Doubly Robust Regression Analysis for Data Fusion

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Supplementary Material

This Supplementary Material contains proofs of the results, as well as additional simulation results.

S1 Derivation of DR linear space

The observed data likelihood is given by

$$L(O) = f(R|V;\eta)\left\{\int f(Y|V,L;\theta)dF(L|V;\alpha)\right\}^{R} f(L|V;\alpha)^{1-R} f(V;\epsilon),$$

where we consider α and ϵ to be possibly infinite-dimensional nuisance parameters and $O = (R, RY, (1-R)L, V)$. The nuisance tangent space is $\Lambda_\eta \oplus \Lambda_\alpha \oplus \Lambda_\epsilon$, where

$$\Lambda_\epsilon = \{B_1 S_\epsilon(V) : E[S_\epsilon(V)] = 0\}$$
$$\Lambda_\alpha = \{B_2 E[S_\alpha(V,L)|O] = B_2 \{RE[S_\alpha(V,L)|Y,V] + (1-R)S_\alpha(V,L)\} :$$
$$E[S_\alpha(V,L)|V] = 0\}$$
$$\Lambda_\eta = \left\{ B_3 \left[\frac{\partial}{\partial \eta} \log f(R|V;\eta) \right] \right\}.$$

Let $\Lambda^\perp$ be the observed-data linear space that is orthogonal to $\Lambda_\epsilon \oplus \Lambda_\alpha$. Then for given $h(O) \in \Lambda^\perp_{\epsilon,\alpha}$ we have

$$E\left[h(O)S_\epsilon(V)\right] = 0 \quad \forall S_\epsilon(V) \in \Lambda_\epsilon,$$
$$E\left\{h(O)E[S_\alpha(V,L)|O]\right\} = E\left\{E[h(O)S_\alpha(V,L)|O]\right\}$$
$$= E\left\{h(O)S_\alpha(V,L)\right\} = 0 \quad \forall S_\alpha(V,L).$$

From the results of Robins et al. (1995) and Hasminskii and Ibragimov (1983), $\Lambda^\perp_{\epsilon,\alpha}$ is given by

$$\Lambda^\perp_{\epsilon,\alpha} = \{Bh(O) : E[h(R,V)|V] = 0 \text{ or } E[h(O)|L,V] = 0\}$$
$$= \left\{ B\left[\frac{R}{\pi(V)}\left[g(Y,V) + k(V)\right] - \frac{1-R}{1-\pi(V)} E[g(Y,V) + k(V)|V,L] \right] : \right.$$
$$\left. g, k \text{ arbitrary, } g(0,x) = 0 \right\}.$$

Therefore, when the data source process is modeled, a typical element in the ortho-complement $\Lambda^{\perp}$ to the nuisance tangent space is given by

$$\left\{h(O) - \Pi\left[h(O)|\Lambda_{\eta}\right] : h(O) \in \Lambda_{\epsilon,\alpha}^{\perp}\right\},$$

where Π denotes the projection operator. For a fixed choice of function $g(Y,V)$, the space of elements in $\Lambda^{\perp}$ is a translation of a linear space away from the origin. Specifically, this linear space is given by $V(g) = x_0 + M$, with the element

$$x_0 = \left\{\frac{R}{\pi(V)} g(Y,V) - \frac{1-R}{1-\pi(V)} E[g(Y,V)|V,L]\right\} - \Pi\left[\{\cdot\}|\Lambda_{\eta}\right]$$

and linear subspace

$$M = \left\{\left[\frac{R}{\pi(V)} - \frac{1-R}{1-\pi(V)}\right] k(V)\right\} - \Pi\left[\{\cdot\}|\Lambda_{\eta}\right] = \Pi[\Omega(V)|\Lambda_{\eta}^{\perp}].$$

It is clear that $\Lambda_{\eta} \subset \Omega(V)$. By Theorem 10.1 of (Tsiatis, 2007), the optimal influence function (in terms of smallest variance) for fixed $g(Y,V)$ is given by

$$\mathbb{IF}^*(g) = \left\{\frac{R}{\pi(V)} g(Y,V) - \frac{1-R}{1-\pi(V)} E[g(Y,V)|V,L]\right\} - \Pi\left[\{\cdot\}|\Omega(V)\right].$$

