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EFFICIENT ESTIMATION AND COMPUTATION IN GENERALIZED VARYING COEFFICIENT MODELS WITH UNKNOWN LINK AND VARIANCE FUNCTIONS FOR LARGE-SCALE DATA

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Abstract: Generalized varying-coefficient models have emerged as a powerful tool for modeling nonlinear interactions between covariates and an index variable when the outcome follows a non-normal distribution. The model often stipulates a link function and a variance function, which may not be valid in practice. For example, a large-scale study of loan payment delinquency related to the purchase of expensive smartphones in China, found that parametric functions may not adequately characterize the data and may yield biased results. We propose a generalized varying-coefficient model with unknown link and variance functions. With a massive data set, the simultaneous estimation of these functions and the large number of varying-coefficient functions poses challenges. Thus, we further propose a global kernel estimator and a series of linear approximations that achieves computational and statistical efficiency. The estimators can be expressed explicitly as a linear function of outcomes and are proven to be semiparametrically efficient. Extensive simulations demonstrate the superiority of the method over other competing methods. Lastly, we apply the proposed method to analyze the aforementioned smartphone loan payment data.

Key words and phrases: Asymptotic properties, generalized varying coefficient models, local linear smoothing, quasi-likelihood, semiparametric efficiency.

1. Introduction

For non-normal response data, generalized varying-coefficient models (GVCMs) are widely used to model the nonlinear interactions between an index variable (or effect modifier) and important covariates; see Hastie and Tibshirani (1993), Xia and Li (1999), Cai, Fan and Li (2000), Zhang and Peng (2010), Kuruwita, Kulasekera and Gallagher (2011), Xue and Wang (2012), Huang et al. (2014), and Zhang, Li and Xia (2015). The models have been applied in *longi-*

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tudinal data analysis (Hoover et al. (1998); Wu, Chiang and Hoover (1998); Fan and Zhang (2000); Lin and Ying (2001); Fan, Huang and Li (2007); Lin, Song and Zhou (2007)), *time series analysis* (Chen and Tsay (1993); Cai, Fan and Li (2000); Huang and Shen (2004)), *survial analysis* (Zucker and Karr (1990); Murphy and Sen (1991); Gamerman (1991); Murphy (1993); Marzec and Marzec (1997); Martinussen, Scheike and Skovgaard (2002); Cai and Sun (2003); Tian, Zucker and Wei (2005); Fan, Lin and Zhou (2006); Chen, Lin and Zhou (2012)), and *functional data analysis* (Ramsay and Silverman (2002)). Like generalized linear models, GVCMs specify link and variance functions to associate the means and variances of outcomes with predictors. The functions are typically specified according to the data type of the outcomes and for mathematical convenience. For binary outcomes, a logit link and a variance $\mu(1-\mu)$ as a function of the mean μ are chosen; for count data, a logarithmic link and an identity variance function of the mean are specified; and for continuous outcomes, an identity link and a constant variance are chosen. However, misspecified link and variance functions may cause biased and inefficient estimates, leading to erroneous conclusions.

Our study is motivated by a large-scale data set on the loan payment delinquency of young customers who have purchased expensive smartphones in a major city in China. The data set consists of payment delinquency records for the period 2015 to 2016 (recorded as $Y = 1$ if the loan was not paid back on time, and zero otherwise) for 105,548 customers, along with each customer's credit score, age, monthly income, downpayment ratio, loan amount, and number of credit cards owned. Preliminary analyses found that the effects of risk factors may depend on the loan amount. For example, the effect of age increases with the loan amount, and the effect of the credit score is significant only when the loan amount is in the range $(2{,}000, 4{,}000)$. We examine whether and how these factors affect loan payment behavior by applying a GVCM. Using the proposed nonparametric methodology, the estimated link and variance functions (see Figures 3 and 4) deviate significantly from the commonly used link and variance functions for binary data, suggesting they are unsuitable for this data set. Furthermore, Table 4 shows that the method with data-driven link and variance functions performs better, with smaller prediction errors, than the logistic varying-coefficient model for the independent testing data. In many applications, the estimation of variance structures is of interest. Recent examples include a study of the variability in propensity-score matching (Austin and Cafri (2020)), an evaluation of the variability in aggregate stock returns (Pyun (2019)), the effects on employment of several state-level policy shifts (Pustejovsky and Tipton (2018); Deriso, Maunder and Skalski (2007)), and analyses of several functional or longitudinal data sets

(Lin, Raz and Harlow (1997); Wang and Lin (2005); Zhang and Paul (2014)).

Two related works nonparametrically estimate link functions for varying-coefficient models (Kuruwita, Kulasekera and Gallagher (2011); Zhang, Li and Xia (2015)). Kuruwita, Kulasekera and Gallagher (2011) consider a model $Y = g\{\mathbf{X}'\boldsymbol{\beta}(U)\} + \epsilon$ for continuous response data with a constant variance. For non-continuous response data, Zhang, Li and Xia (2015) propose a class of GVCMs with an unknown link, but a known variance function. These methods focus on estimating mean functions, while specifying variance functions that are be constant or have a known structure. However, our simulation (see Example 3 in Section 4) shows that misspecifications of variance functions lead to considerably large biases for the link and varying-coefficient functions. In addition, because Zhang, Li and Xia (2015) used a local likelihood method to estimate the link and coefficient functions, the number of parameters to be estimated is of the same order as the sample size. This method is not applicable to our loan payment data set, which has more than 100,000 samples. Moreover, Zhang, Li and Xia (2015) and (Kuruwita, Kulasekera and Gallagher (2011)) estimated $g(\cdot)$ using a two-dimensional kernel, which may not be efficient.

We propose a new class of GVCMs with unspecified link and variance functions (GVULV). Let Y be the response variable, $\mathbf{X} = (X_1, \ldots, X_d)'$ be the vector of covariates, and U be a univariate index variable, for example, the loan amount. The GVULV model is specified as

$$\begin{aligned} \mu &= E(Y|\mathbf{X}, U) = g\{\mathbf{X}'\boldsymbol{\beta}(U)\}, \\ Var(Y|\mathbf{X}, U) &= V(\mu), \end{aligned} \tag{1.1}$$

where $g(\cdot)$ and $V(\cdot)$ are the unknown link and variance functions, respectively, and $\boldsymbol{\beta}(\cdot)$ is a vector of unknown varying-coefficient functions.

Using one-dimensional kernel functions, we propose a quasi-likelihood-based approach to estimate $g(\cdot)$ and $\boldsymbol{\beta}(\cdot)$, and show that the proposed estimators are uniformly consistent, asymptotically normal, and semiparametrically efficient in the sense of Bickel et al. (1998). To the best of our knowledge, semiparametric efficiency has never been established for similar models. In addition, using a series of linear approximations, we propose an iterative algorithm, that is computationally efficient and easily implementable, because each step involves only closed-form one-dimensional smoothing.

The remainder of paper is organized as follows. Section 2 presents the model formulation and introduces the local quasi-likelihood estimation, and Section 3 establishes the asymptotic results. Section 4 gives numerical comparisons with competing methods, and Section 5 applies the proposed method to analyze loan payment data. We conclude the paper with a discussion in Section 6. Technical proofs are relegated to the Supplementary Material. The R code for the proposed method is available at `https://github.com/LinhzLab/gvcm_code`.

2. Estimation of the GVULV Model

2.1. Model formulation

With n random samples from an underlying population, the observed data, $(Y_i, \mathbf{X}_i, U_i)$, for $i = 1, \ldots, n$, are independent and identically distributed(i.i.d.) copies of $(Y, \mathbf{X}, U)$ satisfying (1.1). Following Zhang, Li and Xia (2015), we specify the following identifiability conditions:

$$\beta_1(u) > 0, \quad \text{for any } u, \quad \text{and} \quad \|\boldsymbol{\beta}(U_n)\| = 1, \tag{2.1}$$

where $\|\boldsymbol{\beta}(u)\| = \{\boldsymbol{\beta}(u)^T\boldsymbol{\beta}(u)\}^{1/2}$, and $\beta_1(\cdot)$ is the first component of $\boldsymbol{\beta}(\cdot)$.

