | 1.000 | 0.500 | 1.000 | 1.000 | 0.842 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 300 | 0.144 | 0.986 | 1.000 | 0.316 | 1.000 | 1.000 | 0.500 | 1.000 | 1.000 | 0.883 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.160 | 0.984 | 1.000 | 0.291 | 1.000 | 1.000 | 0.503 | 1.000 | 1.000 | 0.884 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances II | | | | | | | | |
| 10 | 0.157 | 0.915 | 1.000 | 0.253 | 1.000 | 1.000 | 0.405 | 1.000 | 1.000 | 0.692 | 1.000 | 1.000 | 0.990 | 1.000 | 1.000 |
| 50 | 0.161 | 0.982 | 1.000 | 0.290 | 1.000 | 1.000 | 0.502 | 1.000 | 1.000 | 0.845 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 100 | 0.149 | 0.986 | 1.000 | 0.318 | 1.000 | 1.000 | 0.505 | 1.000 | 1.000 | 0.860 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.131 | 0.982 | 1.000 | 0.296 | 1.000 | 1.000 | 0.491 | 1.000 | 1.000 | 0.841 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.134 | 0.979 | 1.000 | 0.302 | 1.000 | 1.000 | 0.491 | 1.000 | 1.000 | 0.850 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances III | | | | | | | | |
| 10 | 0.169 | 0.872 | 1.000 | 0.264 | 0.999 | 1.000 | 0.393 | 1.000 | 1.000 | 0.611 | 1.000 | 1.000 | 0.963 | 1.000 | 1.000 |
| 50 | 0.161 | 0.965 | 1.000 | 0.314 | 1.000 | 1.000 | 0.462 | 1.000 | 1.000 | 0.780 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 100 | 0.172 | 0.976 | 1.000 | 0.300 | 1.000 | 1.000 | 0.504 | 1.000 | 1.000 | 0.826 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 300 | 0.135 | 0.974 | 1.000 | 0.279 | 1.000 | 1.000 | 0.476 | 1.000 | 1.000 | 0.844 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.157 | 0.984 | 1.000 | 0.268 | 1.000 | 1.000 | 0.508 | 1.000 | 1.000 | 0.860 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 14: Empirical powers of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.1^{|r_1-r_2|} I(|r_1-r_2|)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.102 | 0.786 | 1.000 | 0.189 | 0.988 | 1.000 | 0.298 | 1.000 | 1.000 | 0.551 | 1.000 | 1.000 | 0.924 | 1.000 | 1.000 |
| 50 | 0.125 | 0.888 | 1.000 | 0.226 | 1.000 | 1.000 | 0.374 | 1.000 | 1.000 | 0.677 | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 |
| 100 | 0.121 | 0.907 | 1.000 | 0.210 | 1.000 | 1.000 | 0.382 | 1.000 | 1.000 | 0.695 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.131 | 0.944 | 1.000 | 0.222 | 1.000 | 1.000 | 0.359 | 1.000 | 1.000 | 0.725 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.126 | 0.920 | 1.000 | 0.250 | 1.000 | 1.000 | 0.398 | 1.000 | 1.000 | 0.728 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.141 | 0.756 | 1.000 | 0.226 | 0.990 | 1.000 | 0.315 | 1.000 | 1.000 | 0.553 | 1.000 | 1.000 | 0.917 | 1.000 | 1.000 |
| 50 | 0.140 | 0.894 | 1.000 | 0.230 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.694 | 1.000 | 1.000 | 0.992 | 1.000 | 1.000 |
| 100 | 0.142 | 0.903 | 1.000 | 0.223 | 1.000 | 1.000 | 0.365 | 1.000 | 1.000 | 0.696 | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 |
| 300 | 0.123 | 0.922 | 1.000 | 0.247 | 1.000 | 1.000 | 0.389 | 1.000 | 1.000 | 0.731 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 600 | 0.144 | 0.919 | 1.000 | 0.227 | 1.000 | 1.000 | 0.397 | 1.000 | 1.000 | 0.720 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.127 | 0.735 | 1.000 | 0.184 | 0.984 | 1.000 | 0.277 | 1.000 | 1.000 | 0.488 | 1.000 | 1.000 | 0.911 | 1.000 | 1.000 |
| 50 | 0.131 | 0.891 | 1.000 | 0.225 | 1.000 | 1.000 | 0.364 | 1.000 | 1.000 | 0.673 | 1.000 | 1.000 | 0.988 | 1.000 | 1.000 |
| 100 | 0.131 | 0.886 | 1.000 | 0.234 | 1.000 | 1.000 | 0.382 | 1.000 | 1.000 | 0.702 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 300 | 0.121 | 0.923 | 1.000 | 0.218 | 1.000 | 1.000 | 0.364 | 1.000 | 1.000 | 0.686 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.111 | 0.904 | 1.000 | 0.218 | 1.000 | 1.000 | 0.372 | 1.000 | 1.000 | 0.703 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.150 | 0.680 | 1.000 | 0.228 | 0.981 | 1.000 | 0.309 | 1.000 | 1.000 | 0.475 | 1.000 | 1.000 | 0.845 | 1.000 | 1.000 |
| 50 | 0.146 | 0.869 | 1.000 | 0.257 | 1.000 | 1.000 | 0.366 | 1.000 | 1.000 | 0.645 | 1.000 | 1.000 | 0.981 | 1.000 | 1.000 |
| 100 | 0.132 | 0.886 | 1.000 | 0.247 | 1.000 | 1.000 | 0.390 | 1.000 | 1.000 | 0.668 | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 |
| 300 | 0.113 | 0.926 | 1.000 | 0.211 | 1.000 | 1.000 | 0.351 | 1.000 | 1.000 | 0.699 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 600 | 0.136 | 0.915 | 1.000 | 0.216 | 1.000 | 1.000 | 0.393 | 1.000 | 1.000 | 0.660 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |

Table 15: Empirical powers of the sphericity test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1r_2} = 0.15^{|r_1-r_2|}I(|r_1-r_2|)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.193 | 0.995 | 1.000 | 0.455 | 1.000 | 1.000 | 0.724 | 1.000 | 1.000 | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.244 | 1.000 | 1.000 | 0.583 | 1.000 | 1.000 | 0.862 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.221 | 1.000 | 1.000 | 0.582 | 1.000 | 1.000 | 0.892 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.275 | 1.000 | 1.000 | 0.618 | 1.000 | 1.000 | 0.907 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.272 | 1.000 | 1.000 | 0.639 | 1.000 | 1.000 | 0.900 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.246 | 0.997 | 1.000 | 0.481 | 1.000 | 1.000 | 0.712 | 1.000 | 1.000 | 0.944 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.262 | 1.000 | 1.000 | 0.599 | 1.000 | 1.000 | 0.874 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.262 | 1.000 | 1.000 | 0.618 | 1.000 | 1.000 | 0.881 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.250 | 1.000 | 1.000 | 0.621 | 1.000 | 1.000 | 0.904 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.261 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.894 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.215 | 0.995 | 1.000 | 0.417 | 1.000 | 1.000 | 0.664 | 1.000 | 1.000 | 0.926 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.241 | 1.000 | 1.000 | 0.571 | 1.000 | 1.000 | 0.849 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.265 | 1.000 | 1.000 | 0.617 | 1.000 | 1.000 | 0.876 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.249 | 1.000 | 1.000 | 0.614 | 1.000 | 1.000 | 0.895 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.243 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.883 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.235 | 0.989 | 1.000 | 0.440 | 1.000 | 1.000 | 0.624 | 1.000 | 1.000 | 0.906 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 |
| 50 | 0.268 | 1.000 | 1.000 | 0.582 | 1.000 | 1.000 | 0.833 | 1.000 | 1.000 | 0.989 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.258 | 1.000 | 1.000 | 0.589 | 1.000 | 1.000 | 0.884 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.238 | 1.000 | 1.000 | 0.598 | 1.000 | 1.000 | 0.873 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.261 | 1.000 | 1.000 | 0.619 | 1.000 | 1.000 | 0.915 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 16: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\alpha = 0.05$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.049 | 0.060 | 0.042 | 0.039 | 0.054 | 0.044 | 0.055 | 0.061 | 0.057 | 0.064 | 0.051 | 0.050 | 0.050 | 0.049 | 0.050 |
| 50 | 0.050 | 0.051 | 0.056 | 0.048 | 0.068 | 0.055 | 0.065 | 0.057 | 0.043 | 0.063 | 0.049 | 0.042 | 0.055 | 0.056 | 0.051 |
| 100 | 0.070 | 0.055 | 0.062 | 0.053 | 0.050 | 0.052 | 0.068 | 0.059 | 0.051 | 0.053 | 0.054 | 0.048 | 0.041 | 0.053 | 0.059 |
| 300 | 0.059 | 0.055 | 0.055 | 0.054 | 0.056 | 0.050 | 0.046 | 0.063 | 0.057 | 0.050 | 0.048 | 0.048 | 0.050 | 0.051 | 0.038 |
| 600 | 0.053 | 0.050 | 0.049 | 0.048 | 0.058 | 0.045 | 0.058 | 0.046 | 0.050 | 0.055 | 0.049 | 0.052 | 0.045 | 0.052 | 0.049 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.049 | 0.051 | 0.045 | 0.052 | 0.055 | 0.058 | 0.055 | 0.059 | 0.052 | 0.064 | 0.045 | 0.048 | 0.054 | 0.054 | 0.049 |
| 50 | 0.058 | 0.055 | 0.061 | 0.058 | 0.061 | 0.048 | 0.054 | 0.055 | 0.046 | 0.053 | 0.057 | 0.058 | 0.049 | 0.050 | 0.063 |
| 100 | 0.060 | 0.058 | 0.055 | 0.052 | 0.053 | 0.073 | 0.057 | 0.043 | 0.061 | 0.058 | 0.040 | 0.052 | 0.057 | 0.046 | 0.048 |
| 300 | 0.056 | 0.050 | 0.059 | 0.071 | 0.051 | 0.057 | 0.060 | 0.049 | 0.057 | 0.056 | 0.057 | 0.043 | 0.042 | 0.047 | 0.060 |
| 600 | 0.054 | 0.055 | 0.064 | 0.052 | 0.067 | 0.058 | 0.051 | 0.065 | 0.056 | 0.057 | 0.055 | 0.052 | 0.054 | 0.050 | 0.049 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.051 | 0.050 | 0.055 | 0.051 | 0.053 | 0.052 | 0.049 | 0.055 | 0.049 | 0.044 | 0.051 | 0.063 | 0.046 | 0.058 | 0.057 |
| 50 | 0.061 | 0.049 | 0.060 | 0.063 | 0.060 | 0.044 | 0.057 | 0.061 | 0.058 | 0.063 | 0.058 | 0.044 | 0.051 | 0.052 | 0.044 |
| 100 | 0.063 | 0.049 | 0.046 | 0.056 | 0.048 | 0.052 | 0.063 | 0.047 | 0.048 | 0.061 | 0.037 | 0.058 | 0.053 | 0.045 | 0.048 |
| 300 | 0.057 | 0.058 | 0.051 | 0.058 | 0.048 | 0.054 | 0.059 | 0.045 | 0.059 | 0.066 | 0.052 | 0.050 | 0.057 | 0.049 | 0.044 |
| 600 | 0.068 | 0.055 | 0.072 | 0.050 | 0.049 | 0.047 | 0.053 | 0.045 | 0.060 | 0.049 | 0.056 | 0.060 | 0.052 | 0.063 | 0.047 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.051 | 0.044 | 0.054 | 0.049 | 0.048 | 0.047 | 0.051 | 0.046 | 0.052 | 0.046 | 0.043 | 0.051 | 0.038 | 0.046 | 0.051 |
| 50 | 0.054 | 0.071 | 0.054 | 0.056 | 0.057 | 0.048 | 0.051 | 0.051 | 0.056 | 0.060 | 0.050 | 0.052 | 0.058 | 0.058 | 0.048 |
| 100 | 0.063 | 0.051 | 0.045 | 0.048 | 0.049 | 0.055 | 0.056 | 0.050 | 0.040 | 0.055 | 0.052 | 0.056 | 0.041 | 0.050 | 0.054 |
| 300 | 0.055 | 0.060 | 0.054 | 0.052 | 0.055 | 0.055 | 0.041 | 0.051 | 0.062 | 0.046 | 0.065 | 0.046 | 0.050 | 0.055 | 0.054 |
| 600 | 0.058 | 0.053 | 0.068 | 0.066 | 0.059 | 0.052 | 0.048 | 0.056 | 0.044 | 0.055 | 0.057 | 0.049 | 0.052 | 0.052 | 0.048 |