Let

$$\left[\frac{R}{\pi(V)} - \frac{1-R}{1-\pi(V)}\right] k^0(V) \in \Omega(V)$$

be the projection $\Pi\left[\{\cdot\}|\Omega(V)\right]$. Then $k^0(V)$ needs to satisfy

$$E\left\{\left\{\frac{R}{\pi(V)}\left[g(Y,V)-k^0(V)\right]-\frac{1-R}{1-\pi(V)}\left[k^0(V)-E[g(Y,V)|V,L]\right]\right\}\times\right.$$
$$\left.\left\{\left[\frac{R}{\pi(V)}-\frac{1-R}{1-\pi(V)}\right]k(V)\right\}\right\}$$
$$=E\left\{k(V)\left\{\frac{1}{\pi(V)}\left[E[g(Y,V)|V]-k^0(V)\right]+\right.\right.$$
$$\left.\left.\frac{1}{1-\pi(V)}\left[k^0(V)-E[g(Y,V)|V]\right]\right\}\right\}=0\quad\forall k(V).$$

By assumption (A2), since $\delta<\pi(V)<1-\delta$ almost surely, $k^0(V)=E[g(Y,V)|V]$ and the DR linear space is given by

$$\mathcal{L}_{DR}=\left\{\mathbb{IF}^*(g):g(Y,V)\text{ arbitrary}\right\},$$

where

$$\mathbb{IF}^*(g)=\left\{\frac{R}{\pi(V)}\left[g(Y,V)-E[g(Y,V)|V]\right]\right.$$
$$\left.+\frac{1-R}{1-\pi(V)}\left[E[g(Y,V)|V]-E[g(Y,V)|V,L]\right]\right\}.$$

S2 Proofs of Results

In the following, expectations are evaluated at the true parameter values.

Proof of Result 1.

$$E_{\eta,\theta}\left\{U_g(\theta;\eta)\middle|V,L\right\}=E_{\eta,\theta}\left\{\frac{R}{\pi(V)}g(Y,V)-\frac{1-R}{1-\pi(V)}E_\theta[g(Y,V)|V,L]\middle|V,L\right\}$$
$$=E_\theta[g(Y,V)|V,L]-E_\theta[g(Y,V)|V,L]=0.$$

□

Proof of Result 2 (DR property). **Case 1: $\pi(V)$ is correct but $\tilde{t}(L|V)$ is incorrect**

Unbiasedness of DR estimating function follows from Result 1 by taking $g'(V,L) = g(V,L) + k(V)$; the proof does not involve $\tilde{t}(L|V)$.

Case 2: $\tilde{\pi}(V)$ is incorrect but $t(L|V)$ is correct

$$
\begin{aligned}
E_{\theta,\eta,\alpha}\left\{U_g^{DR}(\theta;\eta,\alpha)\middle|V\right\} &= E_{\theta,\eta,\alpha}\left\{\frac{R}{\tilde{\pi}(V)}\left\{g(Y,V) - E_{\theta,\alpha}[g(Y,V)|V]\right\}\right. \\
&\quad\left.+\frac{1-R}{1-\tilde{\pi}(V)}\left\{E_{\theta,\alpha}[g(Y,V)|V] - E_{\theta}[g(Y,V)|V,L]\right\}\middle|V\right\} \\
&=\frac{\pi(V)}{\tilde{\pi}(V)}\left\{E_{\theta,\alpha}[g(Y,V)|V] - E_{\theta,\alpha}[g(Y,V)|V]\right\} \\
&\quad+\frac{1-\pi(V)}{1-\tilde{\pi}(V)}\left\{E_{\theta,\alpha}[g(Y,V)|V] - E_{\theta,\alpha}[g(Y,V)|V]\right\} = 0.
\end{aligned}
$$

□

Proof of Result 3. The proof is based on the following lemma which is part of Theorem 5.3 in Newey and McFadden (1994).

Lemma S1.

If $\exists\tilde{h}(V)$ satisfying

$$
-E\left[h(V)\nabla_\theta M(\theta)\right] = E\left[M^2(\theta)h(V)\tilde{h}(V)^T\right] \quad \forall h(V),
$$

then the estimator indexed by $\tilde{h}(V)$ is most efficient.