We fit model (1.1) using the maximum quasi-likelihood and kernel smoothing. To proceed, let $\mu_i = g\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}$, and write the log quasi-likelihood function of $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$ and $V(\cdot)$ as

$$Q(\boldsymbol{\beta}, g, V) = \sum_{i=1}^{n} L(\mu_i, Y_i), \tag{2.2}$$

with $L(\mu_i, Y_i)$ defined as

$$\frac{\partial L(\mu_i, Y_i)}{\partial \mu_i} = V(\mu_i)^{-1}(Y_i - \mu_i). \tag{2.3}$$

The following three subsections detail the proposed approach, which estimates $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$, and $V(\cdot)$.

2.2. Estimation of $\boldsymbol{\beta}(\cdot)$ when $g(\cdot)$ and $V(\cdot)$ are given

Applying the Taylor expansion to $\boldsymbol{\beta}(\cdot)$ yields

$$\boldsymbol{\beta}(U_i) \approx \boldsymbol{\beta}(u) + \dot{\boldsymbol{\beta}}(u)(U_i - u) \tag{2.4}$$

when U_i is in a small neighborhood of u. With (2.3), the quasi-likelihood estimator of $\boldsymbol{\delta} = (\boldsymbol{\zeta},\ \boldsymbol{\gamma})' \equiv (\boldsymbol{\beta}(u),\ \dot{\boldsymbol{\beta}}(u))'$ solves

$$S_{\boldsymbol{\beta}}(\boldsymbol{\delta}; \mathbf{g}, V) \hat{=} \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g\Big\{\mathbf{X}_i'\big(\boldsymbol{\zeta} + \boldsymbol{\gamma}(U_i - u)\big)\Big\}\Big]\Upsilon_i(u)$$

$$\times\dot{g}\Big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\Big\}\frac{K_{h_1}(U_i-u)}{V(\mu_i)}=0, \tag{2.5}$$

where $\Upsilon_i(u)=(\mathbf{X}_i',\mathbf{X}_i'(U_i-u))'$, $K_h(\cdot)=\mathcal{K}(\cdot/h)/h$, $\mathcal{K}(\cdot)$ is a non-negative symmetric kernel function on $[-1,1]$, and h_1 is a bandwidth.

Using the Newton–Raphson iteration to compute $\boldsymbol{\delta}=(\boldsymbol{\zeta},\ \boldsymbol{\gamma})'$ is intensive because of the repetitions over all u in the support of U_i, given $g(\cdot)$ and $V(\cdot)$. We explore a local linear approximation. Applying Taylor's expansion to $g(\cdot)$ at $\mathbf{X}'\boldsymbol{\beta}(U_i)$ for U_i around u, we have that

$$\begin{aligned}
&g\Big[\mathbf{X}_i'\{\boldsymbol{\zeta}+\boldsymbol{\gamma}(U_i-u)\}\Big]\\
&=g\Big[\mathbf{X}_i'\boldsymbol{\beta}(U_i)+\mathbf{X}_i'\{\boldsymbol{\zeta}+\boldsymbol{\gamma}(U_i-u)\}-\mathbf{X}_i'\boldsymbol{\beta}(U_i)\Big]\\
&\approx g\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}+\dot{g}\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}\Big[\mathbf{X}_i'\{\boldsymbol{\zeta}+\boldsymbol{\gamma}(U_i-u)\}-\mathbf{X}_i'\boldsymbol{\beta}(U_i)\Big].
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5), we obtain an explicit expression for the estimators of $(\boldsymbol{\beta}(u),\dot{\boldsymbol{\beta}}(u))'$,

$$\begin{aligned}
\begin{pmatrix}\hat{\boldsymbol{\beta}}(u)\\ \hat{\dot{\boldsymbol{\beta}}}(u)\end{pmatrix}&=\left\{\sum_{i=1}^n\rho_i^2\Upsilon_i(u)\Upsilon_i(u)'\frac{K_{h_1}(U_i-u)}{V(\mu_i)}\right\}^{-1}\\
&\quad\times\sum_{i=1}^n\Big[Y_i-g\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}+\rho_i\mathbf{X}_i'\boldsymbol{\beta}(U_i)\Big]\Upsilon_i(u)\rho_i\frac{K_{h_1}(U_i-u)}{V(\mu_i)},
\end{aligned} \tag{2.7}$$

where $\rho_i=\dot{g}\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\big\}$.

2.3. Estimation of $g(\cdot)$ when $\beta(\cdot)$ and $V(\cdot)$ are given

A Taylor expansion yields

$$g\Big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\Big\}\approx g(z)+\dot{g}(z)\Big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)-z\Big\} \tag{2.8}$$

when $\mathbf{X}_i'\boldsymbol{\beta}(U_i)$ is in a small neighborhood of z. With (2.3) and (2.8), the quasi-likelihood estimator of $\mathbf{g}=(g_1,g_2)\equiv(g(z),\ \dot{g}(z))'$ solves

$$S_g(\mathbf{g};\boldsymbol{\beta},V)\hat{=}\frac{1}{n}\sum_{i=1}^n\Big\{Y_i-W_i(z;\boldsymbol{\beta})'\mathbf{g}\Big\}\frac{W_i(z;\boldsymbol{\beta})}{V(\mu_i)}K_{h_2}\Big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)-z\Big\}=0, \tag{2.9}$$

where $W_i(z;\ \boldsymbol{\beta})=\big(1,\ \mathbf{X}_i'\boldsymbol{\beta}(U_i)-z\big)'$, and h_2 is the bandwidth. A closed-form expression is available with

$$\Big(\hat{g}(z), \hat{\dot{g}}(z)\Big)' = \left[\sum_{i=1}^{n} \frac{W_i(z;\boldsymbol{\beta}) W_i(z;\boldsymbol{\beta})' K_{h_2}\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i) - z\big\}}{V(\mu_i)} \right]^{-1} \times \sum_{i=1}^{n} \frac{W_i(z;\boldsymbol{\beta}) K_{h_2}\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i) - z\big\} Y_i}{V(\mu_i)}. \tag{2.10}$$

2.4. Estimation of $V(\cdot)$ when $\beta(\cdot)$ and $g(\cdot)$ are given

Because $E(Y_i^2|\mathbf{X}_i, U_i) = Var(Y_i|\mathbf{X}_i, U_i) + E^2(Y_i|\mathbf{X}_i, U_i) = V(\mu_i) + \mu_i^2 \equiv \tilde{V}(\mu_i)$, it suffices to estimate $\tilde{V}(\cdot)$ for $V(\cdot)$. Using the Taylor expansion gives

$$\tilde{V}(\mu_i) \approx \tilde{V}(\omega) + \dot{\tilde{V}}(\omega)(\mu_i - \omega) \tag{2.11}$$

when $\mu_i = g\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\big\}$ is in a small neighborhood of ω. Then, the estimating equation for $\mathbf{V} = (\tilde{V}(\omega),\ \dot{\tilde{V}}(\omega))'$ becomes

$$S_V(\mathbf{V};\boldsymbol{\beta}, g) \hat{=} \frac{1}{n} \sum_{i=1}^{n} \Big[Y_i^2 - \tilde{V}(\omega) - (\mu_i - \omega)\dot{\tilde{V}}(\omega)\Big] \Omega_i(\omega;\boldsymbol{\beta}, g) K_{h_3}(\mu_i - \omega) = 0, \tag{2.12}$$

with $\Omega_i(\omega;\boldsymbol{\beta}, g) = (1,\ \mu_i - \omega)'$, and h_3 being the bandwidth. The estimator for $(\tilde{V}(\omega),\ \dot{\tilde{V}}(\omega))'$ is

$$\Big(\hat{\tilde{V}}(\omega), \hat{\dot{\tilde{V}}}(\omega)\Big)' = \left[\sum_{i=1}^{n} \Omega_i(\omega;\boldsymbol{\beta}, g)\Omega_i(\omega;\boldsymbol{\beta}, g)' K_{h_3}(\mu_i - \omega) \right]^{-1} \times \sum_{i=1}^{n} \Omega_i(\omega;\boldsymbol{\beta}, g) K_{h_3}(\mu_i - \omega) Y_i^2. \tag{2.13}$$

The estimator for $V(\omega)$ is $\hat{V}(\omega) = \hat{\tilde{V}}(\omega) - \omega^2$. Because (2.12) uses the squared observations, Y_i^2, rather than the squared residuals $(Y_i - \mu_i)^2$, the procedure avoids using the unknown mean function, adding robustness to the estimation of $V(\cdot)$ (Lin and Song (2010)).

