A Generalized Heckman Model With Varying Sample Selection Bias and Dispersion Parameters

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S1. Score function and information matrix

To simplify the calculation of the score function and information matrix, we consider a single pair of observation (y, u). Following the notation of the paper, we have $\mu_1 = \boldsymbol{x}^\top\boldsymbol{\beta}$, $\mu_2 = \boldsymbol{w}^\top\boldsymbol{\gamma}$, $\sigma = \exp(\boldsymbol{e}^\top\boldsymbol{\lambda})$, $\rho = \tanh(\boldsymbol{v}^\top\boldsymbol{\kappa})$, $z \equiv \dfrac{y-\mu_1}{\sigma}$, $A_\rho \equiv \dfrac{1}{\sqrt{1-\rho^2}}$, $A_{\rho\rho} \equiv \rho A_\rho$, $\zeta \equiv \mu_2 A_\rho + z A_{\rho\rho}$, where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top \in \mathbb{R}^p$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q)^\top \in \mathbb{R}^q$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)^\top \in \mathbb{R}^r$ and $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_s)^\top \in \mathbb{R}^s$ are vectors of unknown parameters with associated covariate vectors. The log-likelihood function (for a single pair (y, u)) of the generalized Heckman model is

$$\mathcal{L}(\boldsymbol{\theta}) = u\{\log\Phi(\zeta) + \log\phi(z) - \log\sigma\} + (1-u)\log\Phi(-\mu_2), \qquad \text{(S1.1)}$$

where $u = 1$ if y is observed and $u = 0$ otherwise. Consider the following derivatives, for $k = 1, \ldots, p$, $l = 1, \ldots, q$, $m = 1, \ldots, r$, and $n = 1, \ldots, s$.

$$M(x) = \frac{\partial \log\Phi(x)}{\partial x} = \frac{\phi(x)}{\Phi(x)}, \qquad \frac{\partial z}{\partial \beta_k} = -\frac{x_k}{\sigma}, \qquad \frac{\partial z}{\partial \lambda_m} = -z e_m,$$

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$$\frac{\partial A_\rho}{\partial \kappa_n} = \rho A_\rho^3 v_n \operatorname{sech}^2(v^\top \kappa), \qquad \frac{\partial \zeta}{\partial \gamma_l} = w_l A_\rho, \qquad \frac{\partial \zeta}{\partial \lambda_m}, = -z e_m A_{\rho\rho},$$

$$\frac{\partial \zeta}{\partial \beta_k} = -\frac{x_k A_{\rho\rho}}{\sigma}, \qquad \frac{\partial \zeta}{\partial \kappa_n} = v_n \operatorname{sech}^2(v^\top \kappa) A_\rho \left[\mu_2 \rho A_\rho^2 + z(1 + A_{\rho\rho}^2)\right],$$

$$Q_\rho = \mu_2 \rho A_\rho^2 + z(1 + A_{\rho\rho}^2), \qquad \frac{\partial A_{\rho\rho}}{\partial \kappa_k} = v_k A_\rho (1 + A_{\rho\rho}^2) \operatorname{sech}^2(v^\top \kappa),$$

$$Q_{\rho\rho} = \mu_2(1 + 2A_{\rho\rho}^2) + 2z\rho(1 + A_{\rho\rho}^2).$$

Let $S_\alpha = \dfrac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \alpha}$ be the score function corresponding to the parameter α. For $\alpha \in (\boldsymbol{\beta}^\top, \boldsymbol{\gamma}^\top, \boldsymbol{\lambda}^\top, \boldsymbol{\kappa}^\top)$,

$$\begin{aligned}
S_{\beta_k} &= \frac{u x_k}{\sigma}\left(z - M(\zeta) A_{\rho\rho}\right),\ k = 1, \ldots, p, \\
S_{\gamma_l} &= u w_l M(\zeta) A_\rho - (1-u) w_l M(-\mu_2),\ l = 1, \ldots, q, \\
S_{\lambda_m} &= u e_m \left(z^2 - 1 - M(\zeta) A_{\rho\rho} z\right),\ m = 1, \ldots, r, \\
S_{\kappa_n} &= u v_n M(\zeta) A_\rho \operatorname{sech}^2(v^\top \kappa) Q_\rho,\ n = 1, \ldots, s,
\end{aligned}$$