Proof of Lemma S1. If $h(V)$ and $\tilde{h}(V)$ satisfy the equality in lemma S1 then the difference of the asymptotic variances of the respective estimators indexed by them is as follows:

$$\begin{aligned} & E\left[M^2(\theta)h(V)\tilde{h}(V)^T\right]^{-1} E\left[M^2(\theta)h(V)h(V)^T\right] E\left[M^2(\theta)\tilde{h}(V)h(V)^T\right]^{-1} \\ & - E\left[M^2(\theta)\tilde{h}(V)\tilde{h}(V)^T\right]^{-1} \\ =& E\left[M^2(\theta)h(V)\tilde{h}(V)^T\right]^{-1} E\left[UU^T\right] E\left[M^2(\theta)\tilde{h}(V)h(V)^T\right]^{-1}, \end{aligned}$$

where $U = h(V) - E\left[M^2(\theta)h(V)\tilde{h}(V)^T\right] E\left[M^2(\theta)\tilde{h}(V)\tilde{h}(V)^T\right]^{-1} \tilde{h}(V)$ and $E\left[UU^T\right]$ is positive semi-definite. $\square$

We show that if $\tilde{h}(V)$ satisfies the equality in lemma S1 then $\tilde{h}(V) = h^{opt}(V)$.

$$\begin{aligned} & - E\left[h(V)\nabla_\theta M(\theta)\right] = E\left[M^2(\theta)h(V)h^{opt}(V)^T\right] \quad \forall h(V), \\ \iff & E\left\{h(V)\left[M^2(\theta)h^{opt}(V) + \nabla_\theta M(\theta)\right]^T\right\} = 0 \quad \forall h(V), \\ \iff & E\left\{h(V)E\left[M^2(\theta)h^{opt}(V) + \nabla_\theta M(\theta)\middle|V\right]^T\right\} = 0 \quad \forall h(V), \\ \implies & E\left\{E\left[M^2(\theta)h^{opt}(V) + \nabla_\theta M(\theta)\middle|V\right]^{\otimes 2}\right\} = 0, \\ \implies & E\left[M^2(\theta)h^{opt}(V) + \nabla_\theta M(\theta)\middle|V\right] = 0, \\ \iff & h^{opt}(V) = -E\left[\nabla_\theta M(\theta)|V\right] E\left[M^2(\theta)|V\right]^{-1}. \end{aligned}$$

Due to Hájek's representation theorem (Hájek, 1970), the most efficient regular estimator is asymptotically linear and so the existence condition in lemma S1 holds when we consider only RAL estimators. □

S3 Additional Simulation Results

Figure 1: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_0, whose true value of 0.5 is marked by the horizontal line, when $\alpha_3 = 2$.

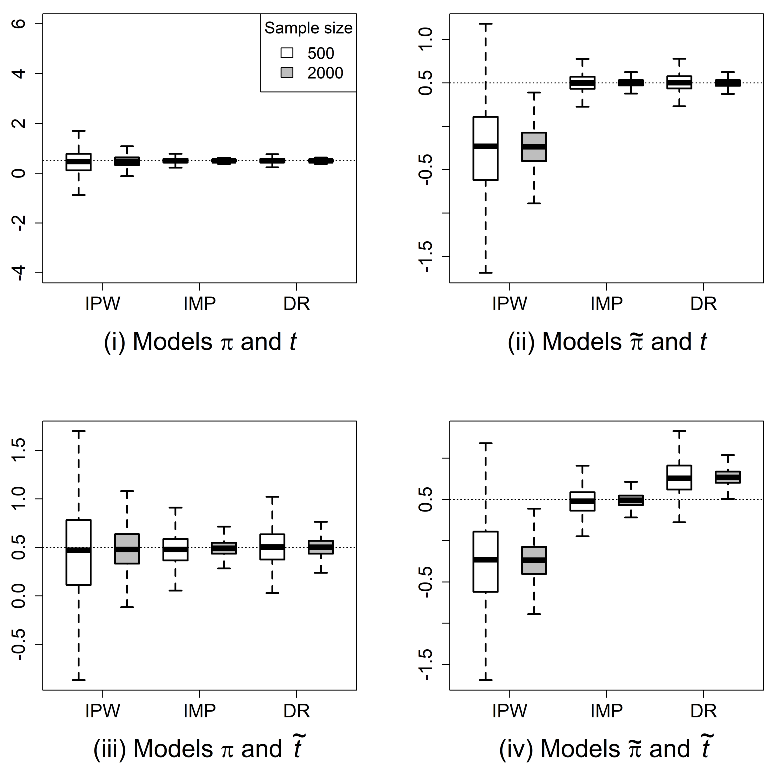


Figure 2: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_1, whose true value of -0.5 is marked by the horizontal line, when $\alpha_3 = 2$.