2.5. An algorithm for estimating $g(\cdot)$, $\beta(\cdot)$, $V(\cdot)$

We choose the initial values of $\boldsymbol{\beta}^{(0)}(u)$, $g^{(0)}(z)$ and $\dot{g}^{(0)}(z)$, with u and z in the support of U and $\mathbf{X}'\boldsymbol{\beta}(U)$, respectively. Because the variance estimation does not affect the asymptotical distribution of the estimator for the mean structure, we choose the initial values based on a model with a constant variance. For the same reason, as long as the estimate of $V^{(0)}(\mu_i^{(0)})$ is consistent, the variance function $V(\mu_i)$ in (2.5) and (2.9) does not need to be updated in the iterative process. The estimate of $V(\cdot)$ only needs to be updated after the final estimates of $g(\cdot)$ and $\boldsymbol{\beta}(\cdot)$ are obtained. This further reduces the computational burden. In addition, because the objective function for estimating $g(\cdot)$ and $\boldsymbol{\beta}(\cdot)$ is different to that for $V(\cdot)$, the iterative algorithm may not guarantee convergence Boyd and Vandenberghe (2004). We conducted simulations by updating $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$, and $V(\cdot)$ iteratively, and found that the algorithm frequently fails to converge.

Using the local linear smoothing technique presented in Section 2.4, we estimate the initial values $V^{(0)}(\omega)$ of $V(\omega)$ for ω in the support of $\mu^{(0)} = g^{(0)}\{\mathbf{X}'\boldsymbol{\beta}^{(0)}(U)\}$, which, by the kernel theory (Fan, Lin and Zhou (2006)), are consistent estimates of $V(g\{\mathbf{X}'\boldsymbol{\beta}(U)\})$. Let $\boldsymbol{\beta}^{(r-1)}(\cdot)$, $g^{(r-1)}(\cdot)$, and $\dot{g}^{(r-1)}(\cdot)$ be estimators of $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$, and $\dot{g}(\cdot)$ respectively, at the $(r-1)$th iteration, and let $\mu_i^{(r-1)} = g^{(r-1)}\{\mathbf{X}_i'\boldsymbol{\beta}^{(r-1)}(U_i)\}$ and $\rho_i^{(r-1)} = \dot{g}^{(r-1)}\{\mathbf{X}_i'\boldsymbol{\beta}^{(r-1)}(U_i)\}$. We obtain the updated values of $\boldsymbol{\beta}(\cdot)$ and $g(\cdot)$ at the rth iteration as follows:

- For each u in the chosen grid points $\{u_1, \ldots, u_{n_1}\}$, we estimate $\boldsymbol{\beta}(u)$ and $\dot{\boldsymbol{\beta}}(u)$ using (2.7). All unknown quantities on the right side of (2.7) are replaced by their updated values at the $(r-1)$th iteration, such as $\boldsymbol{\beta}^{(r-1)}(\cdot)$, $g^{(r-1)}(\cdot), \dot{g}^{(r-1)}(\cdot)$, $\mu_i^{(r-1)}$, and $\rho_i^{(r-1)}$, except that $V(\mu_i)$ is replaced by $V^{(0)}(\mu_i^{(0)})$. We then standardize $\hat{\boldsymbol{\beta}}(u)$ to obtain $\boldsymbol{\beta}^{(r)}(u) = \hat{\boldsymbol{\beta}}(u)/\|\hat{\boldsymbol{\beta}}(U_n)\|$, with $\beta_1^{(r)}(u) > 0$.

- Let $Z_i = \mathbf{X}_i'\boldsymbol{\beta}^{(r)}(U_i)$, for $i = 1, \ldots, n$. We choose n_2 points in the support of Z, denoted as $\{z_1, \ldots, z_{n_2}\}$. For each $z \in \{z_1, \ldots, z_{n_2}\}$, as outlined in Section 2.3, we estimate $(g(z),\ \dot{g}(z))'$ using (2.10). Again, we replace all unknown quantities on the right side of (2.10) with their updated values, except that we replace $V(\mu_i)$ with $V^{(0)}(\mu_i^{(0)})$. We denote the updated estimates of $g(z)$ and $\dot{g}(z)$ as $g^{(r)}(z)$ and $\dot{g}^{(r)}(z)$, respectively.

- The convergence is defined as $\sup_u ||\boldsymbol{\beta}^{(r)}(u) - \boldsymbol{\beta}^{(r-1)}(u)|| < \epsilon_0$ and $\sup_z |g^{(r)}(z) - g^{(r-1)}(z)| < \epsilon_0$, where $\epsilon_0 > 0$ is a prespecified small number. Denote the final estimators for $\boldsymbol{\beta}(u)$ and $g(z)$ as $\hat{\boldsymbol{\beta}}(u)$ and $\hat{g}(z)$, respectively.

- Let $\{\omega_1, \ldots, \omega_{n_3}\}$ be the grid points in the support of $\{\hat{g}(\mathbf{X}_i'\hat{\boldsymbol{\beta}}(U_i)) : i = 1, \ldots, n\}$. For each $\omega \in \{\omega_1, \ldots, \omega_{n_3}\}$, we use (2.13) to obtain the estimate of $V(\omega)$, with $\boldsymbol{\beta}$ and g replaced by $\hat{\boldsymbol{\beta}}$ and $\hat{g}$, respectively.

Remark 1. We calculate $g(\cdot)$, $\boldsymbol{\beta}(\cdot)$, and $V(\cdot)$ at fine grids, and use linear interpolation to fill in the rest. In contrast, Zhang, Li and Xia (2015) needed to estimate

$g(\cdot)$ at all of the observed data points, which is infeasible for a large-scale data set.

Remark 2. If $g(\cdot)$ is known, the estimator of $\hat{\boldsymbol{\beta}}(u)$ based on (2.5) reduces to the existing local quasi-likelihood estimator (Carroll et al. (1997); Chiou and Müller (1998)). If $\boldsymbol{\beta}(\cdot)$ is known, the proposed estimator of $\hat{g}(z)$ is the estimator for the generalized nonparametric regression model. As such, the asymptotic properties could be established using the kernel theory (Fan and Gijbels (1996)). However, because $\boldsymbol{\beta}(\cdot)$ and $g(\cdot)$ are both unknown, our estimator is defined implicitly as the limit of an iterative algorithm, which needs substantial work in order to establish the asymptotic theory.

Remark 3. We substitute the local approximations (2.4) and (2.8) into the quasi-likelihood function, avoiding the use of two-dimensional kernels, and improving the efficiency of the estimator. In fact, the proposed estimator is shown to be semiparametrically efficient in the sense of Bickel et al. (1998). On the other hand, the local approximation (2.6) yields a closed-form expression when updating the estimate of $\boldsymbol{\beta}(\cdot)$, which expedites and simplifies the computation. Hence, the proposed estimators possess theoretical and computational efficiency.