Table 17: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} \sim \mathrm{U}(0.5, 1.5)$ and $\alpha = 0.05$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.047 | 0.053 | 0.050 | 0.045 | 0.057 | 0.044 | 0.060 | 0.053 | 0.049 | 0.060 | 0.049 | 0.053 | 0.053 | 0.048 | 0.047 |
| 50 | 0.046 | 0.056 | 0.048 | 0.051 | 0.059 | 0.047 | 0.056 | 0.052 | 0.039 | 0.058 | 0.046 | 0.052 | 0.051 | 0.057 | 0.057 |
| 100 | 0.058 | 0.056 | 0.064 | 0.048 | 0.061 | 0.058 | 0.055 | 0.056 | 0.049 | 0.052 | 0.059 | 0.053 | 0.040 | 0.048 | 0.057 |
| 300 | 0.060 | 0.064 | 0.054 | 0.053 | 0.053 | 0.047 | 0.047 | 0.050 | 0.045 | 0.054 | 0.053 | 0.047 | 0.051 | 0.044 | 0.044 |
| 600 | 0.051 | 0.057 | 0.056 | 0.057 | 0.043 | 0.053 | 0.063 | 0.048 | 0.043 | 0.053 | 0.042 | 0.064 | 0.042 | 0.052 | 0.052 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.056 | 0.048 | 0.051 | 0.047 | 0.053 | 0.052 | 0.053 | 0.049 | 0.044 | 0.058 | 0.047 | 0.045 | 0.055 | 0.052 | 0.041 |
| 50 | 0.059 | 0.050 | 0.058 | 0.056 | 0.061 | 0.034 | 0.055 | 0.043 | 0.043 | 0.052 | 0.056 | 0.055 | 0.050 | 0.044 | 0.061 |
| 100 | 0.068 | 0.049 | 0.056 | 0.048 | 0.045 | 0.063 | 0.057 | 0.046 | 0.060 | 0.064 | 0.051 | 0.051 | 0.058 | 0.048 | 0.046 |
| 300 | 0.054 | 0.045 | 0.063 | 0.060 | 0.043 | 0.053 | 0.063 | 0.041 | 0.061 | 0.053 | 0.051 | 0.052 | 0.048 | 0.053 | 0.053 |
| 600 | 0.058 | 0.040 | 0.062 | 0.061 | 0.071 | 0.058 | 0.060 | 0.065 | 0.060 | 0.045 | 0.057 | 0.050 | 0.050 | 0.046 | 0.049 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.044 | 0.047 | 0.053 | 0.050 | 0.054 | 0.057 | 0.052 | 0.058 | 0.048 | 0.041 | 0.054 | 0.052 | 0.047 | 0.056 | 0.055 |
| 50 | 0.056 | 0.054 | 0.049 | 0.067 | 0.065 | 0.048 | 0.060 | 0.056 | 0.056 | 0.059 | 0.050 | 0.043 | 0.054 | 0.048 | 0.040 |
| 100 | 0.063 | 0.058 | 0.043 | 0.064 | 0.044 | 0.050 | 0.057 | 0.049 | 0.046 | 0.056 | 0.050 | 0.045 | 0.053 | 0.045 | 0.054 |
| 300 | 0.054 | 0.054 | 0.053 | 0.056 | 0.045 | 0.062 | 0.055 | 0.045 | 0.056 | 0.055 | 0.050 | 0.040 | 0.059 | 0.059 | 0.044 |
| 600 | 0.059 | 0.063 | 0.064 | 0.048 | 0.053 | 0.047 | 0.055 | 0.054 | 0.060 | 0.059 | 0.060 | 0.078 | 0.062 | 0.054 | 0.045 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.058 | 0.044 | 0.056 | 0.046 | 0.044 | 0.043 | 0.038 | 0.045 | 0.052 | 0.039 | 0.043 | 0.047 | 0.047 | 0.051 | 0.046 |
| 50 | 0.053 | 0.061 | 0.049 | 0.064 | 0.058 | 0.051 | 0.048 | 0.048 | 0.059 | 0.070 | 0.055 | 0.054 | 0.055 | 0.063 | 0.060 |
| 100 | 0.072 | 0.057 | 0.043 | 0.047 | 0.052 | 0.053 | 0.050 | 0.037 | 0.049 | 0.057 | 0.061 | 0.056 | 0.056 | 0.047 | 0.056 |
| 300 | 0.045 | 0.065 | 0.069 | 0.045 | 0.055 | 0.050 | 0.048 | 0.054 | 0.050 | 0.047 | 0.060 | 0.053 | 0.050 | 0.045 | 0.054 |
| 600 | 0.056 | 0.056 | 0.052 | 0.068 | 0.060 | 0.040 | 0.056 | 0.052 | 0.070 | 0.049 | 0.050 | 0.050 | 0.059 | 0.049 | 0.060 |