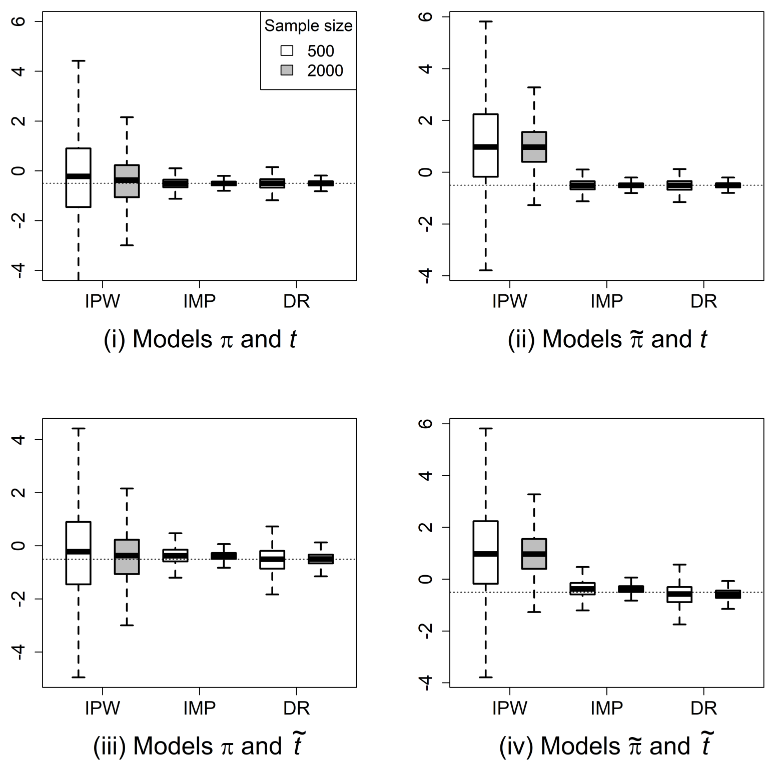


Figure 3: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_2, whose true value of 1.0 is marked by the horizontal line, when $\alpha_3 = 2$.

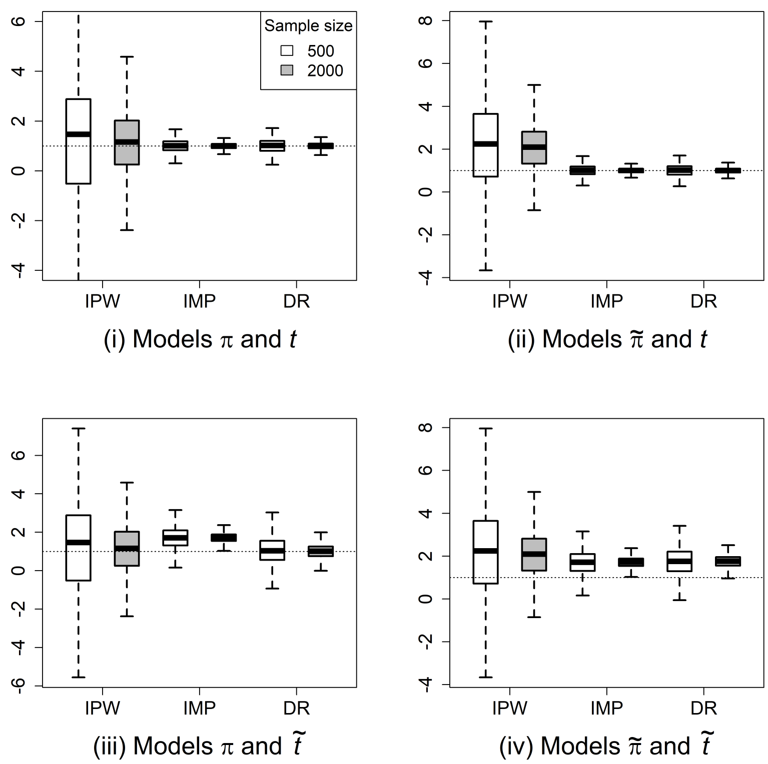


Figure 4: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_0, whose true value of 0.5 is marked by the horizontal line, when $\alpha_3 = 0.5$.