The elements of the Hessian matrix are given by

$$S_{\beta_{k_1}\beta_{k_2}} = -\frac{u x_{k_1} x_{k_2}}{\sigma^2}\left[1 + M(\zeta) A_{\rho\rho}^2 (\zeta + M(\zeta))\right], \quad S_{\beta_{k_1}\gamma_{l_1}} = \frac{u x_{k_1} w_{l_1} \rho A_\rho^2 M(\zeta)}{\sigma}\left(\zeta + M(\zeta)\right),$$

$$S_{\beta_{k_1}\lambda_{m_1}} = \frac{u x_{k_1} e_{m_1}}{\sigma}\left[M(\zeta) A_{\rho\rho} - 2z - z A_{\rho\rho}^2 M(\zeta)(\zeta + M(\zeta))\right],$$

$$S_{\beta_{k_1}\kappa_{n_1}} = \frac{u x_{k_1} v_{n_1} A_\rho M(\zeta) \operatorname{sech}^2(v^\top \kappa)}{\sigma}\left[A_{\rho\rho} Q_\rho (\zeta + M(\zeta)) - 1 - A_{\rho\rho}^2\right],$$

$$S_{\gamma_{l_1}\beta_{k_1}} = \frac{u w_{l_1} x_{k_1} \rho A_\rho^2 M(\zeta)(\zeta + M(\zeta))}{\sigma}, \quad S_{\gamma_{l_1}\lambda_{m_1}} = u w_{l_1} e_{m_1} \rho A_\rho^2 z M(\zeta)(\zeta + M(\zeta)),$$

$$S_{\gamma_{l_1}\gamma_{l_2}} = -uw_{l_1}w_{l_2}A_\rho^2 M(\zeta)(\zeta + M(\zeta)) + (1-u)w_k w_l M(-\mu_2)(M(-\mu_2)+\mu_2),$$

$$S_{\gamma_{l_1}\kappa_{n_1}} = uw_{l_1}v_{n_1}A_\rho^2 M(\zeta)\,\mathrm{sech}(v^\top\kappa)^2\left[A_{\rho\rho} - Q_\rho(\zeta+M(\zeta))\right],$$

$$S_{\lambda_{m_1}\beta_{k_1}} = \frac{ue_{m_1}x_{k_1}}{\sigma}\left[A_{\rho\rho}M(\zeta) - M(\zeta)(\zeta+M(\zeta))zA_{\rho\rho}^2 - 2z\right],$$

$$S_{\lambda_{m_1}\gamma_{l_1}} = ue_{m_1}w_{l_1}z\rho A_\rho^2 M(\zeta)(\zeta+M(\zeta)),$$

$$S_{\lambda_{m_1}\lambda_{m_2}} = ue_{m_1}e_{m_2}z\left[A_{\rho\rho}M(\zeta) - A_{\rho\rho}^2 zM(\zeta)(\zeta+M(\zeta)) - 2z\right],$$

$$S_{\lambda_{m_1}\kappa_{n_1}} = ue_{m_1}v_{n_1}A_\rho M(\zeta)\,\mathrm{sech}^2(v^\top\kappa)\left[zA_{\rho\rho}(\zeta+M(\zeta))Q_\rho - z(1+A_{\rho\rho}^2)\right],$$

$$S_{\kappa_{n_1}\beta_{k_1}} = \frac{uv_{n_1}x_{k_1}M(\zeta)A_\rho\,\mathrm{sech}^2 v^\top\kappa}{\sigma}\left[Q_\rho(\zeta+M(\zeta))A_{\rho\rho}^2 - 1 - A_{\rho\rho}^2\right],$$