Table 18: Empirical sizes of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a diagonal matrix with the diagonal elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_1} = 1 + I(r_1 \leq 0.9r)$ and $\alpha = 0.05$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.047 | 0.051 | 0.046 | 0.045 | 0.060 | 0.045 | 0.050 | 0.062 | 0.055 | 0.058 | 0.049 | 0.054 | 0.060 | 0.052 | 0.046 |
| 50 | 0.048 | 0.049 | 0.055 | 0.048 | 0.061 | 0.057 | 0.066 | 0.055 | 0.045 | 0.051 | 0.042 | 0.043 | 0.055 | 0.056 | 0.049 |
| 100 | 0.065 | 0.057 | 0.063 | 0.057 | 0.049 | 0.048 | 0.063 | 0.050 | 0.056 | 0.055 | 0.056 | 0.051 | 0.047 | 0.045 | 0.064 |
| 300 | 0.063 | 0.053 | 0.059 | 0.046 | 0.054 | 0.044 | 0.051 | 0.061 | 0.052 | 0.050 | 0.054 | 0.041 | 0.051 | 0.046 | 0.048 |
| 600 | 0.056 | 0.049 | 0.055 | 0.050 | 0.057 | 0.058 | 0.053 | 0.046 | 0.051 | 0.051 | 0.047 | 0.060 | 0.047 | 0.052 | 0.047 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.053 | 0.056 | 0.041 | 0.055 | 0.050 | 0.057 | 0.059 | 0.055 | 0.051 | 0.054 | 0.043 | 0.052 | 0.053 | 0.053 | 0.049 |
| 50 | 0.060 | 0.050 | 0.063 | 0.070 | 0.055 | 0.051 | 0.055 | 0.055 | 0.047 | 0.056 | 0.057 | 0.058 | 0.055 | 0.058 | 0.060 |
| 100 | 0.066 | 0.056 | 0.060 | 0.053 | 0.050 | 0.067 | 0.059 | 0.045 | 0.056 | 0.061 | 0.059 | 0.056 | 0.060 | 0.050 | 0.052 |
| 300 | 0.061 | 0.056 | 0.060 | 0.061 | 0.055 | 0.063 | 0.067 | 0.050 | 0.054 | 0.054 | 0.050 | 0.044 | 0.041 | 0.049 | 0.048 |
| 600 | 0.057 | 0.058 | 0.060 | 0.053 | 0.061 | 0.060 | 0.047 | 0.059 | 0.055 | 0.057 | 0.058 | 0.048 | 0.051 | 0.053 | 0.048 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.048 | 0.041 | 0.057 | 0.039 | 0.046 | 0.047 | 0.046 | 0.055 | 0.056 | 0.046 | 0.055 | 0.063 | 0.040 | 0.057 | 0.056 |
| 50 | 0.050 | 0.062 | 0.067 | 0.058 | 0.064 | 0.046 | 0.061 | 0.061 | 0.057 | 0.056 | 0.051 | 0.048 | 0.054 | 0.056 | 0.050 |
| 100 | 0.064 | 0.052 | 0.055 | 0.064 | 0.048 | 0.050 | 0.055 | 0.043 | 0.053 | 0.052 | 0.046 | 0.051 | 0.055 | 0.045 | 0.044 |
| 300 | 0.060 | 0.053 | 0.053 | 0.053 | 0.048 | 0.057 | 0.056 | 0.044 | 0.057 | 0.058 | 0.057 | 0.052 | 0.047 | 0.052 | 0.044 |
| 600 | 0.053 | 0.056 | 0.068 | 0.046 | 0.050 | 0.047 | 0.054 | 0.046 | 0.070 | 0.048 | 0.056 | 0.060 | 0.065 | 0.060 | 0.047 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.053 | 0.050 | 0.053 | 0.045 | 0.046 | 0.048 | 0.047 | 0.038 | 0.047 | 0.047 | 0.038 | 0.049 | 0.044 | 0.050 | 0.052 |
| 50 | 0.049 | 0.067 | 0.052 | 0.064 | 0.049 | 0.047 | 0.055 | 0.054 | 0.059 | 0.048 | 0.039 | 0.048 | 0.057 | 0.060 | 0.048 |
| 100 | 0.059 | 0.056 | 0.043 | 0.051 | 0.053 | 0.049 | 0.056 | 0.042 | 0.047 | 0.055 | 0.052 | 0.059 | 0.045 | 0.054 | 0.057 |
| 300 | 0.043 | 0.046 | 0.050 | 0.055 | 0.049 | 0.049 | 0.048 | 0.053 | 0.057 | 0.054 | 0.065 | 0.064 | 0.057 | 0.056 | 0.048 |
| 600 | 0.065 | 0.049 | 0.058 | 0.052 | 0.059 | 0.054 | 0.046 | 0.052 | 0.044 | 0.055 | 0.060 | 0.056 | 0.048 | 0.051 | 0.060 |