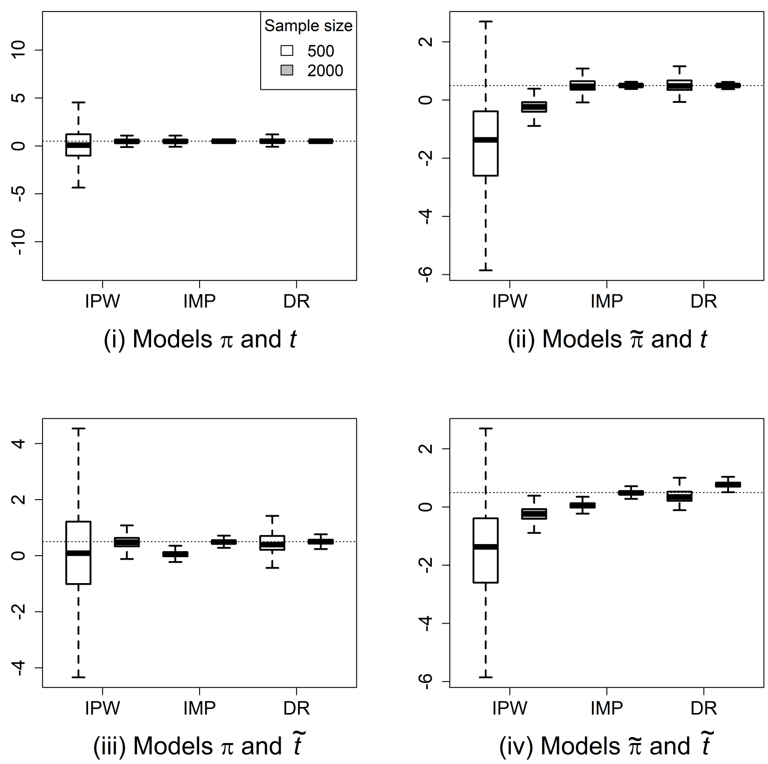


Figure 5: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_1, whose true value of -0.5 is marked by the horizontal line, when $\alpha_3 = 0.5$.

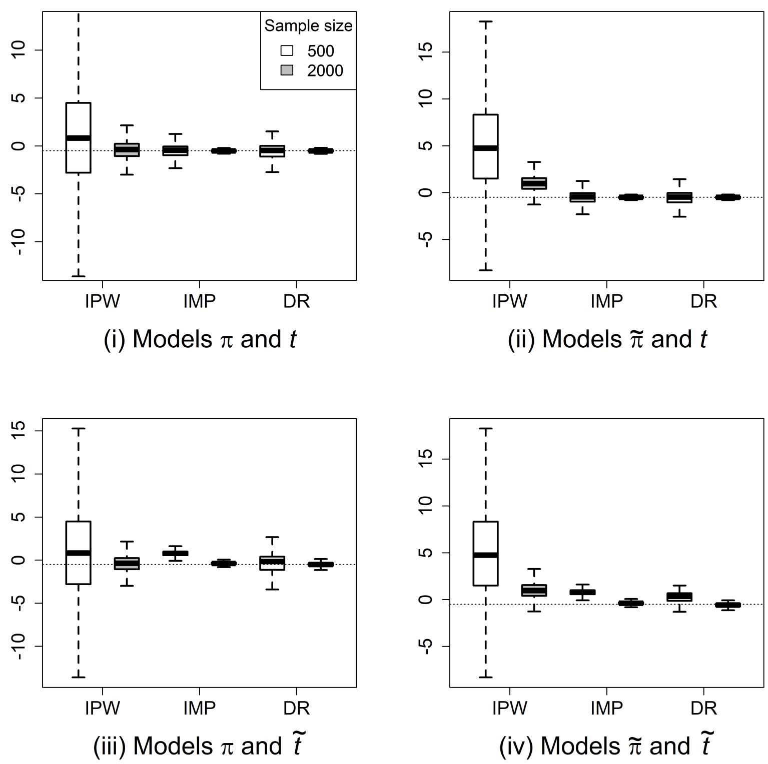
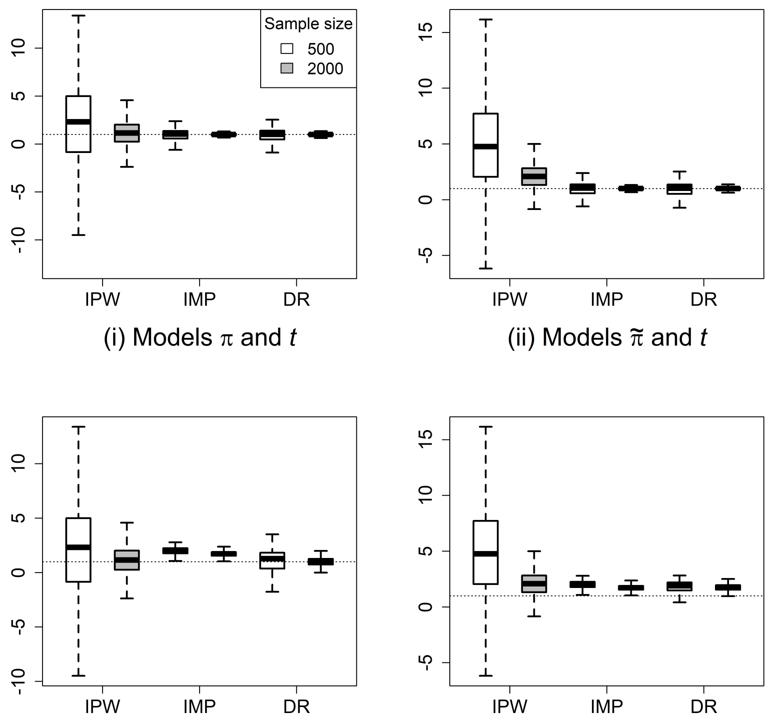