$$S_{\kappa_{n_1}\gamma_{l_1}} = uv_{n_1}w_{l_1}A_\rho^2 M(\zeta)\,\mathrm{sech}^2(v^\top\kappa)\left[A_{\rho\rho} - Q_\rho(\zeta+M(\zeta))\right],$$

$$S_{\kappa_{n_1}\lambda_{m_1}} = uv_{n_1}e_{m_1}zA_\rho M(\zeta)\,\mathrm{sech}^2(v^\top\kappa)\left[A_{\rho\rho}Q_\rho(\zeta+M(\zeta)) - 1 - A_{\rho\rho}^2\right],$$

$$\begin{aligned} S_{\kappa_{n_1}\kappa_{n_2}} = {} & uv_{n_1}v_{n_2}A_\rho M(\zeta)\,\mathrm{sech}^2(v^\top\kappa)\left\{A_\rho\,\mathrm{sech}^2(v^\top\kappa)\left[A_{\rho\rho}Q_\rho - Q_\rho^2(\zeta+M(\zeta)) + A_\rho Q_{\rho\rho}\right]\right. \\ & \left. -2Q_\rho\,\mathrm{tangh}(v^\top\kappa)\right\}, \end{aligned}$$

for $k_1, k_2 = 1, \ldots, p, l_1, l_2 = 1, \ldots, q, m_1, m_2 = 1, \ldots, r$, and $n_1, n_2 = 1, \ldots, s$.

S2. Moments of the response variable

It is known that

$$\begin{aligned} Y_1^*|Y_2^* = y &\sim \mathcal{N}\bigg(\mu_1 + \rho\sigma(y-\mu_2), \sigma^2(1-\rho^2)\bigg), \\ Y_2^*|Y_1^* = y &\sim \mathcal{N}\bigg(\mu_2 + \frac{\rho}{\sigma}(y-\mu_1), 1-\rho^2\bigg). \end{aligned} \tag{S2.2}$$

Therefore,

$$E(Y_1^*|Y_2^* = y) = \mu_1 + \rho\sigma(y-\mu_2) \text{ and } \mathrm{Var}(Y_1^*|Y_2^* = y) = \sigma^2(1-\rho^2),$$

$$E(Y_2^*|Y_1^* = y) = \mu_2 + \frac{\rho}{\sigma}(y-\mu_2) \text{ and } \mathrm{Var}(Y_2^*|Y_1^* = y) = (1-\rho^2).$$

If $X \sim N(\mu, \sigma^2)$, then

$$E(X|X > a) = \mu + \sigma\lambda(\alpha) \text{ and } \mathrm{Var}(X|X > a) = \sigma^2\bigg(1 - \lambda(\alpha)[\lambda(\alpha) - \alpha]\bigg),$$

where $\lambda(\alpha) = \dfrac{\phi(\alpha)}{1-\Phi(\alpha)}$ and $\alpha = \dfrac{a-\mu}{\sigma}$. Using the fact that $\sigma_2 = 1, a = 0$, and to simplify notation $\lambda(\mu_2) = \lambda$, we have that

$$E(Y_2^*|Y_2^* > 0) = \mu_2 + \lambda \quad \text{and} \quad \mathrm{Var}(Y_2^*|Y_2^* > 0) = \bigg(1 - \lambda(\lambda + \mu_2)\bigg) = 1 - \lambda^2 - \mu_2\lambda.$$