Table 19: Empirical powers of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.1^{|r_1 - r_2|} I(|r_1 - r_2|)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Normal Instances | | | | | | | | | | | | | | |
| 10 | 0.098 | 0.818 | 1.000 | 0.185 | 0.992 | 1.000 | 0.319 | 1.000 | 1.000 | 0.574 | 1.000 | 1.000 | 0.945 | 1.000 | 1.000 |
| 50 | 0.116 | 0.900 | 1.000 | 0.222 | 1.000 | 1.000 | 0.368 | 1.000 | 1.000 | 0.688 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 100 | 0.115 | 0.911 | 1.000 | 0.205 | 1.000 | 1.000 | 0.380 | 1.000 | 1.000 | 0.700 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.132 | 0.947 | 1.000 | 0.219 | 1.000 | 1.000 | 0.365 | 1.000 | 1.000 | 0.731 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.129 | 0.920 | 1.000 | 0.248 | 1.000 | 1.000 | 0.402 | 1.000 | 1.000 | 0.730 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances I | | | | | | | | | | | | | | |
| 10 | 0.110 | 0.810 | 1.000 | 0.230 | 0.997 | 1.000 | 0.339 | 1.000 | 1.000 | 0.604 | 1.000 | 1.000 | 0.953 | 1.000 | 1.000 |
| 50 | 0.123 | 0.901 | 1.000 | 0.226 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.719 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 100 | 0.139 | 0.912 | 1.000 | 0.228 | 1.000 | 1.000 | 0.379 | 1.000 | 1.000 | 0.699 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 300 | 0.119 | 0.921 | 1.000 | 0.248 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.728 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 600 | 0.143 | 0.920 | 1.000 | 0.232 | 1.000 | 1.000 | 0.391 | 1.000 | 1.000 | 0.719 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances II | | | | | | | | | | | | | | |
| 10 | 0.105 | 0.807 | 1.000 | 0.205 | 0.995 | 1.000 | 0.319 | 1.000 | 1.000 | 0.568 | 1.000 | 1.000 | 0.956 | 1.000 | 1.000 |
| 50 | 0.116 | 0.903 | 1.000 | 0.241 | 1.000 | 1.000 | 0.371 | 1.000 | 1.000 | 0.681 | 1.000 | 1.000 | 0.991 | 1.000 | 1.000 |
| 100 | 0.132 | 0.892 | 1.000 | 0.244 | 1.000 | 1.000 | 0.381 | 1.000 | 1.000 | 0.702 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 300 | 0.121 | 0.929 | 1.000 | 0.221 | 1.000 | 1.000 | 0.363 | 1.000 | 1.000 | 0.688 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| 600 | 0.117 | 0.907 | 1.000 | 0.223 | 1.000 | 1.000 | 0.374 | 1.000 | 1.000 | 0.711 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 |
| $r$ | Gamma Instances III | | | | | | | | | | | | | | |
| 10 | 0.116 | 0.820 | 1.000 | 0.220 | 0.996 | 1.000 | 0.326 | 1.000 | 1.000 | 0.577 | 1.000 | 1.000 | 0.956 | 1.000 | 1.000 |
| 50 | 0.119 | 0.905 | 1.000 | 0.243 | 1.000 | 1.000 | 0.388 | 1.000 | 1.000 | 0.708 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 |
| 100 | 0.121 | 0.903 | 1.000 | 0.210 | 1.000 | 1.000 | 0.377 | 1.000 | 1.000 | 0.685 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| 300 | 0.109 | 0.924 | 1.000 | 0.218 | 1.000 | 1.000 | 0.360 | 1.000 | 1.000 | 0.707 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |
| 600 | 0.134 | 0.917 | 1.000 | 0.221 | 1.000 | 1.000 | 0.390 | 1.000 | 1.000 | 0.630 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 |