The proposed estimation of $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$, and $V(\cdot)$ involves selecting the bandwidths h_1, h_2, and h_3, respectively, which can be achieved using K-fold cross-validation (Cai, Fan and Li (2000); Fan, Lin and Zhou (2006)). Specifically, denote the full data set by B, and partition the samples into K parts, denoted by B_k, for $k = 1, \ldots, K$. First, for the link function and coefficient functions, we minimize

$$\mathrm{PE}(h_1, h_2) = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{n_k}\sum_{i\in B_k}\left|Y_i - \hat{g}^{(-k)}\{\mathbf{X}_i'\hat{\boldsymbol{\beta}}^{(-k)}(U_i)\}\right|,$$

where n_k is the number of the observations in set B_k, and the estimators $\hat{g}^{(-k)}(\cdot)$ and $\hat{\boldsymbol{\beta}}^{(-k)}(\cdot)$, for $g(\cdot)$ and $\boldsymbol{\beta}(\cdot)$, respectively, are estimated using the training set $B - B_k$. For the variance function, we minimize

$$\mathrm{PE}(h_3) = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{n_k}\sum_{i\in B_k}\left|(Y_i - \hat{\mu}_i^{(-k)})^2 - \hat{V}^{(-k)}(\hat{\mu}_i^{(-k)})\right|,$$

where the estimators $\hat{\mu}_i^{(-k)}$ and $\hat{V}^{(-k)}(\cdot)$, for $\mu_i = g\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}$ and $V(\cdot)$, respectively, are estimated using the training set $\mathbf{B} - \mathbf{B}_k$. The number K is usually chosen to be $K = 5$ or $K = 10$. The bandwidths (h_1, h_2) and h_3 are selected separately, resulting in less computation. In the ensuing simulation studies and real-data analysis, $K = 5$ is used and is found to work well.

3. Large-Sample Properties

We denote by $\boldsymbol{\beta}, g$, and V the true coefficient, link function, and variance function, respectively. This section establishes the uniform consistency, asymptotic normality, and semiparametric efficiency using the following regularity conditions:

(A1) The kernel function $K(\cdot)$ is a symmetric density function with a compact support and a bounded derivative.

(A2) $\mathbf{X}_i$ and U_i are bounded in $\mathbb{R}^d$ and $\mathbb{R}$. Without loss of generality, we assume that $\mathbf{X}_i \in [-1,1]^d$ and $U_i \in [-1,1]$.

(A3) The second derivatives of $\boldsymbol{\beta}(\cdot)$, $g(\cdot)$ and $V(\cdot)$ on $[-1,1]$ are bounded, and the variance function $V(\cdot)$ is bounded away from zero on $[-1,1]$.

(A4) The conditional distribution of Y_i has sub-exponential tails. That is, there exist constants C and $M > 0$ such that $E[|Y_i|^\ell|\mathbf{X}_i] \le C\ell! M^\ell$, for $\forall\, 2 \le \ell \le \infty$.

(A5) Let $\mathbf{g}(z) = (g_1(z), g_2(z))'$ and $\boldsymbol{\delta}(u) = (\boldsymbol{\zeta}(u), \boldsymbol{\gamma}(u))'$, and let f_1 be the density function of U_i, $f_2(\cdot;\boldsymbol{\zeta})$ be the density of the random variable $\mathbf{X}_i'\boldsymbol{\zeta}(U_i)$ associated with $\boldsymbol{\zeta}$, and $f_3(\cdot; g_1, \boldsymbol{\zeta})$ be the density of the random variable $g_1\left\{\mathbf{X}_i'\boldsymbol{\zeta}(U_i)\right\}$. Let

$$
\begin{aligned}
&\mathbf{s}_{\boldsymbol{\beta}}(\boldsymbol{\zeta}, \mathbf{g}, V_1; u) \\
&= E\left(\mathbf{X}_i\Big[g\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\big\} - g_1\{\mathbf{X}_i'\boldsymbol{\zeta}(u)\}\Big]\frac{g_2\big\{\mathbf{X}_i'\boldsymbol{\zeta}(U_i)\big\}}{V_1\big[g_1\left\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\right\}\big]}\Big|U_i = u\right) f_1(u), \\
&\mathbf{s}_g(\boldsymbol{\zeta}, g_1, V_1; z) \\
&= E\left(\frac{\big[g\big\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\big\} - g_1(z)\big]}{V_1\big[g_1\left\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\right\}\big]}\big|\mathbf{X}_i'\boldsymbol{\zeta}(U_i) = z\right) f_2(z;\boldsymbol{\zeta}), \\
&\mathbf{s}_V(\boldsymbol{\zeta}, g_1, V_1; w) = E\bigg(V\big[g\left\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\right\}\big] + g^2\left\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\right\} - V_1(\omega) \\
&\qquad\qquad - \omega^2\Big|g_1\left\{\mathbf{X}_i'\boldsymbol{\zeta}(U_i)\right\} = \omega\bigg) f_3(\omega; g_1, \boldsymbol{\zeta}).
\end{aligned}
$$

Define $\mathbf{s}(\boldsymbol{\zeta}, \mathbf{g}, V_1; u, z, \omega) = (\mathbf{s}_{\boldsymbol{\beta}}(\boldsymbol{\zeta}, \mathbf{g}, V_1; u)', \mathbf{s}_g(\boldsymbol{\zeta}, g_1, V_1; z), \mathbf{s}_V(\boldsymbol{\zeta}, g_1, V_1; \omega))'$. Then, we assume that $\mathbf{s}(\boldsymbol{\zeta}, \mathbf{g}, V_1; u, z, \omega) = 0$ has a unique root over $\boldsymbol{\zeta} \in \mathcal{C}_d$,

$g_1 \in \mathcal{C}_1$, and $V_1 \in \mathcal{C}_2$, where $\mathcal{C}_k, \mathcal{C}_1$, and $\mathcal{C}_2$ are defined in the Supplementary Material.

(A6) $h_j \to 0$ and $nh_j/(\log n) \to \infty$, for $j = 1, 2, 3$, as $n \to \infty$.

(A7) Ψ^{-1} and $(\mathrm{H}_{\boldsymbol{\beta}} - \mathrm{H}_g o \mathrm{H}_{\boldsymbol{\beta} g})^{-1}$ exist and are bounded uniformly, where Ψ is an operator-type matrix, and $\mathrm{H}_{\boldsymbol{\beta}}$, H_g, and $\mathrm{H}_{\boldsymbol{\beta} g}$ are operator-type vectors. The explicit forms of these operators are given in Section 1 of the Supplementary Material.

Conditions (A1)–(A4) are commonly assumed conditions for kernel functions, covariates, functions of interest, and distributions (Fan, Lin and Zhou (2006); Chen et al. (2010); Chen, Lin and Zhou (2012)). The condition of a bounded support for $\mathbf{X}_i$ and U_i simplifies the proof, and is extensively assumed in the nonparametric literature; see for Zhang, Li and Xia (2015), Horowitz and Härdle (1996), Horowitz (2001), Carroll et al. (1997), Chen, Lin and Zhou (2012), and Zhou, Lin and Liang (2018). The condition may be relaxed, as suggested by our simulation studies, where we generate $\mathbf{X}_i$ with unbounded multivariate normal random vectors. Conditions (A5) and (A7) ensure identifiability. Condition (A6) is assumed in the literature for bandwidths (Fan, Lin and Zhou (2006); Chen, Lin and Zhou (2012)).

Theorem 1. *Under Conditions* (A1)–(A6), *as $n \to \infty$, we have*

$$\sup_{u\in[-1,1]} |\widehat{\boldsymbol{\beta}}(u) - \boldsymbol{\beta}(u)| \xrightarrow{p} 0, \quad \sup_{z\in[-1,1]} |\widehat{g}(z) - g(z)| \xrightarrow{p} 0,$$
$$\sup_{\omega\in[-1,1]} |\widehat{V}(\omega) - V(\omega)| \xrightarrow{p} 0.$$

Theorem 1 shows the proposed estimators $\widehat{\boldsymbol{\beta}}(\cdot)$, $\widehat{g}(\cdot)$, and $\widehat{V}(\cdot)$ are all uniformly consistent.