Therefore,

$$E(Y_1^*|Y_2^*>0) = E\Big[E(Y_1^*|Y_2^*)|Y_2^*>0\Big] = E\Big[\mu_1 + \rho\sigma(Y_2^* - \mu_2)|Y_2^*>0\Big]$$
$$= \mu_1 + \rho\sigma\lambda(\mu_2),$$

$$E\left([Y_2^*]^2|Y_2^*>0\right) = \text{Var}\left(Y_2^*|Y_2^*>0\right) + \left[E\left(Y_2^*|Y_2^*>0\right)\right]^2 = 1 + \mu_2\lambda + \mu_2^2,$$

$$E\left[\left(Y_2^* - \mu_2\right)^2|Y_2^*>0\right] = \text{Var}\left(Y_2^*|Y_2^*>0\right) + \left[E\left(Y_2^* - \mu_2|Y_2^*>0\right)\right]^2 = 1 - \mu_2\lambda,$$

$$\begin{aligned} E\left([Y_1^*]^2|Y_2^*>0\right) &= E\left[E\left((Y_1^*)^2|Y_2^*\right)|Y_2^*>0\right] = E\left[\text{Var}(Y_1^*|Y_2^*) + [E(Y_1^*|Y_2^*)]^2|Y_2^*>0\right] \\ &= E[\sigma^2(1-\rho^2) + \mu_1^2 + 2\mu_1\rho\sigma(Y_2^* - \mu_2) + \rho^2\sigma^2(Y_2^* - \mu_2)^2|Y_2^*>0] \\ &= \sigma^2 + \mu_1^2 + 2\mu_1\rho\sigma\lambda - \mu_2\rho^2\sigma^2\lambda. \end{aligned}$$

We have that

$$\text{Var}(Y_1^*|Y_2^*>0) = E((Y_1^*)^2|Y_2^*>0) - [E(Y_1^*|Y_2^*>0)]^2 = \sigma^2 - \mu_2\rho^2\sigma^2\lambda - \rho^2\sigma^2\lambda^2.$$

Hence, it follows that

$$E(Y) = E(Y|Y_2^*>0)P(Y_2^*>0) + E(Y|Y_2^*\leq 0)P(Y_2\leq 0) = (\mu_1 + \rho\sigma\lambda)\Phi(\mu_2)$$

and

$$\text{Var}(Y) = \text{Var}(Y_1^*|Y_2^*>0)P(Y_2^*>0) = (\sigma^2 - \mu_2\rho^2\sigma^2\lambda - \rho^2\sigma^2\lambda^2)\Phi(\mu_2).$$

S3. Additional simulations

We present additional simulations for the generalized Heckman model. The aim of these simulations is to evaluate the behavior of the maximum likelihood estimators in finite-samples of our proposed model, and also to compare to the classic Heckman, skew-normal, and Heckman-t models. The results for the Scenarios 2–6, based on the regression structures (4.1)–(4.4) of the paper, are presented in Tables 1–5, respectively.

- Scenario 2: both varying dispersion and correlation (sample selection bias parameter) without exclusion restriction.
- Scenario 3: constant dispersion ($\lambda_1 = 0$) and varying correlation. In this case, we experienced some numerical problems with the skew-normal model and therefore we do not consider it.
- Scenario 4: varying dispersion and constant correlation ($\kappa_1 = 0$).
- Scenario 5: both constant dispersion and correlation ($\lambda_1 = \kappa_1 = 0$).
- Scenario 6: both varying dispersion and correlation with censorship around 50%.

Table 1: Empirical mean and root mean square error (RMSE) of the maximum likelihood estimates of the parameters based on the generalized Heckman, classic Heckman, Student-t, and skew-normal sample selection models under Scenario 2.