Table 20: Empirical powers of the diagonality test when $\mathbf{\Sigma}_{\mathrm{R}}$ is a tridiagonal matrix with elements $(\mathbf{\Sigma}_{\mathrm{R}})_{r_1 r_2} = 0.15^{|r_1-r_2|} I(|r_1-r_2|)$ and $\mathrm{tr}(\mathbf{\Sigma}_{\mathrm{R}}^2)$ is replaced by its true value.

| $N$ | 20 | | | 40 | | | 60 | | | 100 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | | | | | | | Normal Instances | | | | | | | | |
| 10 | 0.198 | 0.996 | 1.000 | 0.485 | 1.000 | 1.000 | 0.764 | 1.000 | 1.000 | 0.975 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.250 | 1.000 | 1.000 | 0.602 | 1.000 | 1.000 | 0.868 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.222 | 1.000 | 1.000 | 0.584 | 1.000 | 1.000 | 0.888 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.279 | 1.000 | 1.000 | 0.614 | 1.000 | 1.000 | 0.904 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.264 | 1.000 | 1.000 | 0.641 | 1.000 | 1.000 | 0.899 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances I | | | | | | | | |
| 10 | 0.246 | 0.997 | 1.000 | 0.481 | 1.000 | 1.000 | 0.712 | 1.000 | 1.000 | 0.944 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.262 | 1.000 | 1.000 | 0.599 | 1.000 | 1.000 | 0.874 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.262 | 1.000 | 1.000 | 0.618 | 1.000 | 1.000 | 0.881 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.250 | 1.000 | 1.000 | 0.621 | 1.000 | 1.000 | 0.904 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.261 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.894 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances II | | | | | | | | |
| 10 | 0.211 | 0.999 | 1.000 | 0.486 | 1.000 | 1.000 | 0.762 | 1.000 | 1.000 | 0.980 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.258 | 1.000 | 1.000 | 0.595 | 1.000 | 1.000 | 0.885 | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.258 | 1.000 | 1.000 | 0.623 | 1.000 | 1.000 | 0.890 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.241 | 1.000 | 1.000 | 0.620 | 1.000 | 1.000 | 0.895 | 1.000 | 1.000 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.237 | 1.000 | 1.000 | 0.606 | 1.000 | 1.000 | 0.884 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | | | | | | | Gamma Instances III | | | | | | | | |
| 10 | 0.220 | 0.999 | 1.000 | 0.518 | 1.000 | 1.000 | 0.762 | 1.000 | 1.000 | 0.973 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 0.260 | 1.000 | 1.000 | 0.628 | 1.000 | 1.000 | 0.874 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 0.242 | 1.000 | 1.000 | 0.625 | 1.000 | 1.000 | 0.922 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 0.240 | 1.000 | 1.000 | 0.602 | 1.000 | 1.000 | 0.877 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 0.260 | 1.000 | 1.000 | 0.633 | 1.000 | 1.000 | 0.929 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

## S10 Vector-Based Simulations

This section contains tables that display the simulation results for the sphericity test (Table 21) and the identity test (Table 22) proposed by Srivastava, Yanagihara, and Kubokawa (2014) when the correlation between the column features is ignored and each subject specific matrix response is treated as $c$ independent $r$-dimensional vectors.