Theorem 2. *Under Conditions* (A1)–(A7), *we have*

$$\Psi \begin{pmatrix} \widehat{\boldsymbol{\beta}}(u) - \boldsymbol{\beta}(u) \\ \widehat{g}(z) - g(z) \\ \widehat{V}(\omega) - V(\omega) \end{pmatrix} = (nH)^{-1/2}\mathbf{M}(u, z, \omega)^{-1/2}\varphi + H^2\mathrm{B}(u, z, \omega)$$
$$+ o_p\{h_1^2 + h_2^2 + h_3^2 + (nh_1)^{-1/2} + (nh_2)^{-1/2} + (nh_3)^{-1/2}\}$$

uniformly on $u \in [-1,\ 1]$, $z \in [-1,\ 1]$, and $\omega \in [-1,\ 1]$, where $H = diag(h_1 \times \boldsymbol{1}_d, h_2, h_3)$, $\boldsymbol{1}_d$ is a d-dimensional vector with all elements equal to one, φ is *a standard normal random vector, and* $\mathrm{B}(u,z,\omega)$ *and* $\mathbf{M}(u,z,\omega)$ *are defined in Section* 1 *of the Supplementary Material.*

Theorem 2 shows that the asymptotic bias of $\left(\widehat{\boldsymbol{\beta}}(u)',\ \widehat{g}(z),\ \widehat{V}(\omega)\right)'$ is of order $h^2 = (\max\{h_1,h_2,h_3\})^2$, whereas the asymptotic variance is of order $(nh)^{-1}$. Hence, the optimal bandwidth is of order $n^{-1/5}$, and the convergence rate of the estimator is of order $n^{-2/5}$. Theorem 2 implies the following asymptotically normal distribution.

Corollary 1. *Under Conditions* (A1)–(A7)*, for any given* u*,* z*, and* ω *in* $[-1,1]$*, if* $nh^5 = O(1)$*, we have*

$$(nH)^{1/2}\left\{\begin{pmatrix}\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}\\ \widehat{g}-g\\ \widehat{V}-V\end{pmatrix}(u,z,\omega) - H^2\Psi^{-1}(\mathrm{B})(u,z,\omega)\right\} \xrightarrow{d} N(0,\mathbf{V}(u,z,\omega)),$$

where $\mathbf{V}(u,z,\omega) = [\Psi^{-1}(\mathbf{M}^{-1/2})(u,z,\omega)][\Psi^{-1}(\mathbf{M}^{-1/2})(u,z,\omega)]'$.

Linear functionals are pivotal, because any smooth functions can be approximated by linear combinations of orthonormal basis functions $\psi_0,\psi_1,\ldots$ (e.g., Fourier bases). Estimators for $f(\cdot)$ are obtained using a truncated expansion of these bases, with the coefficients being projections of $f(\cdot)$ to $\psi_j(\cdot)$, $\int_{-1}^{1} f(u)\psi_j(u)du$, for $j=0,1,\ldots$. As a result, the properties of $\hat{f}(\cdot)$ can be expressed using those of $(\int_{-1}^{1}\hat{f}(u)\psi_j(u)du, j=0,1,\ldots)'$.

If the conditional distribution of Y_i given $\mathbf{X}_i$ belongs to the exponential family, we prove in the Supplementary Material that $\hat{\tau} = \sum_{j=1}^{d}\int_{-1}^{1}\hat{\beta}_j(u)\psi_j(u)du + \int_{-1}^{1}\widehat{g}(z)\psi_g(z)dz$ for the linear functionals $\tau = \sum_{j=1}^{d}\int_{-1}^{1}\beta_j(u)\psi_j(u)du + \int_{-1}^{1} g(z)$ $\psi_g(z)dz$ has the same asymptotic variance as the maximum likelihood estimator for τ within a family of parametric submodels. This means semiparametric efficiency in the sense of Bickel et al. (1998). More specifically, let

$$\mathcal{D} = \left\{\psi(z)\ \text{have a continous derivative over}\ [-1,1]\ \text{and}\ \int_{-1}^{1}\psi(z)dz = 0\right\}.$$

Theorem 3 presents the results of semiparametric efficiency.

Theorem 3. *Under Conditions* (A1)–(A7)*, if* $nh_k^4 \to 0$*,* $h_k^2 h_j^{-1}\log(n) \to 0$*, and* $nh_kh_j/(log(n))^2 \to \infty$*, for any* $k,j \in \{1,2,3\}$*, then for any functions* $\psi_j(\cdot) \in \mathcal{D}$*, for* $j=1,\ldots,d$*, and* $\psi_g(z)$ *that having a continuous derivative, we have*

$$\sum_{j=1}^{d}\int_{-1}^{1}(\widehat{\beta}_j-\beta_j)(u)\psi_j(u)du+\int_{-1}^{1}(\widehat{g}-g)(z)\psi_g(z)dz\xrightarrow{d}N(0,\sigma_v^2).$$

In particular, $\sum_{j=1}^{d}\int_{-1}^{1}\widehat{\beta}_j(u)\psi_j(u)du+\int_{-1}^{1}\widehat{g}(z)\psi_g(z)dz$ *is an efficient estimator of* $\sum_{j=1}^{d}\int_{-1}^{1}\beta_j(u)\psi_j(u)du+\int_{-1}^{1}g(z)\psi_g(z)dz$ *if the conditional distribution of* Y_i *given* $\mathbf{X}_i$ *and* U_i *belongs to the exponential family, where* σ_v^2 *is defined in Section* 1 *of the Supplementary Material.*

Theorem 3 implies that the estimator of $\sum_{j=1}^{d}\int\beta_j(x)\psi_j(x)dx+\int g(z)\psi_g(z)dz$ is $\sqrt{n}$-consistent with $h=o(n^{-1/4})$, which amounts to undersmoothing. Using undersmoothing to achieve $\sqrt{n}$-consistency is not unusual in semiparametric regression settings (Carroll et al. (1997); Hastie and Tibshirani (1993)).

The quasi-likelihood function is key to for achieving semiparametric efficiency. To see this, consider the estimation of $\mathbf{g}=(g(z),\dot{g}(z))'$. Substitute (2.8) into the quasi-likelihood function

$$\begin{aligned}Q(\boldsymbol{\beta},g,V)&=\sum_{i=1}^{n}L(\mu_i,Y_i)K_{h_2}(Z_i-z)+\sum_{i=1}^{n}L(\mu_i,Y_i)\left\{1-K_{h_2}(Z_i-z)\right\}\\&\approx\sum_{i=1}^{n}L(\bar{\mu}_i,\ Y_i)K_{h_2}(Z_i-z)+\sum_{i=1}^{n}L(\mu_i,\ Y_i)\left\{1-K_{h_2}(Z_i-z)\right\},\end{aligned}\tag{3.1}$$

where $Z_i=\mathbf{X}_i'\boldsymbol{\beta}(U_i)$ and $\bar{\mu}_i=g(z)+\dot{g}(z)(Z_i-z)$. The μ_i in the second term of (3.1) is not approximated by the linear function $\bar{\mu}_i=g(z)+\dot{g}(z)(Z_i-z)$ because Z_i is out of the neighborhood of z, dictated by the weight $1-K_{h_2}(Z_i-z)$. Differentiating the likelihood function $Q(\boldsymbol{\beta},g,V)$ with respect to $\mathbf{g}=(g(z),\dot{g}(z))'$ and setting the derivatives to zero leads to

$$\sum_{i=1}^{n}\Big(Y_i-\bar{\mu}_i\Big)\frac{W_i(z;\boldsymbol{\beta})}{V(\bar{\mu}_i)}K_{h_2}(Z_i-z)=0.\tag{3.2}$$

Because $V(\bar{\mu}_i)\approx V(\mu_i)$ when Z_i is in the neighborhood of z, the proposed estimating equation (2.9) is the same as the score (3.2) for estimating $\mathbf{g}$.