			Generalized Heckman		Classic Heckman		Heckman Skew-Normal		Heckman Student-t	
Parameters		n	mean	RMSE	mean	RMSE	mean	RMSE	mean	RMSE
		500	0.909	0.087	0.868	0.095	1.183	0.344	1.267	0.399
γ_0	0.9	1.000	0.904	0.060	0.858	0.078	1.199	0.343	1.267	0.382
		2.000	0.901	0.040	0.865	0.055	1.251	0.366	1.272	0.380
		500	0.506	0.081	0.600	0.136	0.577	0.108	0.667	0.209
γ_1	0.5	1.000	0.506	0.054	0.596	0.116	0.570	0.087	0.665	0.186
		2.000	0.503	0.037	0.602	0.112	0.565	0.073	0.677	0.188
		500	1.113	0.103	1.029	0.135	0.867	0.268	1.477	0.420
γ_2	1.1	1.000	1.108	0.070	1.022	0.114	0.860	0.257	1.501	0.421
		2.000	1.105	0.052	1.006	0.111	0.829	0.280	1.483	0.395
		500	1.113	0.190	1.308	0.873	-0.027	1.523	1.181	0.384
β_0	1.1	1.000	1.104	0.124	1.217	0.899	0.037	1.394	1.127	0.282
		2.000	1.097	0.082	1.314	0.886	-0.257	1.478	1.095	0.234
		500	0.709	0.079	0.982	0.439	0.391	0.383	0.891	0.233
β_1	0.7	1.000	0.703	0.048	0.964	0.410	0.391	0.346	0.881	0.204
		2.000	0.700	0.031	0.913	0.361	0.325	0.390	0.865	0.177
		500	0.096	0.146	-0.136	0.770	-0.758	0.903	0.055	0.329
β_2	0.1	1.000	0.098	0.089	-0.074	0.745	-0.819	0.939	0.100	0.220
		2.000	0.102	0.056	-0.128	0.745	-0.768	0.878	0.128	0.198
		500	0.406	0.044	0.993	0.595	1.318	0.920	0.355	0.103
λ_0	0.4	1.000	0.401	0.030	0.998	0.599	1.305	0.906	0.345	0.088
		2.000	0.401	0.020	0.998	0.598	1.342	0.943	0.354	0.068
		500	0.694	0.044	–	–	–	–	–	–
λ_1	0.7	1.000	0.699	0.033	–	–	–	–	–	–
		2.000	0.700	0.021	–	–	–	–	–	–
		500	0.311	0.333	0.061	0.998	-0.986	1.451	0.198	0.577
κ_0	0.3	1.000	0.309	0.200	0.137	1.015	-1.085	1.463	0.285	0.409
		2.000	0.311	0.136	0.027	1.021	-1.153	1.490	0.303	0.369
		500	0.565	0.256	–	–	–	–	–	–
κ_1	0.5	1.000	0.521	0.147	–	–	–	–	–	–
		2.000	0.510	0.099	–	–	–	–	–	–

Table 2: Empirical mean and root mean square error (RMSE) of the maximum likelihood estimates of the parameters based on the generalized Heckman, classic Heckman, and Student-t sample selection models under Scenario 3.