We simulated $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ under four matrix-variate normal scenarios with matrix parameters:

1. $\mathbf{M} = \mathbf{0}$, $\boldsymbol{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\boldsymbol{\Sigma}_{\mathrm{C}} = \mathbf{I}_c$.

2. $\mathbf{M}$ with elements $(\mathbf{M})_{r_1 c_1} = r_1 c_1/(rc)$, $\boldsymbol{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\boldsymbol{\Sigma}_{\mathrm{C}} = \mathbf{I}_c$.

3. $\mathbf{M} = \mathbf{0}$, $\boldsymbol{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\boldsymbol{\Sigma}_{\mathrm{C}}$ with elements $(\boldsymbol{\Sigma}_{\mathrm{C}})_{c_1 c_2} = 0.85^{|c_1 - c_2|}$.

4. $\mathbf{M}$ with elements $(\mathbf{M})_{r_1 c_1} = r_1 c_1/(rc)$, $\boldsymbol{\Sigma}_{\mathrm{R}} = \mathbf{I}_r$ and $\boldsymbol{\Sigma}_{\mathrm{C}}$ with elements $(\boldsymbol{\Sigma}_{\mathrm{C}})_{c_1 c_2} = 0.85^{|c_1 - c_2|}$.

To reflect high-dimensional settings, we considered $N = 20, 40, 60$, $r = 10, 50, 100, 300, 600$ and $c = 10, 100, 600$. In each simulation scheme, we used 1000 replicates and we calculated the proportions of rejections at a 5% nominal significance level.

Table 21: Empirical size of the spericity test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}$, $\alpha = 0.05$ and the columns are treated as independent.

| $N$ | 20 | | | 40 | | | 60 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Configuration 1 | | | | | | | | |
| 10 | 0.054 | 0.060 | 0.050 | 0.063 | 0.055 | 0.046 | 0.058 | 0.059 | 0.044 |
| 50 | 0.055 | 0.052 | 0.058 | 0.049 | 0.062 | 0.058 | 0.055 | 0.055 | 0.057 |
| 100 | 0.050 | 0.057 | 0.055 | 0.062 | 0.048 | 0.066 | 0.046 | 0.060 | 0.058 |
| 300 | 0.049 | 0.045 | 0.050 | 0.057 | 0.042 | 0.056 | 0.053 | 0.041 | 0.058 |
| 600 | 0.054 | 0.041 | 0.056 | 0.046 | 0.053 | 0.047 | 0.062 | 0.056 | 0.042 |
| $r$ | Configuration 2 | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Configuration 3. | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Configuration 4 | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 22: Empirical size of the identity test when $\mathbf{\Sigma}_{\mathrm{R}} = \mathbf{I}$, $\alpha = 0.05$ and the columns are treated as independent.

| $N$ | 20 | | | 40 | | | 60 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c$ | 10 | 100 | 600 | 10 | 100 | 600 | 10 | 100 | 600 |
| $r$ | Configuration 1 | | | | | | | | |
| 10 | 0.068 | 0.065 | 0.062 | 0.075 | 0.068 | 0.055 | 0.067 | 0.068 | 0.055 |
| 50 | 0.058 | 0.054 | 0.061 | 0.051 | 0.062 | 0.062 | 0.057 | 0.057 | 0.059 |
| 100 | 0.049 | 0.058 | 0.055 | 0.065 | 0.050 | 0.066 | 0.049 | 0.062 | 0.058 |
| 300 | 0.050 | 0.045 | 0.051 | 0.057 | 0.042 | 0.057 | 0.053 | 0.044 | 0.058 |
| 600 | 0.054 | 0.041 | 0.056 | 0.046 | 0.053 | 0.050 | 0.062 | 0.056 | 0.043 |
| $r$ | Configuration 2 | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Configuration 3 | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $r$ | Configuration 4 | | | | | | | | |
| 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 50 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 100 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 300 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 600 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