4. Simulation Studies

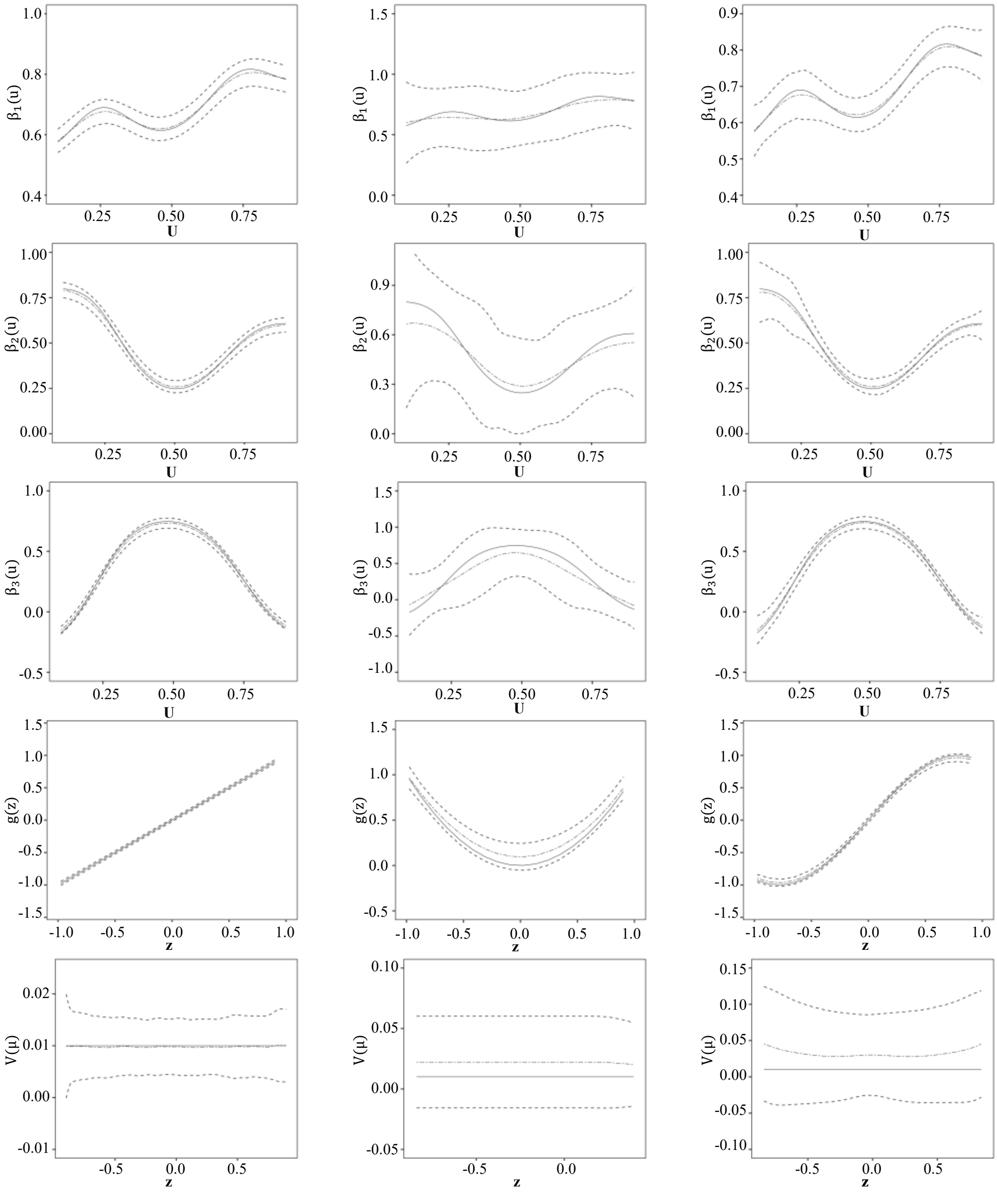
The proposed method is compared with the methods in Zhang, Li and Xia (2015) and Kuruwita, Kulasekera and Gallagher (2011), which are termed ZLX and KKG, respectively. To investigate the impact of misspecifications of the variance functions on estimations, we also compare GVCMs with variance functions that are correctly specified(GVCM-CV) and misspecified(GVCM-MV). The

GVCM-CV and GVCM-MV are implemented using the proposed method with specified variance functions. The Epanechnikov kernel is used in simulations and in the real-data analysis in Section 5. For each configuration, a total of N replications are made. Following Zhang, Li and Xia (2015) and Kuruwita, Kulasekera and Gallagher (2011), the performance of the estimators for $\hat{g}(\cdot)$ and $\hat{\boldsymbol{\beta}}(\cdot)$ is assessed using $MISE_{\beta} = E(\sum_{j=1}^{d}(1/n)\sum_{i=1}^{n}\{\hat{\beta}_j(U_i) - \beta_j(U_i)\}^2)$, and $MISE_g = E((1/n)\sum_{i=1}^{n}[\hat{g}\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\} - g\{\mathbf{X}_i'\boldsymbol{\beta}(U_i)\}]^2)$, respectively. Here, U_i $(i = 1, \ldots, n)$ are the samples of the simulated data, and the expectation is obtained using the sample mean based on the N simulated data sets. We consider three settings, the first two of which were used by Zhang, Li and Xia (2015) and Kuruwita, Kulasekera and Gallagher (2011), respectively. The replication number of simulations is 1,000 for Example 1 and 500 for Examples 2 and 3.

Example 1. (*Normal cases with known variances*). U_i, for $i = 1, \ldots, n$, are independently generated from $Uniform[0, 1]$, and $\mathbf{X}_i$, for $i = 1, \ldots, n$, are independently generated from $N(0_p, I_p)$, with I_p being a $p \times p$ identity matrix, $\varepsilon \sim N(0, 0.01)$. Set $p = 3$ and $\boldsymbol{\beta}(U) = (\beta_1(U), \beta_2(U), \beta_3(U))'$, where $\beta_1(U) = U^2 + 1$, $\beta_2(U) = cos^2(\pi U) + 0.5$, and $\beta_3(U) = 2sin^2(\pi U) - 0.5$. Here, Y is generated as $Y = \mathbf{X}'\boldsymbol{\beta}(U) + \varepsilon$ (Case 1), $Y = (\mathbf{X}'\boldsymbol{\beta}(U))^2 + \varepsilon$ (Case 2), or $Y = sin(2\mathbf{X}'\boldsymbol{\beta}(U)) + \varepsilon$ (Case 3). We set $n = 100, 200$, and 400, and choose the bandwidths to be $h_1 = 0.1, h_2 = 0.3$ for Case 1, $h_1 = 0.2, h_2 = 0.35$ for Case 2, and $h_1 = 0.1, h_2 = 0.25$ for Case 3. With this setup, we aim to investigate the efficiency of our method by assuming a known variance function, as in Zhang, Li and Xia (2015) and Kuruwita, Kulasekera and Gallagher (2011).

Table 1 summarizes the MISEs for the estimators of the functional coefficients obtained using the three methods. Table 1 shows the robustness of the proposed method toward the link function, and its efficiency when the link function is not linear. This is because we use one-dimension smoothing and a quasi-likelihood-based approach, whereas ZLX and KKG both use two kernels. Figure 1 displays the estimates of each unknown function and the 95% pointwise confidence intervals based on the proposed method. Using the estimated link and coefficient functions, we estimate the variance function with $h_3 = 0.1, 0.5, 0.7$ for Cases 1–3, respectively. Figure 1 reveals that the estimates are close to the truth, hinting at the good performance of our proposed method.

Example 2. (*Binary Cases*). U_i and $\mathbf{X}_i$, for$i = 1, \ldots, n$, are generated in the same way as in Example 1. Set $p = 2$, $g(t) = \exp(t)/(1 + \exp(t)), \beta_1(U) = sin(\pi U)$, and $\beta_2(U) = cos(\pi U)$. Here Y_i is independently generated from a Bernoulli distribution with success probability $g\{X_{i1}\beta_1(U_i) + X_{i2}\beta_2(U_i)\}$. We



(a) Case 1. (b) Case 2. (c) Case 3.