			Generalized Heckman		Classic Heckman		Heckman Student-t	
Parameters		n	mean	RMSE	mean	RMSE	mean	RMSE
		500	0.911	0.093	0.910	0.093	0.924	0.099
γ_0	0.9	1.000	0.904	0.063	0.904	0.064	0.917	0.068
		2.000	0.904	0.042	0.902	0.042	0.914	0.046
		500	0.505	0.083	0.500	0.086	0.508	0.089
γ_1	0.5	1.000	0.505	0.057	0.499	0.060	0.507	0.062
		2.000	0.504	0.040	0.499	0.042	0.506	0.043
		500	1.117	0.106	1.117	0.107	1.135	0.115
γ_2	1.1	1.000	1.108	0.074	1.109	0.075	1.125	0.082
		2.000	1.109	0.054	1.108	0.054	1.124	0.060
		500	0.609	0.083	0.609	0.085	0.619	0.090
γ_3	0.6	1.000	0.604	0.064	0.605	0.066	0.614	0.069
		2.000	0.603	0.040	0.604	0.041	0.612	0.044
		500	1.103	0.062	1.111	0.076	1.110	0.077
β_0	1.1	1.000	1.101	0.044	1.113	0.056	1.112	0.056
		2.000	1.100	0.029	1.110	0.039	1.109	0.038
		500	0.699	0.045	0.775	0.084	0.774	0.084
β_1	0.7	1.000	0.699	0.032	0.767	0.073	0.766	0.073
		2.000	0.700	0.022	0.768	0.071	0.768	0.071
		500	0.098	0.051	0.077	0.064	0.077	0.064
β_2	0.1	1.000	0.099	0.037	0.074	0.051	0.074	0.050
		2.000	0.100	0.026	0.079	0.037	0.079	0.037
		500	-0.401	0.042	-0.415	0.044	-0.431	1.050
λ_0	-0.4	1.000	-0.402	0.029	-0.416	0.032	-0.432	1.050
		2.000	-0.401	0.019	-0.418	0.026	-0.432	1.050
		500	-0.003	0.042	–	–	–	–
λ_1	0	1.000	-0.002	0.031	–	–	–	–
		2.000	-0.000	0.020	–	–	–	–
		500	0.318	0.234	0.128	0.313	0.130	0.299
κ_0	0.3	1.000	0.310	0.160	0.105	0.270	0.107	0.265
		2.000	0.308	0.108	0.100	0.238	0.103	0.236
		500	0.563	0.222	–	–	–	–
κ_1	0.5	1.000	0.524	0.130	–	–	–	–
		2.000	0.511	0.086	–	–	–	–

Table 3: Empirical mean and root mean square error (RMSE) of the maximum likelihood estimates of the parameters based on the generalized Heckman, classic Heckman, Student-t, and skew-normal sample selection models under Scenario 4.

Parameters		n	Generalized Heckman		Classic Heckman		Heckman Skew-Normal		Heckman Student-t	
			mean	RMSE	mean	RMSE	mean	RMSE	mean	RMSE
γ_0	0.9	500	0.912	0.093	0.888	0.094	0.913	0.317	1.320	0.454
		1.000	0.905	0.066	0.883	0.069	0.871	0.334	1.310	0.426
		2.000	0.904	0.043	0.886	0.046	0.854	0.302	1.306	0.414
γ_1	0.5	500	0.510	0.086	0.565	0.114	0.538	0.096	0.668	0.214
		1.000	0.506	0.059	0.573	0.097	0.543	0.077	0.679	0.201
		2.000	0.505	0.042	0.568	0.082	0.540	0.061	0.678	0.190
γ_2	1.1	500	1.117	0.104	1.063	0.120	1.016	0.159	1.555	0.493
		1.000	1.109	0.076	1.048	0.099	0.985	0.157	1.532	0.452
		2.000	1.108	0.054	1.043	0.084	0.998	0.133	1.530	0.441
γ_3	0.6	500	0.607	0.086	0.561	0.106	0.539	0.123	0.858	0.299
		1.000	0.605	0.067	0.542	0.097	0.510	0.126	0.838	0.262
		2.000	0.602	0.041	0.544	0.074	0.522	0.098	0.843	0.252
β_0	1.1	500	1.104	0.072	1.021	0.271	0.533	0.731	1.120	0.106
		1.000	1.102	0.046	0.959	0.285	0.558	0.696	1.113	0.081
		2.000	1.099	0.032	0.915	0.244	0.533	0.680	1.106	0.052
β_1	0.7	500	0.699	0.039	0.768	0.149	0.631	0.169	0.715	0.065
		1.000	0.699	0.023	0.770	0.130	0.654	0.158	0.713	0.043
		2.000	0.699	0.017	0.788	0.115	0.683	0.138	0.711	0.031
β_2	0.1	500	0.097	0.052	0.146	0.209	0.058	0.218	0.090	0.080
		1.000	0.098	0.032	0.192	0.215	0.076	0.241	0.094	0.058
		2.000	0.100	0.022	0.221	0.173	0.117	0.193	0.099	0.040
λ_0	-0.4	500	-0.399	0.044	0.204	0.607	0.392	0.797	-0.488	0.120
		1.000	-0.401	0.029	0.202	0.604	0.360	0.763	-0.476	0.097
		2.000	-0.400	0.020	0.202	0.603	0.345	0.748	-0.470	0.080
λ_1	0.7	500	0.701	0.042	–	–	–	–	–	–
		1.000	0.701	0.030	–	–	–	–	–	–
		2.000	0.701	0.020	–	–	–	–	–	–
κ_0	0.3	500	0.309	0.257	0.348	0.642	0.028	0.821	0.219	0.309
		1.000	0.304	0.162	0.483	0.676	0.106	0.910	0.239	0.239
		2.000	0.305	0.116	0.619	0.547	0.219	0.762	0.260	0.156
κ_1	0	500	0.017	0.232	–	–	–	–	–	–
		1.000	0.001	0.140	–	–	–	–	–	–
		2.000	0.005	0.098	–	–	–	–	–	–