# S11 R code

The R code to run the analysis in the mouse example in Section 5 is

```
## Loading required R packages
library(covsep)
library(HDTD)
## Loading data
data(VEGFmouse)
## Assessing the Kronecker Product Assumption
VEGFmouse_arrays <- array(data.matrix(VEGFmouse), c(46, 9, 40))
VEGFmouse_arrays <- aperm(VEGFmouse_arrays, c(3, 1, 2))
set.seed(1)
empirical_bootstrap_test(VEGFmouse_arrays, L1 = 46, L2 = 9)
## Estimating the tissue correlation matrix
Sigma_tissues <- covmat.hat(data = VEGFmouse, N = 40, voi = "columns",
                            shrink = "both")$cols.covmat
Cor_tissues <- cov2cor(Sigma_tissues)
round(Cor_tissues, 4)
## Covariance tests
covmat.ts(data = VEGFmouse, N = 40, voi ="columns")
```

and below are also the details of the R session

```
> sessionInfo()
R version 3.5.0 (2018-04-23)
Platform: x86_64-w64-mingw32/x64 (64-bit)
Running under: Windows 7 x64 (build 7601) Service Pack 1
```

```
Matrix products: default

locale:

[1] LC_COLLATE=English_United Kingdom.1252  LC_CTYPE=English_United Kingdom.1252

LC_MONETARY=English_United Kingdom.1252

[4] LC_NUMERIC=C                            LC_TIME=English_United Kingdom.1252

attached base packages:

[1] stats     graphics  grDevices utils     datasets  methods   base

other attached packages:

[1] HDTD_1.14.0          covsep_1.1.0         BiocInstaller_1.30.0

loaded via a namespace (and not attached):

[1] compiler_3.5.0 tools_3.5.0    yaml_2.1.19    Rcpp_0.12.17   tinytex_0.5
```

The R code to run the analysis in the EEG example in Section 5 is

```
## Load the necessary packages

library(R.matlab)

library(HDTD)


## Read the data. The data files can be downloaded from

## the supplementary material of Xia and Chen (2017).

alcoholic_group <- readMat('./RcodeExamples/X1_t256.mat')

control_group <- readMat('./RcodeExamples/X0_t256.mat')


## Create the data matrices

alcoholic_group <- transposedata(matrix(unlist(alcoholic_group),

                                 nrow = 256, ncol = 77 * 64), N = 77)
```

```
control_group <- transposedata(matrix(unlist(control_group),
                              nrow = 256, ncol = 45 * 64), N = 45)

## Test the temporal covariance matrices
covmat.ts(datamat = alcoholic_group, N = 77, voi="columns")
covmat.ts(datamat = control_group, N = 45, voi="columns")

## Test the covariance matrices of the electrodes
covmat.ts(datamat = alcoholic_group, N = 77, voi="rows")
covmat.ts(datamat = control_group, N = 45, voi="rows")

## Assess the bandness for the alcoholic group
alcoholic_group <- transposedata(alcoholic_group, N=77)
covmat.ts(datamat = alcoholic_group[seq(1, 256, 4), ], N = 77, voi="rows")
covmat.ts(datamat = alcoholic_group[seq(2, 256, 4), ], N = 77, voi="rows")
covmat.ts(datamat = alcoholic_group[seq(3, 256, 4), ], N = 77, voi="rows")

## Assess the bandness for the control group
control_group <- transposedata(control_group, N=45)
covmat.ts(datamat = control_group[seq(1, 256, 4), ], N = 45, voi="rows")
covmat.ts(datamat = control_group[seq(2, 256, 4), ], N = 45, voi="rows")
covmat.ts(datamat = control_group[seq(3, 256, 4), ], N = 45, voi="rows")
```

and below are also the details of the R session

```
sessionInfo()
```

```
R version 3.6.0 (2019-04-26)
Platform: x86_64-w64-mingw32/x64 (64-bit)
Running under: Windows 7 x64 (build 7601) Service Pack 1

Matrix products: default

Random number generation:
 RNG:     Mersenne-Twister
 Normal:  Inversion
 Sample:  Rounding

locale:
[1] LC_COLLATE=English_United Kingdom.1252  LC_CTYPE=English_United Kingdom.1252
[3] LC_MONETARY=English_United Kingdom.1252 LC_NUMERIC=C
[5] LC_TIME=English_United Kingdom.1252

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods   base

other attached packages:
[1] HDTD_1.14.0    R.matlab_3.6.2

loaded via a namespace (and not attached):
[1] compiler_3.6.0    tools_3.6.0       yaml_2.2.0        Rcpp_1.0.1
R.methodsS3_1.7.1
```

```
[6] R.utils_2.8.0     pacman_0.5.1      R.oo_1.22.0
```

# Bibliography

Harville, D. A. (1997). *Matrix Algebra from a Statistician's Perspective.* Springer.

Seber, G. A. F. (2008). *A Matrix Handbook for Statisticians.* John Wiley & Sons.

Srivastava, M. S., Yanagihara, H. and Kubokawa, T. (2014). Tests for covariance matrices in high dimension with less sample size. *Journal of Multivariate Analysis* **130**, 289–309.