Figure 1. (a)–(c): The estimated functions (dotted lines) of $\beta_1(u)$, $\beta_2(u)$, $\beta_3(u)$, $g(z)$, and $V(\omega)$, as well as their 95% pointwise confidence intervals (dashed lines) and the true functions (solid lines) for Example 1 with $n = 400$.

set $n = 800, 1{,}100, 1{,}500$, or $2{,}000$, and choose the bandwidths for our proposed method to be $h_{1,\beta_1} = 0.48, h_{1,\beta_2} = 0.5, h_2 = 1.98$, and $h_3 = 0.1$.

Table 1. MISE for coefficient functions of Example 1.

n	Case 1			Case 2			Case 3		
	ZLX	KKG	Prop.	ZLX	KKG	Prop.	ZLX	KKG	Prop.
100	0.034	0.965	0.004	0.354	2.623	0.311	0.202	2.460	0.186
200	0.018	0.359	0.001	0.228	1.385	0.130	0.098	0.627	0.021
400	0.007	0.177	0.001	0.127	0.360	0.080	0.012	0.241	0.003

Table 2. MISE for Example 2.

	GVULV				GVCM-CV			ZLX		
n	$\beta_1(u)$	$\beta_2(u)$	$g(z)$	$V(\mu)$	$\beta_1(u)$	$\beta_2(u)$	$g(z)$	$\beta_1(u)$	$\beta_2(u)$	$g(z)$
800	0.0784	0.0493	0.0024	0.0017	0.0644	0.0412	0.0019	0.1189	0.0821	0.0142
1,100	0.0656	0.0402	0.0019	0.0014	0.0542	0.0314	0.0014	0.0698	0.0730	0.0048
1,500	0.0505	0.0305	0.0014	0.0013	0.0438	0.0247	0.0012	0.0695	0.0479	0.0036
2,000	0.0479	0.0329	0.0014	0.0012	0.0414	0.0233	0.0012	0.0581	0.0387	0.0025

Example 2 focuses on the impact of the variance function specification on estimation. We compare the MISE of the proposed GVULV with that of the methods with correctly specified variance functions, including ZLX and the GVCM-CV. Table 2 shows that the GVCM-CV is slightly more accurate than the proposed estimator, but that this difference decreases as the sample size grows. In addition, the proposed GVULV outperforms ZLX, with a smaller MISE, even though the variance function is correctly specified in ZLX and is unspecified in the GVULV. Figure 2(a) further shows that the GVULV estimates are close to the truth with reasonable precision, suggesting that the proposed methods work well for the binary case.

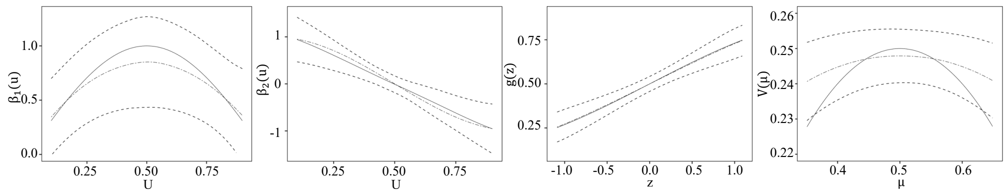
Example 3. (*Normal outcomes with non-constant variances*): U_i, for $i = 1, \ldots, n$, are independently generated from Uniform$[0, 1]$, $\mathbf{X}_i$, for $i = 1, \ldots, n$, are independently generated from $N(0_p, I_p)$, and $\varepsilon \sim N(0, 1)$. Set $p = 2$ and $\boldsymbol{\beta}(U) = (\beta_1(U), \beta_2(U))'$, with $\beta_1(U) = sin(0.5\pi U)$ and $\beta_2(U) = cos(0.5\pi U)$. Here, Y is generated as

$$Y = 5\Phi\{\mathbf{X}'\boldsymbol{\beta}(U)\} + \exp\big[-5\Phi\{\mathbf{X}'\boldsymbol{\beta}(U)\} + 1\big]\varepsilon,$$

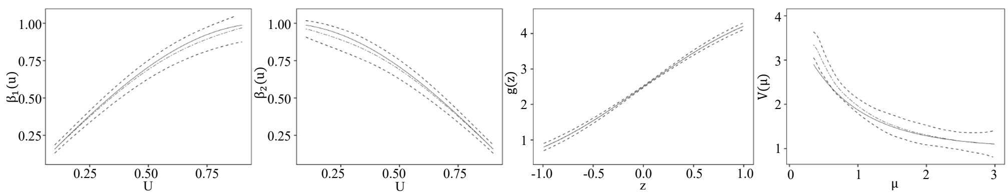
where $\Phi(\cdot)$ is the cumulative distribution function of standard normal. We set the sample size to be $n = 8{,}000, 15{,}000$, and $20{,}000$, and choose the bandwidths to be $(h_1, h_2, h_3) = (0.25, 0.75, 0.45), (0.25, 0.5, 0.38)$, and$(0.25, 0.5, 0.30)$, respectively. We compare the MISE among the proposed GVULV, the GVCM-MV with the variance misspecified as one, and the GVCM-CV. Table 3 shows that GVCM-

Table 3. MISE for Example 3.

	GVULV			GVCM-MV			GVCM-CV		
n	8,000	15,000	20,000	8,000	15,000	20,000	8,000	15,000	20,000
$\beta_1(u)$	0.0027	0.0015	0.0014	0.0200	0.0099	0.0081	0.0025	0.0012	0.0012
$\beta_2(u)$	0.0021	0.0012	0.0013	0.0136	0.0086	0.0074	0.0023	0.0012	0.0012
$g(z)$	0.0034	0.0021	0.0018	0.0226	0.0106	0.0081	0.0025	0.0019	0.0016
$V(\mu)$	0.1258	0.0721	0.0665	–	–	–	–	–	–



(a) Example 2



(b) Example 3

Figure 2. The GVULV estimators (dotted lines) for $\beta_1(u)$, $\beta_2(u)$, $g(z)$, and $V(\omega)$, as well as their 95% pointwise confidence intervals (dashed lines) and the true functions (solid lines) for Examples 2 and 3 with $n = 2{,}000$ and $n = 20{,}000$, respectively.

MV has considerably larger prediction errors, while the proposed estimators are comparable with the GVCM-CV. This suggests that misspecifications of variance functions may bias predictions, and that the uncertainty associated with estimating variance functions decreases as the sample size becomes larger. Figure 2(b) shows $\beta_1(u)$, $\beta_2(u)$, $g(z)$, and $V(\omega)$ estimated using our method, as well as their 95% pointwise confidence intervals. The estimates are close to the truth.