Table 4: Empirical mean and root mean square error (RMSE) of the maximum likelihood estimates of the parameters based on the generalized Heckman, classic Heckman, Student-t, and skew-normal sample selection models under Scenario 5.

Parameters		n	Generalized Heckman		Classic Heckman		Heckman Skew-Normal		Heckman Student-t	
			mean	RMSE	mean	RMSE	mean	RMSE	mean	RMSE
		500	0.913	0.094	0.912	0.094	0.824	0.175	0.932	0.102
γ_0	0.9	1.000	0.906	0.066	0.906	0.066	0.816	0.145	0.923	0.072
		2.000	0.904	0.044	0.904	0.044	0.823	0.122	0.921	0.050
		500	0.509	0.086	0.510	0.086	0.505	0.086	0.521	0.090
γ_1	0.5	1.000	0.505	0.060	0.505	0.060	0.502	0.059	0.515	0.063
		2.000	0.505	0.042	0.505	0.042	0.502	0.042	0.515	0.046
		500	1.120	0.108	1.120	0.108	1.109	0.108	1.145	0.120
γ_2	1.1	1.000	1.109	0.077	1.110	0.077	1.101	0.078	1.131	0.086
		2.000	1.108	0.054	1.108	0.054	1.101	0.055	1.130	0.064
		500	0.607	0.087	0.608	0.086	0.602	0.086	0.622	0.092
γ_3	0.6	1.000	0.605	0.068	0.605	0.068	0.601	0.068	0.617	0.072
		2.000	0.602	0.041	0.602	0.041	0.599	0.041	0.614	0.045
		500	1.109	0.157	1.107	0.153	0.662	0.656	1.107	0.154
β_0	1.1	1.000	1.106	0.110	1.103	0.107	0.662	0.584	1.103	0.107
		2.000	1.099	0.075	1.098	0.074	0.709	0.531	1.097	0.074
		500	0.704	0.111	0.702	0.088	0.701	0.088	0.702	0.088
β_1	0.7	1.000	0.696	0.077	0.696	0.064	0.696	0.064	0.696	0.064
		2.000	0.700	0.053	0.700	0.044	0.699	0.044	0.700	0.044
		500	0.092	0.128	0.095	0.125	0.093	0.125	0.095	0.125
β_2	0.1	1.000	0.095	0.091	0.097	0.088	0.096	0.088	0.097	0.089
		2.000	0.100	0.064	0.101	0.063	0.100	0.062	0.101	0.063
		500	0.400	0.043	0.400	0.042	0.480	0.126	0.377	0.053
λ_0	0.4	1.000	0.399	0.029	0.400	0.029	0.465	0.101	0.380	0.040
		2.000	0.400	0.021	0.400	0.020	0.456	0.084	0.380	0.030
		500	0.000	0.042	–	–	–	–	–	–
λ_1	0	1.000	-0.000	0.030	–	–	–	–	–	–
		2.000	0.000	0.020	–	–	–	–	–	–
		500	0.309	0.255	0.291	0.387	0.305	0.255	0.291	0.238
κ_0	0.3	1.000	0.303	0.174	0.298	0.346	0.313	0.175	0.299	0.163
		2.000	0.308	0.120	0.304	0.328	0.318	0.122	0.306	0.114
		500	-0.003	0.209	–	–	–	–	–	–
κ_1	0	1.000	-0.001	0.126	–	–	–	–	–	–
		2.000	-0.002	0.087	–	–	–	–	–	–