Figure 6: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_2, whose true value of 1.0 is marked by the horizontal line, when $\alpha_3 = 0.5$.



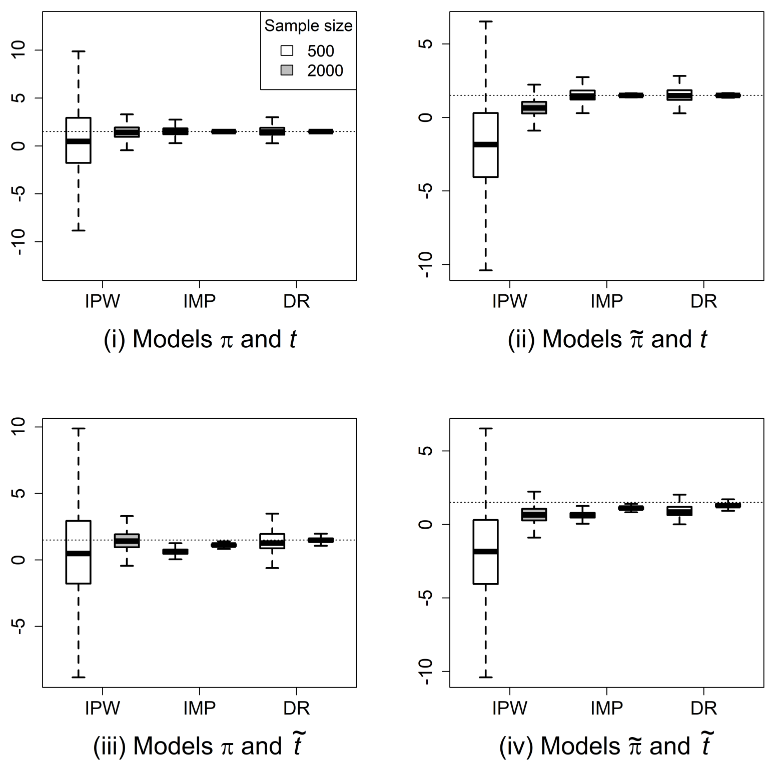
(i) Models π and t

(ii) Models $\tilde{\pi}$ and t

(iii) Models π and $\tilde{t}$

(iv) Models $\tilde{\pi}$ and $\tilde{t}$

Figure 7: Boxplots of inverse probability weighted (IPW), imputation-based (IMP) and doubly-robust (DR) estimators of the regression coefficient β_3, whose true value of 1.5 is marked by the horizontal line, when $\alpha_3 = 0.5$.



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