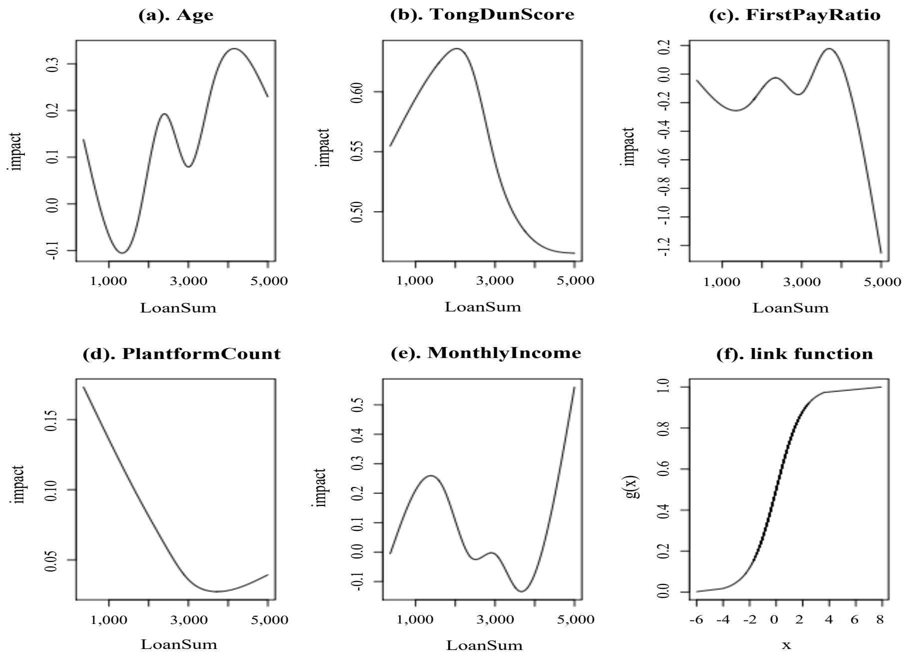
5. Data Analysis

Mobile phones have become an indispensable part of life of young Chinese. To keep pace with the rapidly updated phones or just in pursuit of fashion, some young adults resort to using personal loans to purchase newly marketed phones. Credit checks have become an important step for financial providers before approving a loan. We aim to build a risk prediction model to predict payment delinquency. That is, whether a loanee repays a loan on time, based on personal characteristics collected by the financial provider. The data set we analyze records the personal information of 105,548 borrowers and their repayment status, denoted by Y_i for the ith borrower. In the data set, Y_i takes the value one if the loan was not fully repaid on time, and zero otherwise. The other recorded characteristics are age (X_{i1}), credit score (X_{i2}), the downpayment ratio (X_{i3}), the number of owned credit cards (X_{i4}), monthly income (X_{i5}), and the loan amount (U_i). All covariates are standardized to have mean zero and variance one.

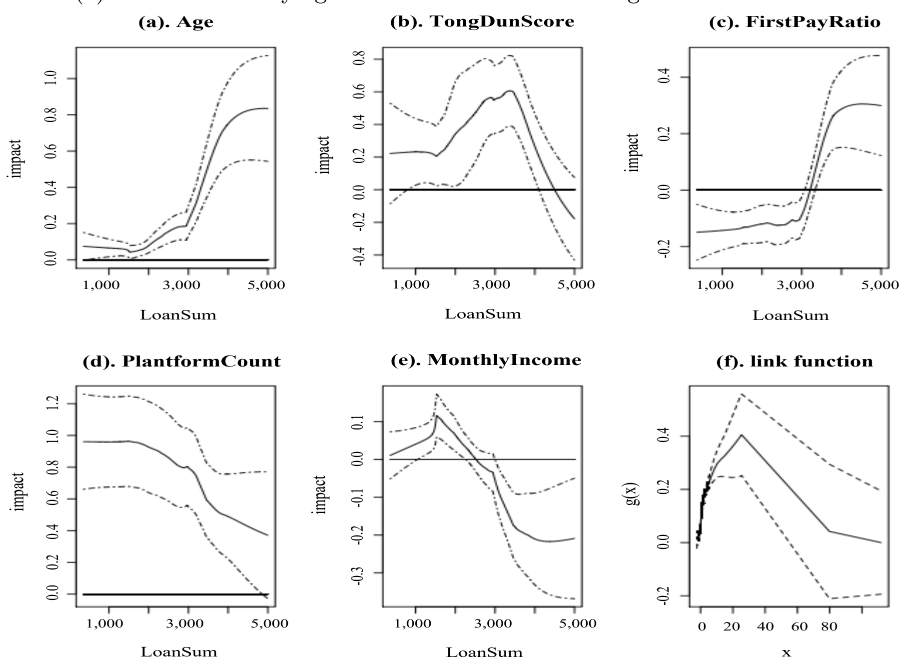
Because the covariates are not uniformly distributed, we use an adaptive approach (Brockmann, Gasser and Herrmann (1993)) to select the bandwidth. Specifically, at each design point, we choose the bandwidth adaptively such that the "window" covers a given portion (q) of neighboring samples. We use five-fold cross-validation, described in Section 2, to determine q, yielding $q = 0.5$.

With the binary response, it is natural to adopt a logistic link function. Figure 3(1) presents the estimates of varying-coefficient functions with a logistic link. Figure 3(2) and Figure 4 show the estimated link, coefficient, and variance functions using the GLULV, revealing that the link and variance functions deviate much from the commonly used link and variance functions for binary responses. In particular, the link function of the GLULV has a unimodal shape with a peak around 35, and differs from the monotone logistic function. Moreover, the prediction error in Table 4 shows that the proposed method outperforms the logistic varying-coefficient model in both the training and the testing data.

Figure 3(2) implies that persons with a combined risk score, $\mathbf{X}_i'\hat{\boldsymbol{\beta}}(U_i)$, around 35 will be most likely to commit payment delinquency. In addition, Figure 3(2) clearly shows nonlinear and significant trends with all the covariates. Specifically, age and the number of owned credit cards are associated with the payment behavior (see Figures 3(2a) and 3(2d)). The age effect increases with the loan sum, and the effect of the number of owned credit cards decreases as the loan amount increases. Figures 3(2b) and 3(2e) suggest quadratic impacts of credit score and monthly income. The former shows that the effect of the credit score increases until the loan amount reaches about RMB 3,800, and then decreases. The latter shows that the impact of monthly income peaks when the loan amount is about RMB 1,800, and becomes statistically insignificant when the loan sum is larger than RMB 2,500. The downpayment ratio acts similarly to age, but the effect switches signs when the loan sum reaches around RMB 3,300.



(1) Generalized varying-coefficient models with a logistic link function.



(2) GVULV and 95% confident interval with $q = 0.5$.

Figure 3. Estimated varying-coefficient and link functions for the mobile phone loan payment data.

Table 4. Prediction accuracy for GVCMs with logistic link and variance functions, and GVCMs with unspecified link and variance functions (GVULV) for the mobile phone microfinance data.

	Logistic prediction error	*GVULV* prediction error
Train set	0.1312094	0.1074576
Test set	0.1312547	0.1074741

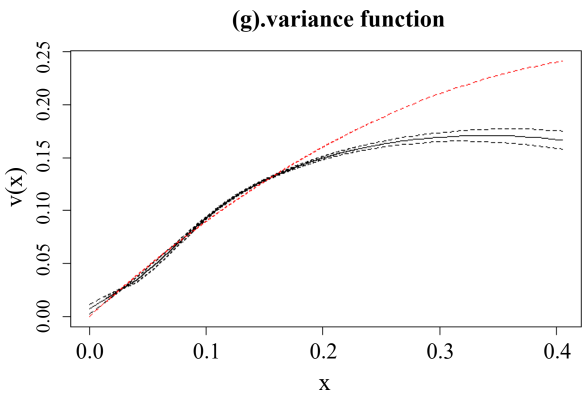


Figure 4. Estimated variance function (solid-black) for the mobile phone microfinance data and its 95% confident interval (dashed-black) using the proposed method with $q = 0.5$. The red-dashed line is the variance function of the logistic method.

6. Discussion

We propose a GVCM for non-normal response data. In contrast to existing methods, our method is a univariate kernel estimator that accounts for heteroscedastic data, and, hence, is more flexible and efficient. Moreover, the proposed estimator has a closed form in the iterative algorithm, which reduces the computational burden. For example, with 105,548 samples in our motivating date set, it is not feasible to apply existing methods, whereas our method converges within a minute. Finally, the proposed method is shown to be uniformly consistent, asymptotically normal, and semiparametrically efficient when the conditional distribution belongs to an exponential family. The simulation study shows that our estimator is more efficient than those obtained using existing methods.

When the covariates outnumber the sample size, we need to estimate the coefficient functions and select the significant covariates simultaneously. A natural approach is to perform a regularized regression by adding a penalty term to the objective function. However, because the proposed method is kernel based and estimates unknown functions pointwise, it may not be straightforward to combine the proposed method with a penalized regression. In this case, using spline approximations may be more feasible. This is left to future research.

Supplementary Material

The online Supplementary Material contains additional notation, lemmas, and proofs.

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