Table 5: Empirical mean and root mean square error (RMSE) of the maximum likelihood estimates of the parameters based on the generalized Heckman, classic Heckman, Student-t, and skew-normal sample selection models under Scenario 6.

Parameters		n	Generalized Heckman		Classic Heckman		Heckman Skew-Normal		Heckman Student-t	
			mean	RMSE	mean	RMSE	mean	RMSE	mean	RMSE
γ_0	0.3	500	0.320	0.143	0.311	0.135	0.171	0.281	0.488	0.282
		1.000	0.307	0.092	0.301	0.086	0.147	0.243	0.489	0.247
		2.000	0.302	0.060	0.300	0.059	0.109	0.261	0.486	0.211
γ_1	1.8	500	1.882	0.286	1.804	0.270	1.754	0.286	2.635	0.965
		1.000	1.839	0.172	1.762	0.169	1.722	0.192	2.653	0.915
		2.000	1.821	0.116	1.740	0.128	1.687	0.170	2.612	0.840
γ_2	4	500	4.193	0.565	4.098	0.530	3.998	0.541	6.129	2.366
		1.000	4.079	0.339	3.944	0.339	3.877	0.376	6.085	2.202
		2.000	4.042	0.230	3.907	0.250	3.813	0.324	5.966	2.017
γ_3	2.6	500	2.722	0.367	2.660	0.350	2.594	0.356	3.978	1.537
		1.000	2.651	0.238	2.557	0.239	2.516	0.262	3.971	1.455
		2.000	2.624	0.151	2.536	0.167	2.476	0.214	3.872	1.305
β_0	1.1	500	1.101	0.064	1.059	0.135	0.573	0.658	1.098	0.084
		1.000	1.100	0.046	1.017	0.123	0.519	0.710	1.105	0.061
		2.000	1.100	0.030	1.011	0.107	0.517	0.720	1.095	0.043
β_1	0.7	500	0.700	0.039	0.792	0.158	0.699	0.125	0.745	0.088
		1.000	0.700	0.030	0.780	0.124	0.698	0.092	0.739	0.066
		2.000	0.700	0.018	0.793	0.114	0.730	0.077	0.741	0.055
β_2	0.1	500	0.101	0.049	0.102	0.122	0.086	0.127	0.102	0.073
		1.000	0.100	0.037	0.137	0.096	0.119	0.094	0.100	0.053
		2.000	0.100	0.026	0.139	0.075	0.127	0.071	0.105	0.040
λ_0	-0.4	500	-0.407	0.047	0.120	0.523	0.269	0.675	-0.421	0.084
		1.000	-0.405	0.033	0.224	0.625	0.354	0.756	-0.389	0.062
		2.000	-0.403	0.021	0.226	0.626	0.352	0.754	-0.381	0.044
λ_1	0.7	500	0.706	0.051	–	–	–	–	–	–
		1.000	0.702	0.034	–	–	–	–	–	–
		2.000	0.703	0.023	–	–	–	–	–	–
κ_0	0.3	500	0.344	0.496	0.487	0.607	0.429	0.450	0.378	0.285
		1.000	0.328	0.351	0.545	0.338	0.487	0.365	0.338	0.168
		2.000	0.311	0.114	0.580	0.318	0.576	0.341	0.383	0.146
κ_1	0.5	500	0.696	0.591	–	–	–	–	–	–
		1.000	0.572	0.246	–	–	–	–	–	–
		2.000	0.524	0.126	–	–	–	–	–	–